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A-posteriori modeling error estimation for hierarchic plate models*

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Summary. Hierarchic plate models are dimensional reductions obtained by semidiscretization of the three-dimensional plate problem in the transverse direction and energy projection. We derive computable a-posteriori estimators for the modeling error thus incurred under the assumption that the resulting two-dimensional plate models are solved exactly. The estimators are valid for homogeneous, monoclinic materials, for plates with unsmooth midsurfaces and for a wide class of variational edge conditions. Computable a-posteriori bounds on the effectivity indices are also derived and sufficient conditions for the asymptotic (i.e. as the plate thickness tends to zero) and the spectral (i.e. as the order of the plate model tends to infinity) exactness of the estimators are given. Numerical examples are presented.

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1. Introduction

A large part of the boundary value problems in three-dimensional elastostatics that are solved in current engineering applications are posed on thin domains or on domains with thin components, such as beams, plates and shells. Classically, such boundary value problems have been *modelled* by dimensionally reduced problems on two- or one-dimensional domains; we mention only the Bernoulli beam, the Kirchhoff-Love [9, 10] or the Reissner-Mindlin [12, 15, 16] plate models. These models have been more amenable to analytical solution, for example by Fourier series, than the fully three dimensional equations. With the advent of the digital computer and in particular the finite element method (FEM), the widespread use of these classical, dimensionally reduced models has continued, now however due

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to the computational savings achieveable with a lower dimensional model. The fact that standard h-Version FE-discretizations of three-dimensional problems on thin domains require excessive mesh refinement and exhibit numerical stiffness also contributed to this trend.

In recent years increasingly accurate FE solutions of the classical lower dimensional models became available and there was a growing awareness that at finite, positive thickness these models merely constitute *approximations* to the underlying three-dimensional problems (see, for example, [8], [3]). Even if solved exactly, lower dimensional models consequently incur a corresponding approximation error, the so-called *modelling error*. Numerous papers have been devoted to the *justification* of lower-dimensional models, i.e. to the proof that the rescaled modeling error vanishes as the thickness of the structure tends to zero, by asymptotic methods (see [7], [8], [11] and the references therein). In practice, however, the thickness of the structure is positive and given, i.e. not at our disposal. Hence there arises the question for an accurate modeling error estimation for a positive, given thickness and given load data. Such estimates which are necessarily *a-posteriori* were for the KL- and the RM-plate model given in [20]. With the availability of such modeling-error estimates the question arises *what one should do if the classical models are not found to be sufficiently accurate*.

To this end it was proposed in [1, 3, 5, 17, 28, 27] to embed the classical models into a *hierarchy* of lower dimensional models of increasing accuracy and complexity. These hierarchies are obtained by constraining the admissible displacements in the three-dimensional variational principle to director fields exhibiting a certain prespecified behaviour in the cross section of the structure. For example, in the case of homogeneous plates and shells, the hierarchy is obtained by constraining the displacements admissible in the three-dimensional variational principle to be polynomials in the transverse direction whereas for laminated materials special, nonpolynomial transverse behaviour must be imposed in order to ensure certain optimality properties of the lower dimensional models [5], [11]. Thus hierarchic modeling is a special case of the method of constraints [2] and leads to a singularly perturbed, elliptic system in the two-dimensional midsurface of the three-dimensional structure. The methodology for constructing a flexible hierarchy of lower dimensional models is general, i.e. it applies to any variational problem on a three-dimensional, thin domain. As with any of the classical lower dimensional models, a *quantitative, computable modelling error estimator* must be available in order to assess the accuracy of the members of the hierarchy for a given boundary value problem.

The present paper is devoted to the derivation and analysis of a-posteriori modeling error estimators for hierarchic models of homogeneous, elastic plates. We consider monoclinic materials which are the most general ones for which bending and membrane effects can be seperated. We derive computable modeling error estimators which are based on the residual tractions on the faces of the plate. We demonstrate numerically that the estimators perform well in applications. The results in the present paper generalize earlier ones [19, 20] by the author.

In [19], an estimator was derived which is valid only for totally clamped edge conditions. The constants defining the error estimators were characterized as smallest eigenvalues of a Steklov eigenvalue problem which had to be solved numerically. In [20], the basic idea in the derivation of modeling error estimators for general edge conditions was presented and the estimators were derived for isotropic materials. In the present paper, we generalize these results to general monoclinic materials. We prove that they are guaranteed upper estimators and, under more restrictive assumptions, that they are asymptotically and spectrally exact. This was previously done only for heat conduction problems in a plate in [6]. Hierarchic models allow also for the simultaneous use of different model orders in various subregions of the plate [6, 18, 20]. In conjunction with modeling error estimators of good quality, this allows for *adaptive hierarchic modeling* of thin structures [4]. Here the contributions to the modeling error from different subdomains of the plate (e.g. the interior and the boundary region) are used as *modelling error indicators* on which the decision where to raise the model order within the hierarchy is based. Although we treat in the present paper explicitly only the case of plates, the techniques introduced here for the derivation of the modelling error estimators, i.e. a covering of the midsurface by a family of axiparallel squares and the asymptotic analysis of a certain variational problem to determine the constants in the estimators, are clearly not limited to plates. The feasibility of hierarchic models for homogeneous shells was demonstrated in [28] where also the notion 'hierarchic model' was first introduced.

The outline of the paper is as follows: in Sect. 2, we introduce notation and present the three-dimensional plate problem with general edge conditions. We describe in particular the decoupling into a bending and a membrane problem. In Sect. 3, we describe the hierarchy of plate models. In Sect. 4, we derive the modeling error estimators and in Sect. 5 we prove its asymptotic exactness and derive a computable bound for its effectivity index. Section 6 contains some numerical results obtained with a recent commercial code where hierarchical plate models and our a-posteriori error estimation are implemented. There each plate model in the hierarchy is also solved approximately by a p-version finite element method which is known to be considerably less susceptible to shear locking than, e.g., lower order finite element methods (see, e.g., [26]). We address briefly some mesh design principles for the p-version applied to plate models which are suggested by the analysis [22, 23]. We demonstrate the practical reliability of our modelling error estimators even for thick, highly anisotropic plates and high model orders.

2. The three-dimensional plate problem

2.1. Governing equations

The plate problem is a boundary value problem of three-dimensional, linearized elasticity in the domain

(2.1) $$\Omega = \omega \times (-d, d)$$

of thickness $t = 2d$ with the lateral boundary

(2.2) $$\Gamma = \gamma \times (-d, d).$$

Here $\omega \subset \mathbb{R}^2$, the *mid-surface* of the plate, denotes a bounded domain with Lipschitz boundary γ. This implies that Ω is locally Lipschitz, too. Therefore the exterior unit normal $\mathbf{n}$ and the unit tangent vector $\mathbf{t}$ on γ (and hence also on Γ) to ω (resp. to Ω) are defined almost everywhere (cf. [13]). Points in Ω are denoted by (x, y) where $x = (x_1, x_2) \in \omega$ and $|y| < d$. Analogously, points in Γ are denoted by (s, y) with s denoting the arclength on γ. We further define the faces of the plate

(2.3) $$R_{\pm} = \{(x, y) \mid x \in \omega, y = \pm d\}.$$

Then the plate problem consists in finding a displacement field $u(x, y) : \Omega \to \mathbb{R}^3$ which satisfies the following governing equations:

1. Equilibrium conditions:

   (2.4) $$\mathbf{L}u = -\mathrm{div}\sigma[u] = f \quad \text{in } \Omega,$$

   with the symmetric stress tensor

   $$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}.$$

2. Constitutive equations (Hooke's Law):

   (2.5) $$\sigma = A\epsilon[u]$$

   with a fourth order tensor A and the linearized strain tensor

   $$\epsilon[u] = \{\epsilon_{ij}[u]\}_{1 \le i,j \le 3}, \qquad \epsilon_{ij}[u] = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

3. Essential and natural edge boundary conditions:

   (2.6) $$B_0 u = 0 \quad \text{and} \quad B_1 u = 0 \quad \text{on } \Gamma$$

   (one of the two boundary operators may vanish).
4. Prescribed normal tractions on the faces:

   (2.7) $$\sigma[u]\mathbf{n} = g^+ \quad \text{on } R_+, \quad \sigma[u]\mathbf{n} = g^- \quad \text{on } R_-$$

   where $\mathbf{n}$ denotes the exterior unit normal vector to $\partial\Omega$.

It will be useful to write the six essential components of σ and ϵ as column vectors in $\mathbb{R}^6$ as

$$\sigma[u] = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})^\top, \quad \epsilon[u] = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23})^\top. \tag{2.8}$$

We admit materials for which Hooke's law (2.5) can be written in the form

$$\sigma[u] = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 \\ a_{12} & a_{22} & a_{23} & a_{24} & 0 & 0 \\ a_{13} & a_{23} & a_{33} & a_{34} & 0 & 0 \\ a_{14} & a_{24} & a_{34} & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{56} & a_{66} \end{pmatrix} \epsilon[u] \tag{2.9}$$

with a symmetric, positive definite matrix $A \in \mathbb{R}^{6\times 6}_{\text{sym}}$. In particular, the thirteen parameters a_{ij} in (2.9) characterize homogeneous materials with monoclinic symmetry.

Isotropic materials are contained in (2.9) as a special case:

$$\begin{aligned} a_{11} &= a_{22} = a_{33} = \lambda + 2\mu = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}, \\ a_{12} &= a_{13} = a_{23} = a_{21} = a_{31} = a_{32} = \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \\ a_{44} &= a_{55} = a_{66} = 2\mu = \frac{E}{1+\nu}, \quad a_{14} = a_{24} = a_{34} = a_{56} = 0. \end{aligned} \tag{2.10}$$

Here λ and μ are the so-called Lamé-constants, $E > 0$ is Young's modulus and $0 \le \nu < 1/2$ the Poisson-coefficient.

Another important special case of (2.9) is an *orthotropic material* where we have with engineering notation

$$A^{-1} = \begin{pmatrix} 1/E_1 & -\nu_{12}/E_2 & -\nu_{13}/E_3 & 0 & 0 & 0 \\ -\nu_{21}/E_1 & 1/E_2 & -\nu_{23}/E_3 & 0 & 0 & 0 \\ -\nu_{31}/E_1 & -\nu_{32}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/(2G_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/(2G_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/(2G_3) \end{pmatrix} \tag{2.11}$$

and the symmetry of A implies

$$\nu_{ij} E_i = \nu_{ji} E_j, \qquad 1 \le i, j \le 3,$$

i.e. an orthotropic material has rhombic symmetry and is determined by nine constants.

To cast the boundary value problem (2.4)–(2.7) into variational form, we use Green's formula

$$\int_{\Omega} v^{\top}\mathbf{L}u dxdy = \mathscr{B}(u,v) - \int_{\partial\Omega} v^{\top}\sigma[u]\mathbf{n} do \quad \forall u \in H_{\mathbf{L}}(\Omega), v \in \left[H^1(\Omega)\right]^3 \tag{2.12}$$

which holds in particular for the mid-surfaces ω under consideration here. In (2.12) *do* denotes the surface measure on $\partial\Omega$ and

$$H_{\mathbf{L}}(\Omega) = \left[H^1(\Omega)\right]^3 \cap \left\{u : \mathbf{L}u \in \left[L^2(\Omega)\right]^3\right\}.$$

The bilinear form $\mathscr{B}(\cdot,\cdot)$ is given by

$$\mathscr{B}(u,v) = (\epsilon[v], \sigma[u])$$

with the inner product (ϵ,σ) defined by

$$(\epsilon,\sigma) = \int_{\Omega} \epsilon : \sigma dxdy \qquad \sigma,\epsilon \in \left[L^2(\Omega)\right]^{3\times 3} \tag{2.13}$$

where

$$\begin{aligned}\epsilon[u] : \sigma[u] &= \sum_{1\le i,j\le 3} \epsilon_{ij}[u]\sigma_{ij}[u] \\ &= \epsilon_{11}\sigma_{11} + \epsilon_{22}\sigma_{22} + \epsilon_{33}\sigma_{33} + 2\left(\epsilon_{12}\sigma_{12} + \epsilon_{13}\sigma_{13} + \epsilon_{23}\sigma_{23}\right).\end{aligned}$$

For prescribed surface tractions $g^+, g^- \in \left[L^2(\omega)\right]^3$ on the faces $R_\pm$ and volume forces $f \in \left[L^2(\Omega)\right]^3$ we define the load functional $\mathscr{F}(u)$ by

$$\begin{aligned}\mathscr{F}(u) &= \int_{\Omega} u(x,y)^{\top} f(x,y) dydx + \int_{R_+} g^+(x)^{\top} u(x,d) ds^+ \\ &\quad + \int_{R_-} g^-(x)^{\top} u(x,-d) ds^- \\ &= \int_{\Omega} u(x,y)^{\top} f(x,y) dydx \\ &\quad + \int_{\omega} \left\{g^+(x)^{\top} u(x,d) - g^-(x)^{\top} u(x,-d)\right\} dx\end{aligned} \tag{2.14}$$

since $ds^+ = -ds^- = dx_1dx_2$. Then Green's formula (2.12) becomes

$$\mathscr{B}(u,v) = \int_{\Gamma} v^{\top}\sigma[u]\mathbf{n} dsdy + \mathscr{F}(v) \quad \forall v \in \left[H^1(\Omega)\right]^3$$

with *ds* denoting the surface measure on the boundary curve $\partial\omega$. The boundary integral

$$\int_{\gamma}\int_{-d}^{d} v^{\top}\sigma[u]\mathbf{n} dyds \tag{2.15}$$

should vanish according to (2.6).

2.2. Edge conditions

In practical applications a variety of *edge conditions* are used since it is well known that the overall solution of the plate problem and in particular its edge behaviour depend strongly on the kind of edge conditions prescribed. We show therefore next how very general traction and displacement resultants can be prescribed as natural and essential edge conditions, respectively, implying (2.15). These conditions include in particular the boundary conditions commonly used in engineering practice. More importantly, any of the edge conditions can be exactly represented by a hierarchic model of sufficiently high order, as we shall see below.

To describe the edge conditions we introduce normal- and tangential components of the displacements and tractions a.e. on Γ:

$$u_n = \mathbf{n}^\top u, \quad u_t = \mathbf{t}^\top u, \quad \sigma_n = \mathbf{n}^\top \sigma[u]\mathbf{n}, \quad \sigma_t = \mathbf{t}^\top \sigma[u]\mathbf{n}. \tag{2.16}$$

The basic idea is now to develop u and $\sigma[u]n$ fiberwise on Γ in Legendre series, i.e.

$$\begin{pmatrix} u_t \\ u_n \\ u_3 \end{pmatrix}(s,y) = \sum_{k=0}^{\infty} L_k\left(\frac{y}{d}\right)\begin{pmatrix} U_{tk}(s) \\ U_{nk}(s) \\ U_{3k}(s) \end{pmatrix}, \tag{2.17}$$

$$\begin{pmatrix} \sigma_t \\ \sigma_n \\ \sigma_3 \end{pmatrix}(s,y) = \sum_{k=0}^{\infty} L_k\left(\frac{y}{d}\right)\begin{pmatrix} T_{tk}(s) \\ T_{nk}(s) \\ T_{3k}(s) \end{pmatrix} \tag{2.18}$$

with the Legendre coefficients of the displacements defined for almost every $s \in \gamma$ by

$$\begin{aligned} U_{tk}(s) &= \frac{2k+1}{2}\int_{-1}^{1} u_t(s,zd)L_k(z)dz, \\ U_{nk}(s) &= \frac{2k+1}{2}\int_{-1}^{1} u_n(s,zd)L_k(z)dz, \\ U_{3k}(s) &= \frac{2k+1}{2}\int_{-1}^{1} u_3(s,zd)L_k(z)dz. \end{aligned}$$

The traction Legendre coefficients $T_{tk}(s), T_{nk}(s)$ and $T_{3k}(s)$ in (2.18) are defined analogously.

The basic idea in specifying compatible pairs of essential boundary conditions B_0 and natural ones B_1 consists in enforcing the vanishing of complementary sets of Legendre coefficients of the boundary displacements and tractions. More precisely, with (2.17), (2.18) we obtain the following possibilities. Let $\mathscr{T}_j \subseteq \mathbb{N}_0$, $j \in \{n,t,3\}$ be index sets. Then we define the *essential boundary conditions*

$$\int_{-d}^{d} u_j(s,y)L_k(\frac{y}{d})dy = 0 \quad \forall k \in \mathscr{T}_j, \quad \text{a.e. } s \in \gamma, \quad j \in \{n,t,3\}. \tag{2.19}$$

Obviously, due to the fiberwise orthogonality of the Legendre polynomials, the complementary *natural boundary conditions* (in the sense of (2.15)) read

$$\sigma_j - \sum_{k\in\mathcal{T}_j} T_{jk}(s)L_k\left(\frac{y}{d}\right) = \sum_{k\in\mathbb{N}_0\setminus\mathcal{T}_j} T_{jk}(s)L_k(\frac{y}{d}) = 0, \quad j\in\{n,t,3\}. \tag{2.20}$$

The most important special cases are obtained with the selections $\mathcal{T}_j=\emptyset$ or $\mathcal{T}_j=\mathbb{N}_0$, i.e. either the displacement or the traction component vanishes identically on Γ. Some examples follow:

1. Dirichlet conditions ("Hard clamped plate"): $\mathcal{T}_j=\mathbb{N}_0, j\in\{n,t,3\}$,

$$B_0u=\gamma_0 u=0 \qquad \text{on } \Gamma, \tag{2.21}$$

2. Neumann conditions ("free plate"): $\mathcal{T}_j=\emptyset, j\in\{n,t,3\}$,

$$B_1u=\gamma_1 u=\sigma[u]\mathbf{n}=0 \qquad \text{on } \Gamma, \tag{2.22}$$

Since we are considering a system of partial differential equations, we can pose Dirichlet- or Neumann conditions also for some components of this system.

3. ("soft simply supported plate"): $\mathcal{T}_n=\mathcal{T}_t=\emptyset$, $\mathcal{T}_3=\mathbb{N}_0$,

$$B_0u=\gamma_0u_3=0, \quad B_1u=\begin{pmatrix}\mathbf{n}^\top\sigma[u]\mathbf{n}\\ \mathbf{t}^\top\sigma[u]\mathbf{n}\end{pmatrix}=0 \quad \text{on } \Gamma. \tag{2.23}$$

Here $\mathbf{t}=\mathbf{t}(s)$ denotes the unit tangent vector to γ. We consider further

4. ("hard simply supported plate"): $\mathcal{T}_t=\mathcal{T}_3=\mathbb{N}_0$ and $\mathcal{T}_n=\emptyset$,

$$B_0u=\begin{pmatrix}u_3\\ \mathbf{t}^\top u\end{pmatrix}=0, \quad B_1u=\mathbf{n}^\top\sigma[u]\mathbf{n}=0 \quad \text{on } \Gamma. \tag{2.24}$$

5. $\mathcal{T}_n=\mathcal{T}_t=\mathbb{N}_0$ and $\mathcal{T}_3=\emptyset$.

$$B_0u=\begin{pmatrix}\mathbf{n}^\top\gamma_0u\\ \mathbf{t}^\top\gamma_0u\end{pmatrix}=0, \quad B_1u=e_3^\top\sigma[u]\mathbf{n}=0 \quad \text{on } \Gamma. \tag{2.25}$$

Examples of nonclassical boundary conditions are obtained by prescribing only certain displacement or stress *resultants* as it is frequently done in engineering.

6. ("supersoft simply supported plate") $\mathcal{T}_n=\mathcal{T}_t=\emptyset$, $\mathcal{T}_3=\{0\}$:

$$\begin{aligned} B_0u &= \int_{-d}^{d} u_3(s,y)dy=0,\\ B_1u &= \begin{pmatrix}\sigma_t(s,y)\\ \sigma_n(s,y)\\ \sigma_3(s,y)-\int_{-d}^{d}\sigma_3(s,y)dy\end{pmatrix}=0 \end{aligned} \tag{2.26}$$

7. (“Prescribed resultant forces”) $F_j(s)$. $\mathscr{T}_j = \mathbb{N}$, $j \in \{t, n, 3\}$.

$$
\begin{aligned}
B_0 u &= u_j(s,y) - \int_{-d}^{d} u_j(s,y)dy = 0,\\
B_1 u &= \int_{-d}^{d} \sigma_j(s,y)dy = F_j(s).
\end{aligned}
\tag{2.27}
$$

8. (“Prescribed edge-moments”) $M_j(s)$. $\mathscr{T}_j = \mathbb{N}_0 \backslash \{1\}$, $j \in \{t, n, 3\}$.

$$
\begin{aligned}
B_0 u &= \int_{-d}^{d} u_j(s,y)dy + \sum_{2}^{\infty} U_{jk}(s) L_k(\frac{y}{d}) = 0,\\
B_1 u &= \int_{-d}^{d} \sigma_j(s,y)ydy = M_j(s).
\end{aligned}
\tag{2.28}
$$

Remark 2.1. We assumed in (2.6) that *one* combination of boundary conditions B_0 and B_1 is posed on all of Γ. This is of course not necessary and (2.6) may also be substituted by a family $\{B_{0j}, B_{1j}\}$ of piecewise defined conditions. To keep the notation simple, we will however continue to work with (2.6).

2.3. Existence of weak solutions

A weak (or variational) solution of the plate problem is a displacement field $u : \Omega \to \mathbb{R}^3$ minimizing the primal energy

$$
\mathscr{G}(u) = \frac{1}{2} \mathscr{B}(u,u) - \mathscr{F}(u)
\tag{2.29}
$$

over a suitable subset

$$
\mathscr{H}(\Omega) \subset \left[H^1(\Omega)\right]^3
$$

of admissible displacement fields. To discuss the uniqueness of minimizers of (2.29) we note that

$$
\sigma[r] = 0 \Longleftrightarrow r \in \mathbf{R} = \left\{ u : \Omega \to \mathbb{R}^3 \mid u(x,y) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_2 y - b_3 x_2 \\ b_3 x_1 - b_1 y \\ b_1 x_2 - b_2 x_1 \end{pmatrix} \right\}.
\tag{2.30}
$$

The set $\mathbf{R}$ is the 6 dimensional set of rigid body motions. We define the set of homogeneous solutions of (2.4)–(2.7)

$$
\mathbf{N} = \{ r \in \mathbf{R} \mid r \,\text{satisfies (2.4)–(2.7) with}\ f = g^{\pm} = 0 \}.
\tag{2.31}
$$

For example, for the Dirichlet condition (2.21) we have $\mathbf{N} = \{0\}$ and for the Neumann condition (2.22) we have $\mathbf{N} = \mathbf{R}$. Thus the set of admissible displacement fields within which a unique minimizer of $\mathscr{G}$ can be expected is

$$\mathscr{H}(\Omega) = \left[H^1(\Omega)\right]^3 \cap \left\{u \mid B_0 u = 0 \text{ and } \int_\Omega r^\top u dx dy = 0 \quad \forall r \in \mathbf{N}\right\}. \tag{2.32}$$

Any minimizer $u \in \mathscr{H}(\Omega)$ of (2.29) satisfies the variational Euler-Lagrange equations

$$u \in \mathscr{H}(\Omega) \qquad B(u, v) = \mathscr{F}(v) \qquad \forall v \in \mathscr{H}(\Omega). \tag{2.33}$$

Theorem 2.1. *1. $\mathscr{H}(\Omega)$ is a closed, linear subspace of $\left[H^1(\Omega)\right]^3$.*
2. Assume that $g^\pm \in \left[L^2(\omega)\right]^3$, $f \in \left[L^2(\Omega)\right]^3$ satisfy the compatibility condition

$$\mathscr{F}(r) = 0 \qquad \forall r \in \mathbf{N}. \tag{2.34}$$

Then there exists a unique weak solution $u \in \mathscr{H}(\Omega)$ of (2.33).

Proof. 1. Since $\gamma = \partial\omega$ is Lipschitz, so is $\partial\Omega$ for $0 < d \leq 1$ (see, e.g., [20, Appendix C]). Consequently, the trace operator $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ is continuous. Therefore the constraint $B_0 u = 0$ on Γ in (2.32) is well-defined.

2. Evidently, $\mathscr{H}(\Omega) \subset \left[H^1(\Omega)\right]^3$ is a linear subspace. We show that it is closed. To this end, let $\{u(j)\}_{j=1}^\infty \subset \mathscr{H}(\Omega)$ be a Cauchy sequence in $\left[H^1(\Omega)\right]^3$. Since $H^1(\Omega)$ is complete, there exists a limit $u \in \left[H^1(\Omega)\right]^3$ such that $\|u(j) - u\|_{H^1(\Omega)} \to 0$ as $j \to \infty$. We claim that $u \in \mathscr{H}(\Omega)$. To show it, we verify (2.32).

First, for every $r \in \mathbf{N}$, we have due to $u(j) \in \mathscr{H}(\Omega)$ that

$$\left|\int_\Omega r^\top u dx\right| = \left|\int_\Omega r^\top (u - u(j))\, dx\right| \leq C(r, d)\, \|u - u(j)\|_{L^2(\Omega)} \to 0,$$

as $j \to \infty$ whence it follows that

$$\int_\Omega r^\top u dx = 0 \qquad \forall r \in \mathbf{N}.$$

Next, the continuity of the trace operator gives

$$\gamma_0(u), \gamma_0(u(j)) \in \left[L^2(\Gamma)\right]^3.$$

By the density of the Legendre polynomials in $L^2(-1, 1)$, we can expand the traces of the component functions of u and $u(j)$ on Γ into fiberwise Legendre series:

$$(\gamma_0 u_i)(s, y) = \sum_{\ell=0}^\infty U_{i\ell}(s) L_\ell(\frac{y}{d}), \quad i = 1, 2, 3,$$

$$(\gamma_0 u_i(j))(s, y) = \sum_{\ell=0}^\infty U_{i\ell}^{(j)}(s) L_l(\frac{y}{d}), \quad i = 1, 2, 3$$

which converge in $L^2(\Gamma) = L^2(\gamma) \otimes L^2(-d, d)$ by Fubini's theorem. Since $\{u(j)\} \subset \mathscr{H}(\Omega)$ we have that $B_0 u(j) = 0$ for all j which implies that

$$U_{k\ell}^{(j)} = 0 \quad \ell \in \mathcal{I}_k, k \in \{n, t, 3\}$$

for all j. From the orthogonality properties of the Legendre polynomials we have for $k \in \{n, t, 3\}$ that

$$\|u_k - u_k(j)\|_{L^2(\Gamma)}^2 = \sum_{\ell \in \mathbb{N}_0 \setminus \mathcal{I}_k} \frac{2d}{2\ell+1} \left\| U_{k\ell} - U_{k\ell}^{(j)} \right\|_{L^2(\gamma)}^2 + \sum_{\ell \in \mathcal{I}_k} \frac{2d}{2\ell+1} \|U_{k\ell}\|_{L^2(\gamma)}^2 . \tag{2.35}$$

By the continuity of the trace operator we have

$$\|u - u(j)\|_{L^2(\Gamma)} \leq C(t) \|u - u(j)\|_{H^1(\Omega)} \to 0 \quad \text{as } j \to \infty$$

and we find from (2.35) that $U_{k\ell} = 0$ on γ for $\ell \in \mathcal{I}_k$, $k \in \{n, t, 3\}$, i.e that $B_0 u = 0$. Hence $u \in \mathcal{H}(\Omega)$ and assertion 1. is proved.

3. Since $\mathcal{H}(\Omega)$ is a closed, linear subspace of $\left[H^1(\Omega)\right]^3$, we have Korn's inequality

$$\mathcal{B}(u, u) \geq C(t, \omega) \|u\|_{H^1(\Omega)}^2 \qquad \forall u \in \mathcal{H}(\Omega)$$

(see, e.g., [14, Theorem 2.5]). By our assumptions on the data the continuity of the trace operator on Lipschitz domains and Korn's inequality imply that

$$\begin{aligned} |\mathcal{F}(v)| &\leq C(t) \left(\|f\|_{L^2(\Omega)}^2 + \|g^+\|_{L^2(\omega)}^2 + \left\|g^-\right\|_{L^2(\omega)}^2 \right)^{1/2} \|v\|_{H^1(\Omega)} \\ &\leq C(t, f, g^{\pm}) \|v\|_{E(\Omega)} \end{aligned}$$

for every $v \in \mathcal{H}(\Omega)$. As usual, the energy norm $\|v\|_{E(\Omega)}$ is defined as $\left(\mathcal{B}(v, v)\right)^{1/2}$. Existence and uniqueness of a weak solution to (2.33) now follow from the Fredholm alternative and the Riesz representation theorem applied to $\mathcal{H}(\Omega)$ with the energy inner product $\mathcal{B}(\cdot, \cdot)$. □

Remark 2.2. The assumption that the data $g^{\pm}(x)$ and $f(x, y)$ are square integrable can be weakened. In what follows it will be sufficient for our purposes.

2.4. Separation of bending and membrane effects

The variational solution $u \in \mathcal{H}(\Omega)$ of the plate problem can be decomposed into a *membrane part* $u^I(x, y)$ and a *bending part* $u^{II}(x, y)$ as follows:

$$u_\alpha^I(x, y) = u_\alpha^I(x, -y), \quad \alpha = 1, 2, \qquad u_3^I(x, y) = -u_3^I(x, -y), \tag{2.36}$$

and

$$u_\alpha^{II}(x, y) = -u_\alpha^{II}(x, -y), \quad \alpha = 1, 2, \qquad u_3^{II}(x, y) = u_3^{II}(x, -y). \tag{2.37}$$

We denote the corresponding sets of admissible displacement fields by $\mathcal{H}^I(\Omega)$ and $\mathcal{H}^{II}(\Omega)$, respectively. These subsets of $\mathcal{H}(\Omega)$ are closed and orthogonal with respect to the inner product induced by the bilinear form $\mathscr{B}(\cdot,\cdot)$ on $\mathscr{H}(\Omega)$, i.e.

$$\mathscr{H}(\Omega)=\mathscr{H}^{I}(\Omega)\oplus\mathscr{H}^{II}(\Omega) \quad\text{or}\quad \mathscr{B}(u,v)=0 \quad \forall u\in\mathscr{H}^{I}(\Omega), v\in\mathscr{H}^{II}(\Omega). \tag{2.38}$$

This is a consequence of the sparsity structure of the constitutive matrix A in (2.9) and the dependence of the strains $\epsilon_{ij}[u]$ on y which is implied by (2.36) and (2.37).

Remark 2.3. The constitutive matrix A in (2.9) is the most general one for which this separation of bending and membrane effects in the plate problem can be achieved. Few materials exist with the still more general, fully anisotropic constitutive relations which are characterized by twenty-one constants. They arise mainly as "effective", i.e. homogenized, models of composite sandwich plates with asymmetric layup. Such materials still appear to be rarely used in practice, since they exhibit some couterintuitive behaviour due to the bending-membrane coupling. For example, such plates *bend* under normal tractions applied to the edges where one would intuitively expect a pure membrane response, i.e. in-plane stretching.

Thus, u^{I} and u^{II} can be obtained independently of each other provided that the load functional $\mathscr{F}(u)$ is also split into bending and membrane parts.

$$\begin{aligned}\mathscr{F}(u)&=\mathscr{F}^{I}(u)+\mathscr{F}^{II}(u)\\ &=\int_{\Omega} f^{I}(x,y)^{\top}v(x,y)dxdy\\ &\quad+\int_{\omega}\left\{g_3^{I}(x)(v_3^{+}-v_3^{-})(x)+g_\alpha^{I}(v_\alpha^{+}+v_\alpha^{-})(x)\right\}dx\\ &\quad+\int_{\Omega} f^{II}(x,y)^{\top}v(x,y)dxdy\\ &\quad+\int_{\omega}\left\{g_3^{II}(x)(v_3^{+}+v_3^{-})(x)+g_\alpha^{II}(x)(v_\alpha^{+}-v_\alpha^{-})(x)\right\}dx\end{aligned} \tag{2.39}$$

where we set $v^{\pm}(x)=v(x,\pm d)$ and summation over repeated indices is implied. The membrane and bending loads are given by

$$\begin{aligned}f_\alpha^{I}(x,y)&=\frac{1}{2}(f_\alpha(x,y)+f_\alpha(x,-y)),\\ f_3^{I}(x,y)&=\frac{1}{2}(f_3(x,y)-f_3(x,-y)),\\ g_3^{I}(x)&=\frac{1}{2}(g_3^{+}(x)+g_3^{-}(x)),\\ g_\alpha^{I}(x)&=\frac{1}{2}(g_\alpha^{+}(x)-g_\alpha^{-}(x)),\quad \alpha=1,2\end{aligned} \tag{2.40}$$

and

$$f_\alpha^{II}(x,y) = \frac{1}{2}(f_\alpha(x,y) - f_\alpha(x,-y)),$$

$$f_3^{II}(x,y) = \frac{1}{2}(f_3(x,y) + f_3(x,-y)),$$

(2.41)

$$g_3^{II}(x) = \frac{1}{2}(g_3^+(x) - g_3^-(x)),$$

$$g_\alpha^{II}(x) = \frac{1}{2}(g_\alpha^+(x) + g_\alpha^-(x)), \quad \alpha = 1,2.$$

3. Hierarchic plate models

Hierarchic plate models are obtained by semidiscretization of the plate problem (2.33) in the transverse direction and energy projection.

Since we deal with an elliptic system, we admit different model orders for the different displacement components. We approximate each component $u_i(x,y)$ of the displacement field $u(x,y)$ by a Legendre series in y of degrees less than or equal to n_i. The maximal "transverse" polynomial degrees n_i, $i = 1,2,3$, are collected in the vector $n \in \mathbb{N}_0^3$. Let, more generally, $\psi_i = \{\psi_{ik}(z)\}_{0\le k\le n_i}$, $i = 1,2,3$, denote vectors of $n_i + 1$ linearly independent basis functions (the so-called *director functions*) in $H^1(-1,1)$. Then the function $u^{\mathbf{n}}(x)$, the solution of the dimensionally reduced plate model of order $\mathbf{n}$, is any minimizer of the total energy $\mathscr{G}(u)$ in (2.29) over the subspace $\mathscr{H}(\mathbf{n}) \subset \mathscr{H}(\Omega)$ of admissible displacement fields of the form

$$u_i^{\mathbf{n}}(x,y) = \sum_{k=0}^{n_i} X_{ik}^{\mathbf{n}}(x)\psi_{ik}\left(\frac{y}{d}\right) = X_i^{\mathbf{n}}(x)^\top \psi_i\left(\frac{y}{d}\right) \quad i = 1,2,3\ . \tag{3.1}$$

Here the coefficient functions $X_{ik}^{\mathbf{n}}(x) \in H^1(\omega)$ may be interpreted as generalized rotations and deflections.

Proposition 3.1. $\mathscr{H}(\mathbf{n}) \subset \mathscr{H}(\Omega)$ *is a closed, linear subspace.*

Proof. Let $\{u(\ell)\}_{\ell=1}^\infty \subset \mathscr{H}(\mathbf{n})$ be a Cauchy sequence in the energy norm. Since $\mathscr{H}(\Omega) \subset [H^1(\Omega)]^3$ is closed by Theorem 2.1, we have $u = \lim_{\ell\to\infty} u(\ell) \in \mathscr{H}(\Omega)$. We claim that in fact $u \in \mathscr{H}(\mathbf{n})$. To prove it, we note that by Korn's inequality in $\mathscr{H}(\Omega)$

$$\|u(\ell) - u(\ell')\|_{H^1(\Omega)} \le C(t,\omega)\|u(\ell) - u(\ell')\|_{E(\Omega)} \to 0 \tag{3.2}$$

as $\ell, \ell' \to \infty$. Now $u(\ell) = (u_1(\ell), u_2(\ell), u_3(\ell))^\top$ with

$$u_i(\ell) = X_i^{(\ell)}(x)^\top \psi_i(y/d)\ , \quad X_i^{(\ell)} \in \left[H^1(\omega)\right]^{n_i+1}\ , \quad i = 1,2,3\ .$$

Further, for every vector fuction $X \in [H^1(\omega)]^n$ and $\varphi \in [H^1(-d,d)]^n$ we have (summation over repeated indices)

$$\left\|X^\top\varphi\right\|_{H^1(\omega)}^2 = d^{-1}\int_\omega \nabla X_j A_{ij} \nabla X_i \, dx + \int_\omega X_j \left(dB_{ij} + d^{-1}A_{ij}\right) X_i \, dx$$

where

$$A_{ij} = \int_{-1}^{1} \varphi_i \varphi_j \, dz \ , \quad B_{ij} = \int_{-1}^{1} \varphi_i' \varphi_j' \, dz \ .$$

If the vector function φ has linearly independent component functions, the matrix A is positive definite and B is positive semidefinite. Hence

$$\left\| X^\top \varphi \right\|^2_{H^1(\omega)} \geq d^{-1} \lambda_{\min}(A) \|X\|^2_{H^1(\omega)} \ .$$

Consequently, from (3.2) we have for $i = 1, 2, 3$

(3.3)
$$\left\| X_i^{(\ell)} - X_i^{(\ell')} \right\|^2_{H^1(\omega)} \leq \frac{d}{\lambda_{\min}(A)} \|u(\ell) - u(\ell')\|^2_{H^1(\Omega)} \leq \frac{dC(t,\omega)}{\lambda_{\min}(A)} \|u(\ell) - u(\ell')\|^2_{E(\Omega)}$$

i.e. that each sequence of coefficient functions $\{X_{ik}^{(\ell)}\}_{\ell=1}^{\infty}$ is Cauchy in $H^1(\omega)$.

Consequently,

$$X_{ik}^{(\ell)} \to X_{ik} \in H^1(\omega) \quad \text{as} \quad \ell \to \infty \tag{3.4}$$

(and $X_{ik} \in H^1(\omega, \gamma)$ if $k \in \mathscr{T}_i$, $i = 1, 2, 3$). Define u componentwise via $u_i(x, y) := X_i(x)^\top \psi_i(y/d)$, $i = 1, 2, 3$. Then $u \in \mathscr{H}(\mathbf{n})$ and, by Korn's inequality,

$$\|u - u(\ell)\|^2_{E(\Omega)} \leq C(t,\omega) \|u - u(\ell)\|^2_{H^1(\Omega)} \leq C(t,\omega) \sum_{i=1}^{3} \|X_i - X_i^{(\ell)}\|^2_{H^1(\omega)} \ .$$

By (3.4), the upper bound tends to zero as $\ell \to \infty$, hence $u = \lim_{\ell\to\infty} u(\ell) \in \mathscr{H}(\mathbf{n})$. The proof is complete. □

The hierarchic plate model is obtained by energy projection onto $\mathscr{H}(\mathbf{n})$ and can consequently be split like the three dimensional solution into a membrane part $u^I(\mathbf{n})$ and a bending part $u^{II}(\mathbf{n})$. They are obtained independently from each other by

$$u^j(\mathbf{n}) \in \mathscr{H}^j(\mathbf{n}) \quad \mathscr{B}(u^j(\mathbf{n}), v) = \mathscr{F}^j(v) \quad \forall v \in \mathscr{H}^j(\mathbf{n}) \ , \quad j \in \{I, II\} \tag{3.5}$$

with $\mathscr{H}^j(\mathbf{n}) = \mathscr{H}(\mathbf{n}) \cap \mathscr{H}^j(\Omega)$ a closed, linear subspace of $[H^1(\Omega)]^3$ by Proposition 3.1. This implies, as in Theorem 2.1, that for every $\mathbf{n}$ there exists a unique, dimensionally reduced solution $u^j(\mathbf{n}) \in \mathscr{H}^j(\mathbf{n})$.

For homogeneous materials with constitutive law (2.4), we select as director functions $\psi_{ik}(z)$, as mentioned above, the Legendre polynomials $L_k(z)$ of degree k. For $\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^3$ we write $\mathbf{n} \succeq \mathbf{m} \Leftrightarrow n_i \geq m_i$, $i = 1, 2, 3$. The relation $\mathbf{n} \preceq \mathbf{m}$ is defined analogously. We write further $\mathbf{n} \succ \mathbf{m} \Leftrightarrow n_i > m_i$, $i = 1, 2, 3$ etc.

Remark 3.1. The selection of polynomial director functions for *homogeneous* materials ensures the asymptotic optimality for the hierarchic models. For *laminated* places, however, this approach is not optimal and the polynomials are replaced by other, material dependent director functions ψ_{ik} with better approximation properties [1, 5, 11, 18, 27].

Remark 3.2. The hierarchic model and the dimensionally reduced solution $u(\mathbf{n})$ depend only on the span of the director functions. The selection of a particular basis for this span becomes important, however, in connection with iterative solution techniques for the hierarchic models. It was shown in [21] that for homogeneous plates the director functions $\{\psi_{ik}(z)\}_{k=0}^{n_i}$ should be orthogonal in $L^2(-1,1)$ *and* in $H^1(-1,1)$. This can be achieved by solving a generalized eigenvalue problem in the plate cross section and ensures a convergence rate proportional to the plate thickness for classical additive and multiplicative Schwarz algorithms.

Equation (3.5) yields, after evaluation of the integrals in the transverse coordinate, a singularly perturbed elliptic system for the unknown vector functions $X_i^{\mathbf{n}}(x) = \{X_{ik}^{\mathbf{n}}\}_{k=0}^{n_i}$. For example, for isotropic material (2.10) we obtain with the Gram-matrices of the director functions

$$\mathbf{A}_{ij} = \int_{-1}^{1} \psi_i \psi_j^{\top} dz \ , \quad \mathbf{B}_{ij} = \int_{-1}^{1} \psi_i' \psi_j'^{\top} dz \ ,$$
$$\mathbf{C}_{ij} = \int_{-1}^{1} \psi_i \psi_j'^{\top} dz \ , \quad 1 \le i,j \le 3 \ .$$

the following strongly elliptic system in ω for the vector functions $X_i^{\mathbf{n}}(x)$, $i = 1,2,3$

$$\Bigg\{ -d \begin{pmatrix} \mu\mathbf{A}_{11} & & \\ & \mu\mathbf{A}_{22} & \\ & & \mu\mathbf{A}_{33} \end{pmatrix} \Delta + d^{-1} \begin{pmatrix} \mu\mathbf{B}_{11} & & \\ & \mu\mathbf{B}_{22} & \\ & & (\lambda+2\mu)\mathbf{B}_{33} \end{pmatrix}$$
$$- \begin{pmatrix} d(\lambda+\mu)\mathbf{A}_{11}\partial_{11}^2 & d(\lambda+\mu)\mathbf{A}_{12}\partial_{12}^2 & (\lambda\mathbf{C}_{13} - \mu\mathbf{C}_{31}^{\top})\partial_1 \\ d(\lambda+\mu)\mathbf{A}_{21}\partial_{21}^2 & d(\lambda+\mu)\mathbf{A}_{22}\partial_{22}^2 & (\lambda\mathbf{C}_{23} - \mu\mathbf{C}_{32}^{\top})\partial_2 \\ (\mu\mathbf{C}_{31} - \lambda\mathbf{C}_{13}^{\top})\partial_1 & (\mu\mathbf{C}_{32} - \lambda\mathbf{C}_{23}^{\top})\partial_2 & 0 \end{pmatrix} \Bigg\} \begin{pmatrix} X_1^{\mathbf{n}} \\ X_2^{\mathbf{n}} \\ X_3^{\mathbf{n}} \end{pmatrix}$$
$$(3.6) \qquad = \begin{pmatrix} \int_{-d}^{d} f_1(x,z)\psi_1(\frac{z}{d})dz\, \psi_1(\frac{y}{d}) \\ \int_{-d}^{d} f_2(x,z)\psi_2(\frac{z}{d})dz\, \psi_2(\frac{y}{d}) \\ \int_{-d}^{d} f_3(x,z)\psi_3(\frac{z}{d})dz\, \psi_3(\frac{y}{d}) \end{pmatrix} + \begin{pmatrix} g_1^+(x)\psi_1(1) - g_1^-(x)\psi_1(-1) \\ g_2^+(x)\psi_2(1) - g_2^-(x)\psi_2(-1) \\ g_3^+(x)\psi_3(1) - g_3^-(x)\psi_3(-1) \end{pmatrix}$$

with the conditions

$$\gamma_0(X_{il}^{\mathbf{n}}) = 0 \quad \text{on} \quad \gamma \ , \quad l \in \mathcal{T}_i \ , \quad i \in \{n,t,3\}$$

where $X_{nl}^{\mathbf{n}} := X_{1l}^{\mathbf{n}} n_1 + X_{2l}^{\mathbf{n}} n_2$, $X_{tl}^{\mathbf{n}} := X_{1l}^{\mathbf{n}} t_1 + X_{2l}^{\mathbf{n}} t_2$.

Generally we have the following relations for the model orders in dependence on the maximal transverse polynomial degree q.

$$(3.7) \qquad \begin{aligned} \mathbf{n} &= (2\lfloor q/2 \rfloor, 2\lfloor q/2 \rfloor, 2\lfloor (q-1)/2 \rfloor + 1) && \text{for} \quad j = I \ , \\ \mathbf{n} &= (2\lfloor (q-1)/2 \rfloor + 1, 2\lfloor (q-1)/2 \rfloor + 1, 2\lfloor q/2 \rfloor) && \text{for} \quad j = II \end{aligned}$$

where $\lfloor x \rfloor$ denotes the largest integer $\le x$. In what follows we shall only consider these model orders. The following proposition collects some basic properties of the hierarchic plate models.

Table 1. Model orders **n** for membrane and bending models in dependence on maximal transverse degree q

j	$q = 1$	$q = 2$	$q = 3$	$q = 4$	$q = 5$	$q = 6$
I	(0,0,1)	(2,2,1)	(2.2.3)	(4,4,3)	(4,4,5)	(6,6,5)
II	(1,1,0)	(1,1,2)	(3,3,2)	(3,3,4)	(5,5,4)	(5,5,6)

Proposition 3.2. *1. Optimality of the n-model,*

(3.8) $$\|u^j - u^j(n)\|_{E(\Omega)} \leq \|u^j - v\|_{E(\Omega)} \quad \forall v \in \mathscr{H}^j(n)\ .$$

2. Let $\mathbf{n} \succeq \mathbf{m}$*. Then we have*

(3.9) $$\|u^j - u^j(\mathbf{n})\|_{E(\Omega)} \leq \|u^j - u^j(\mathbf{m})\|_{E(\Omega)}\ ,$$

i.e. an increase of the model order never increases the modeling error.

3. Convergence of the sequence of **n***-models towards the three-dimensional problem at fixed, positive thickness.*

(3.10) $$\lim_{\mathbf{n}\to\infty} \|u^j - u^j(\mathbf{n})\|_{E(\Omega)} = 0\ .$$

Proof. Statement 1. is a consequence of $u(n)$ being the energy projection of the three-dimensional solution u onto $\mathscr{H}(\mathbf{n})$. Therefore it follows from (2.33) and (3.5) that

$$\mathscr{B}(u - u(\mathbf{n}), v) = 0 \quad \forall v \in \mathscr{H}(\mathbf{n})\ .$$

This implies 1.

Assertion 2. follows from 1. and the inclusion $\mathscr{H}(\mathbf{n}) \supseteq \mathscr{H}(\mathbf{m})$ for $\mathbf{n} \succeq \mathbf{m}$.

The third assertion follows the density of the polynomials in $L^2(-1, 1)$. This implies that the sequence of spaces $\mathscr{H}(\mathbf{n})$ is dense in $\mathscr{H}(\Omega)$ which, with 2., yields 3. □

Remark 3.3. We remark that the Reissner-Mindlin (RM) plate model for homogeneous and isotropic plates is, for $\nu > 0$, *not* contained in this hierarchy. This can be readily verified by comparing (3.6) for $\mathbf{n} = (1, 1, 0)$ with the equations for the RM model. The RM model can be derived, however, by suitably modifying the elastic moduli used in the energy minimization, see [20] and [3] for more. In this case, however, the estimators to be derived below will not allow to estimate the modeling error. Nevertheless, other estimators based on dual variational principles can then be used to obtain computable a-posteriori modelling error estimates [20, Sect. 3.4].

4. A-posteriori estimation of the modeling error

In the present section we will derive computable *a-posteriori* estimators for the modeling error

$$e(\mathbf{n}) = u - u(\mathbf{n}) \tag{4.1}$$

of the hierarchic plate models in the energy norm.

For the error estimation we will utilize the decomposition of the modeling error into a membrane part $e^I(\mathbf{n})$ and a bending part $e^{II}(\mathbf{n})$ which can be estimated independently of each other due to their orthogonality in energy, i.e.

$$e^j(\mathbf{n}) = u^j - u^j(\mathbf{n}), \quad j \in \{I, II\}, \qquad \mathcal{B}(e^I, e^{II}) = 0.$$

This implies in particular

$$\|e(\mathbf{n})\|^2_{E(\Omega)} = \left\|e^I(\mathbf{n})\right\|^2_{E(\Omega)} + \left\|e^{II}(\mathbf{n})\right\|^2_{E(\Omega)}. \tag{4.2}$$

Our modeling error estimators will be based on the residual tractions

$$r^j_{\mathbf{n}}(x) := g^j(x) - g^j_{\mathbf{n}}(x), \qquad j \in \{I, II\}, \tag{4.3}$$

with $g^j_{\mathbf{n}}(x) = \sigma[u^j(\mathbf{n})]e_3\,|_{y=d}$ denoting the normal tractions corresponding to the $\mathbf{n}$-plate model obtained directly (i.e. without reference to the equilibrium equations) from $u(\mathbf{n})$ via Hooke's law (2.9). We recall further that $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$.

The main result on a-posteriori modeling error estimation is as follows.

Theorem 4.1. *Let the material be homogeneous with a positive definite constitutive matrix* (2.9) *and let the model order* $\mathbf{n}$ *be given in dependence on* q *as in* (3.7). *Assume further that the surface and volume forces are square integrable and that the volume forces* $f^j(x, y)$ *are, for almost every* $x \in \omega$, *polynomials with respect to* y *of degree* $\mathbf{m} \succeq \mathbf{n}$. *Let the midsurface* $\omega \subset \mathbb{R}^2$ *be an open, bounded Lipschitz domain.*

Then for any of the variational edge conditions of Sect. 2 *there hold the a-posteriori modeling error estimations:*

$$\begin{aligned}\left\|e^I(\mathbf{n})\right\|^2_{E(\Omega)} \leq t\Bigg\{ & a(q)\int_\omega \begin{pmatrix} r^I_{\mathbf{n}1} \\ r^I_{\mathbf{n}2}\end{pmatrix}^\top \begin{pmatrix} a_{55} & a_{56} \\ a_{56} & a_{66}\end{pmatrix}^{-1} \begin{pmatrix} r^I_{\mathbf{n}1} \\ r^I_{\mathbf{n}2}\end{pmatrix} dx \\ & + \frac{b(q)}{2a_{33}} \int_\omega \left(r^I_{\mathbf{n}3}\right)^2 dx \Bigg\}\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\left\|e^{II}(\mathbf{n})\right\|^2_{E(\Omega)} \leq t\Bigg\{ & b(q)\int_\omega \begin{pmatrix} r^{II}_{\mathbf{n}1} \\ r^{II}_{\mathbf{n}2}\end{pmatrix}^\top \begin{pmatrix} a_{55} & a_{56} \\ a_{56} & a_{66}\end{pmatrix}^{-1} \begin{pmatrix} r^{II}_{\mathbf{n}1} \\ r^{II}_{\mathbf{n}2}\end{pmatrix} dx \\ & + \frac{a(q)}{2a_{33}} \int_\omega \left(r^{II}_{\mathbf{n}3}\right)^2 dx \Bigg\}\end{aligned} \tag{4.5}$$

where $a(q)$ and $b(q)$ are defined by

$$(4.6) \qquad a(q) = \left(2\left\lfloor \frac{q}{2} \right\rfloor + \frac{3}{2}\right)^{-1}, \qquad b(q) = \left(2\left\lfloor \frac{q+1}{2} \right\rfloor + \frac{1}{2}\right)^{-1}$$

with $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$.

The remainder of the present section is devoted to the proof of Theorem 4.1. Prior to giving it, however, we mention some special cases of the general bound where the constants simplify.

Corollary 4.1. *Under the assumptions of Theorem* 4.1 *with the additional condition* $a_{56} = 0$ *there hold the a-posteriori modeling error estimations*

1. *Membrane part: for* $q \geq 1$

$$\left\|e^I(\mathbf{n})\right\|^2_{E(\Omega)} \leq t \left\{ \frac{1}{2\left[\frac{q}{2}\right] + \frac{3}{2}} \left[\frac{\left\|r^I_{\mathbf{n}1}\right\|^2_{L^2(\omega)}}{a_{55}} + \frac{\left\|r^I_{\mathbf{n}2}\right\|^2_{L^2(\omega)}}{a_{66}} \right] + \frac{1}{2\left[\frac{q+1}{2}\right] + \frac{1}{2}} \frac{\left\|r^I_{\mathbf{n}3}\right\|^2_{L^2(\omega)}}{2a_{33}} \right\},$$

2. *Bending part: for* $q \geq 2$

$$\left\|e^{II}(\mathbf{n})\right\|^2_{E(\Omega)} \leq t \left\{ \frac{1}{2\left[\frac{q+1}{2}\right] + \frac{1}{2}} \left[\frac{\left\|r^{II}_{\mathbf{n}1}\right\|^2_{L^2(\omega)}}{a_{55}} + \frac{\left\|r^{II}_{\mathbf{n}2}\right\|^2_{L^2(\omega)}}{a_{66}} \right] + \frac{1}{2\left[\frac{q}{2}\right] + \frac{3}{2}} \frac{\left\|r^{II}_{\mathbf{n}3}\right\|^2_{L^2(\omega)}}{2a_{33}} \right\}.$$

Remark 4.1. Corollary 4.1 applies in particular to *orthotropic* materials: the constants a_{33}, a_{55} and a_{66} that are relevant for the error estimate can be expressed in terms of the engineering moduli (2.11). Obviously,

$$a_{55} = 2G_2, \qquad a_{66} = 2G_3$$

and a short calculation yields

$$a_{33} = E_3 \frac{1 - \nu_{12}\nu_{21}}{1 - \nu_{12}\nu_{21} - \nu_{31}(\nu_{13} + \nu_{12}\nu_{23}) - \nu_{32}(\nu_{23} + \nu_{13}\nu_{21})} \quad .$$

Another important special case of Theorem 4.1 occurs for *isotropic* materials. Here we get

Corollary 4.2. *Under the assumptions of Theorem* 4.1 *the a-posteriori error estimate for isotropic materials* (2.10) *reads*

1. *Membrane part:*

$$\left\|e^I(\mathbf{n})\right\|^2_{E(\Omega)} \leq \frac{t}{2\mu}\left\{\frac{\left\|r^I_{\mathbf{n}1}\right\|^2_{L^2(\omega)} + \left\|r^I_{\mathbf{n}2}\right\|^2_{L^2(\omega)}}{2\lfloor\frac{q}{2}\rfloor + \frac{3}{2}} + \frac{1-2\nu}{2(1-\nu)}\,\frac{\left\|r^I_{\mathbf{n}3}\right\|^2_{L^2(\omega)}}{2\lfloor\frac{q+1}{2}\rfloor + \frac{1}{2}}\right\},$$

2. *Bending part:*

$$\left\|e^{II}(\mathbf{n})\right\|^2_{E(\Omega)} \leq \frac{t}{2\mu}\left\{\frac{\left\|r^{II}_{\mathbf{n}1}\right\|^2_{L^2(\omega)} + \left\|r^{II}_{\mathbf{n}2}\right\|^2_{L^2(\omega)}}{2\lfloor\frac{q+1}{2}\rfloor + \frac{1}{2}} + \frac{1-2\nu}{2(1-\nu)}\,\frac{\left\|r^{II}_{\mathbf{n}3}\right\|^2_{L^2(\omega)}}{2\lfloor\frac{q}{2}\rfloor + \frac{3}{2}}\right\}.$$

It remains to prove Theorem 4.1. This will be done in several steps in the remainder of this section. First, we show that the residual tractions $r^j_{\mathbf{n}}(x)$ in (4.3) are always square integrable over ω and we prove a variational representation for the modeling error in terms of the residual tractions alone. The modeling error estimate follows then directly with the Schwarz inequality. The constant in the estimate will be analyzed by means of a covering $\mathscr{C}$ of ω by a family of small closed, axiparallel squares $\mathbf{q}$ with disjoint interior. The contribution to the constant from each “box” $\mathbf{q}\times(-d,d)$, $\mathbf{q}\in\mathscr{C}$ is then estimated. This is followed by an asymptotic analysis of the constants in these local estimates as the size of $\mathbf{q}$ tends to zero.

Remark 4.2. The basic idea of the proof has been first obtained in [20, Chap. 2]. There, however, only orthotropic materials were treated. For orthotropic plates with totally clamped edge Γ an a-posteriori modeling error estimate with *numerically computed constants* has been presented in [19]. Due to the *asymptotic exactness* of the estimators in Theorem 4.1, these bounds are generally not sharper than the ones in Theorem 4.1. This is evidenced by the isotropic case, where the computed bounds in [19] were very close to those in Corollary 4.2 above.

4.1. Variational characterization of the modeling error

We investigate the regularity of the residual tractions $r^j_{\mathbf{n}}(x)$ on which our modeling-error estimates will be based.

Lemma 4.1. *Assume that the given surface tractions $g^+(x)$, $g^-(x)$ are square integrable over the faces $R_\pm$. Then, for any admissible midsurface ω and any variational edge condition, the residual tractions $r^j_{\mathbf{n}}$ in* (4.3)*are square integrable.*

Proof. Due to our assumptions, $g^j(x)\in L^2(\omega)$. By (4.3) it remains to show that

$$g^j_{\mathbf{n}}(x) = \sigma[u^j(\mathbf{n})]e_3\,|_{y=d}\in L^2(\omega).$$

This follows from the representation (3.1) with the smoothness of the director functions $\psi_{ik}(z) = L_k(z)$ and the fact that the coefficient functions $X^{\mathbf{n}}_{ik}(x)\in H^1(\omega)$. □

Remark 4.3. The natural space for the residual tractions is actually larger than $L^2(\omega)$, namely a suitable subspace of $H^{-1/2}(\omega)$. However, the corresponding norm is more difficult to evaluate which is why our estimators will be based on the $L^2(\omega)$-norm of the residual tractions. Lemma 4.1 ensures that the estimators will be well-defined for all $0 < d \leq 1$ and all model orders $\mathbf{n}$. It should be borne in mind, however, that the *convergence* $\left\| r^j_{\mathbf{n}} \right\|_{L^2(\omega)} \to 0$ as $d \to 0$ or $\mathbf{n} \to \infty$ may fail due to a lack of regularity of the three dimensional solution caused by the edge and vertex singularities of the plate problem.

We derive next a variational characterization of $e(\mathbf{n})$ essential to the development of the a-posteriori modeling error estimators.

Lemma 4.2. *Let the model order* $\mathbf{n}$ *be uniform and as in* (3.7). *Then the membrane- and bending part of the modeling error satisfy the residual equation*

$$e^j(\mathbf{n}) \in \mathscr{H}^j(\Omega) \quad \mathscr{B}(e^j(\mathbf{n}), v) = \mathscr{R}^j_{\mathbf{n}}(v) \quad \forall v \in \mathscr{H}^j(\omega), \quad j \in \{I, II\} \tag{4.7}$$

with

$$\mathscr{R}^j_{\mathbf{n}}(v) = \int_\omega r^j_{\mathbf{n}}(x)^\top \Phi^j_{\mathbf{n}}[v](x)dx + \int_\Omega R^j_{\mathbf{n}}[f](x,y)^\top v(x,y)dydx \tag{4.8}$$

and

$$\Phi^I_{\mathbf{n}}[v] = \begin{pmatrix} \int_{-d}^{d} \frac{\partial}{\partial y} v_1(x,y) L_{2\lfloor q/2 \rfloor+1}(\frac{y}{d})dy \\ \int_{-d}^{d} \frac{\partial}{\partial y} v_2(x,y) L_{2\lfloor q/2 \rfloor+1}(\frac{y}{d})dy \\ \int_{-d}^{d} \frac{\partial}{\partial y} v_3(x,y) L_{2\lfloor (q+1)/2 \rfloor}(\frac{y}{d})dy \end{pmatrix}, \tag{4.9}$$

$$\Phi^{II}_{\mathbf{n}}[v] = \begin{pmatrix} \int_{-d}^{d} \frac{\partial}{\partial y} v_1(x,y) L_{2\lfloor (q+1)/2 \rfloor}(\frac{y}{d})dy \\ \int_{-d}^{d} \frac{\partial}{\partial y} v_2(x,y) L_{2\lfloor (q+1)/2 \rfloor}(\frac{y}{d})dy \\ \int_{-d}^{d} \frac{\partial}{\partial y} v_3(x,y) L_{2\lfloor q/2 \rfloor+1}(\frac{y}{d})dy \end{pmatrix} \tag{4.10}$$

and the residual volume forces are given by

$$R^I_{\mathbf{n}}[f] = \begin{pmatrix} f^I_1(x,y) - \sum_{k=0}^{\lfloor q/2 \rfloor} \frac{4k+1}{2} \int_{-1}^{1} f^I_1(x,zd) L_{2k}(z)dz L_{2k}\left(\frac{y}{d}\right) \\ f^I_2(x,y) - \sum_{k=0}^{\lfloor q/2 \rfloor} \frac{4k+1}{2} \int_{-1}^{1} f^I_2(x,zd) L_{2k}(z)dz L_{2k}\left(\frac{y}{d}\right) \\ f^I_3(x,y) - \sum_{k=0}^{\lfloor (q-1)/2 \rfloor} \frac{4k+3}{2} \int_{-1}^{1} f^I_3(x,zd) L_{2k+1}(z)dz L_{2k+1}\left(\frac{y}{d}\right) \end{pmatrix}, \tag{4.11}$$

$$R^{II}_{\mathbf{n}}[f] = \begin{pmatrix} f^{II}_1(x,y) - \sum\limits_{k=0}^{\lfloor (q-1)/2 \rfloor} \frac{4k+3}{2} \int_{-1}^{1} f^{II}_1(x,zd) L_{2k+1}(z) dz L_{2k+1}\left(\frac{y}{d}\right) \\ f^{II}_2(x,y) - \sum\limits_{k=0}^{\lfloor (q-1)/2 \rfloor} \frac{4k+3}{2} \int_{-1}^{1} f^{II}_2(x,zd) L_{2k+1}(z) dz L_{2k+1}\left(\frac{y}{d}\right) \\ f^{II}_3(x,y) - \sum\limits_{k=0}^{\lfloor q/2 \rfloor} \frac{4k+1}{2} \int_{-1}^{1} f^{II}_3(x,zd) L_{2k}(z) dz L_{2k}\left(\frac{y}{d}\right) \end{pmatrix}. \tag{4.12}$$

Proof. Due to the homogeneity of the material there holds

$$\left(\operatorname{div}\sigma[u^j(\mathbf{n})]\right)_i(x,y) = \sum_{k=0}^{q} A^j_{ik}(x) L_k\left(\frac{y}{d}\right), \quad i = 1,2,3, \quad j \in \{I, II\} \tag{4.13}$$

with certain $A^j_{ik} \in H^{-1}(\omega)$. To determine them, we employ Green's identity (valid since Ω is a Lipschitz domain) as follows:

$$\begin{aligned} \mathscr{B}(e^j(\mathbf{n}), v) &= \mathscr{R}^j_{\mathbf{n}}(v) := \mathscr{F}^j(v) - \mathscr{B}(u^j(\mathbf{n}), v) \\ &= \int_\Omega v^\top \left(f^j + \operatorname{div}\sigma[u^j(\mathbf{n})]\right) dydx + \int_{R_+ \cup R_-} \gamma_0 v^\top \left(g^j - \sigma[u^j(\mathbf{n})]\mathbf{n}\right) do \\ &\quad + \int_\Gamma (\gamma_0 v^\top \sigma[u^j - u^j(\mathbf{n})]\mathbf{n}) do \qquad j \in \{I, II\}. \end{aligned}$$

The edge term vanishes for $v \in \mathscr{H}^j(\mathbf{n})$ due to $B_0 v = 0$ and $B_1 u^j = B_1 u^j(\mathbf{n}) = 0$ for the boundary operators B_i in (2.21), (2.22).

Let now $j = I$. Then there holds

$$\mathscr{B}(e^I(\mathbf{n}), v) = 0 \qquad \forall v \in \mathscr{H}^{II}(\Omega), v \in \mathscr{H}^I(\mathbf{n}).$$

For these v we have in particular (with summation over repeated indices)

$$\begin{aligned} 0 = \mathscr{R}^I_{\mathbf{n}}(v) &= \int_\Omega v_i \left(f^I_i + \sum_{k=0}^{q} A^I_{ik}(x) L_k\left(\frac{y}{d}\right)\right) dydx \\ &\quad + \int_\omega \left\{ r^I_{\mathbf{n}\alpha}(x)(v^+_\alpha + v^-_\alpha)(x) + r^I_{\mathbf{n}3}(x)(v^+_3 - v^-_3)(x) \right\} dx. \end{aligned} \tag{4.14}$$

We select now $v_1(x,y) = V_1(x) L_\ell\left(\frac{y}{d}\right)$ and $v_2 = v_3 \equiv 0$ for odd values of ℓ and an arbitrary $V_1(x) \in \overset{\circ}{H}{}^1(\omega)$. This implies

$$0 = \int_\omega V_1(x) \left\{ \int_{-d}^{d} f^I_1(x,y) L_\ell\left(\frac{y}{d}\right) dy + A^I_{1\ell}(x) \frac{2d}{2\ell+1} \right\} dx.$$

Since ℓ was assumed odd, L_ℓ is an odd function. According to (2.40), however, $f^I_1(x,y)$ is an even function of y. Thus we find

$$A^I_{1\ell}(x) \equiv 0 \qquad \ell \text{ odd}.$$

By analogous reasoning we find also

$$A^I_{2\ell}(x) \equiv 0 \qquad \ell \text{ odd}, \quad A^I_{3\ell}(x) \equiv 0 \qquad \ell \text{ even}$$

and

$$A^{II}_{\alpha\ell}(x) \equiv 0 \qquad \ell \text{ even}, \alpha = 1,2 \quad A^{II}_{3\ell}(x) \equiv 0 \qquad \ell \text{ odd.}$$

We consider the remaining cases. Let first $j = I$, $i = 1$ and ℓ even. We obtain from (4.14) with $v_1(x,y) = V_1(x)L_\ell\left(\frac{y}{d}\right)$ and arbitrary $V_1(x) \in H^1(\omega,\gamma)$ that

$$0 = \int_\omega V_1(x)\int_{-d}^{d}\left\{f_1^I(x,y)L_\ell\left(\frac{y}{d}\right) + \sum_{k=0}^{q} A^I_{1k}(x)L_k\left(\frac{y}{d}\right)L_\ell\left(\frac{y}{d}\right)\right\}dxdy$$
$$+2\int_\omega V_1(x)r^I_{\mathbf{n}1}(x)dx$$

from where we find that

$$A^I_{1\ell}(x) = -\frac{2\ell+1}{2d}\left\{2r^I_{\mathbf{n}1}(x) + \int_{-d}^{d} f_1^I(x,y)L_\ell\left(\frac{y}{d}\right)dy\right\}.$$

Analogously we obtain for $j = I$ and $i = 1,2$ or $j = II$ and $i = 3$

$$A^j_{i\ell}(x) = \begin{cases} -\dfrac{2\ell+1}{d}\left\{r^j_{\mathbf{n}i}(x) + \dfrac{1}{2}\displaystyle\int_{-d}^{d} f_i^j(x,y)L_\ell\left(\frac{y}{d}\right)dy\right\}, & \ell \text{ even,}\\ 0 & \ell \text{ odd}\end{cases}$$

and for $j = II$ and $i = 1,2$ or $j = I$ and $i = 3$ that

$$A^j_{i\ell}(x) = \begin{cases} 0 & \ell \text{ even,}\\ -\dfrac{2\ell+1}{d}\left\{r^j_{\mathbf{n}i}(x) + \dfrac{1}{2}\displaystyle\int_{-d}^{d} f_i^j(x,y)L_\ell\left(\frac{y}{d}\right)dy\right\}, & \ell \text{ odd.}\end{cases}$$

For $j = I$ we arrive at

$$\operatorname{div}\sigma[u^I(\mathbf{n})]$$
$$= \begin{cases} -\displaystyle\sum_{k=0}^{\lfloor q/2\rfloor}\frac{4k+1}{d}\left\{r^I_{\mathbf{n}1}(x) + \frac{1}{2}\int_{-d}^{d} f_1^I(x,y)L_{2k}\left(\frac{y}{d}\right)dy\right\}L_{2k}\left(\frac{y}{d}\right)\\ -\displaystyle\sum_{k=0}^{\lfloor q/2\rfloor}\frac{4k+1}{d}\left\{r^I_{\mathbf{n}2}(x) + \frac{1}{2}\int_{-d}^{d} f_2^I(x,y)L_{2k}\left(\frac{y}{d}\right)dy\right\}L_{2k}\left(\frac{y}{d}\right)\\ -\displaystyle\sum_{k=0}^{\lfloor (q-1)/2\rfloor}\frac{4k+3}{d}\left\{r^I_{\mathbf{n}3}(x) + \frac{1}{2}\int_{-d}^{d} f_3^I(x,y)L_{2k+1}\left(\frac{y}{d}\right)dy\right\}L_{2k+1}\left(\frac{y}{d}\right)\end{cases}$$

Inserting this into $\mathscr{R}^I_{\mathbf{n}}(v)$ and utilizing

$$L'_{2m+1}(z) = \sum_{k=0}^{m}(4k+1)L_{2k}(z), \quad L'_{2m+2}(z) = \sum_{k=0}^{m}(4k+3)L_{2k+1}(z)$$

results in (again with summation over repeated indices)

$$\begin{aligned}\mathscr{R}^I_{\mathbf{n}}(v) = \int_\Omega \Bigg\{ & v_\alpha \left[f^I_\alpha(x,y) - \sum_{k=0}^{[q/2]} \frac{4k+1}{2d} \int_{-d}^{d} f^I_\alpha(x,z) L_{2k}\left(\frac{z}{d}\right) dz L_{2k}\left(\frac{y}{d}\right)\right] \\ & + v_3 \left[f^I_3(x,y) - \sum_{k=0}^{[(q-1)/2]} \frac{4k+3}{2d} \int_{-d}^{d} f^I_3(x,z) L_{2k}\left(\frac{z}{d}\right) dz L_{2k+1}\left(\frac{y}{d}\right)\right] \Bigg\} dydx \\ & + \int_\omega \Bigg\{ r^I_{\mathbf{n}\alpha}(x) \left[(v^+_\alpha + v^-_\alpha)(x) - \int_{-d}^{d} v_\alpha(x,y) \frac{d}{dy}\left(L_{2[q/2]+1}\left(\frac{y}{d}\right)\right) dy \right] \\ & + r^{II}_{\mathbf{n}3}(x) \left[(v^+_3 - v^-_3)(x) - \int_{-d}^{d} v_3(x,y) \frac{d}{dy}\left(L_{2[(q-1)/2]+2}\left(\frac{y}{d}\right)\right) dy \right] \Bigg\} dx .\end{aligned}$$

An integration by parts with respect to y in the volume integral yields the assertion for $j = I$ since $2\lfloor (q-1)/2 \rfloor + 2 = 2\lfloor (q+1)/2 \rfloor$. For $j = II$ the proof is completely analogous. □

Remark 4.4. We point out that the densities $R^j_{\mathbf{n}}[f]$ of the volume residual forces are the remainders of the (fiberwise) Legendre expansions for the volume forces. Thus $R^j_{\mathbf{n}}[f]$ vanishes in particular for volume forces $f^j(x,y)$ that are, for every $x \in \omega$, polynomials of degree $\mathbf{m} \succeq \mathbf{n}$ in y. In this case

$$\begin{aligned} 0 &= \int_\omega r^I_{\mathbf{n}}(x)^\top \Phi^I_{\mathbf{n}}[v](x)dx \\ &= \int_\Omega R^I_{\mathbf{n}}[f^I](x,y)^\top v(x,y)dydx \qquad \forall v \in \mathscr{H}^I(\mathbf{n}), \\ 0 &= \int_\omega r^{II}_{\mathbf{n}}(x)^\top \Phi^{II}_{\mathbf{n}}[v](x)dx \\ &= \int_\Omega R^{II}_{\mathbf{n}}[f^{II}](x,y)^\top v(x,y)dydx \qquad \forall v \in \mathscr{H}^{II}(\mathbf{n}). \end{aligned} \tag{4.15}$$

4.2. Basic error estimate

Based on Lemma 4.2 it is now straightforward to derive the modelling error estimate.

Lemma 4.3. *Let $M \in \mathbb{R}^{3\times 3}$ be an arbitrary, nonsingular matrix. Assume that the volume forces $f^j(x,y)$ are polynomials of degree $\mathbf{m} \succeq \mathbf{n}$ in the variable y for a.e. $x \in \omega$.*

Then we have for every $0 < d \leq 1$ the modelling error estimate

(4.16) $$\left\|e^{j}(\mathbf{n})\right\|_{E(\Omega)} \leq C_j \left\|Mr^{j}_{\mathbf{n}}\right\|_{L^2(\omega)} \qquad j \in \{I, II\}.$$

Here the constants C_j are given by

$$\left(C_j(d,M)\right)^2 = \sup_{0 \neq v \in \mathscr{H}^{j}_{*}(\Omega)} \frac{\int_\omega \left|M^{-\top}\Phi^{j}_{\mathbf{n}}[v](x)\right|^2 dx}{\mathscr{B}(v,v)}$$

and the supremum is taken over the subset $\mathscr{H}^{j}_{*}(\Omega) \subset \mathscr{H}^{j}(\Omega)$ of admissible displacement fields for which

(4.17) $$\int_{-d}^{d} v_i(x,y) L_j\left(\frac{y}{d}\right) dy = 0 \text{ a.e. } x, j = 0, \ldots, n_i, i = 1, 2, 3$$

holds.

Proof. Since, by assumption, the volume forces are polynomial over each fiber with degree $\mathbf{m} \succeq \mathbf{n}$ the volume residuals $R^{j}_{\mathbf{n}}[f]$ drop out of the error estimate according to Lemma 4.2 and Remark 4.4. Thus we get from Lemma 4.2 for $j = I, II$

(4.18) $$\|e^{j}(\mathbf{n})\|_{E(\Omega)} = \sup_{0 \neq v \in \mathscr{H}^{j}(\Omega)} \frac{\mathscr{B}(e^{j}(\mathbf{n}), v)}{\|v\|_{E(\Omega)}} = \sup_{0 \neq v \in \mathscr{H}^{j}(\Omega)} \frac{\mathscr{R}^{j}_{\mathbf{n}}(v)}{\|v\|_{E(\Omega)}}$$

with

$$\mathscr{R}^{j}_{\mathbf{n}}(v) = \int_\omega r^{j}_{\mathbf{n}}(x)^\top \Phi^{j}_{\mathbf{n}}[v](x)dx = \int_\omega \left(Mr^{j}_{\mathbf{n}}(x)\right)^\top \Phi^{j}_{\mathbf{n}}[M^{-\top}v](x)dx$$

where $M \in \mathbb{R}^{3\times 3}$ is arbitrary, nonsingular. By the Schwarz inequality,

$$\left(\mathscr{R}^{j}_{\mathbf{n}}(v)\right)^2 \leq \left\|Mr^{j}_{\mathbf{n}}\right\|^2_{L^2(\omega)} \int_\omega \left|\Phi^{j}_{\mathbf{n}}[M^{-\top}v](x)\right|^2 dx$$

and we get from (4.18) the a-posteriori estimate

$$\|e^{j}(\mathbf{n})\|_{E(\Omega)} \leq C_j \left\|Mr^{j}_{\mathbf{n}}\right\|_{L^2(\omega)}$$

as asserted. □

It remains therefore to estimate the constants $C_j(d,M)$ in (4.16). To this end, we cover the midsurface ω by families of small, axiparallel squares. They are defined as follows.

Let $\varepsilon > 0$. By $\mathscr{M}(\varepsilon)$ we denote a covering of $\mathbb{R}^2$ by closed, axiparallel squares $\mathbf{q}$ of edgelength 2ε with the origin being a vertex. Then we define

$$\mathscr{C}(\varepsilon) = \{\mathbf{q} \in \mathscr{M}(\varepsilon) : \mathbf{q} \cap \omega \neq \emptyset\}$$

and

$$\omega_\varepsilon := \text{interior}\left(\bigcup_{\mathbf{q}\in\mathscr{C}(\varepsilon)} \mathbf{q}\right) \supseteq \omega.$$

Due to the way ω_ε was constructed there exists a constant $C(\gamma)$, independent of ε, such that

$$|\omega_\varepsilon \backslash \omega| \leq C(\gamma)\varepsilon. \tag{4.19}$$

By Ω_ε we mean the set

$$\Omega_\varepsilon := \omega_\varepsilon \times (-d, d) \supseteq \Omega.$$

For $\mathbf{q} \in \mathscr{C}(\varepsilon)$ we denote by Q the set $\mathbf{q} \times (-d, d)$.

We will also need prolongations $\tilde{v}$ of $v \in \mathscr{H}_*^j(\Omega)$ to $\mathscr{H}_*^j(\Omega_\varepsilon)$.

Lemma 4.4. *Assume that $\partial\Omega = \partial(\omega \times (-d, d))$ is Lipschitz. Let $v \in \left[H^1(\Omega)\right]^3$ be such that* (4.17) *holds. Let further $\tilde{\omega} \supseteq \omega$ be open and bounded and denote by $\tilde{\Omega} := \tilde{\omega} \times (-d, d)$.*

Then v has an extension $\tilde{v}$ to $\left[H^1(\tilde{\Omega})\right]^3$ satisfying (4.17).

Proof. Since $\partial\Omega$ is Lipschitz, we can use the extension Theorem of [25]. We extend v to V in $\left[H^1_{comp}(\mathbb{R}^3)\right]^3$. Then we define for a.e. $(x, y) \in \tilde{\Omega}$

$$\tilde{v}_i(x, y) := V_i(x, y) - \sum_{k=0}^{n_i} \frac{2k+1}{2} \int_{-1}^{1} V_i(x, zd) L_k(z) dz L_k(\frac{y}{d}).$$

One verifies that $\tilde{v}$ has the desired properties. $\square$

Remark 4.5. The norm of the extension operator $\mathscr{E} : v \to \tilde{v}$ in Lemma 4.4 in general depends on the Lipschitz constant of γ and on d. For clamped plates with homogeneous Dirichlet conditions, the zero extension can be used.

We are now in position to derive an estimate for the constants $C_j(M, d)$ in (4.16).

Let $0 < \theta \leq 1$ be a parameter and denote, for a given $v \in \mathscr{H}^j(\Omega)$, by $\tilde{v} \in \mathscr{H}_*^j(\Omega_d)$ its extension to Ω_d as constructed in Lemma 4.4. When dealing with the bilinear form $\mathscr{B}$ with the integration taken over sets $\Omega' \neq \Omega$, we write $\mathscr{B}(\Omega'; u, v)$ etc. If no set is specified, the meaning is as before. Then

$$\int_\omega \left|M^{-\top}\Phi^j_{\mathbf{n}}[v]\right|^2 dx \leq \int_{\omega_{\theta d}} \left|M^{-\top}\Phi^j_{\mathbf{n}}[\tilde{v}]\right|^2 dx = \sum_{\mathbf{q}\in\mathscr{C}(\theta d)} \int_{\mathbf{q}} \left|M^{-\top}\Phi^j_{\mathbf{n}}[\tilde{v}]\right|^2 dx. \tag{4.20}$$

Now, for every $\mathbf{q} \in \mathscr{C}(\theta d)$

$$\int_{\mathbf{q}} \left|M^{-\top}\Phi^j_{\mathbf{n}}[\tilde{v}]\right|^2 dx \leq D.\mathscr{B}(Q; v, v)$$

where $Q := \mathbf{q} \times (d, -d)$ and

$$D = \sup_{0 \neq v \in \mathscr{H}_*^j(Q)} \frac{\int_{\mathbf{q}} \left| M^{-\top} \Phi_{\mathbf{n}}^j[\tilde{v}] \right|^2 dx}{\mathscr{B}(Q; v, v)} .$$

Since the elasticity tensor A in the bilinear form does not depend on x the supremum D is independent of the particular cube $\mathbf{q}$. Moreover, scaling the variables of integration by $1/d$ we find that $D = d/\Lambda^j(M, \theta)$ where the constants $\Lambda^j(M, \theta)$ are defined by

$$\Lambda^j(M, \theta) = \inf_v \frac{\int_{\mathbf{K}(\theta)} \epsilon[v] : \sigma[v] d\overline{x} d\overline{y}}{\int_{\mathbf{k}(\theta)} (\overline{\Phi}_{\mathbf{n}}^j[v])^\top M^{-\top} M^{-1} \overline{\Phi}_{\mathbf{n}}^j[v] d\overline{x}} \tag{4.21}$$

with $\mathbf{k}(\theta) = (-\theta, \theta)^2$ and $\mathbf{K}(\theta) = \mathbf{k}(\theta) \times (-1, 1)$. The functionals $\overline{\Phi}_{\mathbf{n}}^j[v]$ are as in (4.9), (4.10) with $d = 1$ and the infimum in (4.21) is taken over all $v \in \mathscr{H}_*^j(\mathbf{K}(\theta))$.

Now we have with (4.20) that

$$\int_\omega \left| M^{-\top} \Phi_{\mathbf{n}}^j[v] \right|^2 dx \leq \frac{d}{\Lambda^j(M, \theta)} \sum_{\mathbf{q} \in \mathscr{C}(\theta d)} \mathscr{B}(Q; \tilde{v}, \tilde{v}) = \frac{d}{\Lambda^j(M, \theta)} \mathscr{B}(\Omega_{\theta d}; \tilde{v}, \tilde{v}) \tag{4.22}$$

which implies that

$$\left(C_j\right)^2 \leq \frac{d}{\Lambda^j(M, \theta)} \sup_{0 \neq v \in \mathscr{H}_*^j(\Omega)} \frac{\mathscr{B}(\Omega_{\theta d}; \tilde{v}, \tilde{v})}{\mathscr{B}(v, v)} . \tag{4.23}$$

The objective is now to let θ tend to zero for fixed v in the ratio of bilinear forms in (4.23). To this end we investigate the infima $\Lambda^j(M, \theta)$ in (4.21) in detail.

4.3. Analysis of $\Lambda^j(M, \theta)$

We investigate the constants $\Lambda^j(M, \theta)$ for $\theta \in (0, 1]$. Our first result shows that for $\theta \in (0, 1]$ the infimum in (4.21) is positive and indeed attained. It also characterizes the functions v on which it is attained.

Since (4.21) is formally a Rayleigh-quotient, we consider the eigenvalue problem associated with it. To this end, we define the bilinear forms

$$b(u, v) := \int_{\mathbf{K}(\theta)} \epsilon[v] : \sigma[u] dx dy \tag{4.24}$$

and

$$c^{j\mathbf{n}}(u, v) := \int_{\mathbf{k}(\theta)} (\overline{\Phi}_{\mathbf{n}}^j[v])^\top M^{-\top} M^{-1} (\overline{\Phi}_{\mathbf{n}}^j[u]) dx . \tag{4.25}$$

Then we consider the eigenvalue problem: Find $0 \neq u \in W$ and $\Lambda \in \mathbb{R}_0^+$ such that

$$b(u,v) = \Lambda c^{j\mathbf{n}}(u,v) \qquad \forall v \in W \tag{4.26}$$

where the admissible displacements W are given by

$$W = [H^1(\mathbf{K}(\theta))]^3 \cap \{u \mid \overline{\Phi}^j_{\mathbf{n}}[u] \neq 0\}.$$

Lemma 4.5. *Let $q \geq 1$ in* (3.7) *and $0 < \theta \leq 1$. Then we have for every model order $\mathbf{n}$ in* (3.7)

1°. *The spectrum of* (4.26) *is discrete and consists of a sequence $\{\Lambda^j_k\}_{k=1}^{\infty}$ of real eigenvalues which accumulate only at infinity,*
2°. *The eigenfunctions $u^j_k(x,y)$ corresponding to Λ^j_k are of the form*

$$u^I_k(x,y) = \begin{pmatrix} U^I_{1k}(x)Q_{2\lfloor q/2\rfloor+1}(y) \\ U^I_{2k}(x)Q_{2\lfloor q/2\rfloor+1}(y) \\ U^I_{3k}(x)Q_{2\lfloor (q+1)/2\rfloor}(y) \end{pmatrix}, \tag{4.27}$$
$$u^{II}_k(x,y) = \begin{pmatrix} U^{II}_{1k}(x)Q_{2\lfloor (q+1)/2\rfloor}(y) \\ U^{II}_{2k}(x)Q_{2\lfloor (q+1)/2\rfloor}(y) \\ U^{II}_{3k}(x)Q_{2\lfloor q/2\rfloor+1}(y) \end{pmatrix}$$

where $Q_k(y) := (L_{k+1} - L_{k-1})/(2k+1)$ is an antiderivative of the Legendre-polynomial of degree k and the $U^j_{ik}(x)$ are certain functions in $H^1(\mathbf{k}(\theta))$.

Proof. Since the bilinear forms $b(u,v)$ and $c^{j\mathbf{n}}(u,v)$ are both real and symmetric, it follows that the eigenvalues Λ in (4.26) are real, too.

To show 1°. we prove that the bilinear forms $c^{j\mathbf{n}}(u,v)$ are compact on $[H^1(\mathbf{K}(\theta))]^3 \times [H^1(\mathbf{K}(\theta))]^3$. To this end we integrate the expression (4.9) by parts with respect to y thus rewriting it in the form

$$\Phi^I_{\mathbf{n}}[v] = \begin{pmatrix} (v_1^+ + v_1^-)(x) - \int_{-d}^{d} v_1(x,y)\frac{d}{dy}L_{2\lfloor q/2\rfloor+1}(\frac{y}{d})dy \\ (v_2^+ + v_2^-)(x) - \int_{-d}^{d} v_2(x,y)\frac{d}{dy}L_{2\lfloor q/2\rfloor+1}(\frac{y}{d})dy \\ (v_3^+ - v_3^-)(x) - \int_{-d}^{d} v_3(x,y)\frac{d}{dy}L_{2\lfloor (q+1)/2\rfloor}(\frac{y}{d})dy \end{pmatrix} \tag{4.28}$$

where $v_i^{\pm}(x) = v_i(x,\pm d)$, $i = 1,2,3$.

Now we apply the Schwarz inequality and obtain

$$|c^{I\mathbf{n}}(u,v)| \leq |c^{I\mathbf{n}}(u,u)|^{1/2}|c^{I\mathbf{n}}(v,v)|^{1/2}.$$

Let $v \in \mathscr{H}^I(\Omega)$. Then we estimate as follows:

$$\begin{aligned} |c^{I\mathbf{n}}(v,v)| &\leq C(M)\int_{\mathbf{k}(\theta)} (\overline{\Phi}^I_{\mathbf{n}}[v])^{\top}(\overline{\Phi}^I_{\mathbf{n}}[v])dx \\ &= C(M)\sum_{\alpha=1}^{2}\int_{\mathbf{k}(\theta)} \left\{(v_\alpha^+ + v_\alpha^-)(x) - \int_{-1}^{1} v_\alpha(x,y)\frac{d}{dy}L_{2\lfloor q/2\rfloor+1}(y)dy\right\}^2 dx \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathbf{k}(\theta)} \left\{ (v_3^+ - v_3^-)(x) - \int_{-1}^{1} v_3(x,y) \frac{d}{dy} L_{2\lfloor (q+1)/2 \rfloor}(y) dy \right\}^2 dx \\
&\le 2C(M) \int_{\mathbf{k}(\theta)} \left\{ (v_1^+ + v_1^-)^2 + (v_2^+ + v_2^-)^2 + (v_3^+ - v_3^-)^2 \right\} dx \\
&\quad + C_{1q} \int_{\mathbf{K}(\theta)} v^\top v dx dy \\
&= C(M) \sum_{j=1}^{3} 8 \| v_j(\cdot, 1) \|^2_{L^2(\mathbf{k}(\theta))} + C_{1q} \| v_j \|^2_{L^2(\mathbf{K}(\theta))} \\
&\le C(M, \theta, d) \sum_{j=1}^{3} \| v_j \|^2_{H^{1/2+\varepsilon}(\mathbf{K}(\theta))}
\end{aligned}
$$

for $0 < \varepsilon \le 1/2$ due to the continuity of the trace operator. The assertion follows now from the compactness of the embedding

$$
H^1(\mathbf{K}(\theta)) \hookrightarrow H^s(\mathbf{K}(\theta)), \qquad 0 \le s < 1.
$$

The proof for $j = II$ is analogous.

To show 2° we consider the functions $\overline{\Phi}^j_{\mathbf{n}}$ in the form (4.9), (4.10). A necessary condition for the existence of eigenvalues Λ^j_k is that $c^{j\mathbf{n}}(u,u) \neq 0$ for the corresponding eigenfunction. Now $c^{j\mathbf{n}}(u,u) = 0$ happens if and only if $\frac{\partial u_i}{\partial y}$ equals the product of the Legendre polynomials in the definition of the functional $\overline{\Phi}^j_{\mathbf{n}}[v]$ and of a function $U_{ik}(x) \in H^1(\mathbf{k}(\theta))$, whence we arrive at (4.27).

Evidently we have $\Lambda^j_k \ge 0$. To verify that $\Lambda^j_k > 0$, we assume the contrary which implies that $b(u^j_k, u^j_k) = 0$ for the eigenfunction corresponding to Λ^j_k. This implies that $\epsilon[u^j_k] = 0$ which is the case if and only if u^j_k is a rigid body motion. Since, however, u^j_k is of the form (4.27), this is not possible for $q \ge 1$ and we arrive at a contradiction to $\Lambda^j_k = 0$. □

The constants $\Lambda^j(M,\theta)$ could now, for fixed $0 < \theta \le 1$, be approximated numerically in the usual way, i.e. by discretizing the eigenvalue problem (4.26) via restriction of both forms to a finite dimensional subspace and numerically solving the resulting finite dimensional, generalized eigenvalue problem. This suffices in certain cases to obtain a modeling error estimator. For details, see [19].

Since $\Lambda^j(M,\theta)$ is a continuous and positive function of $\theta \in (0,1]$ due to Lemma 4.5, we investigate whether $\lim_{\theta \to 0^+} \Lambda^j(M,\theta)$ exists and, if so, whether it is positive. This is the purpose of the following Lemma.

Lemma 4.6. *Under the assumptions of Theorem* 4.1, *and with*

$$
M = \begin{pmatrix} m_1 \begin{pmatrix} a_{55} & a_{56} \\ a_{56} & a_{66} \end{pmatrix} & 0 \\ 0 & m_2 a_{33} \end{pmatrix}^{-1/2} \tag{4.29}
$$

for arbitrary $m_1, m_2 > 0$ *the limits*

$$\Lambda^j(M,0) = \lim_{\theta\to 0^+} \Lambda^j(M,\theta)$$

exist and are equal to

$$\Lambda^I(M,0) = \min\left\{\frac{1}{2m_1 a(q)}, \frac{1}{m_2 b(q)}\right\} \tag{4.30}$$

and

$$\Lambda^{II}(M,0) = \min\left\{\frac{1}{2m_1 b(q)}, \frac{1}{m_2 a(q)}\right\} \tag{4.31}$$

respectively, where $a(q)$ *and* $b(q)$ *are as in* (4.6).

Proof. Let $j = I$ and set $\ell = 2\lfloor \frac{q}{2} \rfloor + 1$, $m = 2\lfloor \frac{q+1}{2} \rfloor$. Further, throughout this proof we set

$$\tilde{A} = \begin{pmatrix} a_{55} & a_{56} \\ a_{56} & a_{66} \end{pmatrix}. \tag{4.32}$$

Finally, for $0 < \theta \leq 1$ let $u^I(\overline{x}, \overline{y})$ be the eigenfunction corresponding to the smallest eigenvalue of the spectral problem (4.26). Then, according to Lemma 4.5,

$$u^I(\overline{x}, \overline{y}) = \begin{pmatrix} U_1(\overline{x}) Q_\ell(\overline{y}) \\ U_2(\overline{x}) Q_\ell(\overline{y}) \\ U_3(\overline{x}) Q_m(\overline{y}) \end{pmatrix}$$

and

$$\epsilon[u^I] = \left(\partial_1 U_1 Q_\ell, \partial_2 U_2 Q_\ell, U_3 Q_m', \frac{\partial_2 U_1 + \partial_1 U_2}{2} Q_\ell, \frac{U_1 Q_\ell' + \partial_1 U_3 Q_m}{2}, \frac{U_2 Q_\ell' + \partial_2 U_3 Q_m}{2}\right)^\top.$$

Thus, using that

$$\epsilon_{ij}[u^I]\sigma_{ij}[u^I] = \epsilon_{11}\sigma_{11} + \epsilon_{22}\sigma_{22} + \epsilon_{33}\sigma_{33} + 2\left(\epsilon_{12}\sigma_{12} + \epsilon_{13}\sigma_{13} + \epsilon_{23}\sigma_{23}\right),$$

the form of $u^I(\overline{x}, \overline{y})$ and performing the scaling $x = \overline{x}/\theta$, $dx = \theta^{-2} d\overline{x}$ we get with $U = (U_1, U_2, U_3)^\top$ that

$$\mathscr{B}(\mathbf{k}(\theta); u^I, u^I) = \mathscr{B}_0(\mathbf{k}; U, U) + \theta \mathscr{B}_1(\mathbf{k}; U, U) + \theta^2 \mathscr{B}_2(\mathbf{k}; U, U). \tag{4.33}$$

Here the bilinear forms $\mathscr{B}_i(\mathbf{k}; U, U)$ are independent of θ and explicitly given by

$$\mathscr{B}_0(\mathbf{k};U,U) = \int_{-1}^{1}(Q_\ell)^2 dz \int_{\mathbf{k}} \Big\{ a_{11}(\partial_1 U_1)^2 + 2a_{12}\partial_1 U_1 \partial_2 U_2 + a_{22}(\partial_2 U_2)^2$$
$$+\frac{3}{2}(\partial_1 U_2 + \partial_2 U_1)(a_{14}\partial_1 U_1 + a_{24}\partial_2 U_2) + \frac{a_{44}}{2}(\partial_2 U_1 + \partial_1 U_2)^2 \Big\} dx$$
$$+ \int_{-1}^{1}(Q_m)^2 dz \int_{\mathbf{k}} \Big\{ \frac{a_{55}}{2}(\partial_1 U_3)^2 + a_{56}\partial_1 U_3 \partial_2 U_3 + \frac{a_{66}}{2}(\partial_2 U_3)^2 \Big\} dx,$$

$$\mathscr{B}_1(\mathbf{k};U,U) =$$
$$\int_{-1}^{1} Q_\ell Q_m' dz \int_{\mathbf{k}} U_3 \Big\{ 2a_{13}\partial_1 U_1 + 2a_{23}\partial_2 U_2 + \frac{3}{2} a_{34}(\partial_1 U_2 + \partial_2 U_1) \Big\} dx$$
$$+ \int_{-1}^{1} Q_\ell' Q_m dz \int_{\mathbf{k}} \{ a_{55} U_1 \partial_1 U_3 + a_{56}(U_1 \partial_2 U_3 + U_2 \partial_1 U_3) + a_{66} U_2 \partial_2 U_3 \} dx,$$

$$\mathscr{B}_2(\mathbf{k};U,U) = a_{33} b(q) \int_{\mathbf{k}} (U_3)^2 dx$$
$$+ \frac{a(q)}{2} \int_{\mathbf{k}} \{ a_{55}(U_1)^2 + 2a_{56} U_1 U_2 + a_{66}(U_2)^2 \} dx$$

with $a(q) = \int_{-1}^{1}(Q_\ell')^2 dz$, $b(q) = \int_{-1}^{1}(Q_m')^2 dz$. Further, we find with

$$\overline{\Phi}^I_{\mathbf{n}}[u^I] = \begin{pmatrix} a(q)U_1(\overline{x}) \\ a(q)U_2(\overline{x}) \\ b(q)U_3(\overline{x}) \end{pmatrix}$$

and $\tilde{U} = (U_1, U_2)^\top$ that

$$c^{j\mathbf{n}}(u^I, u^I) = \theta^2 \mathscr{C}(U,U)$$

with the form $\mathscr{C}(\cdot,\cdot)$ given by

$$\mathscr{C}(U,U) = m_1(a(q))^2 \int_{\mathbf{k}} \Big\{ \tilde{U}^\top \left(\tilde{A}^{1/2}\right)^\top \tilde{A}^{1/2} \tilde{U} \Big\} dx + m_2 a_{33} b(q) \int_{\mathbf{k}} (U_3)^2 dx.$$

Hence

$$\Lambda^I(M,\theta) = \inf_U \frac{\theta^{-2}\mathscr{B}_0(\mathbf{k};U,U) + \theta^{-1}\mathscr{B}_1(\mathbf{k};U,U) + \mathscr{B}_2(\mathbf{k};U,U)}{\mathscr{C}(U,U)} .$$

For the uniform boundedness of $\Lambda^j(M,\theta)$ as $\theta \to 0$ we must have necessarily $\mathscr{B}_0(\mathbf{k};U,U) \to 0$. Therefore in the limit $\theta = 0$ the minimization is constrained to

$$U \in \mathscr{N} := \left\{ U : \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right\}, U_3 = \text{const.} \right\}$$

For $U \in \mathscr{N}$ we have also $\mathscr{B}_1(\mathbf{k};U,U) = 0$ as is easily verified. Thus, since $\dim \mathscr{N} = 4$, we get in the limit $\theta = 0$

$$\Lambda^I(M,0) = \inf_{U \in \mathscr{N}} \frac{\mathscr{B}_2(\mathbf{k};U,U)}{\mathscr{C}(U,U)}$$

which is a symmetric, generalized eigenvalue problem in $\mathbb{R}^4$. Since

$$\mathcal{N} = \mathcal{N}_1 \times \{1\}, \quad \mathcal{N}_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right\},$$

we get, setting $\tilde{V} = (m_1)^{1/2} \tilde{A}^{1/2} \tilde{U}$ and $V_3 = (m_2 a_{33})^{1/2} U_3$, that

$$\Lambda^I(M, 0) = \min \left\{ \frac{1}{2m_1 a(q)}, \frac{1}{m_2 b(q)} \right\}.$$

as claimed.

The proof for $j = II$ is analogous and yields the same expression for $\Lambda^{II}(M, 0)$, with $a(q)$ and $b(q)$ exchanged. □

4.4. Proof of Theorem 4.1

We are now in position to give the proof of Theorem 4.1. We recall that we have (4.16) with the constant $C_j(M, \theta)$ bounded by (4.23).

Let the matrix M be as in (4.29). We will show that for fixed $v \in \mathcal{H}^j(\Omega)$ and $d > 0$ we have

$$\lim_{\theta \to 0^+} \frac{\mathcal{B}(\Omega_{\theta d}; \tilde{v}, \tilde{v})}{\Lambda^j(M, \theta)} = \frac{\mathcal{B}(\Omega; v, v)}{\Lambda^j(M, 0)}. \tag{4.34}$$

To see it, we use that $\tilde{v}\,|_\Omega = v$ and write

$$\frac{\mathcal{B}(\Omega_{\theta d}; \tilde{v}, \tilde{v})}{\Lambda^j(M, \theta)} - \frac{\mathcal{B}(\Omega; v, v)}{\Lambda^j(M, 0)} = \frac{\mathcal{B}(\Omega_{\theta d} \backslash \Omega; \tilde{v}, \tilde{v})}{\Lambda^j(M, \theta)} + \mathcal{B}(\Omega; v, v) \frac{\Lambda^j(M, 0) - \Lambda^j(M, \theta)}{\Lambda^j(M, 0) \Lambda^j(M, \theta)}.$$

By (4.19) we have $|\Omega_{\theta d} \backslash \Omega| \le 2C(\gamma)\theta d^2$. Therefore $\mathcal{B}(\Omega_{\theta d} \backslash \Omega; \tilde{v}, \tilde{v}) \to 0$ as $\theta \to 0$ for fixed v and d. The statement (4.34) follows then with Lemma 4.6. We can now pass to the limit in (4.23) and get in (4.16) that

$$\left\| e^j(\mathbf{n}) \right\|^2_{E(\Omega)} \le \frac{d}{\Lambda^j(M, 0)} \left\| M r^j_{\mathbf{n}} \right\|^2_{L^2(\omega)} \qquad j \in \{I, II\}$$

where the constants $\Lambda^j(M, 0)$ are defined in (4.30), (4.31). Thus we obtain for $j = I$ the estimate

$$\left\| e^I(\mathbf{n}) \right\|^2_{E(\Omega)} \le d \, \frac{\dfrac{1}{m_1} \displaystyle\int_\omega \begin{pmatrix} r^I_{\mathbf{n}1} \\ r^I_{\mathbf{n}2} \end{pmatrix}^\top \begin{pmatrix} a_{55} & a_{56} \\ a_{56} & a_{66} \end{pmatrix}^{-1} \begin{pmatrix} r^I_{\mathbf{n}1} \\ r^I_{\mathbf{n}2} \end{pmatrix} dx + \dfrac{1}{m_2 a_{33}} \displaystyle\int_\omega \left(r^I_{\mathbf{n}3} \right)^2 dx}{\min \left\{ \dfrac{1}{2m_1 a(q)}, \dfrac{1}{m_2 b(q)} \right\}}.$$

This estimate holds for *any* $m_1, m_2 > 0$. We can therefore minimize the bound with respect to m_i. With the estimate

$$\min_{m\in\mathbb{R}^2_+}\left\{\frac{a/m_1 + b/m_2}{\min\{\alpha/m_1, \beta/m_2\}}\right\} \le \frac{a}{\alpha} + \frac{b}{\beta}.$$

(which follows with the choice $m_2 = m_1\beta/\alpha$) we obtain (4.4). This completes the proof of Theorem 4.1 for $j = I$. The proof for $j = II$ is analogous, with $a(q)$ and $b(q)$ interchanged.

5. Asymptotic exactness of the estimator

In this section we will prove that the modelling error estimator obtained in Theorem 4.1 is asymptotically, i.e. as $t \to 0$, and spectrally, i.e. as $q \to \infty$, exact provided some extra conditions on the data are met. A similar result for the heat conduction problem in a plate was first obtained in [6].

To state the result, we denote by EST^j a computable quantity constituting the estimator for the modeling error $e^j(\mathbf{n})$ in energy norm, i.e. the right hand sides of (4.4), (4.5) are equal to $(EST^j)^2$, $j = I, II$. Then we define the *effectivity index* of the estimator EST^j in the usual way

$$\Theta^j_{eff} = \frac{EST^j}{\|e^j(\mathbf{n})\|_{E(\Omega)}} \qquad j \in \{I, II\}.$$

Theorem 5.1. *Let the assumptions of Theorem* 4.1 *hold and assume in addition that*

$$0 \neq r^j_{\mathbf{n}}(x) \in \left[H^1(\omega)\right]^3.$$

Then there holds with

$$D = \begin{pmatrix} \frac{1}{2}\begin{pmatrix} a_{55} & a_{56} \\ a_{56} & a_{66}\end{pmatrix} & 0 \\ 0 & a_{33}\end{pmatrix}^{-1}$$

and the bilinear forms $\mathcal{B}_i$ defined in (4.33) *that*

$$\begin{aligned} 1 &\le \left(\Theta^j_{eff}\right)^2 \\ &\le 1 + d\,\frac{\mathcal{B}_1(\omega; Dr^j_{\mathbf{n}}, Dr^j_{\mathbf{n}}) + d\,\mathcal{B}_0(\omega; Dr^j_{\mathbf{n}}, Dr^j_{\mathbf{n}})}{\mathcal{B}_2(\omega; Dr^j_{\mathbf{n}}, Dr^j_{\mathbf{n}})} \quad j \in \{I, II\}. \end{aligned} \tag{5.1}$$

Proof. The lower bound for the effectivity index is evident from Theorem 4.1. It remains therefore to prove the upper bound.

To this end we assume that $j = I$ and omit in this proof the index j. We set, as in the proof of Theorem 4.1, $\ell = 2\lfloor\frac{q}{2}\rfloor + 1$, $m = 2\lfloor\frac{q+1}{2}\rfloor$ and use again $\tilde{A}$ as in (4.32). Since the volume forces are assumed to be polynomials of degree $\mathbf{m} \succeq \mathbf{n}$

in the transverse variable y, we have the variational characterization (4.7) of the modeling error with

$$\mathscr{R}_{\mathbf{n}}(v) = \int_{\omega} r_{\mathbf{n}}(x)^{\top} \Phi_{\mathbf{n}}[v](x) dx .$$

Now we select $v = v^*$ such that $\mathscr{R}_{\mathbf{n}}(v^*) = (EST)^2$ with EST denoting the computable modelling error estimator in Theorem 4.1. One verifies that this is the case for the selection

$$v^*(x,y) = dD \begin{pmatrix} Q_\ell\left(\frac{y}{d}\right) r_{\mathbf{n}1}(x) \\ Q_\ell\left(\frac{y}{d}\right) r_{\mathbf{n}2}(x) \\ Q_m\left(\frac{y}{d}\right) r_{\mathbf{n}3}(x) \end{pmatrix} .$$

This implies that

$$r_{\mathbf{n}}(x)^{\top} \Phi_{\mathbf{n}}[v^*](x) = d\left\{2a(q)\,(\tilde{r}_{\mathbf{n}}(x))^{\top} \tilde{A}^{-1} \tilde{r}_{\mathbf{n}}(x) + b(q)\,(r_{\mathbf{n}3}(x))^2 / a_{33}\right\}$$

with $a(q)$ and $b(q)$ as in (4.6) and $\tilde{r}_{\mathbf{n}} = (r_{\mathbf{n}1}, r_{\mathbf{n}2})^{\top}$. Comparison with (4.4) shows that $\mathscr{R}(v^*) = (EST)^2$. Now we estimate for any $\varepsilon > 0$

$$\mathscr{R}_{\mathbf{n}}(v^*) = (EST)^2 = \mathscr{B}(e(\mathbf{n}), v^*) \leq \frac{\varepsilon}{2} \|e(\mathbf{n})\|^2_{E(\Omega)} + \frac{1}{2\varepsilon} \|v^*\|^2_{E(\Omega)} .$$

Selecting $\varepsilon = \varepsilon_0$ such that

$$\frac{1}{2\varepsilon_0} \|v^*\|^2_{E(\Omega)} \leq \frac{1}{2}(EST)^2,$$

we arrive at

$$(EST)^2 \leq \varepsilon_0 \|e(\mathbf{n})\|^2_{E(\Omega)} .$$

It remains therefore to estimate ε_0. Due to the way v^* was chosen, we get

$$\varepsilon_0 = \frac{\|v^*\|^2_{E(\Omega)}}{\mathscr{R}_{\mathbf{n}}(v^*)} .$$

Now

$$\begin{aligned} \|v^*\|^2_{E(\Omega)} = \mathscr{B}(v^*, v^*) &= d^3 \mathscr{B}_0(\omega; Dr_{\mathbf{n}}, Dr_{\mathbf{n}}) \\ &\quad + d^2 \mathscr{B}_1(\omega; Dr_{\mathbf{n}}, Dr_{\mathbf{n}}) + d\mathscr{B}_2(\omega; Dr_{\mathbf{n}}, Dr_{\mathbf{n}}) \end{aligned}$$

and we find with the definition of $\mathscr{B}_2(\omega; U, U)$, i.e. with

$$\mathscr{B}_2(\omega; U, U) = a_{33} b(q) \int_{\omega} (U_3)^2\, dx + \frac{a(q)}{2} \int_{\omega} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\top} \tilde{A} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} dx ,$$

that

$$d\mathscr{B}_2(\omega; Dr_{\mathbf{n}}, Dr_{\mathbf{n}}) = \mathscr{R}_{\mathbf{n}}(v^*) = (EST)^2 .$$

From this follows the assertion for $j = I$. The proof for $j = II$ is completely analogous. □

The main significance of this result lies in that it gives a computable upper bound on the effectivity index of the modeling error estimator. The explicitly given bilinear forms $\mathscr{B}_i(\omega; U, U)$ allow moreover to formulate sufficient conditions for the spectral exactness of the estimator.

To this end, we define the class

$$T(\overline{A}, \varepsilon) = \Big\{(g^+, g^-): \text{ either } r^j_{\mathbf{n}} = 0 \text{ or} \tag{5.2}$$
$$\left|r^j_{\mathbf{n}}\right|_{H^1(\omega)} / \left\|r^j_{\mathbf{n}}\right\|_{L^2(\omega)} \le \overline{A}(q/t)^{1-\varepsilon}\Big\}$$

for some $\overline{A} > 0$ and $\varepsilon \ge 0$ independent of d. Then we have

Corollary 5.1. *Let $(g^+, g^-) \in T(\overline{A}, \varepsilon)$ for some $\overline{A} > 0$ and $\varepsilon > 0$. Then the modelling error estimator in Theorem* 4.1 *is asymptotically and spectrally exact.*

If $(g^+, g^-) \in T(\overline{A}, 0)$ for some $\overline{A} > 0$ the estimator in Theorem 4.1 *is asymptotically and spectrally uniform, i.e. its effectivity index is uniformly bounded with respect to t and q.*

Proof. Throughout the proof, C denotes a generic, positive constant depending only on the elastic moduli a_{ik}. We observe that

$$\int_{-1}^{1} (Q_\ell)^2\, dz = \frac{4}{(2\ell+1)((2\ell+1)^2 - 4)},$$
$$\int_{-1}^{1} Q_\ell Q'_m dz = \begin{cases} \dfrac{1}{(2\ell+1)(2\ell\pm 1)} & \text{if } m = \ell \pm 1, \\ 0 & \text{else.} \end{cases}$$

Thus we find with

$$\frac{\left|r^j_{\mathbf{n}}\right|^2_{H^1(\omega)}}{\left\|r^j_{\mathbf{n}}\right\|^2_{L^2(\omega)}} \le \overline{A}^2 (q/t)^{2-2\varepsilon}$$

that

$$\begin{aligned}\mathscr{B}_0(\omega; Dr^j_{\mathbf{n}}, Dr^j_{\mathbf{n}}) &\le Cq^{-3} \left|r^j_{\mathbf{n}}\right|^2_{H^1(\omega)} \\ &\le C\overline{A}^2 q^{-3+2-2\varepsilon} t^{-2+2\varepsilon} \left\|r^j_{\mathbf{n}}\right\|^2_{L^2(\omega)}\end{aligned}$$

and that

$$\begin{aligned}\mathscr{B}_1(\omega; Dr^j_{\mathbf{n}}, Dr^j_{\mathbf{n}}) &\le Cq^{-2} \left\{ \left\|r^j_{\mathbf{n}3}\right\|_{L^2(\omega)} \left|\tilde{r}^j_{\mathbf{n}}\right|_{H^1(\omega)} + \left\|\tilde{r}^j_{\mathbf{n}}\right\|_{L^2(\omega)} \left|r^j_{\mathbf{n}3}\right|_{H^1(\omega)} \right\} \\ &\le C\overline{A} q^{-2+1-\varepsilon} t^{-1+\varepsilon} \left\|r^j_{\mathbf{n}}\right\|^2_{L^2(\omega)}.\end{aligned}$$

Here we have set $\tilde{r}^j_{\mathbf{n}} = \left(r^j_{\mathbf{n}1}, r^j_{\mathbf{n}2}\right)^\top$. Furthermore, there also holds

$$\mathscr{B}_2(\omega; Dr^j_{\mathbf{n}}, Dr^j_{\mathbf{n}}) \geq Cq^{-1} \left\| r^j_{\mathbf{n}} \right\|^2_{L^2(\omega)} .$$

Consequently, we find that

$$d \frac{\mathscr{B}_1(\omega; Dr^j_{\mathbf{n}}, Dr^j_{\mathbf{n}})}{\mathscr{B}_2(\omega; Dr^j_{\mathbf{n}}, Dr^j_{\mathbf{n}})} + d^2 \frac{\mathscr{B}_0(\omega; Dr^j_{\mathbf{n}}, Dr^j_{\mathbf{n}})}{\mathscr{B}_2(\omega; Dr^j_{\mathbf{n}}, Dr^j_{\mathbf{n}})} \leq C\overline{A}(1 + \overline{A})(t/q)^{\varepsilon}$$

from where the assertion follows. □

Let us comment on the assumption $(g^+, g^-) \in T(\overline{A}, \varepsilon)$. It implies in particular that the residual tractions $r^j_{\mathbf{n}} \in \left[H^1(\omega)\right]^3$. This assumption holds whenever the coefficient functions $X^{\mathbf{n}}_i(x)$ are in $H^2(\omega)$. This is so, for example, if the given data (g^+, g^-) belong to $H^1(\omega)$ componentwise and the edge $\partial\omega$ is smooth. Nevertheless, in this case the $X^{\mathbf{n}}_i(x)$ contain boundary layers, i.e. solution components which behave like $\exp(-a\,\mathrm{dist}(x, \partial\omega)/t)$. Here the constant $a > 0$ is independent of t but depends on the model order and the elastic moduli [24]. Owing to the form of the boundary layers, we get

Corollary 5.2. *Assume that $\partial\omega$ is smooth and that the surface tractions $g^{\pm}(x) \in \left[H^1(\omega)\right]^3$. Then for fixed model order q the data belong to $T(\overline{A}, 0)$ for some $\overline{A} > 0$ independent of t.*

6. A-posteriori control of discretization and modelling error

Our purpose in the present section is to investigate the computational performance of the modeling error estimators derived above. Since the estimators were derived under the assumption that the plate models in the hierarchy are solved exactly, the question arises how well they will perform in the case that only an approximate solution of the plate models, for example by finite elements, is available. Therefore we will also address the approximate finite element solution of the models in the hierarchy by *hp*-FEM.

Owing to the elliptic systems constituting the plate models being singularly perturbed, their solutions exhibit boundary layers, i.e. solution components that are exponentially decaying off $\partial\omega$ which, together with the phenomenon of shear locking, renders the accurate numerical solution of the plate models nontrivial. Nevertheless, the boundary layers can be resolved [22] and the shear locking overcome by the use of high order p-FEM discretizations [23, 26]. In [22] it was shown that in the context of the *hp*-FEM a single element of width $O(tp)$ near the edge of the plate suffices to resolve the boundary layer with an exponential rate of convergence *uniformly in t*.

In the present section we will we consider the following example: consider the pure bending of a clamped square plate with midsurface $\omega = (-a/2, a/2)^2 \backslash \{x : |x| \leq 0.1a\}$ and thickness t which is subject to uniform unit normal loads on the faces. The $\mathbf{n}$ models with $\mathbf{n}$ as in Table 1 were discretized using a p-hierarchic finite element method ($1 \leq p \leq 8$) based on the mesh depicted in Fig. 1 (for $q = 1$ the modification mentioned in Remark 3.2 was made, i.e. model

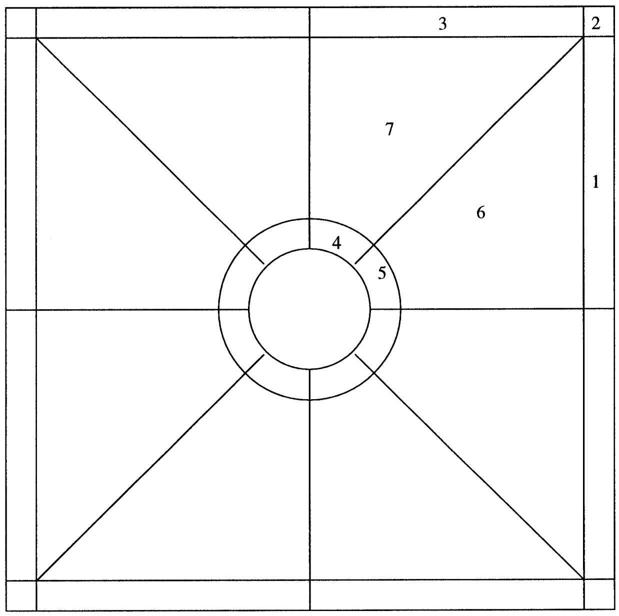

Fig. 1. Finite element mesh for the plate with central hole (not drawn to scale).

1 is the Reissner-Mindlin model). These capabilities are available in the finite element code STRESSCHECK[1] with which the computations in this section were done. This software allows for separate computation of the bending and membrane models and also contains a unified implementation for homogeneous and laminated plates (see [1, 27]). Let us also mention that in STRESSCHECK all element mappings are done exactly using transfinite blending maps, an essential feature of p-hierarchic finite element methods in curvilinear geometries.

Specifically, we considered a plate where $a = 2$, $d = 0.05$ and hence $a/t = 20$. The material was orthotropic with the constitutive parameters in (2.11) given by

$$E_1 = 25 \cdot 10^6, E_2 = E_3 = 10^6, G_1 = G_3 = 0.5 \cdot 10^6, G_2 = 0.2 \cdot 10^6 \tag{6.1}$$
$$\nu_{12} = \nu_{23} = 0.25, \nu_{31} = 0.01.$$

Due to material symmetry, it suffices to discretize only the quarter of the plate located in the first quadrant; all data below refer to this discretization.

Since boundary layers will occur near the (free) perimeter of the hole and the (clamped) outer edge of the plate, thin elements parallel to these boundaries were inserted (elements 1 and 3 near the clamped edge and 4 and 5 near the perimeter of the hole). We denote the normal distance of the mesh lines defining elements 1 and 3, respectively 4 and 5 from the nearest edge by $r1$. The discrete

[1] STRESSCHECK is a registered trademark of ESRD Inc., St. Louis, Mo, USA

potential energies corresponding to (uniform) polynomial degree p on the mesh in Fig. 1 and the model of order q (in the sense of (3.7)) will be denoted by $\mathscr{G}_{q,p}$ and the finite element approximation of $u^{\mathbf{n}}$ by $u_p^{\mathbf{n}}$ in what follows. Table 2 shows a solution run where p increases hierarchically from 1 to 8 for the $(3,3,2)$ model (i.e. $q = 3$ in (3.7)). Due to the symmetry of the problem, only one quarter of the plate was discretized in these calculations. Correspondingly, the potential energies refer only to this subdomain. The discretization error was estimated from the discrete total potential energies $\mathscr{G}_{3,p}$ by extrapolation, i.e. by fitting them to the asymptotic convergence estimate $\mathscr{G}_{3,p} - \mathscr{G}_{3,\infty} = Cp^{-\alpha}$. The unknowns $\mathscr{G}_{3,\infty}$, C and the convergence rate α are determined from $\mathscr{G}_{q,p}$ corresponding to three successive p-levels. Throughout, the mesh in Fig. 1 was

Table 2. p-hierarchic solution of the $(3,3,2)$ model. $r1 = 0.04a$

p	DOF	$\mathscr{G}_{3,p}$	α	$\left\Vert u_p^{\mathbf{n}} - u^{\mathbf{n}}\right\Vert_{E(\Omega)} / \Vert u^{\mathbf{n}}\Vert_{E(\Omega)}$ [%]
1	42	-4.3338026E-6	0.00	53.60
2	168	-5.7561015E-6	0.61	23.12
3	378	-6.0150995E-6	0.98	10.42
4	672	-6.0691653E-6	1.49	4.43
5	1050	-6.0774790E-6	1.34	2.44
6	1512	-6.0793402E-6	0.99	1.70
7	2058	-6.0802827E-6	1.24	1.16
8	2688	-6.0806806E-6	1.24	0.84
∞		-6.0811048023E-6		

used with $p = 8$ at each stage and the full tensor product polynomial set was used (this proved to be advantageous for anisotropic materials).

Let us now turn to the a-posteriori estimation of the modelling error based on the residual tractions on the faces $R_{\pm}$. The element contributions to the estimator can be used as *modelling error indicators* for a local adaptive selection of the model orders [4] and are listed in Table 3. Due to symmetry, only elements located in the first quadrant are shown.

Table 3. Elemental contributions to the error estimate ($p = 8$, $a/t = 20$)

q	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
1	1.9526E-7	4.2715E-10	8.9515E-8	4.8366E-10	3.0273E-09	5.3390E-7	1.1288E-7
2	1.9306E-7	3.1999E-10	4.6037E-8	2.5070E-10	2.1393E-09	5.0902E-7	8.2697E-8
3	4.4595E-8	1.0706E-10	1.0795E-8	4.8627E-11	1.7647E-10	2.9366E-9	2.0092E-10
4	4.2741E-8	1.0068E-10	8.4492E-9	4.3075E-11	1.7199E-10	2.6772E-9	1.2776E-11
5	1.4035E-8	4.2413E-11	4.3844E-9	9.0109E-13	1.8191E-11	3.6851E-11	6.0261E-13
6	1.3733E-8	4.0699E-11	3.9150E-9	6.1306E-13	1.7534E-11	2.5526E-11	6.4987E-13

The convergence as the model order q is increased is clearly visible. We also observe the different rates of convergence in the interior, i.e. for ω_6 and ω_7 and near the clamped edge of the plate. The failure of the error corresponding to the edge elements to decrease sufficiently fast is due to the *edge singularities* of the three dimensional problem near the sets $\gamma \times \{\pm d\}$ which are approximated poorly by (3.1). In this respect, it is important to note that the hierarchic plate models in ω_6 and ω_7 can be coupled conformingly to a fully three dimensional *hp* finite element discretization in the boundary layer subregions $\omega_j \times (-d, d)$, $j = 1, 2, 3, 4, 5$ [18].

Finally, we compare the residual modelling error estimates according to Theorem 4.1 with the ones obtained by extrapolation in Table 4 where we used that

$$0 < 2(\mathscr{G}(u^{\mathbf{n}}) - \mathscr{G}(u)) = \mathscr{E}(u) - \mathscr{E}(u^{\mathbf{n}})$$
$$= \|u\|^2 - \|u^{\mathbf{n}}\|^2 = \|u^{\mathbf{n}} - u\|^2 = \|e^{\mathbf{n}}\|^2.$$

Table 4. Residual and extrapolation based estimation of the modelling error

q	2	3	4
$EST^2 / \|u\|^2_{E(\Omega)}$	6.8428E-2	4.8321E-3	4.4481E-3
$\|u^{\mathbf{n}}_8 - u\|^2_{E(\Omega)} / \|u\|^2_{E(\Omega)}$	5.5944E-2	1.6210E-3	1.3192E-3
$\Theta_{eff} = EST / \|u^{\,n}_{\,8} - u\|_{E(\Omega)}$	1.1059	1.7265	1.8363

We see that the residual based estimators are guaranteed upper estimators which follow the modelling error accurately as the model order increases.

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