Numer. Math. 74: 153–176 (1996)

Numerische Mathematik

Mixed finite element methods in fluid structure systems

Lucia Gastaldi⋆

Dipartimento di Matematica, Università di Pavia, Italy

Received October 20, 1992 / Revised version received May 23, 1995

Summary. The eigenvalue problem describing the frequencies of a fluid vibrating in a rigid cavity or within moving boundaries is considered. Based on the method of Lagrange multipliers, a three field mixed formulation is introduced in order to avoid the spurious circulating modes. Stability and optimal error bounds are proved for two choices of finite element spaces.

Mathematics Subject Classification (1991): 65N30, 73K70

1. Introduction

The interaction between fluids and structures can, in many practical engineering problems, significantly affect the response of the structures and hence must be considered properly in the analysis.

Here we study mixed finite element methods to calculate the frequencies of a fluid vibrating in a rigid cavity or within moving boundaries. Mathematically the problem is an eigenvalue problem with irrotational eigenfunctions.

In the literature (see [12], [7]), standard finite element methods based on the displacement formulation have received considerable attention due to its simplicity and to the potential applicability to the resolution of a broad range of problems (specifically nonlinear problems). Unfortunately, these methods exhibit spurious rotational modes with nonzero frequencies. Since, in the frequency analysis, only the irrotational modes are to be considered, Bathe, Nitikitpaiboon and Wang proposed, in [11], a modified formulation such that these spurious circulation modes are suppressed from the solution. Namely, they developed a

⋆ Partially supported by IAN-CNR (Pavia), MURST 40% Analisi Numerica e Matematica Computazionale (Italy) and C.N.R. under contracts n.92.00052.CT12, n.92.00526.CT01, n.93.00564.CT01, n.93.01899.CT12, n. 94.00139.CT01

three-field mixed displacement-pressure formulation based on the method of Lagrange multipliers, to enforce the irrotationality condition. Then they tested the finite element discretization of such formulation into various fluid-structure problems, with different choices for the finite element spaces, obtaining satisfactory results in some cases.

In this paper, we present a new mixed variational formulation of the constrained problem and analyze theoretically the corresponding finite element discretization. We consider two categories of fluid-structure interaction problems: natural frequencies of fluids in rigid cavities and fluids vibrating in moving boundaries (see Fig. 1 and 2). The corresponding mathematical formulations differ only for the boundary conditions. Applying known results on the eigenvalues approximation by mixed finite element methods (see [10]), we obtain the error estimates for the methods under consideration. More precisely, the fundamental ingredients in the proof are V-ellipticity and the inf-sup condition. While the V-ellipticity is an easy task for our problem, provided V is suitably chosen, the discrete inf-sup condition can give some troubles. In fact, we have that the triple (Q_2, P_1, P_0) satisfy this latter condition, but the three spaces do not provide the same accuracy. Enriching the third space with piecewise linear functions, we obtain a triple for which we are not able to obtain the inf-sup condition. We circumvent this difficulty by presenting an augmented formulation via a stabilization procedure.

Therefore we obtain optimal error estimates for the eigenvalues for both the categories of problem and both the finite element methods considered.

An outline of the paper follows: in Sect. 2 we describe the problem and we present the constrained variational formulation and its discretization; in Sect. 3 the error estimates are stated under quite general abstract assumptions on the bilinear form and on the finite element spaces; the validity of these assumptions is checked in Sects. 4 and 5 for the two choices of finite element spaces and for the two categories of problems.

We end this section with the list of the basic notation used in the paper.

Let A be a bounded open set in $\mathbb{R}^2$. For any real number $s \geq 0$, $\mathbf{H}^s(A)$ will denote the usual $\mathbf{L}^2$-based Sobolev spaces, ($\mathbf{H}^0(A) = \mathbf{L}^2(A)$) and $\| \ \|_{s,A}$ will stand for their norms. Moreover $(f, g)_A = \int_A f\, g$ is the inner product in $\mathbf{L}^2(A)$. When no confusion may arise we drop the subscript A. Hereafter we shall denote vector valued functions and operators in bold face.

Letter C stands for constants which are not necessarily the same in any two occurrences.

Finally, we recall the following standard differential operators for any scalar function r and any vector valued function $\mathbf{u} = (u_1, u_2)$:

$$\operatorname{grad} r = \begin{pmatrix} \frac{\partial r}{\partial x} \\ \frac{\partial r}{\partial y} \end{pmatrix}, \quad \operatorname{rot} r = \begin{pmatrix} -\frac{\partial r}{\partial y} \\ \frac{\partial r}{\partial x} \end{pmatrix},$$

$$\operatorname{div} \mathbf{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}, \quad \operatorname{rot} \mathbf{u} = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}.$$

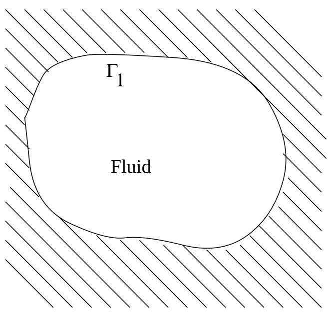

Fig. 1. The rigid cavity

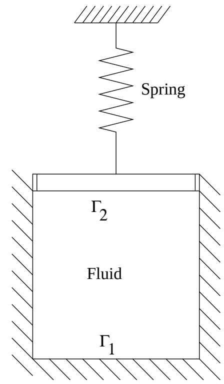

Fig. 2. The piston container

2. Statement of the problem

This section is devoted to the mathematical formulation of the problem and to its discretization. We consider two categories of fluid-structure interaction problems: natural frequencies of fluids in rigid cavities (see Fig. 1) and fluid vibrating within moving boundaries. As an example for the second category we chose the piston/container/spring arrangement shown in Fig. 2. We shall present the two problems under a common formulation, which takes into account of the differences in the boundary conditions.

Let us denote by Ω the open, bounded, simply-connected region of $\mathbb{R}^2$ occupied by the fluid. We assume that the boundary of Ω is $\mathbf{C}^{1,1}$ or piecewise smooth with no reentrant corners.

The partial differential equations governing the motion of the fluid in the cavity are the same in the two situations. They can be derived using the classical acoustic approximation for small amplitude motions of an ideal inviscid barotropic fluid contained in the cavity Ω. Then the basic equations written in Lagrangian form are

$$\rho\ddot{\mathbf{U}} + \operatorname{grad} P = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$\beta \operatorname{div} \mathbf{U} + P = 0 \quad \text{in } \Omega, \tag{2.2}$$

where $\mathbf{U}$ is the displacement of the fluid and P is the fluid pressure. The density ρ and the bulk modulus β are given constants. The superposed dots indicate partial time derivative of second order.

In order to write the boundary conditions we distinguish between the two cases we are considering.

In the case depicted in Fig.1, where the natural frequencies of a fluid in a rigid cavity are studied, we denote by Γ_1 the boundary of Ω and we suppose that the fluid is constrained to move only tangentially to the boundary, that is:

$$\mathbf{U} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_1, \tag{2.3}$$

where **n** is the outward normal.

In Fig. 2 we have a rectangular container fulfilled by a fluid and closed at the top by a plate. The plate moves in a rigid way in the vertical direction and it is fixed through a spring which produces a traction proportional to the vertical displacement with oppposite direction. Then denoting by Γ_1 the fixed walls of the container Ω and by Γ_2 the surface of the plate at the top of the container, we have the following boundary conditions

$$\mathbf{U} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_1, \tag{2.4}$$

$$m\ddot{\mathbf{U}} \cdot \mathbf{n} = -K\mathbf{U} \cdot \mathbf{n} + \int_{\Gamma_2} P\, d\gamma \quad \text{on } \Gamma_2. \tag{2.5}$$

The first condition expresses the fact that the fluid can move only tangentially along the fixed walls, Γ_1. Instead, along Γ_2 we impose the continuity of the normal displacements of the fluid and of the rigid piston and the equilibrium of the surface forces. The term in the left hand side stands for the internal forces, where m is the mass of the piston. The two terms in the right hand side represent the external forces: the traction force exerted by the spring, with stiffness K, and the resultant of the pressure forces exerted by the fluid on the plate, given by $\int_{\Gamma_2} P\, d\gamma$. Moreover, since the motion of the piston takes place in a rigid way, the displacement of the fluid along its surface results constant.

In order to study the vibration frequencies, we write problem (2.1), (2.2), (2.3) and problem (2.1), (2.2), (2.4), (2.5) in the frequency domain. Putting together the two situations, with the convention that in the first case Γ_2 is empty, we have the following eigenvalue problem:

$$\rho\omega^2\mathbf{u} - \operatorname{grad} p = 0 \quad \text{in } \Omega, \tag{2.6}$$

$$\beta \operatorname{div} \mathbf{u} + p = 0 \quad \text{in } \Omega, \tag{2.7}$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_1, \tag{2.8}$$

$$(K - m\omega^2)\mathbf{u} \cdot \mathbf{n} = \int_{\Gamma_2} p\, d\gamma \quad \text{on } \Gamma_2. \tag{2.9}$$

The unknowns are the real numbers ω, the vibration frequencies, **u** the amplitude of the vibrations and p the pressure. We recall that ρ, β, K and m are given positive constants. We remark also that the boundary condition (2.9) implies that $\mathbf{u} \cdot \mathbf{n}$ is constant along Γ_2.

It is easy to obtain from (2.6)

$$\rho\omega^2 \operatorname{rot} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.10}$$

hence we have two types of solutions: one corresponding to the vortex motions ($\operatorname{rot} \mathbf{u} \neq 0$), for which the frequencies are zero and another one corresponding to the irrotational motions ($\operatorname{rot} \mathbf{u} = 0$) for which the frequencies are not zero. Since in the frequency analysis, only the irrotational modes are interesting, we require that $\operatorname{rot} \mathbf{u} = 0$. To enforce this constraint, we consider the following modified formulation, which can be obtained as the Euler equations associated to the stationarity conditions of the augmented Lagrangian functional: find $\omega,\ \mathbf{u},\ p,\ \lambda$ such that

$$\rho\omega^2\mathbf{u} - \operatorname{grad} p - \operatorname{rot}\lambda = 0 \quad \text{in } \Omega, \tag{2.11}$$

$$\beta \operatorname{div}\mathbf{u} + p = 0 \quad \text{in } \Omega, \tag{2.12}$$

$$\alpha \operatorname{rot}\mathbf{u} + \lambda = 0 \quad \text{in } \Omega, \tag{2.13}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_1, \tag{2.14}$$

$$(K - m\omega^2)\mathbf{u}\cdot\mathbf{n} = \int_{\Gamma_2} p\, d\gamma \quad \text{on } \Gamma_2, \tag{2.15}$$

$$\lambda = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2, \tag{2.16}$$

where λ is the Lagrange multiplier associated to the irrotationality constraint and α is a penalization parameter which has to be chosen sufficiently larger than β.

Remark 2.1. We observe that if $(\omega, \mathbf{u}, p)$ is an eigensolution of problem (2.6)–(2.9), with the irrotationality constraint, then $(\omega, \mathbf{u}, p, \lambda = 0)$ is an eigensolution of problem (2.11)-(2.16).

Viceversa, in order to show that an eigensolution $(\omega, \mathbf{u}, p, \lambda)$ of (2.11)-(2.16) is an eigensolution of (2.6)-(2.9), it is necessary to prove that $\lambda = 0$.

Let us take the divergence of (2.11) and eliminate $\mathbf{u}$, then we have

$$-\Delta p = \frac{\rho\omega^2}{\beta} p \quad \text{in } \Omega, \tag{2.17}$$

$$\frac{\partial p}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma_1, \tag{2.18}$$

$$(K - m\omega^2)\frac{\partial p}{\partial \mathbf{n}} = \rho\omega^2 \int_{\Gamma_2} p\, d\gamma \quad \text{on } \Gamma_2. \tag{2.19}$$

Similarly, let us apply the rotational to (2.11) and eliminate $\mathbf{u}$; we obtain

$$-\Delta\lambda = \frac{\rho\omega^2}{\alpha}\lambda \quad \text{in } \Omega, \tag{2.20}$$

$$\lambda = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2. \tag{2.21}$$

For α much bigger than β, the first eigenvalues of (2.11)-(2.16) are eigenvalues of (2.17)-(2.19), but not of (2.20)-(2.21); hence for such values of ω it results $\lambda = 0$.

In particular, it is easy to prove that for $\alpha > \beta$ the first eigenvalue of (2.11)-(2.16) is the first eigenvalue of (2.17)-(2.19) and that for such value of ω, the unique solution of (2.20)-(2.21) is $\lambda = 0$.

In order to discretize problem (2.11)-(2.16), by means of a finite element method, let us introduce a variational formulation. We shall obtain it formally in the present section and we shall specify the functional framework in Sect. 4 and 5 below.

Let Q and M be two subspaces of $\mathbf{L}^2(\Omega)$, we multiply (2.12) by $q \in Q$ and (2.13) by $\mu \in M$ and integrate over Ω, so we have:

$$\beta(\operatorname{div}\mathbf{u}, q) + (p, q) = 0 \quad \forall q \in Q, \tag{2.22}$$

$$\alpha(\operatorname{rot}\mathbf{u}, \mu) + (\lambda, \mu) = 0 \quad \forall \mu \in M. \tag{2.23}$$

Due to (2.14) and (2.15), we consider the space $\mathbf{V}$ of sufficiently regular vector valued functions $\mathbf{v}$ such that $\mathbf{v}\cdot\mathbf{n} = 0$ on Γ_1 and $\mathbf{v}\cdot\mathbf{n}$ is constant along Γ_2; then multiply (2.11) by $\mathbf{v} \in \mathbf{V}$, integrate by parts, use (2.16) and obtain

$$-(\operatorname{div}\mathbf{v}, p) + \int_{\Gamma_2} p\,\mathbf{v}\cdot\mathbf{n}\,d\gamma - (\operatorname{rot}\mathbf{v}, \lambda) = \rho\omega^2(\mathbf{u}, \mathbf{v}).$$

Since $\mathbf{v}\cdot\mathbf{n}$ is constant along Γ_2 we can easily substitute the condition (2.15) and we arrive at

$$\begin{aligned} &K\mathbf{u}\cdot\mathbf{n}_{|\Gamma_2}\,\mathbf{v}\cdot\mathbf{n}_{|\Gamma_2} - (\operatorname{div}\mathbf{v}, p) - (\operatorname{rot}\mathbf{v}, \lambda)\\ &\quad = \rho\omega^2(\mathbf{u}, \mathbf{v}) + m\omega^2\mathbf{u}\cdot\mathbf{n}_{|\Gamma_2}\,\mathbf{v}\cdot\mathbf{n}_{|\Gamma_2} \quad \forall \mathbf{v} \in \mathbf{V}. \end{aligned} \tag{2.24}$$

In view of the numerical discretization, it is convenient to take the following linear combination of (2.24), (2.22) with $q = \operatorname{div}\mathbf{v}$ and (2.23) with $\mu = \operatorname{rot}\mathbf{v}$:

$$\begin{aligned} &\gamma_1(\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) + \left(\frac{\gamma_1}{\beta} - 1\right)(\operatorname{div}\mathbf{v}, p) + \gamma_2(\operatorname{rot}\mathbf{u}, \operatorname{rot}\mathbf{v}) + \left(\frac{\gamma_2}{\alpha} - 1\right)(\operatorname{rot}\mathbf{v}, \lambda)\\ &\quad + K\mathbf{u}\cdot\mathbf{n}_{|\Gamma_2}\,\mathbf{v}\cdot\mathbf{n}_{|\Gamma_2} = \omega^2[\rho(\mathbf{u}, \mathbf{v}) + m\mathbf{u}\cdot\mathbf{n}_{|\Gamma_2}\,\mathbf{v}\cdot\mathbf{n}_{|\Gamma_2}] \quad \forall \mathbf{v} \in \mathbf{V}, \end{aligned}$$

where γ_1 and γ_2 are positive numbers such that $1 \le \gamma_1 \le \beta$ and $1 \le \gamma_2 \le \alpha$.

Summarizing, let

$$\begin{aligned} &a(\mathbf{u}, \mathbf{v}) = \gamma_1(\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) + \gamma_2(\operatorname{rot}\mathbf{u}, \operatorname{rot}\mathbf{v}) + K\mathbf{u}\cdot\mathbf{n}_{|\Gamma_2}\,\mathbf{v}\cdot\mathbf{n}_{|\Gamma_2}\\ &\forall \mathbf{u}, \mathbf{v} \in \mathbf{V}, \end{aligned} \tag{2.25}$$

then we have the following problem:

Problem 2.2 Find $\omega \in \mathbb{R}$, $\mathbf{u} \in \mathbf{V}$, $p \in Q$ and $\lambda \in M$ such that

$$\begin{aligned} &a(\mathbf{u}, \mathbf{v}) + \left(\frac{\gamma_1}{\beta} - 1\right)(\operatorname{div}\mathbf{v}, p) + \left(\frac{\gamma_2}{\alpha} - 1\right)(\operatorname{rot}\mathbf{v}, \lambda)\\ &\quad = \rho\omega^2(\mathbf{u}, \mathbf{v}) + m\omega^2\mathbf{u}\cdot\mathbf{n}_{|\Gamma_2}\,\mathbf{v}\cdot\mathbf{n}_{|\Gamma_2}\\ &\qquad \forall \mathbf{v} \in \mathbf{V}, \end{aligned} \tag{2.26}$$

$$\beta(\operatorname{div}\mathbf{u}, q) + (p, q) = 0 \quad \forall q \in Q, \tag{2.27}$$

$$\alpha(\operatorname{rot}\mathbf{u}, \mu) + (\lambda, \mu) = 0 \quad \forall \mu \in M. \tag{2.28}$$

Now let us briefly introduce the finite element discretization of Problem 2.2.

Let us consider a family $\mathscr{T}_h$ of regular and quasi-uniform meshes of Ω (h is the meshsize). Let $\mathbf{V}_h \subseteq \mathbf{V}$, $Q_h \subseteq Q$ and $M_h \subseteq M$ be finite dimensional spaces, which will be defined more precisely later. Then the discrete counterpart of Problem 2.2 reads:

Problem 2.3 Find $\omega_h \in \mathbb{R}$, $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in Q_h$ and $\lambda_h \in M_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + \left(\frac{\gamma_1}{\beta} - 1\right)(\operatorname{div} \mathbf{v}_h, p_h) + \left(\frac{\gamma_2}{\alpha} - 1\right)(\operatorname{rot} \mathbf{v}_h, \lambda_h)$$

$$= \rho\omega_h^2(\mathbf{u}_h, \mathbf{v}_h) + m\omega_h^2 \mathbf{u}_h \cdot \mathbf{n}_{|\Gamma_2} \quad \mathbf{v}_h \cdot \mathbf{n}_{|\Gamma_2}$$

$$\forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.29}$$

$$\beta(\operatorname{div} \mathbf{u}_h, q_h) + (p_h, q_h) = 0 \quad \forall q_h \in Q_h, \tag{2.30}$$

$$\alpha(\operatorname{rot} \mathbf{u}_h, \mu_h) + (\lambda_h, \mu_h) = 0 \quad \forall \mu_h \in M_h. \tag{2.31}$$

In the next sections we shall study the stability and the convergence properties of this method for suitable choices of $\mathbf{V}_h$, Q_h, and M_h.

3. Abstract results

In this section we recall some basic results on the approximation of eigenvalue problems (see for example [10]). Let us consider the eigenvalue problem of the form: find $\chi \in \mathbb{R}$, $U \in \mathbf{H}$ satisfying

$$A(U, V) = \chi R(U, V) \quad \forall\, V \in \mathbf{H}, \tag{3.1}$$

where $\mathbf{H}$ is a Hilbert space and $A : \mathbf{H} \times \mathbf{H} \mapsto \mathbb{R}$ and $R : \mathbf{H} \times \mathbf{H} \mapsto \mathbb{R}$ are symmetric continuous bilinear forms. We assume that there exists a positive constant k_1 such that

$$\inf_{U \in \mathbf{H}} \sup_{V \in \mathbf{H}} \frac{|A(U, V)|}{\|U\|_{\mathbf{H}} \|V\|_{\mathbf{H}}} \geq k_1 \quad \forall\, U,\ V \in \mathbf{H} \tag{3.2}$$

and that

$$T : \mathbf{H} \mapsto \mathbf{H} \text{ is compact,} \tag{3.3}$$

where T satisfies

$$A(TF, V) = R(F, V) \quad \forall\, V \in \mathbf{H}. \tag{3.4}$$

Assumptions (3.2) and (3.3) give that the eigenvalues of (3.1) exist in $\mathbb{R}$, since A and R are symmetric; in addition, if R is positive definite the eigenvalues are all strictly positive. Let χ be an eigenvalue of geometric multiplicity n, we denote by E the corresponding eigenspace with $\dim(E) = n$.

To approximate problem (3.1), we consider a family of finite dimensional subspaces $\mathbf{H}_h \subseteq \mathbf{H}$, $0 < h \leq 1$. Then the approximate eigenvalue problem is: find $\chi_h \in \mathbb{R}$, $U_h \in \mathbf{H}_h$ satisfying

$$A(U_h, V_h) = \chi_h R(U_h, V_h) \quad \forall\, V_h \in \mathbf{H}_h. \tag{3.5}$$

Concerning (3.5) we assume that

$$\lim_{h\to 0} \inf_{V_h\in\mathbf{H}_h} \|U - V_h\|_{\mathbf{H}} = 0 \quad \forall\, U \in \mathbf{H}, \tag{3.6}$$

and that there exists a positive constant k_2 independent of h, such that

$$\inf_{U_h\in\mathbf{H}_h} \sup_{V_h\in\mathbf{H}_h} \frac{|A(U_h, V_h)|}{\|U_h\|_{\mathbf{H}}\|V_h\|_{\mathbf{H}}} \geq k_2 \quad \forall\, U_h,\ V_h \in \mathbf{H}_h. \tag{3.7}$$

Then let us consider the operator

$$T_h : \mathbf{H} \mapsto \mathbf{H} \tag{3.8}$$

defined as follows: for all $F \in \mathbf{H}$, $T_h F \in \mathbf{H}_h \subseteq \mathbf{H}$ and satisfies

$$A(T_h F, V_h) = R(F, V_h) \quad \forall\, V_h \in \mathbf{H}_h. \tag{3.9}$$

Clearly T_h is compact, since its range is contained in $\mathbf{H}_h$ which is a finite dimensional subspace of $\mathbf{H}$. Moreover, the assumptions (3.6) and (3.7) imply that T_h converges to T uniformly in $\mathbf{H}$ as $h \to 0$. Hence the following theorem holds:

Theorem 3.1 *The eigenvalues of problem* (3.5) *exist in* $\mathbb{R}$ *and, if R is positive definite, they are strictly positive, bounded away from zero independently of h.*

As a consequence of that, there are exactly n eigenvalues χ_{ih} $i = 1, \ldots, n$, of problem (3.5) converging to χ as $h \to 0$ and the direct sum E_h of the eigenspaces corresponding to χ_{ih}, $i = 1, \ldots, n$, has dimension n.

Let us set

$$\epsilon_h = \sup_{\substack{U\in E\\ \|U\|_{\mathbf{H}}=1}} \inf_{V_h\in\mathbf{H}_h} \|U - V_h\|_{\mathbf{H}}, \tag{3.10}$$

and for N, M two subspaces of $\mathbf{H}$, let us define, $\hat{\delta}(N, M)$, the gap between N and M with respect to the norm of $\mathbf{H}$, by

$$\hat{\delta}(N, M) = \max[\delta(N, M), \delta(M, N)], \tag{3.11}$$

where

$$\delta(N, M) = \sup_{\substack{U\in N\\ \|U\|_{\mathbf{H}}=1}} \inf_{V\in M} \|U - V\|_{\mathbf{H}}.$$

Then we have the following error estimates, see [10]:

Theorem 3.2 *There are constants C and $h_0 > 0$ such that, for* $0 < h \leq h_0$

$$|\chi - \chi_{ih}| \leq C\epsilon_h^2, \quad i = 1, \ldots, n, \tag{3.12}$$

$$\hat{\delta}(E, E_h) \leq C\epsilon_h. \tag{3.13}$$

We state now the fundamental results on the approximation of the eigenvalues and of the eigenvectors of Problem 2.2. These results are based on the theory developped in [10] on the eigenvalue approximation by mixed finite element methods.

Let $\mathbf{H} = \mathbf{V} \times Q \times M$ be the space of the triples $V = (\mathbf{v}, q, \mu)$, endowed with the graph norm $\|V\|_{\mathbf{H}} = (\|\mathbf{v}\|^2_{\mathbf{V}} + \|q\|^2_Q + \|\mu\|^2_M)^{\frac{1}{2}}$. Then setting

$$\begin{aligned} A(U,V) = {} & a(\mathbf{u},\mathbf{v}) + \left(\tfrac{\gamma_1}{\beta} - 1\right)(\operatorname{div}\mathbf{v}, p) + \left(\tfrac{\gamma_2}{\alpha} - 1\right)(\operatorname{rot}\mathbf{v}, \lambda) \\ & + \left(\tfrac{\gamma_1}{\beta} - 1\right)(\operatorname{div}\mathbf{u}, q) + \tfrac{1}{\beta}\left(\tfrac{\gamma_1}{\beta} - 1\right)(p, q) \\ & + \left(\tfrac{\gamma_2}{\alpha} - 1\right)(\operatorname{rot}\mathbf{u}, \mu) + \tfrac{1}{\alpha}\left(\tfrac{\gamma_2}{\alpha} - 1\right)(\lambda, \mu) \end{aligned} \tag{3.14}$$

and

$$R(U,V) = \rho(\mathbf{u},\mathbf{v}) + m\mathbf{u}\cdot\mathbf{n}_{|\Gamma_2}\,\mathbf{v}\cdot\mathbf{n}_{|\Gamma_2}, \tag{3.15}$$

Problem 2.2 can be reduced in the form (3.1).

In order to apply Theorem 3.1, we introduce the following assumptions on the form a, the Hilbert spaces $\mathbf{V}$, Q and M and the finite dimensional subspaces $\mathbf{V}_h$, Q_h and M_h, which imply that A and R given by (3.14) and (3.15), satisfy (3.2)-(3.7).

Let $B : \mathbf{V} \mapsto Q' \times M'$ be the operator given by $B\mathbf{v} = (\operatorname{div}\mathbf{v}, \operatorname{rot}\mathbf{v})$ for all $\mathbf{v} \in \mathbf{V}$ and $B^{\mathrm{T}} : Q \times M \mapsto \mathbf{V}'$ its transpose, then set

$$\operatorname{Ker} B = \{\mathbf{v} \in \mathbf{V} : (\operatorname{div}\mathbf{v}, q) = 0\ \forall q \in Q,\ (\operatorname{rot}\mathbf{v}, \mu) = 0\ \forall \mu \in M\} \tag{3.16}$$

and

$$\operatorname{Ker} B^{\mathrm{T}} = \{(q,\mu) \in Q \times M :\ (\operatorname{div}\mathbf{v}, q) + (\operatorname{rot}\mathbf{v}, \mu) = 0\ \forall \mathbf{v} \in \mathbf{V}\}. \tag{3.17}$$

Let B_h be the discrete operator corresponding to B and B_h^{T} its tranpose, then

$$\operatorname{Ker} B_h = \{\mathbf{v}_h \in \mathbf{V}_h : (\operatorname{div}\mathbf{v}_h, q_h) = 0\ \forall q_h \in Q_h,\ (\operatorname{rot}\mathbf{v}_h, \mu_h) = 0\ \forall \mu_h \in M_h\} \tag{3.18}$$

and

$$\operatorname{Ker} B_h^{\mathrm{T}} = \{(q_h,\mu_h) \in Q_h \times M_h :\ (\operatorname{div}\mathbf{v}_h, q_h) + (\operatorname{rot}\mathbf{v}_h, \mu_h) = 0\ \forall \mathbf{v}_h \in \mathbf{V}_h\}. \tag{3.19}$$

Let $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$1 \le \gamma_1 < \beta \qquad 1 \le \gamma_2 < \alpha, \tag{3.20}$$

then we assume:

(A1)- The bilinear form a is symmetric and continuous on $\mathbf{V} \times \mathbf{V}$ and there exists $\kappa_1 > 0$ such that

$$a(\mathbf{v},\mathbf{v}) \ge \kappa_1 \|\mathbf{v}\|^2_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}, \tag{3.21}$$

(A2)- There exists a constant $\kappa_2 > 0$ such that

$$\sup_{\mathbf{u}\in\mathbf{V}} \frac{|\left(1-\frac{\gamma_1}{\beta}\right)(\operatorname{div}\mathbf{u},q)+\left(1-\frac{\gamma_2}{\alpha}\right)(\operatorname{rot}\mathbf{u},\mu)|}{\|\mathbf{u}\|_{\mathbf{V}}} \geq \kappa_2\left(\left(1-\frac{\gamma_1}{\beta}\right)^2\|q\|^2_{Q/\operatorname{Ker}B^{\mathrm{T}}}+\left(1-\frac{\gamma_2}{\alpha}\right)^2\|\mu\|^2_{M/\operatorname{Ker}B^{\mathrm{T}}}\right)^{\frac{1}{2}} \quad \forall(q,\mu)\in Q\times M/\operatorname{Ker}B^{\mathrm{T}}. \tag{3.22}$$

(A3)- There exists a positive constant κ_3 such that

$$\rho(\mathbf{u},\mathbf{v})+m\mathbf{u}\cdot\mathbf{n}_{|\Gamma_2}\mathbf{v}\cdot\mathbf{n}_{|\Gamma_2}\leq\kappa_3\|\mathbf{u}\|_{\mathbf{V}}\|\mathbf{v}\|_{\mathbf{V}} \quad \forall\mathbf{u},\mathbf{v}\in\mathbf{V} \tag{3.23}$$

(A4)- There exists $\kappa_4 > 0$, independent of h, such that

$$\sup_{\mathbf{u}_h\in\mathbf{V}_h} \frac{|\left(1-\frac{\gamma_1}{\beta}\right)(\operatorname{div}\mathbf{u}_h,q_h)+\left(1-\frac{\gamma_2}{\alpha}\right)(\operatorname{rot}\mathbf{u}_h,\mu_h)|}{\|\mathbf{u}_h\|_{\mathbf{V}}} \geq \kappa_4\left(\left(1-\frac{\gamma_1}{\beta}\right)^2\|q_h\|^2_{Q/\operatorname{Ker}B_h^{\mathrm{T}}}+\left(1-\frac{\gamma_2}{\alpha}\right)^2\|\mu_h\|^2_{M/\operatorname{Ker}B_h^{\mathrm{T}}}\right)^{\frac{1}{2}} \quad \forall(q_h,\mu_h)\in Q_h\times M_h/\operatorname{Ker}B_h^{\mathrm{T}}. \tag{3.24}$$

(A5)- The finite element spaces enjoy the following approximation property

$$\lim_{h\to 0}\inf_{\substack{(\mathbf{v}_h,q_h,\mu_h)\in\\ \mathbf{V}_h\times Q_h\times M_h}}(\|\mathbf{v}-\mathbf{v}_h\|_{\mathbf{V}}+\|q-q_h\|_Q+\|\mu-\mu_h\|_M)=0 \tag{3.25}$$

for each $(\mathbf{v},q,\mu)\in\mathbf{V}\times Q\times M$.

Notice that (A4) is the classical discrete inf-sup condition, which together with (A1)-(A3) and (A5) ensure the stability and the optimal error estimates for the source problem associated to Problem 2.3 (see for example [1]).

We can now prove the main theorems:

Theorem 3.3 *Under assumptions* (A1)-(A3)*, and* (3.3) *with A and R defined by* (3.14) *and* (3.15)*, the eigenvalues of Problem 2.2 exist in* $\mathbb{R}$ *and are strictly positive.*

Proof. From (A1) and (A2), one deduces that the bilinear form A defined in (3.14) satisfies (3.2); while, from (A3), one gets the continuity of R given by (3.15). Since A and R are symmetric and T is compact, Problem 2.2 admits a sequence of real eigenvalues diverging to plus infinity. Let us prove that they are strictly positive: let us take $\mathbf{v}=\mathbf{u}$ in (2.26), $q=\frac{1}{\beta}\left(1-\frac{\gamma_1}{\beta}\right)p$ in (2.27) and $\mu=\frac{1}{\alpha}\left(1-\frac{\gamma_2}{\alpha}\right)\lambda$ in (2.28) and sum the equations, then we have

$$a(\mathbf{u},\mathbf{u})+\frac{1}{\beta}\left(1-\frac{\gamma_1}{\beta}\right)(p,p)+\frac{1}{\alpha}\left(1-\frac{\gamma_2}{\alpha}\right)(\lambda,\lambda)=\omega^2[\rho\|\mathbf{u}\|_0^2+m|\mathbf{u}\cdot\mathbf{n}_{|\Gamma_2}|^2].$$

From which, we obtain the following bound for the eigenvalues, using also (3.21) and (3.23):

$$\omega^2 \geq \frac{\kappa_1}{\kappa_3}. \quad \square$$

We end this section with the following theorem regarding the eigenvalues of the discrete problem and their approximation properties.

Theorem 3.4 *Under assumptions* (A1)-(A5) *and* (3.3), *the eigenvalues of Problem* 2.3 *exist in* $\mathbb{R}$ *and are strictly positive, bounded away from zero uniformly with respect to h. Moreover, there exist constants C and* $h_0 > 0$ *such that the following error estimates hold for* $0 < h \leq h_0$

$$|\omega - \omega_{ih}| \leq C\epsilon_h^2, \quad i = 1, \dots, n, \tag{3.26}$$

$$\hat{\delta}(E, E_h) \leq C\epsilon_h, \tag{3.27}$$

where $\hat{\delta}$, *defined in* (3.11), *is the gap between E and* E_h *with respect to the norm of* $\mathbf{V} \times Q \times M$ *and*

(3.28)

$$\epsilon_h = \sup_{\substack{(\mathbf{u},p,\lambda)\in E \\ \|\mathbf{u}\|_{\mathbf{V}}+\|p\|_Q+\|\lambda\|_M=1}} \inf_{\substack{(\mathbf{v}_h,q_h,\mu_h)\in \\ \mathbf{V}_h\times Q_h\times M_h}} (\|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{V}} + \|p-q_h\|_Q + \|\lambda-\mu_h\|_M).$$

Proof. The existence is an obvious consequence of Theorem 3.1, since assumptions (A1)-(A5) imply that (3.6) and (3.7) are satisfied. Let us show that the eigenvalues of Problem 2.3 are strictly positive. We take in (2.29) $\mathbf{v}_h = \mathbf{u}_h$, in (2.30) $q_h = \frac{1}{\beta}\left(1 - \frac{\gamma_1}{\beta}\right)p_h$ and in (2.31) $\mu_h = \frac{1}{\alpha}\left(1 - \frac{\gamma_2}{\alpha}\right)\lambda_h$, sum the three equations and we get

$$a(\mathbf{u}_h, \mathbf{u}_h) + \frac{1}{\beta}\left(1 - \frac{\gamma_1}{\beta}\right)(p_h, p_h) + \frac{1}{\alpha}\left(1 - \frac{\gamma_2}{\alpha}\right)(\lambda_h, \lambda_h)$$
$$= {\omega_h}^2[\rho\|\mathbf{u}_h\|_0^2 + m|\mathbf{u}_h \cdot \mathbf{n}_{|\Gamma_2}|^2].$$

Then (3.21) and (3.23) give

$${\omega_h}^2 \geq \frac{\kappa_1}{\kappa_3}$$

At the end, using (A5) and Theorem 3.2, we obtain also (3.26) and (3.27). $\square$

In the next sections we shall specify some finite element spaces and check the validity of (A1)-(A5) and of (3.3).

4. The rigid cavity problem

Throughout this section, we consider the problem of natural frequencies of a fluid in a rigid cavity (see Fig.1), hence the part of boundary denoted by Γ_2 is empty. Therefore we define

(4.1)
$$\mathbf{V}_0 = \{\mathbf{v} \in [\mathbf{L}^2(\Omega)]^2 : \operatorname{div}\mathbf{v} \in \mathbf{L}^2(\Omega), \ \operatorname{rot}\mathbf{v} \in \mathbf{L}^2(\Omega) \text{ and } \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_1\}$$

endowed with the norm

$$|||\mathbf{v}|||_0 = (\|\operatorname{div}\mathbf{v}\|_0^2 + \|\operatorname{rot}\mathbf{v}\|_0^2)^{\frac{1}{2}} \tag{4.2}$$

and

$$\widehat{\mathbf{L}}^2(\Omega) = \{q \in \mathbf{L}^2(\Omega) | \int_\Omega q\,dx = 0\}. \tag{4.3}$$

Remark 4.1 Due to the assumptions on Ω and to the boundary constraint in the definition (4.1), the elements of $\mathbf{V}_0$ belong to $[\mathbf{H}^1(\Omega)]^2$ and the norm (4.2) is equivalent to the norm of $[\mathbf{H}^1(\Omega)]^2$, see [5] Sect. 3.1. Moreover the analogous of the Poincaré inequality holds, that is

$$\|\mathbf{v}\|_0^2 \le C_0 |||\mathbf{v}|||_0^2. \quad \Box \tag{4.4}$$

Since Γ_2 is empty in the case at the hand, we have some simplifications: the bilinear form a becomes

$$a_0(\mathbf{u},\mathbf{v}) = \gamma_1(\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}) + \gamma_2(\operatorname{rot}\mathbf{u}, \operatorname{rot}\mathbf{v}) \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{V}_0, \tag{4.5}$$

so that

$$a_0(\mathbf{v},\mathbf{v}) \ge \min(\gamma_1,\gamma_2) |||\mathbf{v}|||_0^2 \qquad \forall \mathbf{v} \in \mathbf{V}_0. \tag{4.6}$$

Moreover from (2.7) we have that $p \in \widehat{\mathbf{L}}^2(\Omega)$. Hence Problem 2.2 takes the following form:

Problem 4.2 Find $\omega \in \mathbb{R}$, $\mathbf{u} \in \mathbf{V}_0$, $p \in \widehat{\mathbf{L}}^2(\Omega)$ and $\lambda \in \mathbf{L}^2(\Omega)$ such that

$$a_0(\mathbf{u},\mathbf{v}) + \left(\frac{\gamma_1}{\beta} - 1\right)(\operatorname{div}\mathbf{v}, p) + \left(\frac{\gamma_2}{\alpha} - 1\right)(\operatorname{rot}\mathbf{v}, \lambda)$$
$$= \rho\omega^2(\mathbf{u},\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_0, \tag{4.7}$$
$$\beta(\operatorname{div}\mathbf{u}, q) + (p, q) = 0 \quad \forall q \in \widehat{\mathbf{L}}^2(\Omega), \tag{4.8}$$
$$\alpha(\operatorname{rot}\mathbf{u}, \mu) + (\lambda, \mu) = 0 \quad \forall \mu \in \mathbf{L}^2(\Omega). \tag{4.9}$$

Due to (4.1)-(4.6), it can be easily seen that (3.21) and (3.23) are satisfied. The decomposition theorem for two dimensional vector fields belonging to $\mathbf{V}_0$ (see e.g. [5] Sect. 3.1) gives that $\operatorname{Ker} B = \{\mathbf{0}\}$. Moreover it is clear that $\operatorname{Ker} B^{\mathrm{T}} = \{(0,0)\}$. On the other hand we have that $\operatorname{Im} B = \{(q,\mu) \in [\mathbf{L}^2(\Omega)]^2 : \int_\Omega q = 0\}$.

Hence B possesses a continuous lifting and the inf-sup condition can be written as follows for every fixed γ_1 and γ_2 satisfying (3.20)

$$\sup_{\mathbf{u}\in\mathbf{V}_0}\frac{|\left(1-\frac{\gamma_1}{\beta}\right)(\operatorname{div}\mathbf{u},q)+\left(1-\frac{\gamma_2}{\alpha}\right)(\operatorname{rot}\mathbf{u},\mu)|}{|||\mathbf{u}|||_0}$$
$$\geq \kappa_2(\left(1-\frac{\gamma_1}{\beta}\right)^2\|q\|_0^2+\left(1-\frac{\gamma_2}{\alpha}\right)^2\|\mu\|_0^2)^{\frac{1}{2}}$$
$$\forall (q,\mu)\in \widehat{\mathbf{L}}^2(\Omega)\times\mathbf{L}^2(\Omega). \tag{4.10}$$

In order to apply Theorem 3.3, it remains to show that the operator $T:\mathbf{V}_0\times\widehat{\mathbf{L}}^2(\Omega)\times\mathbf{L}^2(\Omega)\mapsto\mathbf{V}_0\times\widehat{\mathbf{L}}^2(\Omega)\times\mathbf{L}^2(\Omega)$, defined by (3.4), (3.14), (3.15) and (4.5), is compact. Due to the compactness of the inclusion $\mathbf{V}_0\times\widehat{\mathbf{L}}^2(\Omega)\times\mathbf{L}^2(\Omega)\hookrightarrow[\mathbf{L}^2(\Omega)]^2\times[\mathbf{H}^{-1}(\Omega)]^2$, it is enough to prove the continuity of T from $[\mathbf{L}^2(\Omega)]^2\times[\mathbf{H}^{-1}(\Omega)]^2$ into $\mathbf{V}_0\times\widehat{\mathbf{L}}^2(\Omega)\times\mathbf{L}^2(\Omega)$. For this, let us consider the source problem associated to Problem (4.2): for every $\mathbf{f}\in[\mathbf{L}^2(\Omega)]^2$, find $\mathbf{u}^f\in\mathbf{V}_0$, $p^f\in\widehat{\mathbf{L}}^2(\Omega)$, $\lambda^f\in\mathbf{L}^2(\Omega)$, such that

$$a_0(\mathbf{u}^f,\mathbf{v})+\left(\frac{\gamma_1}{\beta}-1\right)(\operatorname{div}\mathbf{v},p^f)+\left(\frac{\gamma_2}{\alpha}-1\right)(\operatorname{rot}\mathbf{v},\lambda^f)$$
$$=\rho(\mathbf{f},\mathbf{v})\qquad\forall\mathbf{v}\in\mathbf{V}_0, \tag{4.11}$$
$$\beta(\operatorname{div}\mathbf{u}^f,q)+(p^f,q)=0\qquad\forall q\in\widehat{\mathbf{L}}^2(\Omega), \tag{4.12}$$
$$\alpha(\operatorname{rot}\mathbf{u}^f,\mu)+(\lambda^f,\mu)=0\qquad\forall\mu\in\mathbf{L}^2(\Omega). \tag{4.13}$$

We need a priori estimates for $\mathbf{u}^f$, p^f and λ^f in term of the norm of $\mathbf{f}$ in $[\mathbf{L}^2(\Omega)]^2$.

Lemma 4.3 *Let* $\mathbf{u}^f\in\mathbf{V}_0$, $p^f\in\widehat{\mathbf{L}}^2(\Omega)$ *and* $\lambda^f\in\mathbf{L}^2(\Omega)$ *be the solution of* (4.11)-(4.13)*, then for every* γ_1 *and* γ_2 *satisfying* (3.20) *the following a priori estimate holds*

$$|||\mathbf{u}^f|||_0+\|p^f\|_0+\|\lambda^f\|_0\leq C\|\mathbf{f}\|_0,$$

where C is a positive constant depending only on α*,* β*,* ρ *and* Ω *and not on* γ_1 *and* γ_2*.*

Proof. For the sake of brevity, we drop the superscript f all along the proof.

Let us take $\mathbf{v}=\mathbf{u}$ in (4.11) and substitute (4.12) with $q=\operatorname{div}\mathbf{u}$ and (4.13) with $\mu=\operatorname{rot}\mathbf{u}$, so we obtain

$$\beta\|\operatorname{div}\mathbf{u}\|_0^2+\alpha\|\operatorname{rot}\mathbf{u}\|_0^2=\rho(\mathbf{f},\mathbf{u})\leq\rho\|\mathbf{f}\|_0\|\mathbf{u}\|_0, \tag{4.14}$$

and, thanks to (4.4), we arrive at

$$|||\mathbf{u}|||_0\leq C\|\mathbf{f}\|_0. \tag{4.15}$$

Then we take $q=p$ in (4.12) and $\mu=\lambda$ in (4.13) and we get

$$\|p\|_0\leq C\beta\|\operatorname{div}\mathbf{u}\|_0,\quad \|\lambda\|_0\leq C\alpha\|\operatorname{rot}\mathbf{u}\|_0$$

which together with (4.14) and (4.15) gives the desired a priori estimate. □

So we have proved:

Proposition 4.4 *Problem* 4.2 *has a countable set of real and strictly positive eigenvalues.*

Remark 4.5 The result of Proposition 4.4 can be extended to the case of a general polygonal domain Ω. The crucial point is the compactness of the imbedding of $\mathbf{V}_0$ into $[\mathbf{L}^2(\Omega)]^2$. This can be obtained following the outline of the proof of Proposition 3.1 in [5]. The main idea consists in applying a decomposition theorem. Every element $\mathbf{v}$ of $\mathbf{V}_0$ can be split as follows (see also the proof of Lemma 5.1 below): $\mathbf{v} = \operatorname{grad} r + \operatorname{rot} \phi$, where r and ϕ are the solutions of a homogeneous Neumann's problem and of a non-homogeneous Dirichlet's problem for the Laplace operator, respectively. When Ω is a polygon with reentrant corners the solutions r and ϕ of these second order elliptic problems are not in $\mathbf{H}^2(\Omega)$, but they still belong to some $\mathbf{H}^{1+\epsilon}(\Omega)$ with $\epsilon > 0$, depending on the width of the angles of Ω, (see [6]). Therefore $\mathbf{v}$ belongs to $[\mathbf{H}^{\epsilon}(\Omega)]^2$ for $\epsilon > 0$, which is compactly embedded into $[\mathbf{L}^2(\Omega)]^2$. See also [14] for an analogous result in the framework of Maxwell's equations. □

Let us consider a finite element approximation of Problem 4.2.

Let us suppose that $\mathcal{T}_h$ is built by rectangles, then set

$$\text{(4.16)} \qquad \mathbf{V}_{0h} = \{\mathbf{v}_h \in [\mathbf{H}^1(\Omega)]^2 |\ \mathbf{v}_{h|K} \in [Q_2(K)]^2\ \forall K \in \mathcal{T}_h\},$$

$$\text{(4.17)} \qquad Q_h = \{q_h \in \mathbf{L}^2(\Omega) |\ q_{h|K} \in P_1(K)\ \forall K \in \mathcal{T}_h\},$$

$$\text{(4.18)} \qquad M_h^0 = \{\mu_h \in \mathbf{L}^2(\Omega) |\ \mu_{h|K} \in P_0(K)\ \forall K \in \mathcal{T}_h\},$$

where $P_k(K)$ is the set of polynomials of degree k on K and $Q_2(K)$ is the set of the polynomials which are quadratic separately with respect each variable.

In addition we shall denote by $\widehat{Q}_h$ the subset of the elements of Q_h with null mean value, that is

$$\text{(4.19)} \qquad \widehat{Q}_h = \{q_h \in Q_h |\ \int_\Omega q_h\, dx = 0\}\ .$$

The spaces defined above enjoy the following approximation properties:

$$\text{(4.20)} \qquad \inf_{\mathbf{v}_h \in \mathbf{V}_{0h}} |||\mathbf{v} - \mathbf{v}_h|||_0 \le Ch^2 \|\mathbf{v}\|_3 \quad \forall \mathbf{v} \in \mathbf{V}_0 \cap [\mathbf{H}^3(\Omega)]^2,$$

$$\text{(4.21)} \qquad \inf_{q_h \in Q_h} \|q - q_h\|_0 \le Ch^2 \|q\|_2 \quad \forall q \in \mathbf{H}^2(\Omega),$$

$$\text{(4.22)} \qquad \inf_{\mu_h \in M_h^0} \|\mu - \mu_h\|_0 \le Ch \|\mu\|_1 \quad \forall \mu \in \mathbf{H}^1(\Omega).$$

Problem 2.3 becomes in this case:

Problem 4.6 Find $\omega_h \in \mathbb{R}$, $\mathbf{u}_h \in \mathbf{V}_{0h}$, $p_h \in \widehat{Q}_h$ and $\lambda_h \in M_h^0$ such that

$$a_0(\mathbf{u}_h, \mathbf{v}_h) + \left(\frac{\gamma_1}{\beta} - 1\right)(\operatorname{div} \mathbf{v}_h, p_h) + \left(\frac{\gamma_2}{\alpha} - 1\right)(\operatorname{rot} \mathbf{v}_h, \lambda_h)$$

$$= \rho\omega_h^2(\mathbf{u}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_{0h}, \tag{4.23}$$

$$\beta(\operatorname{div} \mathbf{u}_h, q_h) + (p_h, q_h) = 0 \quad \forall q_h \in \widehat{Q}_h, \tag{4.24}$$

$$\alpha(\operatorname{rot} \mathbf{u}_h, \mu_h) + (\lambda_h, \mu_h) = 0 \quad \forall \mu_h \in M_h^0. \tag{4.25}$$

The approximation assumption (A5) is satisfied. Hence it remains to verify the discrete inf-sup condition (A4). To this aim, we can prove the following equivalent Fortin condition (see [1]):

Lemma 4.7 *There exists an operator $\Pi : \mathbf{V}_0 \mapsto \mathbf{V}_{0h}$ such that for every $\mathbf{v} \in \mathbf{V}_0$*

$$(\operatorname{div}(\mathbf{v} - \Pi\mathbf{v}), q_h) = 0 \quad \forall q_h \in \widehat{Q}_h, \tag{4.26}$$

$$(\operatorname{rot}(\mathbf{v} - \Pi\mathbf{v}), \mu_h) = 0 \quad \forall \mu_h \in M_h^0, \tag{4.27}$$

$$|||\Pi\mathbf{v}|||_0 \le C\,|||\mathbf{v}|||_0, \tag{4.28}$$

where C is a positive constant independent of h.

Proof. We observe that the pair $(\mathbf{V}_{0h}, \widehat{Q}_h)$ is stable for the Stokes problem, hence we can consider the operator $\Pi : \mathbf{V}_0 \mapsto \mathbf{V}_{0h}$, introduced in that framework, see [1], Ex. II.4.2.

Let us denote by $B_4(K)$ the subset of $Q_2(K)$ containing the bubble functions, that is the elements of $Q_2(K)$ which vanish along ∂K. Then we define

$$\Pi\mathbf{v} = \Pi_1\mathbf{v} + \Pi_2(\mathbf{v} - \Pi_1\mathbf{v}), \tag{4.29}$$

where for all $K \in \mathscr{T}_h$

$$\begin{cases} \Pi_1\mathbf{v}_{|K} \in [Q_2(K)\backslash B_4(K)]^2, \\ \Pi_1\mathbf{v}(M) = \mathbf{v}(M) \quad \forall M \text{ vertex of } K, \\ \int_e \Pi_1\mathbf{v} = \int_e \mathbf{v} \quad \forall e \text{ edge of } K, \end{cases} \tag{4.30}$$

and, for $\mathbf{v} \in \mathbf{V}_0$ such that $\int_K \operatorname{div}\mathbf{v} = 0\ \forall K \in \mathscr{T}_h$, $\Pi_2\mathbf{v}_{|K} \in [B_4(K)]^2$ and satisfies

$$\int_K \operatorname{div}(\mathbf{v} - \Pi_2\mathbf{v})q_h = 0 \quad \forall q_h \in P_1(K). \tag{4.31}$$

We intend that the second condition in (4.30), which has no meaning for $\mathbf{v}$ belonging only to $\mathbf{V}_0$, is modified via the interpolation operator of Clement, [2], in order to become meaningful. The operator Π defined in (4.29)-(4.31), satisfies obviously (4.26). Moreover, since $\Pi_2\mathbf{v}$ vanishes along the boundaries of the elements, we obtain from (4.30) for all $K \in \mathscr{T}_h$

$$\int_K \operatorname{rot}(\mathbf{v} - \Pi\mathbf{v}) = \int_{\partial K} (\mathbf{v} - \Pi\mathbf{v}) \cdot \tau = 0,$$

where τ is the counterclockwise oriented tangential vector to the edges of K. Finally, (4.28) can be achieved via a standard scaling argument. □

Concluding, due to Lemma 4.7, we can apply Theorem 3.4 and we get the following results:

Proposition 4.8 *Problem* 4.6 *has real and positive eigenvalues bounded away from zero independently of h. Moreover, there exist constants C and h_0 such that for all* $0 < h \leq h_0$ *it results*

$$|\omega - \omega_{ih}| \leq C(h^2 + h)^2 \quad \forall i = 1, \dots, n, \tag{4.32}$$

$$\hat{\delta}(E, E_h) \leq C(h^2 + h), \tag{4.33}$$

where E is the eigenspace corresponding to the eigenvalue ω, E_h is the direct sum of the eigenspaces corresponding to the eigenvalues ω_{ih}, $i = 1, \dots, n$ and $\hat{\delta}$ is the gap between E and E_h with respect to the norm of $\mathbf{V}_0 \times \widehat{\mathbf{L}}^2(\Omega) \times \mathbf{L}^2(\Omega)$.

Notice that when ω is a simple eigenvalue (4.33) means

$$|||\mathbf{u} - \mathbf{u}_h|||_0 + \|p - p_h\|_0 + \|\lambda - \lambda_h\|_0 \leq C(h^2 + h).$$

These results are optimal with respect to the choice of the spaces, but due to the poorness of M_h^0, they are not very appealing because there is a sort of loss of accuracy with respect to the approximation properties of $\mathbf{V}_{0h}$ and Q_h. Therefore let us examine what happens when M_h^0 is substituted by

$$M_h^1 = \{\mu_h \in \mathbf{L}^2(\Omega) | \ \mu_{h|K} \in P_1(K) \ \forall K \in \mathscr{T}_h\}. \tag{4.34}$$

The space M_h^1 has the following approximation property:

$$\inf_{\mu_h \in M_h^1} \|\mu - \mu_h\|_0 \leq Ch^2 \|\mu\|_2 \quad \forall \mu \in \mathbf{H}^2(\Omega). \tag{4.35}$$

As a drawback, the operator Π defined in (4.29)-(4.31) does no longer satisfy (4.27) and (4.28). To circumvent this difficulty, we propose a stabilization procedure following the main ideas of the method by Hughes and Franca, introduced in connection with Stokes flow (see [8] and [9]).

Problem 4.9 Find $\omega_h \in \mathbb{R}$, $\mathbf{u}_h \in \mathbf{V}_{0h}$, $p_h \in \widehat{Q}_h$ and $\lambda_h \in M_h^1$ such that

$$\begin{aligned} a_0(\mathbf{u}_h, \mathbf{v}_h) + \left(\frac{\gamma_1}{\beta} - 1\right)(\operatorname{div} \mathbf{v}_h, p_h) + \left(\frac{\gamma_2}{\alpha} - 1\right)(\operatorname{rot} \mathbf{v}_h, \lambda_h) \\ = \rho\omega_h^2(\mathbf{u}_h, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_{0h}, \end{aligned} \tag{4.36}$$

$$\beta(\operatorname{div} \mathbf{u}_h, q_h) + (p_h, q_h) = 0 \qquad \forall q_h \in \widehat{Q}_h, \tag{4.37}$$

$$\begin{aligned} \alpha(\operatorname{rot} \mathbf{u}_h, \mu_h) + (\lambda_h, \mu_h) + \alpha h^2 \sum_{K \in \mathscr{T}_h} (\operatorname{rot} \lambda_h, \operatorname{rot} \mu_h)_K = 0 \\ \forall \mu_h \in M_h^1. \end{aligned} \tag{4.38}$$

We remark that the solutions of the continuous problem (2.11)-(2.16) still satisfy this new formulation because the term added in (4.38) is the irrotational constraint $\lambda = -\alpha \operatorname{rot} \mathbf{u} = 0$.

Defining $\mathbf{H}_h = \mathbf{V}_{0h} \times \widehat{Q}_h \times M_h^1$, whose elements are the triples $V_h = (\mathbf{v}_h, q_h, \mu_h)$, the Problem 4.9 can be written in the form (3.5) with

$$\begin{aligned} A(U_h, V_h) &= a_0(\mathbf{u}_h, \mathbf{v}_h) + \Big(\frac{\gamma_1}{\beta} - 1\Big)(\operatorname{div} \mathbf{v}_h, p_h) + \Big(\frac{\gamma_2}{\alpha} - 1\Big)(\operatorname{rot} \mathbf{v}_h, \lambda_h) \\ &+ \Big(\frac{\gamma_1}{\beta} - 1\Big)(\operatorname{div} \mathbf{u}_h, q_h) + \frac{1}{\beta}\Big(\frac{\gamma_1}{\beta} - 1\Big)(p_h, q_h) + \Big(\frac{\gamma_2}{\alpha} - 1\Big)(\operatorname{rot} \mathbf{u}_h, \mu_h) \\ &+ \frac{1}{\alpha}\Big(\frac{\gamma_2}{\alpha} - 1\Big)(\lambda_h, \mu_h) + \Big(\frac{\gamma_2}{\alpha} - 1\Big) h^2 \sum_{K \in \mathcal{T}_h} (\operatorname{rot} \lambda_h, \operatorname{rot} \mu_h)_K, \end{aligned} \tag{4.39}$$

and

$$R(U_h, \mathbf{V}_h) = \rho(\mathbf{u}_h, \mathbf{v}_h). \tag{4.40}$$

Hence we must check (3.7). To this aim we use an argument introduced by Franca and Stenberg in [4].

As a first step let us prove the following lemma:

Lemma 4.10 *There exist two positive constants C_1 and C_2 which do not depend on h, such that it results:*

$$\begin{aligned} &\sup_{\mathbf{v}_h \in \mathbf{V}_{0h}} \frac{\left|\Big(1 - \frac{\gamma_1}{\beta}\Big)(\operatorname{div} \mathbf{v}_h, q_h) + \Big(1 - \frac{\gamma_2}{\alpha}\Big)(\operatorname{rot} \mathbf{v}_h, \mu_h)\right|}{|||\mathbf{v}_h|||_0} \\ &\qquad \geq C_1 \left(\Big(1 - \frac{\gamma_1}{\beta}\Big)^2 \|q_h\|_0^2 + \Big(1 - \frac{\gamma_2}{\alpha}\Big)^2 \|\mu_h\|_0^2\right)^{\frac{1}{2}} - C_2 h \|\operatorname{rot} \mu_h\|_h, \\ &\qquad \forall (q_h, \mu_h) \in \widehat{Q}_h \times M_h, \end{aligned} \tag{4.41}$$

where $\|\operatorname{rot} \mu_h\|_h = (\sum_{K \in \mathcal{T}_h} \|\operatorname{rot} \mu_h\|_{0,K}^2)^{\frac{1}{2}}$.

Proof. The proof is quite simple. In fact, we have for all $\mathbf{v}_h \in \mathbf{V}_{0h}$, $q_h \in \widehat{Q}_h$ and $\mu_h \in M_h^1$

$$\begin{aligned} &\Big(1 - \frac{\gamma_1}{\beta}\Big)(\operatorname{div} \mathbf{v}_h, q_h) + \Big(1 - \frac{\gamma_2}{\alpha}\Big)(\operatorname{rot} \mathbf{v}_h, \mu_h) \\ &\quad = \Big(1 - \frac{\gamma_1}{\beta}\Big)(\operatorname{div} \mathbf{v}_h, q_h) + \Big(1 - \frac{\gamma_2}{\alpha}\Big)(\operatorname{rot} \mathbf{v}_h, P_0 \mu_h) \\ &\quad + \Big(1 - \frac{\gamma_2}{\alpha}\Big)(\operatorname{rot} \mathbf{v}_h, \mu_h - P_0 \mu_h), \end{aligned}$$

where $P_0 \mu_h$ is the $\mathbf{L}^2$-projection of μ_h onto M_h^0.

Now using the fact that the triple $(\mathbf{V}_h, \widehat{Q}_h, M_h^0)$ satisfies the discrete inf-sup condition (3.24) (see Lemma 4.7) and that the interpolation error estimate $\|\mu_h - P_0 \mu_h\|_0 \leq Ch \|\operatorname{rot} \mu_h\|_h$ holds for all $\mu_h \in M_h^1$, we arrive at (4.41). $\square$

Let us define the norm in $\mathbf{H}$ as follows

(4.42) $$\|V_h\|_{\mathbf{H}} = \left(|||\mathbf{v}_h|||_0^2 + \left(1-\frac{\gamma_1}{\beta}\right)^2 \|q_h\|_0^2 + \left(1-\frac{\gamma_2}{\alpha}\right)^2 \|\mu_h\|_0^2 \right)^{\frac{1}{2}},$$

then we have (see [13]):

Lemma 4.11 *There exists a positive constant k_2 independent of h, such that for all $U_h \in \mathbf{H}_h$ it results*

(4.43) $$\sup_{V_h \in \mathbf{H}_h} \frac{|A(U_h, V_h)|}{\|V_h\|_{\mathbf{H}}} \geq k_2 \|U_h\|_{\mathbf{H}}.$$

Proof. It is enough to prove that for a properly chosen V_h, the inequality in (4.43) is valid. Given $U_h = (\mathbf{u}_h, p_h, \lambda_h)$, let us consider $V_h = (\mathbf{u}_h - \delta\mathbf{w}_h, -p_h, -\lambda_h)$, where $\mathbf{w}_h$ realizes the supremum in (4.41) and is such that $|||\mathbf{w}_h|||_0 = \left(\left(1-\frac{\gamma_1}{\beta}\right)^2 \|p_h\|_0^2 + \left(1-\frac{\gamma_2}{\alpha}\right)^2 \|\lambda_h\|_0^2 \right)^{\frac{1}{2}}$. Then we have

(4.44)
$$A(U_h, V_h) = A((\mathbf{u}_h, p_h, \lambda_h); (\mathbf{u}_h, -p_h, -\lambda_h)) + \delta A((\mathbf{u}_h, p_h, \lambda_h); (-\mathbf{w}_h, 0, 0)).$$

It is easy to see that

$$\begin{aligned} &A((\mathbf{u}_h, p_h, \lambda_h); (\mathbf{u}_h, -p_h, -\lambda_h)) \\ &= |||\mathbf{u}_h|||_0^2 + \frac{1}{\beta}\left(1-\frac{\gamma_1}{\beta}\right)\|p_h\|_0^2 + \frac{1}{\alpha}\left(1-\frac{\gamma_2}{\alpha}\right)\|\lambda_h\|_0^2 + \left(1-\frac{\gamma_2}{\alpha}\right)h^2 \|\operatorname{rot}\lambda_h\|_h^2. \end{aligned}$$
(4.45)

Moreover due to the definition of $\mathbf{w}_h$, we have

$$\begin{aligned} &A((\mathbf{u}_h, p_h, \lambda_h); (-\mathbf{w}_h, 0, 0)) \\ &= A((\mathbf{u}_h, 0, 0); (-\mathbf{w}_h, 0, 0)) + A((0, p_h, \lambda_h); (-\mathbf{w}_h, 0, 0)) \\ &= -a_0(\mathbf{u}_h, \mathbf{w}_h) + \left(1-\frac{\gamma_1}{\beta}\right)(\operatorname{div}\mathbf{w}_h, p_h) + \left(1-\frac{\gamma_2}{\alpha}\right))(\operatorname{rot}\mathbf{w}_h, \lambda_h). \end{aligned}$$

Then the continuity of a_0 and (4.41) give

$$\begin{aligned} A((\mathbf{u}_h, p_h, \lambda_h); (-\mathbf{w}_h, 0, 0)) &\geq -|||\mathbf{u}_h|||_0 |||\mathbf{w}_h|||_0 \\ &+ \left[C_1 \left(\left(1-\frac{\gamma_1}{\beta}\right)^2 \|p_h\|_0^2 + \left(1-\frac{\gamma_2}{\alpha}\right)^2 \|\lambda_h\|_0^2 \right)^{\frac{1}{2}} - C_2 h \|\operatorname{rot}\lambda_h\|_h \right] |||\mathbf{w}_h|||_0 \\ &\geq -C_3 |||\mathbf{u}_h|||_0^2 + C_4 \left(\left(1-\frac{\gamma_1}{\beta}\right)^2 \|p_h\|_0^2 + \left(1-\frac{\gamma_2}{\alpha}\right)^2 \|\lambda_h\|_0^2 \right) \end{aligned}$$
(4.46) $$-C_5 h^2 \|\operatorname{rot}\lambda_h\|_h^2.$$

Substituting the inequalities (4.45) and (4.46) in (4.44) we obtain

$$\begin{aligned} A(U_h, V_h) &\geq (1-\delta C_3)|||\mathbf{u}_h|||_0^2 + \delta C_4 \left(\left(1-\frac{\gamma_1}{\beta}\right)^2 \|p_h\|_0^2 + \left(1-\frac{\gamma_2}{\alpha}\right)^2 \|\lambda_h\|_0^2 \right) \\ &+ \frac{1}{\beta}\left(1-\frac{\gamma_1}{\beta}\right)\|p_h\|_0^2 + \frac{1}{\alpha}\left(1-\frac{\gamma_2}{\alpha}\right)\|\lambda_h\|_0^2 + (1-\frac{\gamma_2}{\alpha} - \delta C_5)h^2 \|\operatorname{rot}\lambda_h\|_h^2; \end{aligned}$$

so that choosing $\delta = \frac{1}{2}\min(1/C_3, (1-\frac{\gamma_2}{\alpha})/C_5)$, we arrive at

$$A(U_h, V_h) \geq C_6 \|U_h\|^2_{\mathbf{H}}. \tag{4.47}$$

On the other hand we have

$$\begin{aligned}\|V_h\|^2_{\mathbf{H}} &= |||\mathbf{u}_h - \delta\mathbf{w}_h|||^2_0 + \left(1-\frac{\gamma_1}{\beta}\right)^2\|p_h\|^2_0 + \left(1-\frac{\gamma_2}{\alpha}\right)^2\|\lambda_h\|^2_0 \\ &\leq 2|||\mathbf{u}_h|||^2_0 + (1+2\delta^2)\left(1-\frac{\gamma_1}{\beta}\right)^2\|p_h\|^2_0 + (1+2\delta^2)\left(1-\frac{\gamma_2}{\alpha}\right)^2\|\lambda_h\|^2_0 \\ &\leq C_7\|U_h\|^2_{\mathbf{H}}\end{aligned}$$

which combined with (4.47) yields the desired estimate. □

Concluding, Lemma 4.11 implies that the assumptions of Theorem 3.2 are satisfied, so that we have:

Proposition 4.12 *Problem* 4.9 *has real and positive eigenvalues, bounded away from zero uniformly with respect to h. Moreover, the following error estimates hold*

$$|\omega - \omega_{ih}| \leq Ch^4 \quad \forall i = 1, \ldots, n, \tag{4.48}$$

$$\hat{\delta}(E, E_h) \leq Ch^2, \tag{4.49}$$

where $\hat{\delta}$ is the gap between E and E_h with respect to the norm (4.42).

Remark 4.13 Let us take a triangular mesh and, instead of the space $\mathbf{V}_{0h}$ defined in (4.16), let us consider the space

$$V'_{0h} = \{\mathbf{v}_h \in \mathbf{V}_0 |\ \mathbf{v}_{h|K} \in [P_2(K) + B_3(K)]^2\ \forall K \in \mathcal{T}_h\},$$

where $B_3(K)$ is the set of the bubble functions of degree 3 on K.

Then the results of Propositions 4.8 and 4.12 can be easily extended to the approximation of Problem 2.2 by means of $(V'_{0h}, \widehat{Q}_h, M^0_h)$ and $(V'_{0h}, \widehat{Q}_h, M^1_h)$. In fact, the couple $(V'_{0h}, \widehat{Q}_h)$ is the well-known Crouzeix-Raviart element to approximate the Stokes problem (see [3]). Moreover, we recall that the proofs of Propositions 4.8 and 4.12 are based on the fact that the couple $(V_{0h}, \widehat{Q}_h)$ is stable for the Stokes problem and that the triple $(V_{0h}, \widehat{Q}_h, M^0_h)$ is stable for Problem 2.2.

5. The piston container problem

Let us discuss the second category of problems we have introduced in Sect. 2: fluids vibrating in moving boundaries, (see Fig. 2). Without loss of generality we take $\Omega = (0,1) \times (0,1)$. Hence $\Gamma_2 = \{(x,1); x \in (0,1)\}$. In the present case, since Γ_2 is not empty, we set

$$\begin{aligned}\mathbf{V}_1 = \{\mathbf{v} \in [\mathbf{L}^2(\Omega)]^2 : \ &\operatorname{div}\mathbf{v} \in \mathbf{L}^2(\Omega),\ \ \operatorname{rot}\mathbf{v} \in \mathbf{L}^2(\Omega), \\ &\mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_1 \ \text{ and } \ \mathbf{v}\cdot\mathbf{n} \text{ constant along } \Gamma_2\}.\end{aligned} \tag{5.1}$$

Lemma 5.1 *There exists a positive constant C_0 depending only on the domain Ω such that for all $\mathbf{v} \in \mathbf{V}_1$*

$$\|\mathbf{v}\|_0^2 \le C_0(\|\operatorname{div}\mathbf{v}\|_0^2 + \|\operatorname{rot}\mathbf{v}\|_0^2 + |\mathbf{v}\cdot\mathbf{n}_{|\Gamma_2}|^2). \tag{5.2}$$

Proof. We apply a decomposition theorem in $[\mathbf{L}^2(\Omega)]^2$ (see [5] Th.3.2), hence every element $\mathbf{v}$ of $\mathbf{V}_1$ can be written as the sum

$$\mathbf{v} = \operatorname{grad} r + \operatorname{rot}\phi, \tag{5.3}$$

where $r \in \mathbf{H}^1(\Omega)$ is the unique solution of

$$(\operatorname{grad} r, \operatorname{grad} s) = (\mathbf{v}, \operatorname{grad} s) \qquad \forall\, s \in \mathbf{H}^1(\Omega), \qquad (r, 1) = 0 \tag{5.4}$$

and $\phi \in \mathbf{H}_0^1(\Omega)$ is the only solution of

$$(\operatorname{rot}\phi, \operatorname{rot}\psi) = (\mathbf{v} - \operatorname{grad} r, \operatorname{rot}\psi) \qquad \forall\psi \in \mathbf{H}_0^1(\Omega). \tag{5.5}$$

Taking $s = r$ in (5.4) and integrating by parts, we have

(5.6)
$$\|\operatorname{grad} r\|_0^2 = (-\operatorname{div}\mathbf{v}, r) + \mathbf{v}\cdot\mathbf{n}_{|\Gamma_2}\int_{\Gamma_2} r\, d\gamma \le C(\|\operatorname{div}\mathbf{v}\|_0 + |\mathbf{v}\cdot\mathbf{n}_{|\Gamma_2}|)\|\operatorname{grad} r\|_0.$$

To obtain the last inequality, we used also the Poincaré inequality (for functions with null mean value) and a trace theorem in $\mathbf{H}^1(\Omega)$.

Next, let us take $\psi = \phi$ in (5.5) and integrate by parts, then we get

$$\|\operatorname{rot}\phi\|_0^2 = |(\operatorname{rot}\mathbf{v}, \phi)| \le C\|\operatorname{rot}\mathbf{v}\|_0\|\operatorname{rot}\phi\|_0. \tag{5.7}$$

The inequalities (5.6) and (5.7) give (5.2), therefore the proof of the lemma is completed. □

By Lemma (5.1) we endow the space $\mathbf{V}_1$ with the following norm

$$|||\mathbf{v}|||_1 = (\|\operatorname{div}\mathbf{v}\|_0^2 + \|\operatorname{rot}\mathbf{v}\|_0^2 + |\mathbf{v}\cdot\mathbf{n}_{|\Gamma_2}|^2)^{\frac{1}{2}}. \tag{5.8}$$

Taking $\mathbf{V} = \mathbf{V}_1$ in Problem 2.2 we obtain the following Problem:

Problem 5.2 Find $\omega \in \mathbb{R}$, $\mathbf{u} \in \mathbf{V}_1$, $p \in \mathbf{L}^2(\Omega)$ and $\lambda \in \mathbf{L}^2(\Omega)$ such that

$$\begin{aligned} &a(\mathbf{u},\mathbf{v}) + \Big(\frac{\gamma_1}{\beta} - 1\Big)(\operatorname{div}\mathbf{v}, p) + \Big(\frac{\gamma_2}{\alpha} - 1\Big)(\operatorname{rot}\mathbf{v}, \lambda) \\ &\quad = \rho\omega^2(\mathbf{u},\mathbf{v}) + m\omega^2\mathbf{u}\cdot\mathbf{n}_{|\Gamma_2}\,\mathbf{v}\cdot\mathbf{n}_{|\Gamma_2} \\ &\quad \forall\mathbf{v} \in \mathbf{V}_1, \end{aligned} \tag{5.9}$$

$$\beta(\operatorname{div}\mathbf{u}, q) + (p, q) = 0 \quad \forall q \in \mathbf{L}^2(\Omega), \tag{5.10}$$

$$\alpha(\operatorname{rot}\mathbf{u}, \mu) + (\lambda, \mu) = 0 \quad \forall\mu \in \mathbf{L}^2(\Omega). \tag{5.11}$$

We recall that a is defined by (2.25).

In order to deal with the constraint that $\mathbf{v}\cdot\mathbf{n}_{|\Gamma_2}$ is constant introduced in the definition of the space $\mathbf{V}_1$, we characterize $\mathbf{V}_1$ as follows:

Lemma 5.3 *Let* $\mathbf{w} = (0, y)$*, then*

$$\mathbf{V}_1 = \mathbf{V}_0 \bigoplus \operatorname{span}\{\mathbf{w}\} \tag{5.12}$$

where $\mathbf{V}_0$ *is given by* (4.1).

Proof. It is easy to see that $\mathbf{w}$ belongs to $\mathbf{V}_1$, with $\mathbf{w}\cdot\mathbf{n} = 1$ on Γ_2. Then if we take an element of the direct sum in the right hand side of (5.12), it is contained in $\mathbf{V}_1$. Viceversa, let $\mathbf{v} \in \mathbf{V}_1$. We set $v_c = \mathbf{v}\cdot\mathbf{n}_{|\Gamma_2}$, then $\mathbf{v}_0 = \mathbf{v} - v_c\mathbf{w}$ is in $\mathbf{V}_0$. In fact it is sufficiently regular, $\mathbf{v}_0\cdot\mathbf{n}$ vanishes along Γ_1 since $\mathbf{v}\cdot\mathbf{n}$ and $\mathbf{w}\cdot\mathbf{n}$ are both zero there; along Γ_2 we have $\mathbf{v}_0\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{n} - v_c = 0$, by definition of v_c. □

Then, since $\operatorname{div}\mathbf{w} = 1$ and $\operatorname{rot}\mathbf{w} = 0$ Problem 5.2 reduces to:

Problem 5.4 Find $\omega \in \mathbb{R}$, $\mathbf{u}_0 \in \mathbf{V}_0$, $u_c \in \mathbb{R}$, $p \in \mathbf{L}^2(\Omega)$ and $\lambda \in \mathbf{L}^2(\Omega)$ such that

$$\begin{aligned} a_0(\mathbf{u}_0, \mathbf{v}) + \left(\frac{\gamma_1}{\beta} - 1\right)(\operatorname{div}\mathbf{v}, p) + \left(\frac{\gamma_2}{\alpha} - 1\right)(\operatorname{rot}\mathbf{v}, \lambda) & \\ = \rho\omega^2(\mathbf{u}_0 + u_c\mathbf{w}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_0, & \qquad (5.13) \end{aligned}$$

$$(K + \gamma_1)u_c + \left(\frac{\gamma_1}{\beta} - 1\right)(1, p) = \rho\omega^2(\mathbf{u}_0 + u_c\mathbf{w}, \mathbf{w}) + m\omega^2 u_c \tag{5.14}$$

$$\beta(\operatorname{div}\mathbf{u}_0, q) + \beta u_c(1, q) + (p, q) = 0 \quad \forall q \in \mathbf{L}^2(\Omega), \tag{5.15}$$

$$\alpha(\operatorname{rot}\mathbf{u}_0, \mu) + (\lambda, \mu) = 0 \quad \forall \mu \in \mathbf{L}^2(\Omega). \tag{5.16}$$

Problems 5.2 and 5.4 are equivalent.

Let us check if the assumptions (A1)-(A3) are verified for Problem 5.4: the ellipticity (A1) corresponds to

(5.17)

$$a_0(\mathbf{v}, \mathbf{v}) + v_c{}^2(K + \gamma_1) \geq \min(\gamma_1, \gamma_2, K + \gamma_1)(|||\mathbf{v}|||_0^2 + v_c{}^2) \quad \forall \mathbf{v} \in \mathbf{V}_0,\ v_c \in \mathbb{R}$$

analogously the continuity (A3) is given by

(5.18)

$$\rho(\mathbf{u}_0 + u_c\mathbf{w}, \mathbf{v}) + \rho(\mathbf{u}_0 + u_c\mathbf{w}, v_c\mathbf{w}) + m u_c v_c \leq \kappa_2(|||\mathbf{u}|||_0^2 + u_c{}^2)^{\frac12}(|||\mathbf{v}|||_0^2 + v_c{}^2)^{\frac12}\ .$$

The operator B becomes $\tilde{B}(\mathbf{u}_0, u_c) = (\operatorname{div}(\mathbf{u}_0 + u_c\mathbf{w}), \operatorname{rot}\mathbf{u}_0)$. Hence we have that $\operatorname{Ker}\tilde{B}^{\mathrm{T}} = \{(0,0)\}$ and $\operatorname{Im}\tilde{B} = [\mathbf{L}^2(\Omega)]^2$, which is equivalent to the inf-sup condition (3.22), that is

$$\begin{aligned} \sup_{\substack{\mathbf{v}\in\mathbf{V}_0 \\ v_c\in\mathbb{R}}} \frac{\left|\left(1 - \frac{\gamma_1}{\beta}\right)(\operatorname{div}(\mathbf{v} + v_c\mathbf{w}), q) + \left(1 - \frac{\gamma_2}{\alpha}\right)(\operatorname{rot}\mathbf{v}, \mu)\right|}{(|||\mathbf{v}|||_0^2 + v_c{}^2)^{\frac12}} & \\ \geq \kappa_2\left(\left(1 - \frac{\gamma_1}{\beta}\right)^2 \|q\|_0^2 + \left(1 - \frac{\gamma_2}{\alpha}\right)^2 \|\mu\|_0^2\right)^{\frac12} & \\ \forall (q, \mu) \in [\mathbf{L}^2(\Omega)]^2. & \qquad (5.19) \end{aligned}$$

Moreover, the operator T associated to Problem 5.4 defined as in (3.4) with (3.14) and (3.15) is compact. For the proof one can apply the same argument used in sect. 4; namely T is the composition of the compact inclusion of $\mathbf{V}_0 \times \mathbb{R} \times [\mathbf{L}^2(\Omega)]^2$ into $[\mathbf{L}^2(\Omega)]^2 \times \mathbb{R} \times [\mathbf{H}^{-1}(\Omega)]^2$ and the continuous operator which associates to each function $(\mathbf{f}, g) \in [\mathbf{L}^2(\Omega)]^2 \times \mathbb{R}$ the solution of the source problem corresponding to Problem 5.4.

Therefore we can apply Theorem 3.3 and we have that Problem 5.4 admits a countable set of diverging eigenvalues which are strictly positive bounded away from zero as follows:

$$\omega^2 \geq \frac{\min(\beta, K + \gamma_1)}{\max(2\rho C_0, 2\rho\|\mathbf{w}\|_0^2 + m)}. \tag{5.20}$$

To obtain this inequality it is enough to take $q = \operatorname{div}\mathbf{u}_0$ in (5.15) and $\mu = \operatorname{rot}\mathbf{u}_0$ in (5.16) and to substitute properly into (5.13) with $\mathbf{v} = \mathbf{u}_0$ plus (5.14) multiplied by u_c.

Let us turn to the discretization of Problem 5.4. We use again $\mathbf{V}_0$, Q_h, M_h^0 and M_h^1 defined in (4.16), (4.17), (4.18) and (4.34) respectively to approximate $\mathbf{V}_0$ and $\mathbf{L}^2(\Omega)$ (see also (4.21), (4.22) and (4.35) for the approximation estimates).

Hence the discretization of Problem 5.4 is:

Problem 5.5 Find $\omega_h \in \mathbb{R}$, $\mathbf{u}_{0h} \in \mathbf{V}_{0h}$, $u_{ch} \in \mathbb{R}$, $p_h \in Q_h$ and $\lambda_h \in M_h$ such that

$$a_0(\mathbf{u}_{0h}, \mathbf{v}_h) + \left(\frac{\gamma_1}{\beta} - 1\right)(\operatorname{div}\mathbf{v}_h, p_h) + \left(\frac{\gamma_2}{\alpha} - 1\right)(\operatorname{rot}\mathbf{v}_h, \lambda_h)$$

$$= \rho\omega_h^2(\mathbf{u}_{0h} + u_{ch}\mathbf{w}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_{0h}, \tag{5.21}$$

$$(K + \gamma_1)u_{ch} + \left(\frac{\gamma_1}{\beta} - 1\right)(1, p_h) = \rho\omega_h^2(\mathbf{u}_{0h} + u_{ch}\mathbf{w}, \mathbf{w}) + m\omega_h^2 u_{ch} \tag{5.22}$$

$$\beta(\operatorname{div}\mathbf{u}_{0h}, q_h) + \beta\, u_{ch}(1, q_h) + (p_h, q_h) = 0 \quad \forall q_h \in Q_h, \tag{5.23}$$

$$\alpha(\operatorname{rot}\mathbf{u}_{0h}, \mu_h) + (\lambda_h, \mu_h) = 0 \quad \forall \mu_h \in M_h. \tag{5.24}$$

It is evident that the assumption (A5) is satisfied.

Let us consider $M_h = M_h^0$. The discrete inf-sup condition (A4) can be obtained easily, in fact

$$\sup_{\substack{\mathbf{v}_h \in \mathbf{V}_{0h} \\ v_{ch} \in \mathbb{R}}} \frac{|\left(1 - \frac{\gamma_1}{\beta}\right)(\operatorname{div}(\mathbf{v}_h + v_{ch}\mathbf{w}), q_h) + \left(1 - \frac{\gamma_2}{\alpha}\right)(\operatorname{rot}\mathbf{v}_h, \mu_h)|}{(|||\mathbf{v}_h|||_0^2 + v_{ch}^2)^{\frac{1}{2}}}$$

$$\geq \frac{1}{2}\sup_{\mathbf{v}_h \in \mathbf{V}_{0h}} \frac{|\left(1 - \frac{\gamma_1}{\beta}\right)(\operatorname{div}\mathbf{v}_h, q_h - ⨍ q_h) + \left(1 - \frac{\gamma_2}{\alpha}\right)(\operatorname{rot}\mathbf{v}_h, \mu_h)|}{|||\mathbf{v}_h|||_0}$$

$$+\frac{1}{2}\left(1 - \frac{\gamma_1}{\beta}\right)|\int_\Omega q_h|, \tag{5.25}$$

where $⨍ q_h$ stands for the mean value of q_h over Ω.

Then using Lemma 4.7 and the fact that $\|q_h\|_0^2 = \|q_h - ⨍ q_h\|_0^2 + \|⨍ q_h\|_0^2$, we obtain the desired inequality: there exists κ_4 independent of h such that

$$\sup_{\substack{\mathbf{v}_h\in\mathbf{V}_{0h}\\ v_{ch}\in\mathbb{R}}} \frac{|\left(1-\frac{\gamma_1}{\beta}\right)(\operatorname{div}(\mathbf{v}_h+v_{ch}\mathbf{w}),q_h)+\left(1-\frac{\gamma_2}{\alpha}\right)(\operatorname{rot}\mathbf{v}_h,\mu_h)|}{(|||\mathbf{v}_h|||_0^2+v_{ch}^2)^{\frac{1}{2}}}$$
$$\geq \kappa_2(\left(1-\frac{\gamma_1}{\beta}\right)^2\|q_h\|_0^2+\left(1-\frac{\gamma_2}{\alpha}\right)^2\|\mu_h\|_0^2)^{\frac{1}{2}}$$

(5.26) $\qquad \forall (q_h,\mu_h)\in Q_h\times M_h^0.$

Therefore we can extend to the present case the results of Proposition 4.8, that is:

Proposition 5.6 *Problem 5.5 has real, positive eigenvalues bounded away from zero, independently of h and the following error estimates hold*

(5.27) $\qquad |\omega-\omega_{ih}| \leq C(h^2+h)^2 \quad \forall i=1,\dots,n,$

(5.28) $\qquad \hat{\delta}(E,E_h) \leq C(h^2+h),$

where $\hat{\delta}$ *is the gap between* E *and* E_h *with respect to the norm of* $\mathbf{V}_0\times\mathbb{R}\times[\mathbf{L}^2(\Omega)]^2$.

Let us now discretize Problem 5.2 by means of the triple $(\mathbf{V}_{1h},Q_h,M_h^1)$. Then we can apply the argument of Sect. 4 and introduce the augmented formulation analogous to Problem 4.9:

Problem 5.7 Find $\omega_h\in\mathbb{R}$, $\mathbf{u}_h\in\mathbf{V}_{1h}$, $u_{ch}\in\mathbb{R}$, $p_h\in Q_h$ and $\lambda_h\in M_h^1$ such that

$$a_0(\mathbf{u}_{0h},\mathbf{v}_h)+\left(\frac{\gamma_1}{\beta}-1\right)(\operatorname{div}\mathbf{v}_h,p_h)+\left(\frac{\gamma_2}{\alpha}-1\right)(\operatorname{rot}\mathbf{v}_h,\lambda_h)$$

(5.29) $\qquad = \rho\omega_h^2(\mathbf{u}_{0h}+u_{ch}\mathbf{w},\mathbf{v}_h)\quad \forall \mathbf{v}_h\in\mathbf{V}_{0h},$

(5.30) $\qquad (K+\gamma_1)u_{ch}+\left(\frac{\gamma_1}{\beta}-1\right)(1,p_h)=\rho\omega_h^2(\mathbf{u}_{0h}+u_{ch}\mathbf{w},\mathbf{w})+m\omega_h^2u_{ch}$

(5.31) $\qquad \beta(\operatorname{div}\mathbf{u}_{0h},q_h)+\beta\, u_{ch}(1,q_h)+(p_h,q_h)=0\quad \forall q_h\in Q_h,$

$$\alpha(\operatorname{rot}\mathbf{u}_{0h},\mu_h)+(\lambda_h,\mu_h)+\alpha h^2\sum_{K\in\mathcal{T}_h}(\operatorname{rot}\lambda_h,\operatorname{rot}\mu_h)_K=0$$

(5.32) $\qquad \forall \mu_h\in M_h^1.$

It is possible to define suitable bilinear forms A and R, see (4.39) and (4.40), in order to see that this problem fits into the setting of the discrete eigenvalue Problem (3.5). Hence we have:

Proposition 5.8 *The eigenvalues of Problem* 5.7 *are real, positive and bounded away from zero independently of h and the following error estimates are valid:*

(5.33) $\qquad |\omega-\omega_{ih}| \leq Ch^4 \quad \forall i=1,\dots,n,$

(5.34) $\qquad \hat{\delta}(E,E_h) \leq Ch^2,$

where $\hat{\delta}$ *is the gap between* E *and* E_h *with respect to the norm of* $\mathbf{V}_0\times\mathbb{R}\times[\mathbf{L}^2(\Omega)]^2$.

Remark 5.9 A remark analogous to Remark 4.13 can be done also for the category of problems considered in this section.

References

1. Brezzi, F., Fortin, M. (1991): Mixed and hybrid finite element methods. Springer, New York
2. Clement, P. (1975): Approximation by finite element functions using local regularization. R.A.I.R.O. Anal. Numer. **9**, 77–84
3. Crouzeix, M., Raviart, P.A. (1973): Conforming and non-conforming finite element methods for solving the stationary Stokes equations. R.A.I.R.O. Anal. Numer. **7**, 33–76
4. Franca, L.P., Stenberg, R. (1991): Error analysis of some Galerkin least-squares methods of the elasticity equations. SIAM J. Numer. Anal. **28**, 1680–1697
5. Girault, V., Raviart, P.A. (1986): Finite element approximation of Navier-Stokes equations, theory and algorithms. Springer, Berlin
6. Grisvard, P. (1985): Elliptic problems in non-smooth domains. Pitman, London
7. Hamdi, M.A., Ousset, Y. (1978): A displacement method for the analysis of vibrations of coupled fluid-structure systems. Int. J. Numer. Meth. Eng. **13**, 139–150
8. Hughes, T.J.R., Franca, L.P. (1987): A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions: symmetric formulations that converge for all velocity/pressure spaces. Comp. Meths. Appl. Mech. Engrg. **65**, 85–96
9. Hughes, T.J.R., Franca, L.P., Balestra, M. (1986): A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuška-Brezzi condition: a stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations. Comp. Meths. Appl. Mech. Engrg. **59**, 85–99
10. Mercier, B., Osborn, J., Rappaz, J., Raviart, P.A. (1981): Eigenvalue approximation by mixed and hybrid methods. Math. Comp. **36**, 427–453
11. Bathe, K.J., Nitikitpaiboon, C., Wang, X. (1995): A mixed displacement-based finite element formulation for acoustic fluid-structure interaction. Computers & Structures **56**, 2/3, 225–237
12. Olson, L.G., Bathe, K.J. (1983): A study of displacement-based finite elements for calculating frequencies of fluid and fluid-structure systems. Nuclear Engineering and Design **76**, 137–151
13. Verfurth, R. (1984): Error estimates for a mixed finite element approximation of the Stokes equation. RAIRO, Numer. Anal. **18**, 2, 175–182
14. Wang, X., Bathe, K.J.: On mixed elements for acoustic fluid-structure interactions. Submitted to M^3AS
15. Weber, C. (1980): A local compactness theorem for Maxwell's equations. Math. Meth. in the Appl. Sci. **2**, 12–25