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Adaptive finite elements for exterior domain problems

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Summary. We present an adaptive finite element method for solving elliptic problems in exterior domains, that is for problems in the exterior of a bounded closed domain in $\mathbb{R}^d$, $d \in \{2,3\}$. We describe a procedure to generate a sequence of bounded computational domains Ω_h^k, $k = 1, 2, ...$, more precisely, a sequence of successively finer and larger grids, until the desired accuracy of the solution u_h is reached. To this end we prove an a posteriori error estimate for the error on the unbounded domain in the energy norm by means of a residual based error estimator. Furthermore we prove convergence of the adaptive algorithm. Numerical examples show the optimal order of convergence.

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0. Introduction

We consider the numerical treatment of elliptic partial differential equations in *exterior domains* of $\mathbb{R}^d$ for $d \in \{2,3\}$, that is, the (open) complement of a bounded (simply connected) domain ω.

Besides the differential equation, we also have to impose boundary conditions. Here we want to prescribe the function values on the finite boundary and a homogeneous condition at infinity which we will formally write as $\lim_{|x|\to\infty} u(x) = 0$. Note that the well-posedness of such a boundary condition at infinity is not a trivial task [MS] for general elliptic operators. In this work we will analyse Laplace's equation and the correct boundary con-

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dition is part of the variational formulation (for example, we have that some L^p-norm ($p \in [1, \infty)$) of the solution will exist).

One possibility to solve such a problem are *boundary element methods* [Ha]. Here one arrives at an integral equation on a function space on the finite boundary $\partial\omega$ and the solution can be computed pointwise by evaluating an integral over the surface. Numerically, the advantage of such an algorithm is that it works on a finite computational domain which is of lower dimension than the original problem. The drawback, however, is that one has to know the correct integral kernel, the arising matrices are dense, and the evaluation of the solution outside the boundary is costly.

Another approach is to combine a usual finite element discretisation with *infinite elements* (FEM/IEM) outside a given ball B containing $\overline{\omega}$. These infinite elements are given as a product of a shape function on ∂B times a radial function. h-p finite–infinite elements for the Helmholtz equation in $\mathbb{R}^3$ have been considered for example in [DI, GD].

The most natural idea, however, is to work on a finite approximation of the exterior domain, since one can use standard finite element techniques. This makes sense because due to some decay properties of the solution, the corresponding error will tend to zero if the computational domain increases. The main problem is to precisely estimate the error introduced by cutting the domain. This will be the major point of this article.

In [Ba] the equation $-\Delta u + u = f$ in $\mathbb{R}^d$ (subject to homogeneous Dirichlet conditions) is considered. It is shown that on a uniformly discretised (bounded) computational domain with (small enough) grid size h and diameter $R(h) = h^{-s}$ (for arbitrary $s > 0$) the error in the energy norm decreases at a rate almost like in the case of a bounded domain.

Since solutions of these boundary value problems have certain decay properties (which is reflected by a priori estimates in weighted Sobolev spaces), one would expect that a grid with radially increasing step size would allow to compute an approximate solution of a given accuracy with much less unknowns than in the uniform approach. This situation has been analysed in [SC] for a class of elliptic operators including $-\Delta$. It has been proved that on a correctly spaced grid the error outside a given ball decreases at the optimal rate with respect to the number of unknowns.

However, what is still lacking is an adaptive procedure that constructs a large enough and a fine enough discretised computational domain to obtain a discrete solution of prescribed accuracy (in some norm). Such a procedure should only use a posteriori error estimates and a priori information on data.

In this work we present an adaptive algorithm for Laplace's equation that meets these requirements. We first discretise the domain $B \backslash \overline{\omega}$ (B some ball containing $\overline{\omega}$). After computing a discrete solution subject to homogeneous boundary conditions at the outer boundary, we estimate the error on the whole exterior domain (in the energy norm) by means of an a posteriori error estimate (that is, in terms of computable quantities such as the discrete solution and data). Using this estimate as a local error indicator, we will enrich the discrete space both by refining the current grid and expanding the computational domain. By expanding we mean that we add elements (subject to a regular structure) at the outer boundary. From layer to layer the size of these new elements will increase by a certain factor κ, say $\kappa = 2$, so that their diameter are of comparable size to their distance from ω.

Our paper is organised as follows: in Sect. 1 we formulate the main assumptions on data and state Hardy's inequality which is the counterpart to Poincaré's inequality on bounded domains. In Sect. 2 we formulate the model problem (Laplace's equation), and prove existence, uniqueness, regularity as well as decay properties. The discretisation of the problem is described in Sect. 3. Here, we also prove a priori estimates and convergence. In Sect. 4 we derive a posteriori error estimates in the energy norm. A description of the discretisation and the adaptive procedure is subject of Sect. 5. Numerical examples in two and three dimensions are presented in Sect. 6.

1. Notations and preliminaries

For $\Omega \subseteq \mathbb{R}^d$, $m \in I\!N_0$, and $p \in [1, \infty]$ let $H^{m,p}(\Omega)$ ($H^{0,p}(\Omega) := L^p(\Omega)$) denote the Sobolev spaces [Ad] with the usual norms. The L^p–norm on Ω will be denoted by $||u||_{p;\Omega}$. Let $C_0^\infty(\Omega)$ be the space of infinitely differentiable functions with compact support in Ω.

For $G \subset \mathbb{R}^d$ let $d_G := \text{diam}(G)$ denote the *diameter* of G and $\text{dist}(x, G) := \inf\{|x - y| \,:\, y \in G\}$ the *distance* of x from G. $B_r(x)$ denotes the ball with radius r and center x.

Assumptions

We call $\Omega \subset \mathbb{R}^d$ an *exterior domain*, if there is a bounded domain $\omega \subset \mathbb{R}^d$ such that $\Omega = \mathbb{R}^d \backslash \overline{\omega}$. In the following, we will make the additional assumptions that ω is simply connected with C^∞–boundary and that (without loss of generality) $0 \in \omega$.
Concerning data functions f, q (used in the model problem below) we will make the following general assumptions. Fix some $k \geq 1$ (later this will be

the order of our finite element space) and assume

$$
\begin{aligned}
\text{Case I}: \quad & d \in \{2,3\},\ f \in H^{k+1,2}(\Omega) \text{ with compact support},\\
& q \geq q_0 > 0, \text{ and } q \text{ is of the form } q = q_* - \tilde{q}\\
& \text{for some constant } q_* \geq q_0 > 0\\
& \text{and } \tilde{q} \in H^{k+1,2}(\Omega) \text{ with compact support}\\
\text{Case II}: \quad & d = 3, \text{ for } \nu_f, \nu_q \geq 0 \text{ and } l \in \{0,\dots,k\}\\
& |x|^{l+1+\nu_f} \nabla^l f \in L^2(\Omega),\\
& q \geq 0,\ |x|^{l+1+\nu_q} \nabla^l q \in L^\infty(\Omega),\\
& \text{if } \nu_q = 0: \ \lim_{R\to\infty} \|\, |x|^3 \nabla q \|_{\infty;\Omega\setminus B_R(0)} \ = \ 0.
\end{aligned}
$$

Note that for $p \in [1,\infty]$ and $0 \leq \mu \leq \nu$: $|x|^\nu \phi \in L^p(\Omega)$ implies $|x|^\mu \phi \in L^p(\Omega)$.

Remark 1. i) If, in Case II, f and q are given by $f(x) = |x|^{\alpha_1}$, $q(x) = |x|^{\alpha_2}$, this implies $\alpha_1 < -2.5$ and $\alpha_2 < -2$ (for $\nu_f = \nu_q = 0$).
ii) Not all our assertions on f and q will need all the requirements stated in the assumptions above, but this will be apparent from the respective proofs.

The counterpart to *Poincaré's inequality* on bounded domains (that is, $\|v\|_{2;\Omega} \leq C_{\mathrm{P}} \|\nabla v\|_{2;\Omega}$ for all $v \in C_0^\infty(\Omega)$) will be the following inequality on unbounded domains.

Lemma 1. *(Hardy's inequality) Let $\Omega \subset \mathbb{R}^3$, $0 \notin \overline{\Omega}$, an exterior domain. Then for any $v \in C_0^\infty(\Omega)$ and $\mu \in (-\frac{1}{2}, \frac{1}{2})$*

$$
\int_\Omega |x|^{2\mu-2}\,|v|^2 \ \leq \ \frac{4}{1-4\mu^2} \int_\Omega |x|^{2\mu}\,|\nabla v|^2.
$$

Proof. The proof is a slight modification of the arguments in [BGH] that lead to the same inequality, but with $|x|$ replaced by $\sqrt{1+|x|^2}$. □

2. The model problem

As a model problem, we consider the boundary value problem

$$
\begin{aligned}
-\Delta u + qu \ &= \ f \qquad \text{in } \Omega,\\
u \ &= \ u_{\mathrm{D}} \quad \text{on } \partial\Omega,\\
\lim_{|x|\to\infty} u(x) \ &= \ 0,
\end{aligned}
$$

where $u_{\mathrm{D}} \in H^{k+2,2}(\Omega)$ has compact support. Introducing $u - u_{\mathrm{D}}$ as a new dependent variable, we end up with a problem with homogeneous boundary conditions and modified right hand side (again satisfying the previous assumptions).

We now want to formulate our problem in a variational setting to which end we introduce the following spaces and norms (depending on the space dimension considered)

$$S := \overline{C_0^\infty(\Omega)}^{||\,\cdot\,||_S} \text{ where}$$

$$\text{Case I} \;:\quad ||u||_S^2 := ||\nabla u||_2^2 + ||\sqrt{q}u||_2^2,$$

$$\text{Case II} :\quad ||u||_S^2 := ||\nabla u||_2^2.$$

Here and in the following we let $||\,.\,||_p \equiv ||\,.\,||_{p;\Omega}$.

The *weak formulation* of our problem reads as follows:

$$(\mathrm{P})\;\; \text{find } u \in S \text{ such that} \qquad \int_\Omega \nabla u \cdot \nabla \phi + q u \phi \;=\; \int_\Omega f \phi \qquad \forall \phi \in S.$$

Theorem 1. *(Existence and uniqueness) Let $\Omega \subset \mathbb{R}^d$ $(d \in \{2, 3\})$ be an exterior domain and f, q as in the assumptions in Sect.* 1. *Then problem* (P) *admits a unique solution $u \in S$ satisfying the bounds*

$$\text{Case I} \;:\; ||u||_S \;\leq\; ||\tfrac{f}{\sqrt{q}}||_2,$$

$$\text{Case II} :\; ||u||_S \;\leq\; 2\,||\,|x| f||_2.$$

Proof. Consider Case I. Since

$$|\int_\Omega q u \phi\,| \;\leq\; ||u||_S\,||\phi||_S, \quad |\int_\Omega f \phi\,| \;\leq\; ||\frac{f}{\sqrt{q}}||_2\,||\phi||_S,$$

all integrals in (P) are well defined and the right hand side is a linear functional on S. In Case II, we obtain using Hardy's inequality (cf. Lemma 1 for $\mu = 0$)

$$|\int_\Omega q u \phi\,| \;\leq\; ||\,|x|^2 q||_\infty\,||\tfrac{u}{|x|}||_2\,||\tfrac{\phi}{|x|}||_2 \;\leq\; 4\,||\,|x|^2 q||_\infty\,||u||_S\,||\phi||_S,$$

$$|\int_\Omega f \phi\,| \;\;\leq\; ||\,|x| f||_2\,||\tfrac{\phi}{|x|}||_2 \;\leq\; 2\,||\,|x| f||_2\,||\phi||_S.$$

This shows the same result as above and since in both cases

$$\int_\Omega \nabla u \cdot \nabla u + q u u \;\geq\; ||u||_S^2,$$

we can conclude the existence of a unique solution by the Lax–Milgram theorem [GT; Ch. 5]. The bound for u follows easily using the test function $\phi = u$. $\square$

Remark 2. i) Note that it is crucial to define S by the closure of $C_0^\infty(\Omega)$ (with respect to $||\,.\,||_S$) and *not* by $\{v \in H^{1,2}_{\mathrm{loc}}(\Omega) \;:\; ||v||_S < \infty\}$. The second choice would lead to non–uniqueness in Case II: for $\Omega = \mathbb{R}^3 \backslash B_1(0)$ both $u_1 := 0$ and $u_2 := 1 - 1/|x|$ solve the homogeneous boundary value problem. However, with the first definition we have $u_2 \notin S$ [SS; pp. 12].
ii) It is possible to treat also $d = 2$ within Case II. One has to work in weighted Sobolev spaces using an estimate cited in [SS; p. 95] (which replaces Hardy's inequality).
iii) Instead of using $||\,|x|f||_2$ in Case II, we may refer to the embedding $S \hookrightarrow L^6(\Omega)$ (that is $||v||_6 \le C_* ||v||_S$ for all $v \in S$ [BF; 4-7]) and obtain (using Hölder's inequality)

$$|\int_\Omega qu\phi| \;\le\; ||q||_{\frac{3}{2}}\, ||u||_6\, ||\phi||_6 \;\le\; C_*^2\, ||q||_{\frac{3}{2}}\, ||u||_S\, ||\phi||_S,$$
$$|\int_\Omega f\phi| \;\;\le\; ||f||_{\frac{6}{5}}\, ||\phi||_6 \;\le\; C_*\, ||f||_{\frac{6}{5}}\, ||\phi||_S.$$

Obviously, in this situation we have to impose the conditions: $q \in L^{\frac{3}{2}}(\Omega)$ and $f \in L^{\frac{6}{5}}(\Omega)$. Generalisations of these inequalities to weighted Sobolev spaces can be found in [BGH].

Theorem 2. *(Regularity and decay properties) Let data f, q, Ω, and u as in Theorem 1. Then $u \in H^{k+1,2}_{\mathrm{loc}}(\Omega)$ and for $l \in \{0, \dots, k+1\}$*

$$\textit{Case I}\;:\quad |\nabla^l u(x)| \;=\; O(\mathrm{e}^{-q_0|x|}) \quad \textit{for } |x| \to \infty,$$

$$\textit{Case II}:\quad ||\,|x|^{l-1+\nu}\nabla^l u||_{2;\Omega} \quad \textit{is bounded for } \nu \in [0, \min\{\nu_f, \tfrac{1}{2}\}).$$

Proof. To prove the asymptotic behaviour, we start with Case I. Inserting $q = q_* - \tilde{q}$ we see that u satisfies an equation with constant coefficients and compactly supported right hand side, namely,

$$-\Delta u + q_* u \;=\; \tilde{q} u + f \;\text{ in } \Omega, \quad u \;=\; 0 \;\text{ on } \partial\Omega.$$

From [DL; p. 636] we obtain that $u = O(\mathrm{e}^{-q_0|x|})$ for $|x| \to \infty$.

Differentiating the differential equation we obtain that $w := \nabla u$ solves the boundary value problem

$$-\Delta w + q_* w \;=\; \tilde{q} w + \nabla\tilde{q}\, u + \nabla f \;\text{ in } \Omega, \quad w \;=\; \nabla u \;\text{ on } \partial\Omega.$$

Since ∇u is bounded on $\partial\Omega$ (note that $\nabla u \in H^{k+1}_{\mathrm{loc}}(\Omega) \subset H^2_{\mathrm{loc}}(\Omega) \subset L^\infty_{\mathrm{loc}}(\Omega)$ (for $d \in \{2, 3\}$ and $k \ge 1$) by elliptic estimates) and the right hand side is compactly supported, we again conclude that w, hence $|\nabla u|$, is exponentially decaying. Further differentiating gives the assertion for $|\nabla^l u|$, $l = 2, \dots, k+1$.

Now we consider Case II. It is shown in [BGH; p. 1023] that the variational problem (P) admits a unique solution for which $|| \, |x|^\mu \nabla u||_2$ is bounded by $|| \, |x|^{1+\mu} f||_2$ for $\mu \in [0, \frac{1}{2})$. This shows the assertion for $l = 1$. The case $l = 0$ then follows immediately from Lemma 1 and the cases $l = 2, \dots, k+1$ will follow from an iterative argument as before.

To prove for $l = 2$, we refer to a result of [MO1, MO2], where it has been proved that Δ is an isomorphism

$$\Delta \, : \, M^2_{2,-1+\mu}(\Omega) \; \longrightarrow \; M^2_{0,1+\mu}(\Omega)$$

for $\mu \in (-\frac{1}{2}, \frac{1}{2})$, where $M^2_{s,\delta}(\Omega)$ is defined as the completion of $\{v \in C^\infty(\Omega) \, : \, v|_{\partial\Omega} = 0, \, \mathrm{supp}(v) \subset\subset \mathbb{R}^3\}$ under the norm

$$||v||_{M^2_{s,\delta}(\Omega)} \; := \; \sum_{m \le s} || \, |x|^{m+\delta} \nabla^m v||_2$$

(we could derive this result also from the case $l = 1$, but we prefer to show one iteration for the strong form of the differential equation). Fix $\rho > d_\omega$. Let $\phi \in C^\infty(\Omega)$ with $0 \le \phi \le 1$, $\phi(x) = 0$ for $x \in B_\rho(0)$ and $\phi(x) = 1$ for $x \in \Omega \backslash B_{2\rho}(0)$. Since $w := \phi u$ solves the differential equation:

$$-\Delta w \; = \; \phi(f - qu) - u\Delta\phi - 2\nabla u \cdot \nabla \phi \; =: \; \tilde{f} \;\; \text{in } \Omega, \quad w \; = \; 0 \;\; \text{on } \partial\Omega,$$

we have

$$|| \, |x|^{1+\nu} \nabla^2 w||_2 \; \le \; C_0 \, || \, |x|^{1+\nu} \tilde{f}||_2.$$

$|| \, |x|^{1+\nu} \tilde{f}||_2$ can be shown to be bounded (depending on data ν and ρ; observe that $|| \, |x|^{1+\nu} qu||_2 \; \le \; || \, |x|^2 q||_\infty \, || \, |x|^{-1+\nu} u||_2$ and that $\nabla\phi$, $\Delta\phi$ have compact support). Now take $W := |x| \nabla w$. Again

$$|| \, |x|^{1+\nu} \nabla^2 W||_2 \; \le \; C_0 \, || \, |x|^{1+\nu} \Delta W||_2.$$

Observing that

$$\begin{aligned} || \, |x|^{2+\nu} \nabla^3 u||_{2;\Omega \backslash \Omega_{2\rho}} \; &\le \; || \, |x|^{2+\nu} \nabla^3 w||_2 \\ &\le \; || \, |x|^{1+\nu} \nabla^2 W||_2 \\ &\quad\; + 2 \, || \, |x|^{1+\nu} \nabla^2 w||_2 + || \, |x|^\nu \nabla w||_2, \end{aligned}$$

we obtain the assertion by computing

$$|| \, |x|^{1+\nu} \Delta W||_2 \; \le \; 2 \, || \, |x|^{1+\nu} \nabla^2 w||_2 + || \, |x|^\nu \nabla w||_2 + || \, |x|^2 \nabla \tilde{f}||_2$$

and using the assumptions on data and previous results. $\square$

3. Discretisation

The idea of our approach is to construct a *bounded computational domain* Ω_h such that the corresponding finite element solution u_h is close to the continuous solution u up to a given tolerance. This is expected to be possible since u decays towards infinity at some rate (cf. Theorem 2). Thus we will need Ω_h both discretised fine enough and large enough.

To simplify our presentation (that will focus on the disretisation of the unbounded domain), we assume that we use *exact finite elements* of Lagrangian type at the finite part of the boundary of $\partial\Omega$. These are *curved finite elements* that exactly represent the boundary. For a detailed analysis see [Be]. For error estimates including the approximation of the boundary see [BK] (a priori) and [DR] (a posteriori).

Let $\Omega_h \subset \Omega$ be composed of generalised simplices (curved or straight) in the usual way [Be, Ci (Ch. 2)]. In fact, every simplex having fewer than two vertices on $\partial\omega$ will be straight. By $\mathcal{T}_h$ we denote this set of simplices. For each $T \in \mathcal{T}_h$ we fix a mapping F_T (of class C^∞) that maps the unit simplex $\hat{T}$ onto T (cf. [Be]). A finite element space of order k on Ω_h is given by

$$\begin{aligned} S_h^k &:= \{v_h \in C^0(\Omega) \,:\, \forall T \in \mathcal{T}_h \;\; v_h \circ F_T^{-1}|_T \in I\!P_k, \\ v_h &= 0 \text{ on } \partial\Omega_h \text{ and in } \Omega\backslash\Omega_h\} \subset S. \end{aligned}$$

We assume that data q, f are approximated by q_h, f_h respectively, where q_h, f_h are set to zero outside Ω_h. Then the *discrete weak formulation* reads as follows

$$(\mathrm{P_h}) \qquad \text{find } u_h \in S_h^k \text{ such that} \qquad \begin{aligned} &\int_\Omega \nabla u_h \cdot \nabla \phi_h + q_h u_h \phi_h \\ &\quad = \int_\Omega f_h \phi_h \quad \forall \phi_h \in S_h^k. \end{aligned}$$

Since the left hand side of $(\mathrm{P_h})$ defines a strictly positive definite bilinear form, a discrete solution u_h of this problem exists.

Lemma 2. *(A priori error estimate) Let u, u_h be the solutions of problems (P), $(\mathrm{P_h})$, respectively. Then*

$$\textit{Case I} \;:\quad \begin{aligned} ||u-u_h||_S \le \inf_{\phi_h \in S_h^k} ||u-\phi_h||_S + ||\tfrac{f-f_h}{\sqrt{q}}||_{2;\Omega_h} \\ +||\tfrac{q-q_h}{q}||_{\infty;\Omega_h}\, ||\tfrac{f_h}{\sqrt{q}}||_{2;\Omega_h}, \end{aligned}$$

$$\textit{Case II}: \quad ||u-u_h||_S \le \Big(1+4\,||\,|x|^2 q||_\infty\Big) \inf_{\phi_h\in S_h^k} ||u-\phi_h||_S$$
$$+2\Big(||\,|x|(f-f_h)||_{2;\Omega_h}$$
$$+4\,||\,|x|^2(q-q_h)||_{\infty;\Omega_h}\,||\,|x|f_h||_{2;\Omega_h}\Big).$$

Proof. We define the bilinear form $a : S\times S\to\mathbb{R}$ by

$$a(u,v) := \int_\Omega \nabla u\cdot\nabla v + quv.$$

a is continuous and coercive, that is, there is a number $\Lambda > 0$ such that for all $u, v\in S$

$$a(u,v) \le \Lambda\,||u||_S\,||v||_S, \quad a(u,u) \ge ||u||_S^2.$$

This is easily proved using estimates given in the proof of Theorem 1 and we obtain $\Lambda = 1$ in Case I and $\Lambda = 1+4\,||\,|x|^2 q||_\infty$ in Case II. Applying the first Strang Theorem [Ci; Theorem 4.1.1] we obtain

$$||u-u_h||_S \le \Lambda \inf_{\phi_h\in S_h^k} ||u-\phi_h||_S$$
$$+ \sup_{v_h\in S_h^k\backslash\{0\}} \frac{1}{||v_h||_S}\,|\int_\Omega (f-f_h)v_h + (q_h-q)u_h v_h|.$$

The second term on the right can be estimated by (as in the proof of Theorem 1)

$$\text{Case I}: \quad ||\tfrac{f-f_h}{\sqrt{q}}||_{2;\Omega_h} + ||\tfrac{q-q_h}{q}||_{\infty;\Omega_h}\,||u_h||_S,$$
$$\text{Case II}: \quad 2\,||\,|x|(f-f_h)||_{2;\Omega_h} + 4\,||\,|x|^2(q-q_h)||_{\infty;\Omega_h}\,||u_h||_S.$$

The assertion follows, since $||u_h||_S$ can be estimated as in Theorem 1. □

For the following we need quantities measuring geometrical properties of a given discretisation. The first is given by

$$\sigma_{\text{int}} := \max\left\{\frac{d_T}{\rho_T} : T\in\mathcal{T}_h,\ B_{\rho_T} \text{ the largest ball inscribed } T\right\} \tag{3.1}$$

and controls the shape regularity of the simplices of $\mathcal{T}_h$. The second condition concerns the size of the simplices at the exterior boundary $\partial_{\text{ext}}\Omega_h := \partial\Omega_h\backslash\partial\omega$. Let $\mathcal{T}_h^\partial$ be defined by

$$\mathcal{T}_h^\partial := \{T\in\mathcal{T}_h : \overline{T}\cap\partial_{\text{ext}}\Omega_h \ne \emptyset\},$$

and define

$$\sigma_{\text{ext}} := \max\left\{\frac{\text{dist}(0,T)}{d_T} : T\in\mathcal{T}_h^\partial\right\}. \tag{3.2}$$

Later, we will make the assumption that $\sigma_{\rm int},\ \sigma_{\rm ext}$ will be uniformly bounded on sequences of refined and enlarged discretisations. (For $\sigma_{\rm int}$ this requirement is known as *uniform shape condition*).

We will further use the notation

$$\Omega_h^\partial := \bigcup_{T\in\mathcal{T}_h^\partial} \overline{T}.$$

The *grid size function* $h \in L^\infty(\Omega)$ is defined almost everywhere by

$$h|_T := h_T := d_T \quad \forall T \in \mathcal{T}_h.$$

Let $\mathcal{N}_h$ be a set of points in Ω_h with the following property: for each $p \in \mathcal{N}_h$ there is a function $\phi_p \in S_h^k$ such that $\{\phi_p\}_{p\in\mathcal{N}_h}$ forms a basis of S_h^k (*Lagrange elements*). In addition, the functions ϕ_p should have the properties

$$\text{supp}(\phi_p) \subset M_p := \bigcup_{T\in\mathcal{T}_h\,:\,p\in\overline{T}} \overline{T}, \qquad \phi_p(q) = \delta_{pq} \quad \forall p,q \in \mathcal{N}_h.$$

The next Lemma refers to the projection operator $P_h^0 : S \to S_h^k$ defined in [Cl] by

$$P_h^0 v := \sum_{p\in\mathcal{N}_h} c_p(v)\phi_p \tag{3.3}$$

where the c_p are continuous functionals on $L^2(M_p)$.

Lemma 3. *(Interpolation estimates) The projection operator P_h^0 defined in* (3.3) *fulfils*

$$\textit{Case I} \;:\; ||v - P_h^0 v||_S \le C\Bigg(||h^k\nabla^{k+1}v||_{2;\Omega_h} + \sqrt{q_1}\,||h^{k+1}\nabla^{k+1}v||_{2;\Omega_h} + \Big(1 + \frac{1}{\sqrt{q_0}R_h} + \sqrt{\frac{q_1}{q_0}}\Big)\,||v||_{S;\Omega_h^\partial\cup(\Omega\setminus\Omega_h)}\Bigg),$$

with $q_1 := ||q||_\infty$ and $R_h := \text{dist}(0,\Omega_h^\partial)$,

$$\textit{Case II} :\; ||v - P_h^0 v||_S \le C\left(||h^k\nabla^{k+1}v||_{2;\Omega_h} + ||\nabla v||_{2;\Omega\setminus\Omega_h} + ||\frac{v}{|x|}||_{2;\Omega_h^\partial}\right)$$

for all $v \in S \cap H^{k+1,2}_{\rm loc}(\Omega)$. The constants depend on $\partial\omega$, $\sigma_{\rm int}$, $\sigma_{\rm ext}$ (Case II*), and k only.*

Proof. Let $\mathcal{N}_h^\partial$ be the set of nodes located on $\partial_{\rm ext}\Omega_h$. We can write

$$||v - P_h^0 v||_S \;\le\; ||v - P_h v||_{S;\Omega_h} + ||P_h v - P_h^0||_{S;\Omega_h^\partial} + ||v||_{S;\Omega\setminus\Omega_h},$$

where P_h is the projection operator from [Cl; Theorem 1]. It remains to consider the last two terms of this inequality.
Exploiting the fact that the coefficients c_p obey the estimate (cf. [Dö2])

$$|c_p| \;\le\; C\,\frac{||v||_{2;M_p}}{||\phi_p||_2},$$

we obtain

$$||(P_h - P_h^0)v||_2^2 \;\le\; C \sum_{p\in\mathcal{N}_h^\partial} ||v||^2_{2;M_p} \;\le\; C\,\tfrac{1}{q_0}\,||v||^2_{S;\Omega_h^\partial}, \quad \text{(Case I)},$$

and similarly, using $|x| \le 2\sigma_{\rm ext} d_T$ for $T \in \mathcal{T}_h^\partial$,

$$\begin{aligned} ||\nabla(P_h - P_h^0)v||_2^2 \;&\le\; C \sum_{p\in\mathcal{N}_h^\partial} \tfrac{1}{d^2_{M_p}} ||v||^2_{2;M_p} \\ &\le\; C\,\tfrac{1}{q_0 R_h^2}\,||v||^2_{S;\Omega_h^\partial}, \quad \text{(Case I)}, \\ &\le\; C\,||\tfrac{v}{|x|}||^2_{2;\Omega_h^\partial}, \qquad \text{(Case II)}. \end{aligned}$$

Therefore, in Case II we obtain (with $m = k+1$, $l = 0$)

$$\begin{aligned} ||v - P_h^0 v||_{S;\Omega_h} + ||v||_{S;\Omega\setminus\Omega_h} \;\le\; & C\,||h^k \nabla^{k+1} v||_{2;\Omega_h} \\ & + C\,||\frac{v}{|x|}||_{2;\Omega_h^\partial} + ||v||_{S;\Omega\setminus\Omega_h} \end{aligned}$$

and in Case I the result follows analogously. □

Theorem 3. *(Convergence) Let u be the solution of* (P) *under the assumptions of Theorem* 2*. Choose a finite computational domain Ω_h as described before such that* (*without loss of generality*) $B_R(0) \subset \Omega_h\setminus\Omega_h^\partial$ *for some* $R > d_\omega$*. We define* $\Omega_{h,R} := \cup\{T \in \mathcal{T}_h \,:\, T \cap B_R(0) \ne \emptyset\}$*. If f_h and q_h are k-th order approximations of f and q, respectively, on Ω_h $(k \ge 1)$, the difference between u and the discrete solution $u_h \in S_h^k$ is estimated by*

Case I : $\quad ||u - u_h||_S \;\le\; C\,h^k + O(\mathrm{e}^{-\frac{1}{2}q_0 R})$

$\qquad$ for $R \ge R_0,\ \mathrm{supp}(f) \cup \mathrm{supp}(\tilde q) \subset B_{R_0}(0)$,

Case II : $\quad ||u - u_h||_S \;\le\; C \max_{T\,:\,T\subset\Omega_{h,R}} \left\{ \left(\frac{h_T}{|x_T|}\right)^k \frac{1}{|x_T|^\nu} \right\} \;+\; o(R^{-\nu}),$

where x_T denotes the barycentre of a simplex T and with $\nu \in [0, \min\{\nu_f, \nu_q, \frac{1}{2}\})$. The constants in this estimate depend on constants from interpolation estimates and bounded norms of the solution and data, but not on R or h. They are not uniformly bounded with respect to $\nu \to \frac{1}{2}$.

Proof. We consider Case II only, Case I is similar. To derive the result, we have to consider the error terms appearing in Lemma 2 and Lemma 3.
Let $R > d_\omega$ be given. By assumption on data and Theorem 2 we obtain for ν as defined above

$$\max\Big\{||\,|x|^{l-1}\nabla^l u||_{2;\Omega\setminus B_R(0)} \,:\, l \in \{0,1,k+1\},\, ||\,|x|^2\nabla f||_{2;\Omega\setminus B_R(0)}, \\ ||\,|x|^3\nabla q||_{\infty;\Omega\setminus B_R(0)}\Big\} \,=\, o(R^{-\nu}).$$

For the given R we choose Ω_h that satisfies the requirements stated above. Now we will use that, for example,

$$||\,|x|(f-f_h)||_{2;\Omega_h} \,\le\, c\Big(||\,|x|h^k\nabla^k f||_{2;\Omega_{h,R}} + ||\,|x|^2\nabla f||_{2;\Omega\setminus B_R(0)}\Big).$$

A similar estimate can be obtained for the terms with $q - q_h$ and we get

$$\begin{aligned} ||u-u_h||_S \,\le\, & c\,(1+4\,||\,|x|^2 q||_\infty)\,||h^k\nabla^{k+1}u||_{2;\Omega_{h,R}} \\ & +2c\,||\,|x|h^k\nabla^k f||_{2;\Omega_{h,R}} \\ & +8c\,||\,|x|f_h||_{2;\Omega_h}\,||\,|x|^2h^k\nabla^k q||_{\infty;\Omega_{h,R}} \\ & +o(R^{-\nu}). \end{aligned}$$

For the first term on the left hand side we further compute

$$\begin{aligned} &||h^k\nabla^{k+1}u||^2_{2;\Omega_{h,R}} \,=\, \textstyle\sum_{T\,:\,T\subset\Omega_{h,R}} ||h^k\nabla^{k+1}u||^2_{2;T} \\ &\le\, C\, \textstyle\sum_{T\,:\,T\subset\Omega_{h,R}} \left(\frac{h_T}{|x_T|}\right)^{2k} \frac{1}{|x_T|^{2\nu}}\,||\,|x|^{k+\nu}\nabla^{k+1}u||^2_{2;T} \\ &\le\, C \max_{T\,:\,T\subset\Omega_{h,R}} \left\{\left(\frac{h_T}{|x_T|}\right)^{2k}\frac{1}{|x_T|^{2\nu}}\right\}||\,|x|^{k+\nu}\nabla^{k+1}u||^2_{2;\Omega_{h,R}}. \end{aligned}$$

This can be done similarly also for the two data error terms and this proves the assertion. Choosing first R large enough and then h small enough on $\Omega_{h,R}$ the error will become arbitrarily small. □

Remark 3. The purpose of the following considerations is to show that the number of unknowns required for a numerical solution with a prescribed accuracy $\epsilon > 0$ on an exterior domain in $\mathbb{R}^3$ is comparable to the case of a bounded domain. To this end we will assume that we are in the worst case for an adaptive procedure (of fixed polynomial degree k) that is, the a priori error terms are equally distributed.

i) *The case of a bounded domain* $\Omega \subset \mathbb{R}^3$. In this case we can compute the step size that is necessary to achieve the accuracy ϵ by

$$h^k \, ||\nabla^{k+1} u||_2 \, \leq \, \epsilon$$

and the number of necessary degrees of freedoms N is then of the order

$$N \, \sim \, \epsilon^{-3/k}.$$

ii) *The case of an exterior domain* $\Omega \subset \mathbb{R}^3$. Here, we will refer to the estimate given in Theorem 3.ii. Let $\omega = B_1(0)$, $R_j := \kappa^j$ for some $\kappa > 1$, and $S_j := B_{R_{j+1}}(0) \backslash B_{R_j}(0)$ ($j = 0, 1, \ldots$). To get an error bound ϵ we will need

$$R \, \sim \, \epsilon^{-1/\nu} \quad \text{and} \quad \Big(\frac{h_T}{|x_T|}\Big)^k \frac{1}{|x_T|^\nu} \, \sim \, \epsilon.$$

Let $T \in S_j$. Then $|x_T| \sim \kappa^j$ and we choose (due to the second relation) $h_T \sim h_j$ with $h_j/\kappa_j \sim (\epsilon \kappa^{j\nu})^{1/k}$. On each layer S_j we thus have a number of unknowns proportional to $(\kappa^j/h_j)^3$ and the number of layers is $l \sim \log(R)$. Outside $B_R(0)$ we need additional layers to ensure a bounded σ_{ext}. For this we assume that outside $B_R(0)$ the computational domain Ω_h is covered by rings of increasing width but with geometrical decreasing number of unknowns (so that σ_{int} stays bounded). Hence the total amount of unknowns outside $B_R(0)$ will be bounded by a fixed number times the number of unknowns in the last layer in $B_R(0)$. Therefore we obtain

$$N \, \sim \, \sum_{j=1}^{l} \Big(\frac{\kappa^j}{h_j}\Big)^3 \, = \, \epsilon^{-3/k} \, \sum_{j=1}^{l} (\kappa^{-3\nu/k})^j.$$

For fixed k and $\nu > 0$ the last sum is bounded independently of ϵ and thus N depends in the same way on ϵ as in the case of a bounded domain. In case $\nu = 0$, the error is only $o(1)$ for $R \to \infty$. In this case one needs $N \sim \epsilon^{-3/k} \, \log(R(\epsilon))$ to achieve an accuracy of ϵ.

Remark 4. i) One can prove Theorem 3 also for the case $\Omega = \mathbb{R}^3$. We only have to replace the weight $|x|$ by $\sqrt{1 + |x|^2}$.
ii) For $\nu_q = 0$ Theorem 3 can be proved with given $\nu \in [0, \min\{\nu_f, \frac{1}{2}\})$. For this we apply the stability estimate $||\,|x|^\mu u_h||_2 \leq C||\,|x|^\mu f_h||_2$ for $\mu \in [0, \frac{1}{2})$ (this holds in the continuous case [BGH]; the discrete version is technical and requires a decomposition of Ω_h in annular domains as in Remark 3) and replace the last term in Lemma 2 by $||\,|x|^{2-\nu} q_h||_{\infty;\Omega_h} \, ||\,|x|^\nu f_h||_{2;\Omega_h}$.

4. A posteriori error estimates for exterior domains

Definition 1.
i) For $v_h \in S_h$ we define the *continuous residual* with respect to problem $(\mathrm{P_h})$ by

$$r_c|_T := (-\Delta v_h + q_h v_h - f_h)|_T \quad \forall T \in \mathcal{T}_h.$$

ii) Let $\mathcal{E}_h$ be the set of all faces of simplices in the discretisation $\mathcal{T}_h$ excluded those on $\partial\omega$. For $x \in E \in \mathcal{E}_h$ we define

$$r_s(x) = [\partial_n v_h]_E(x) := \lim_{s \to 0+} n \cdot (\nabla u_h(x+sn) - \nabla u_h(x-sn)),$$

where n is a given normal vector on E (note that the definition above does not depend on the orientation of n). r_s is called *singular residual*.
iii) The *local (residual) error indicator* η_T is defined for each $T \in \mathcal{T}_h$ by

$$\eta_T^2 := h_T^2 ||r_c||_{2;T}^2 + h_T ||r_s||_{2;\partial T \setminus \partial\omega}^2. \tag{4.1}$$

Note that this estimator differs from the usual definition by additional jump terms on the exterior boundary. For any subset Σ_h of $\mathcal{T}_h$ we let

$$\eta_{\Sigma_h}^2 := \sum_{T \in \Sigma_h} \eta_T^2.$$

The following theorem shows that the error can be estimated by the computable quantity $\eta_{\mathcal{T}_H}$ and data errors. The corresponding result for bounded domains has been proved by [Ve].

Theorem 4. *(A posteriori error estimates) Let u, u_h be the solutions of* (P), $(\mathrm{P_h})$, *respectively and let $e_h := u - u_h$.*
Case I: Assume that R_h (defined in Lemma 3) is such that $q_0 R_h^2 \geq 1$. Then the error is estimated by

$$||e_h||_S \leq C\,\eta_{\mathcal{T}_h} + ||\frac{f - f_h}{\sqrt{q}}||_2 + ||\frac{q - q_h}{\sqrt{q}} u_h||_2.$$

Case II:

$$||e_h||_S \leq C\,\eta_{\mathcal{T}_h} + 2\left(||\,|x|(f - f_h)||_2 + ||\,|x|(q - q_h)u_h||_\infty\right).$$

The constants C depend on σ_{int}, σ_{ext} (Case II), k, and $\partial\omega$.

Proof. Let $v \in S$ be arbitrary and $v_h := P_h^0 v$ with P_h^0 defined in Lemma 3. Then

$$\begin{aligned}\int_\Omega \nabla e_h \cdot \nabla v + q e_h v \;&=\; \sum_{E\in\mathcal{E}_h} \int_E [\partial_n u_h]_E \,(v - v_h) \\ &\quad - \int_\Omega (-\Delta u_h + q_h u_h - f_h)\,(v - v_h) \\ &\quad + \int_\Omega (f - f_h) v + \int_\Omega (q_h - q) u_h v \\ &=: \langle r_s, v - v_h\rangle - \int_{\Omega_h} r_c\,(v - v_h) \\ &\quad + \int_\Omega (f - f_h) v + \int_\Omega (q_h - q) u_h v.\end{aligned}$$

The last two terms can be estimated as in Lemma 2. From the proof of Lemma 3 we obtain (cf. also [Dö2; Lemma 2])

$$|\int_{\Omega_h} r_c\,(v - v_h)| \;\le\; C\Big(||h r_c||_{2;\Omega_h}\,||\nabla v||_2 + ||r_c||_{2;\Omega_h^\partial}\,||v||_{2;\Omega_h^\partial}\Big).$$

Note that we have $|x| \le 2\sigma_{\mathrm{ext}} h$ on Ω_h^∂ (and $1/\sqrt{q_0} R_h \le 1$ in Case I). Therefore

$$\begin{aligned}|\int_{\Omega_h} r_c\,(v - v_h)| &\le^{\text{Case I}} C\Big(||h r_c||_{2;\Omega_h} + \tfrac{1}{\sqrt{q_0}}\,||r_c||_{2;\Omega_h^\partial}\Big)\,||v||_S \\ &\le\; C\,||h r_c||_{2;\Omega_h}\,||v||_S, \\ &\le^{\text{Case II}} C\Big(||h r_c||_{2;\Omega_h} + ||\,|x| r_c||_{2;\Omega_h^\partial}\Big)\,||v||_S \\ &\le\; C\,||h r_c||_{2;\Omega_h}\,||v||_S.\end{aligned}$$

In [Dö2; Lemma 2] it has been shown that

$$\begin{aligned}|\langle r_s, v - v_h\rangle| \;\le\; C\Big(&\sum_{T\in\mathcal{T}_h} h_T\,||[\partial_n u_h]||^2_{2;\partial T}\,||\nabla v||^2_{2;\Omega_h} \\ &+ \sum_{T\in\mathcal{T}_h^\partial} h_T^{-1}||[\partial_n u_h]||^2_{2;\partial T}\,||v||^2_{2;\Omega_h^\partial}\Big)^{\frac{1}{2}}.\end{aligned}$$

For the last term we proceed as before. □

Remark 5. i) Theorem 4 allows to estimate the error on the *unbounded domain* based on the actual discrete solution u_h (here assumed to be known exactly) and data. The constant C depends on the quality of our grid only and we can assign a reasonable value (of order one) by numerical experience. η_T can be computed exactly for each $T \in \mathcal{T}_h$. Note that in contrast to a

problem on a bounded domain it contains also jumps of the derivative on the exterior boundary. The computation of the remaining terms requires some a priori information on data since we have to integrate over large elements (with $h_T \sim \text{dist}(0,T)$) or to estimate its influence from $\Omega\backslash\Omega_h$. These terms should not be neglected as "higher order terms", as often done in a posteriori error estimation.
ii) We approximated the exterior domain problem by a problem on a finite domain with homogeneous boundary conditions at the exterior boundary. In the theory it turns out that the decay of the error with the diameter R of the computational domain will, in Case II, in general be not better than $O(R^{-1/2+s})$ ($s > 0$ small). This can be improved to be $O(R^{-3/2+s})$ if we let the exterior boundary be a large sphere provided with a boundary condition of mixed type [SC]. However, our numerical experiments show that even with the simple Dirichlet boundary condition only few degrees of freedom have to be invested in the far field, cp. Remark 8.i, Sect. 6. Thus we do not see any need for replacing this simple Dirichlet boundary condition.

5. Numerical method

In this section we develop the mechanism to adaptively solve the exterior domain problem. We start with a coarse and "small" macro–discretisation. According to information provided by the error estimator introduced in (4.1) this discretisation is successively refined and expanded. More precisely, if the local estimation of the error on an element is too big, this element will be refined if it lies in the interior of the domain. If the element is located at the outer boundary, the computational domain Ω_h has to be expanded, i. e. an additional element has to be added to the discretisation. Thus we iterate the procedure

$$\text{solve} \rightarrow \text{estimate} \rightarrow \text{refine/expand.}$$

To this end, we need two numerical devices:

– a *marking strategy* that decides, where to refine and expand according to the local error estimation,
– an algorithm that actually *refines* and *expands* a given discretisation.

5.1. Marking strategy

In our computations we use the *guaranteed error reduction strategy* proposed in [Dö1]. For that let u_h be a discrete solution on the grid $\mathcal{T}_h$. Then we choose the set $\Sigma \subseteq \mathcal{T}_h$ of elements to be refined such that

$$\eta_\Sigma \geq \vartheta\, \eta_{\mathcal{T}_h}$$

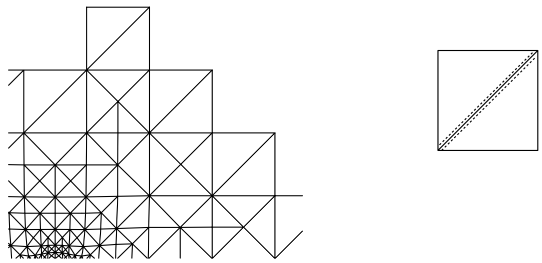

Fig. 1. "Cartesian" structure in 2D and one macro block

for some parameter $\vartheta \in (0,1)$. The set Σ should be as small as possible fulfilling the above inequality. See [Dö1] on how to construct Σ and choose ϑ.

5.2. Refinement/expansion strategy

The algorithm we present is an extension of a refinement algorithm based on the *bisection method*, see [Bä, Ma, Mi]. In order to get an efficient and simple method we make the following two fundamental assumptions, that appear to be quite natural:

- there is $R > 0$ such that $\Omega_h \backslash B_R(0)$ is of "regular" structure, i. e. consists of layers of macro blocks, each layer having the same topological and geometrical structure and is divided into macro blocks in the same kind,
- all elements at the outer boundary of Ω_h are of coarsest level.

By the first assumption an additional block of macro elements can easily be matched to the current discretisation at the outer boundary. Moreover this assumption allows for using a simple data structure to keep track of the expansion process.

The second assumption guarantees that there will be no hanging nodes after having added a macro block.

We consider a grid expansion strategy that generate meshes with bounded σ_{ext} for $d \in \{2,3\}$. Besides that this property was required in Case II, an exponentially growing grid seems to be advantageous from a numerical point of view also in Case I: only few steps of the expansion strategy are needed to reach the final domain.

5.2.1. 2D case The simplest example of a regular structure is the Cartesian one, see Fig. 1. Note that in this case the constant σ_{int} defined by (3.1) trivially stays bounded for the macro elements in a macro block while σ_{ext} (cf. (3.2)) blows up.

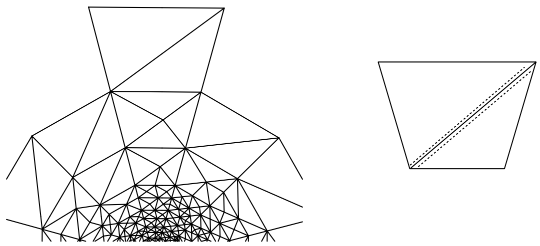

Fig. 2. "Ring" structure in 2D and one macro block

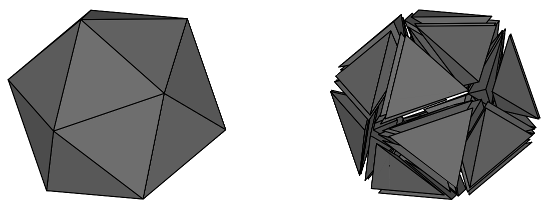

Fig. 3. Icosahedron and one layer of the discretisation

To insure the boundedness of $\sigma_{\rm ext}$ we choose a second possibility. Here, the regular structure consists of "rings" of stretched quadrilaterals divided into two triangles, see Fig. 2. Without loss of generality the centre of the rings is the origin. The inner radius r_j and the outer radius r_{j+1} of the j-th ring are given by

$$r_{j+1} = \kappa r_j = \kappa^{j+1} r_0,$$

respectively, with some $\kappa > 1$. It is easy do see that in this case $\sigma_{\rm int}$ and $\sigma_{\rm ext}$ stay bounded during expansion.

5.2.2. 3D case In analogy to the ring structure in 2D we define the j-th layer in 3D as a discretisation of the annular region with inner radius $r_j = \kappa^j r_0$ and outer radius $r_{j+1} = \kappa^{j+1} r_0$ for some $\kappa > 1$. The inner and outer boundaries of these discretisations are icosahedra, i. e. consisting of 20 triangles. The macro blocks of one layer are the 20 prisms corresponding to the 20 triangular faces. Each prism is subdivided in 3 tetrahedra in such a way that the resulting discretisation is conforming, see Fig. 3.

5.2.3. Refinement/expansion algorithm As mentioned above we incorporate expansion into the bisection method. We briefly recall the bisection method. For details see for instance [Bä, Ma, Mi].

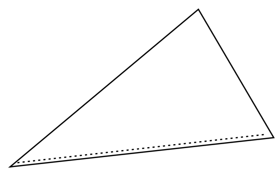

Fig. 4. Triangle with refinement edge

Fig. 5. Bisection of a single triangle

Before starting the refinement process one edge of every triangle of the macro–discretisation is marked, see Fig. 4. This edge is called *refinement edge*. To divide a single triangle, it is cut through the midpoint of the refinement edge and the vertex opposite to the refinement edge. The refinement edges for the new triangles are chosen as in Fig. 5. This kind of bisection is called *newest vertex bisection*, because the new refinement edges lie opposite to the newest vertex.

There is also a natural generalization of this approach to 3D, see [Bä].

This local operation of bisecting an element is used to refine a given discretisation (locally). For that let $\mathcal{T}_k$ be a given regular discretisation and a subset $\Sigma \subseteq \mathcal{T}_k$ of elements which should be divided. Then one proceeds as follows:

Refinement algorithm:

while $\Sigma \neq \emptyset$ **do**
  for all $T \in \Sigma$
    bisect T
  $\Sigma := \{T \in \mathcal{T} \,:\, T$ has a non-conforming node$\}$
endwhile

It can easily be shown that this algorithm terminates in a finite number of steps and that for a sequence $\{\mathcal{T}_k\}_k$ of successively refined discretisations the constant σ_{int} is bounded.

This algorithm can be modified to take into account also expansion. This can be done in the following way. Let $\mathcal{T}_k$ be a given regular discretisation and a subset $\Sigma \subseteq \mathcal{T}_k$ of elements which should be divided or where the discretisation has to be expanded. Then apply the following procedure:

Refinement/expansion algorithm:

while $\Sigma \neq \emptyset$ **do**
  for all $T \in \Sigma$
    if (T is a boundary element)
      add a macro block
    bisect T
  $\Sigma := \{T \in \mathcal{T} \,:\, T$ has a non-conforming node$\}$
endwhile

"Add a macro block" is understood in the way explained above for the ring structure in 2D and 3D.

If (for the 3D case) the decomposition of the prisms into 3 tetrahedra is chosen properly, it is easy to show the following result:

Proposition 1. *The above algorithm stops in a finite number of steps, the resulting discretisations are conforming and for a sequence of successively refined/expanded discretisations* $\sigma_{\rm ext}$ *and* $\sigma_{\rm int}$ *are bounded.*

Remark 6. From our construction and the proof in [Bä] it is clear that $\sigma_{\rm ext}$ and $\sigma_{\rm int}$ stay bounded. The only point to show is the termination of the algorithm. In addition to the pure refinement algorithm one has to prove that the algorithm does not enforce infinitely many macro blocks to be added. This is simple to see in 2D for the regular structures introduced above. In 3D one can easily check this for a given decomposition of the 20 prisms into tetrahedra.

5.2.4. Convergence of the adaptive iteration We will summarize the previous description and establish convergence of the adaptive iteration in Case II with $q = 0$ and $k = 1$ (for simplicity). Let ϵ be the stopping criterion for the estimated global error and assume that a macro–discretisation is given such that the inner boundary is $\partial\omega$ and the outer boundary $\partial_{\rm ext}\Omega_h$ has a regular structure as described in 5.2.2.

Adaptive iteration. 1) Construct an initial mesh $\mathcal{T}_0 := \mathcal{T}_{h_0}$, both fine and large enough, such that

$$\max\Big\{||\,|x|(f - f_{h_*})||_2\,,\, ||h_* f_{h_*}||_2\Big\} \;\leq\; \mu\epsilon \tag{5.1}$$

for some given $\mu > 0$ and all refinements/expansions $\mathcal{T}_{h_*}$ of $\mathcal{T}_0$.

2) Given a discretisation $\mathcal{T}_l := \mathcal{T}_{h_l}$ for $l \geq 0$, we compute the (exact) discrete solution u_l, the local error indicators η_T for $T \in \mathcal{T}_l$, and $\eta_l := \eta_{\mathcal{T}_l}$. If $\eta_l \leq \epsilon$, then stop. Otherwise mark a set Σ of elements in $\mathcal{T}_l$ as described in 5.1 and refine/expand the grid as in 5.2. Denote the new grid by $\mathcal{T}_{l+1}$ and continue.

Theorem 5. *Assume that we have a sequence of discretisations and discrete solutions in S_h^1, constructed by the algorithm above. Then there are numbers $\mu_* > 0$ and $\rho \in (0,1)$, depending on $\sigma_{\rm int}$, $\sigma_{\rm ext}$ and ϑ only (cf. Proposition 1), such that the following holds: If we start from a macro–discretisation $\mathcal{T}_0$ such that (5.1) holds for some $\mu \leq \mu_*$, we have for subsequent errors*

$$||\nabla e_{l+1}||_2 \;\leq\; \rho\, ||\nabla e_l||_2$$

as long as $\eta_l > \epsilon$.

Proof. The proof follows the lines of [Dö1] with minor changes. We give a short sketch. Let Ω_l be a given computational domain with finite element space S_l^1. Let Ω_{l+1} be obtained from Ω_l by the refinement/expansion process described before. Obviously, $\Omega_l \subseteq \Omega_{l+1}$ and $S_l^1 \subseteq S_{l+1}^1$. If u is the exact solution and e_l and e_{l+1} are the errors for the exact discrete solutions, we obtain using (5.1)

$$||\nabla e_l||_2^2 \;\geq\; ||\nabla e_{l+1}||_2^2 + ||\nabla(u_l - u_{l+1})||_2^2 - 8\mu^2\epsilon^2.$$

If $\mathcal{E}_l^0$ denotes the set of all faces of simplices in $\mathcal{T}_l$ that have been divided to get $\mathcal{T}_{l+1}$ or have become inner faces after expansion, we get

$$\sum_{E\in\mathcal{E}_l^0} d_E\, ||\nabla[\partial_n u_l]_E||_{2;E}^2 \;\leq\; c_1\,(||\nabla(u_l - u_{l+1})||_2^2 + \mu^2\epsilon^2)$$

(here and in the subsequel all constants will depend on $\sigma_{\rm int}, \sigma_{\rm ext}$ only). Using in addition the marking and refinement/expansion strategy (5.1, 5.2) as well as Theorem 4 we conclude

$$||\nabla e_l||_2^2 - ||\nabla e_{l+1}||_2^2 \;\geq\; c_2(\vartheta^2 - c_3\mu^2)||\nabla e_l||_2^2$$

for μ small enough, as long as $\eta_{\mathcal{T}_l} > \epsilon$. Note that the convergence factor $1 - c_2(\vartheta^2 - c_3\mu^2)$ does not depend on l. □

Remark 7. i) The first step in our adaptive algorithm needs a priori information on the contribution of $|x|f$ outside the given computational domain. The construction of the initial mesh (that is, a mesh that satisfies (5.1) for some given μ) is done as in part 2) of the adaptive iteration, but with local error indicators $\eta_T^0 := \max\{||\,|x|(f - f_{h_0})||_{2;T}\,,\, ||h_0 f_{h_0}||_{2;T}\}$ and stopping criterion $\mu\epsilon$.
ii) In our analysis and the numerical algorithm we made the idealization that we have computed the exact numerical solution u_l. In practice, however, we accept the outcome of some iterative linear equation solver with some stopping criterion. In our examples we used a stopping criterion which is strict enough to ensure that the corresponding error is negligible compared to the estimated error. Connections between iteration errors and errors in the finite element solution have been studied in [BD].

Table 1. Results for the 2D problem, discretisation by linear elements

k	1	2	3	4	5	6
N_k	504	576	840	1752	5064	13848
max. rad.	1.69	3.80	8.54	12.81	12.81	19.22
$\|\|e_k\|\|_S$	0.730	0.197	0.130	0.090	0.049	0.031
EOC	—	18.49	2.12	0.97	1.14	0.91
EOC_h	—	3.84	2.20	1.21	0.98	0.95

6. Numerical results

In order to get a measure for the numerical efficiency we define the experimental order of convergence $\text{EOC}(k)$ in the k-th iteration step in terms of the number of unknowns N_k:

$$\text{EOC}(k) := d\,\frac{\log(||e_{k-1}||_S/||e_k||_S)}{\log(N_k/N_{k-1})}$$

and the numerical experimental order of convergence $\text{EOC}_h(k)$:

$$\text{EOC}_h(k) := d\,\frac{\log(\eta_{k-1}/\eta_k)}{\log(N_k/N_{k-1})}.$$

6.1. 2D case

Set $q \equiv 1$, $u_\text{D} = const.$ and $\omega = B_1(0)$, $\Omega = \mathbb{R}^2 \backslash \overline{\omega}$, so we solve

$$\begin{aligned} -\Delta u + u &= 0 && \text{in } \Omega, \\ u &= \text{const.} && \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} u(x) &= 0. \end{aligned}$$

Choosing u_D appropriately, u is given by $u(x) = K_0(x)$, K_0 the modified Bessel (or Macdonald) function. In Table 1 the results using the refinement/expansion strategy are listed. Figure 6 shows the triangulations and Fig. 7 the graph of u_h on the final grid.

6.2. 3D case

We consider the following problem:

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega, \\ u &= u_\text{D} && \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} u(x) &= 0, \end{aligned}$$

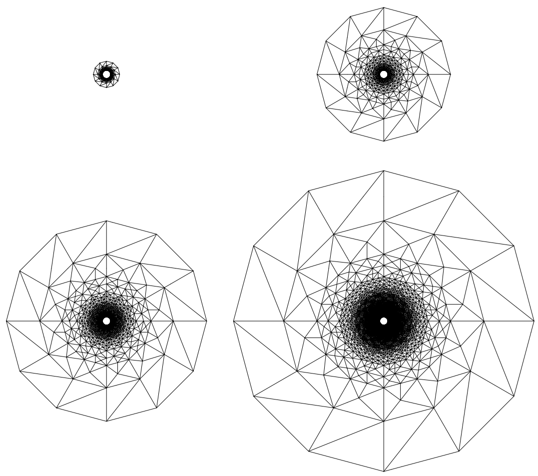

Fig. 6. Computational domains Ω_h; initial domain and domains after 3, 4 and 6 iterations respectively

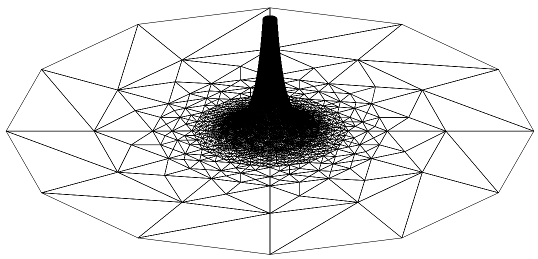

Fig. 7. Graph of u_h after 6 iterations

with $\omega = B_1(0)$, $\Omega = \mathbb{R}^3 \backslash \overline{\omega}$. The boundary values are defined such that the solution is a dipole field:

$$u_{\mathrm{D}}(x) = u(x) = \frac{x_1}{r^3}.$$

Table 2 shows the results using linear elements, Table 3 reports the results for quadratic elements. Figures 8 and 9 display the discrete solutions and corresponding grids.

Remark 8. i) As expected, the experimental order of convergence is 1 for linear elements and 2 for quadratics, thus confirming our considerations in Remark 3. Furthermore, most of the unknowns are located in the interior of the domain. This means that there is no need for replacing the simple condition $u_h = 0$ at the exterior boundary by a more sophisticated one.
ii) The advantage using quadratic elements compared to linear elements for

Table 2. Results for the 3D problem, discretisation by linear elements

k	1	2	3	4	5	6	7	8	9	10	11
N_k	520	652	1032	1900	3758	6686	13166	23152	43658	88096	163128
max. rad.	4.0	8.0	16.0	32.0	64.0	64.0	128.0	128.0	256.0	512.0	512.0
$\|\|e_k\|\|_S$	1.962	1.584	1.341	1.066	0.882	0.719	0.574	0.474	0.380	0.303	0.244
EOC	—	2.34	0.97	1.05	0.80	1.07	0.95	1.02	1.06	0.93	1.06
EOC$_h$	—	2.81	1.33	1.24	0.75	1.10	1.07	0.93	1.09	0.98	1.00

Table 3. Results for the 3D problem, discretisation by quadratic elements

k	1	2	3	4	5	6	7	8	9
N_k	3538	3860	4570	5956	8482	14650	28100	47492	66676
max. rad.	4.0	8.0	16.0	32.0	64.0	64.0	128.0	256.0	256.0
$\|\|e_k\|\|_S$	0.903	0.635	0.435	0.338	0.233	0.170	0.121	0.082	0.065
EOC	—	12.14	6.72	2.87	3.16	1.73	1.55	2.22	2.05
EOC$_h$	—	14.12	6.11	3.07	3.41	1.68	1.62	2.46	2.11

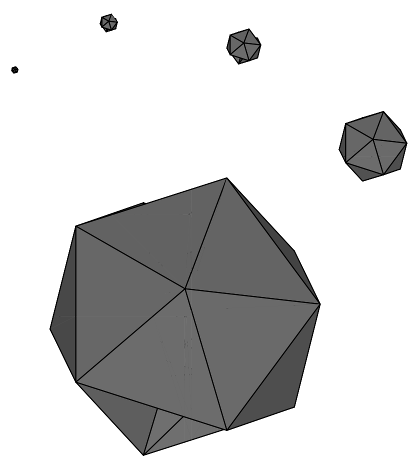

Fig. 8. Computational domains Ω_h; initial domain and domains after 3, 4, 5 and 9 iterations respectively, quadratic elements

our concrete numerical example can clearly be seen from Tables 2 and 3. To achieve e. g. an accuracy of about 0.2 one has to spend about 20-times the number of unknowns using linear elements, resulting in a CPU time, which is 22-times higher.

iii) In order to solve the resulting systems efficiently we use multilevel preconditioning. Some care has to be taken due to the fact that the macro elements are not of equal size. So the method of choice in $\mathbb{R}^3$ is the MDS

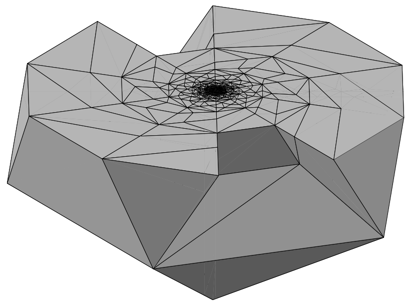

Fig. 9. Clip at $x_3 = 0$ into the final triangulation, quadratic elements

Table 4. Numbers of PCG-iterations, Iter1 = diagonal scaling, Iter2 = MDS, linear elements

k	1	2	3	4	5	6	7	8	9	10	11
N_k	520	652	1032	1900	3758	6686	13166	23152	43658	88096	163128
Iter1	17	31	38	48	51	61	74	83	88	106	114
Iter2	24	31	34	33	33	41	39	37	42	41	42

preconditioner, see [Ys]. See also Table 4 for a comparison using simple diagonal scaling and MDS preconditioning.

7. Conclusions

We proved a priori and a posteriori error estimates (using only a priori information on data in the latter case) for the numerical solution of Poisson's equation with homogeneous boundary conditions on an exterior domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. Thereby the numerical solution is computed on a finite computational domain with homogeneous Dirichlet data. By a refinement/expansion technique for the grid in connection with the use of an a posteriori error estimator we get an efficient and reliable method. The strategy may easily be incorporated into existing adaptive finite element codes on bounded domains. We are able to prove convergence for our adaptive procedure. Numerical examples confirm the expected optimal order of convergence in terms of number of unknowns.

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