Numer. Math. 72: 1–20 (1995)

Numerische
Mathematik

An error bound for the finite element approximation of the Cahn-Hilliard equation with logarithmic free energy

John W. Barrett[1], **James F. Blowey**

[1] Department of Mathematics, Imperial College, London, SW7 2BZ, U.K.
[2] Department of Mathematical Sciences, University of Durham, DH1 3LE, U.K.

Received October 12, 1994

Summary. An error bound is proved for a fully practical piecewise linear finite element approximation, using a backward Euler time discretization, of the Cahn-Hilliard equation with a logarithmic free energy.

Mathematics Subject Classification (1991): 65M60

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^d, d \leq 3$, with a Lipschitz boundary $\partial\Omega$. We consider the Cahn-Hilliard equation with logarithmic free energy:
Find $\{u(x,t), w(x,t)\}$ such that

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta w \qquad \text{in } \ \Omega_T := \Omega \times (0,T), \\ w &= \Psi'(u) - \gamma \Delta u \qquad \text{in } \ \Omega_T, \\ u(x,0) &= u_0(x) \qquad \forall\, x \in \Omega, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \ \partial\Omega \times (0,T); \end{aligned}$$

where ν is normal to $\partial\Omega$, the free energy $\Psi : [-1,1] \to \mathbb{R}$ is given by

$$\Psi(s) := \psi(s) + \frac{\theta_c}{2}(1-s^2) := \frac{\theta}{2}\left[(1+s)\ln[\frac{1+s}{2}] + (1-s)\ln[\frac{1-s}{2}]\right] + \frac{\theta_c}{2}(1-s^2)$$

and γ, θ and θ_c are positive constants with $\theta < \theta_c$. We define the monotone function $\phi : (-1,1) \to \mathbb{R}$ to be

$$\phi(s) := \psi'(s) \equiv \tfrac{\theta}{2}[\ln(1+s) - \ln(1-s)].$$

The above problem was proposed by Cahn (1961) and Cahn and Hilliard (1958) to model phase separation of a binary mixture, which is quenched into an unstable state. Here $u := X_B - X_A \in [-1,1]$, where X_A, $X_B \in [0,1]$ are the mass fractions of the two components A and B. When the quench is shallow, that is θ is close to θ_c, then the free energy, Ψ, is usually approximated by a quartic polynomial. The majority of the mathematics literature has concentrated on this case. However, this approximation is invalid if the quench is deep, i.e. $\theta \ll \theta_c$. For a fuller discussion of the model, see Copetti and Elliott (1992) and the references therein.

We introduce a weak formulation of the above problem:
(P) Find $\{u, w\}$ such that $u(\cdot,0) = u_0(\cdot)$ and for a.e. $t \in (0,T)$

$$\langle \tfrac{\partial u}{\partial t}, \eta \rangle + (\nabla w, \nabla \eta) = 0 \quad \forall\, \eta \in H^1(\Omega), \tag{1.1a}$$

$$\gamma(\nabla u, \nabla \eta) + (\Psi'(u), \eta) = (w, \eta) \quad \forall\, \eta \in H^1(\Omega). \tag{1.1b}$$

We have adopted the standard notation for Sobolev spaces, denoting the norm of $H^m(\Omega)$ ($m \in \mathbb{N}$) by $\|\cdot\|_m$ and the semi-norm by $|\cdot|_m$. Throughout $(\cdot,\cdot)$ denotes the standard L^2 inner product over Ω and $\langle \cdot,\cdot \rangle$ denotes the duality pairing between $\left(H^1(\Omega)\right)'$ and $H^1(\Omega)$; In addition we define

$$⨍ \eta := \tfrac{1}{|\Omega|}(\eta, 1) \quad \forall\, \eta \in L^2(\Omega).$$

The major difficulty in problem (P) is that $\psi'(s)$ is singular at $s = \pm 1$ and therefore has no meaning if $u = \pm 1$ in an open set of non-zero measure. By studying a regularized problem, see (P_ε) in the next section, Elliott and Luckhaus (1991) proved the following result; see also Copetti and Elliott (1992):

Theorem 1.1. *Given $u_0 \in H^1(\Omega)$ and $\delta \in (0,1)$ such that $\|u_0\|_{L^\infty(\Omega)} \le 1$ and $|⨍ u_0| < 1 - \delta$, then there exists a unique solution $\{u, w\}$ to (P) such that*

$$\begin{gathered} u \in L^\infty(0,T; H^1(\Omega)) \cap H^1(0,T; (H^1(\Omega))'), \\ w - ⨍ w \in L^2(0,T; H^1(\Omega)), \end{gathered} \tag{1.2}$$

$$\begin{gathered} \sqrt{t}\, \tfrac{\partial u}{\partial t} \in L^2(0,T; H^1(\Omega)),\ \sqrt{t}\,\phi(u) \in L^\infty(0,T; L^2(\Omega)), \\ \sqrt{t}\, w \in L^\infty(0,T; H^1(\Omega)) \end{gathered} \tag{1.3}$$

and

$$|u| < 1 \qquad \text{a.e. } in \ \ \Omega_T. \tag{1.4}$$

We note that the integral assumption on the initial data only excludes the physically uninteresting case of $u_0 \equiv \pm 1$, when only one component is present.

In addition to the above, Elliott and Luckhaus (1991) proved that as $\theta/\theta_c \to 0$ the solution $\{u, w\}$ of (P) converges to the free boundary limit problem studied by Blowey and Elliott (1991) and Blowey and Elliott (1992).

The finite element approximation of (P) has been considered by Copetti and Elliott (1992) under the following assumptions:

(A) Let Ω be convex polyhedral. Let $\mathscr{T}^h$ be a quasi-uniform partitioning of Ω into disjoint open simplices κ with $h_\kappa := \mathrm{diam}(\kappa)$ and $h := \max_{\kappa\in\mathscr{T}^h} h_\kappa$, so that $\overline{\Omega} = \cup_{\kappa\in\mathscr{T}^h}\overline{\kappa}$. In addition, it is assumed that $\mathscr{T}^h$ is an acute partitioning; that is for (i) $d = 2$ the angle of any triangle does not exceed $\pi/2$, (ii) $d = 3$ the angle between any two faces of the same tetrahedron does not exceed $\pi/2$. In fact the case $d = 2$ can be relaxed to weakly acute, see Nochetto (1991); that is, the sum of opposite angles relative to any side does not exceed π.

Associated with $\mathscr{T}^h$ is the finite element space

$$S^h := \{\chi \in C(\overline{\Omega}) : \chi\,|_\kappa \ \text{ is linear } \forall\ \kappa \in \mathscr{T}^h\} \subset H^1(\Omega).$$

Let $\pi^h : C(\overline{\Omega}) \to S^h$ be the interpolation operator such that $\pi^h\eta(x_j) = \eta(x_j)$ ($j = 1 \to J$), where $\{x_j\}_{j=1}^J$ is the set of nodes of $\mathscr{T}^h$. A discrete inner product on $C(\overline{\Omega})$, is then defined by

$$(\eta_1,\eta_2)^h := \int_\Omega \pi^h(\eta_1(x)\eta_2(x))dx \equiv \sum_{j=1}^J M_j\eta_1(x_j)\eta_2(x_j), \tag{1.5}$$

where $M_j > 0$.

Given N, a positive integer, let $\Delta t := T/N$ denote the time step and $t_n := n\Delta t$, $n = 1 \to N$. Then Copetti and Elliott (1992) considered the following fully practical finite element approximation of (P):

($\mathbf{P}^{h,\Delta t}$) For $n = 1 \to N$ find $\{U^n, W^n\} \in S^h \times S^h$ such that

$$\left(\tfrac{U^n-U^{n-1}}{\Delta t},\chi\right)^h + (\nabla W^n,\nabla\chi) = 0 \quad \forall\ \chi\in S^h, \tag{1.6a}$$

$$\gamma(\nabla U^n,\nabla\chi) + (\Psi'(U^n),\chi)^h = (W^n,\chi)^h \quad \forall\ \chi\in S^h, \tag{1.6b}$$

where

$$\left(U^0,\chi\right)^h = (u_0,\chi) \quad \forall\ \chi\in S^h. \tag{1.7}$$

By studying a regularized problem, similar to ($\mathrm{P}_\varepsilon^{h,\Delta t}$) in Sect. 4, Copetti and Elliott (1992) proved the following result:

Theorem 1.2. *Let the assumptions on u_0 of Theorem 1.1 and the assumptions (A) hold. Let $\Delta t < 4\gamma/\theta_c^2$. Then there exists a unique solution $\{U^n, W^n\}_{n=1}^N$ to ($P^{h,\Delta t}$) such that for $n = 1 \to N$*

$$|U^n|_1^2 + \sum_{k=1}^n |U^k - U^{k-1}|_1^2 + \Delta t\sum_{k=1}^n |W^k|_1^2$$

$$+\Delta t\sum_{k=1}^n t_k\left|\tfrac{U^k-U^{k-1}}{\Delta t}\right|_1^2 + t_n|\pi^h\phi(U^n)|_0^2 + t_n|W^n|_1^2 \le C \tag{1.8}$$

and

$$\|U^n\|_{L^\infty(\Omega)} < 1. \tag{1.9}$$

Furthermore, $(\mathrm{P}^{h,\Delta t}) \to (\mathrm{P})$ *and* $U \to u$ *in* $L^2(\tau, T; L^2(\Omega))$ *for all* $\tau > 0$ *as* $h, \Delta t \to 0$*; where*

$$U(t) := \tfrac{t-t_{n-1}}{\Delta t} U^n + \tfrac{t_n - t}{\Delta t} U^{n-1} \qquad t \in [t_{n-1}, t_n]\ n \geq 1. \tag{1.10}$$

In addition, Copetti and Elliott (1992) discuss two iterative methods for solving the resulting nonlinear algebraic equations at each time level in $(\mathrm{P}^{h,\Delta t})$ and report on some numerical results for $d = 1$.

It is the main purpose of this paper to prove the following error bound for the approximation $(\mathrm{P}^{h,\Delta t})$:

Theorem 1.3. *Let the assumptions on* u_0 *of Theorem 1.1 and the assumptions (A) hold. Let* $\Delta t \equiv Ch$ *for any fixed constant* C*. Then for all* $h > 0$ *such that* $\Delta t \leq 4\gamma/\theta_c^2$*, we have that*

$$\|u - \hat{U}\|^2_{L^2(0,T;H^1(\Omega))} + \|u - U\|^2_{L^\infty(0,T;(H^1(\Omega))')} \leq Ch, \tag{1.11}$$

where $U(t)$ *is defined by (1.10) and* $\hat{U}(t) := U^n \quad t \in (t_{n-1}, t_n)\ n \geq 1$.

Throughout we assume the same assumptions on the partitioning $\mathcal{T}^h$ as Copetti and Elliott (1992), see (A) above. The layout of this paper is as follows. In the next section we study the regularized problem (P_ε), introduced by Elliott and Luckhaus (1991). Firstly we prove some ε independent stability bounds for the solution $\{u_\varepsilon, w_\varepsilon\}$, extending on those given by Elliott and Luckhaus (1991). We then use these to infer more regularity for $\{u, w\}$ and to prove an error bound for this regularization procedure. In Sect. 3 we prove an error bound for a continuous in time finite element approximation $(\mathrm{P}_\varepsilon^h)$ of (P_ε). In Sect. 4 we take time discretization into account and prove an error bound between $(\mathrm{P}_\varepsilon^h)$ and $(\mathrm{P}_\varepsilon^{h,\Delta t})$, a regularized version of $(\mathrm{P}^{h,\Delta t})$. In addition we prove an error bound for this discrete regularization procedure. By combining all the above error bounds and choosing the regularization parameter, ε, and the time step, Δt, in terms of the mesh spacing, h, we obtain the error bound Theorem 1.3. Throughout C denotes a generic constant independent of these three parameters. Finally in Sect. 5 we present a numerical experiment.

We end this section by noting that the results in Theorem 1.1 above have been proved in Elliott and Luckhaus (1991) for a multicomponent version of (P). Recently, results similar to those of Copetti and Elliott (1992) in Theorem 1.2 above have been proved by Blowey et al. (1995) for this multi-component problem. In a forthcoming paper we intend to extend the error bound in this paper to this case.

2. A regularized problem

We use the Elliott and Luckhaus (1991) regularization for problem (P) with the logarithmic free energy $\Psi(u)$ replaced by the twice continuously differentiable function $\Psi_\varepsilon(s) := \psi_\varepsilon(s) + \frac{\theta_c}{2}(1 - s^2)$ where $\varepsilon \in (0, 1)$ and

$$\psi_\varepsilon(s) := \begin{cases} \frac{\theta}{2}(1+s)\ln\left[\frac{1+s}{2}\right] + \frac{\theta}{4\varepsilon}(1-s)^2 + \frac{\theta}{2}(1-s)\ln\left[\frac{\varepsilon}{2}\right] - \frac{\theta\varepsilon}{4} & s \geq 1-\varepsilon \\ \psi(s) & |s| \leq 1-\varepsilon \\ \frac{\theta}{2}(1-s)\ln\left[\frac{1-s}{2}\right] + \frac{\theta}{4\varepsilon}(1+s)^2 + \frac{\theta}{2}(1+s)\ln\left[\frac{\varepsilon}{2}\right] - \frac{\theta\varepsilon}{4} & s \leq -1+\varepsilon \end{cases} . \tag{2.1}$$

The monotone function

$$\phi_\varepsilon(s) := \psi_\varepsilon'(s) = \begin{cases} \frac{\theta}{2}\left(1+\ln(1+s)\right) - \frac{\theta}{2\varepsilon}(1-s) - \frac{\theta}{2}\ln\varepsilon & s \geq 1-\varepsilon \\ \phi(s) & |s| \leq 1-\varepsilon \\ -\frac{\theta}{2}\left(1+\ln(1-s)\right) + \frac{\theta}{2\varepsilon}(1+s) + \frac{\theta}{2}\ln\varepsilon & s \leq -1+\varepsilon \end{cases} \tag{2.2}$$

has the properties: For all $\varepsilon > 0$

$$\begin{aligned} \phi(s) &\geq \phi_\varepsilon(s) \ \text{ if } 1 > s \geq 1-\varepsilon \\ \phi_\varepsilon(s) &\geq \phi(s) \ \text{ if } -1+\varepsilon \geq s > -1 \end{aligned} . \tag{2.3}$$

For all r, s

$$\begin{aligned} \Psi_\varepsilon'(s)(r-s) = \psi_\varepsilon'(s)(r-s) - \theta_c s(r-s) &\leq \psi_\varepsilon(r) - \psi_\varepsilon(s) + \theta_c s(s-r) \\ &= \Psi_\varepsilon(r) - \Psi_\varepsilon(s) + \tfrac{\theta_c}{2}(r-s)^2 \end{aligned} \tag{2.4}$$

where we have used the identity

$$2s(s-r) = s^2 - r^2 + (s-r)^2 \quad \forall\ r, s. \tag{2.5}$$

For $\varepsilon \leq 1/2$ and for all r, s

$$\theta(r-s)^2 \leq (\phi_\varepsilon(r) - \phi_\varepsilon(s))(r-s) \tag{2.6}$$

$$\text{and} \qquad \frac{\varepsilon}{\theta}(\phi_\varepsilon(r) - \phi_\varepsilon(s))^2 \leq (\phi_\varepsilon(r) - \phi_\varepsilon(s))(r-s). \tag{2.7}$$

In addition, if $r, s > 1-\varepsilon$ or $r, s < -1+\varepsilon$ then

$$\tfrac{\theta}{2\varepsilon}(r-s)^2 \leq (\phi_\varepsilon(r) - \phi_\varepsilon(s))(r-s). \tag{2.8}$$

Furthermore, it is a simple matter to show that Ψ_ε is bounded below for ε sufficiently small; e.g. if $\varepsilon \leq \varepsilon_0 := \theta/(8\theta_c)$ then

$$\Psi_\varepsilon(s) \geq \tfrac{\theta}{8\varepsilon}\left([s-1]_+^2 + [-1-s]_+^2\right) - \theta_c \geq -\theta_c \qquad \forall\ s, \tag{2.9}$$

where $[\cdot]_+ := \max\{\cdot, 0\}$. To see this, we note firstly for $|s| \leq 1$ that $\Psi_\varepsilon(s) \geq \psi_\varepsilon(s) \geq \psi_\varepsilon(0) \equiv -\theta \ln 2 \geq -\theta_c$. Secondly, for $s \geq 1$ we have under the stated assumption on ε that $\Psi_\varepsilon(s) \geq \frac{\theta}{4\varepsilon}(s-1)^2 - \frac{\theta\varepsilon}{4} + \frac{\theta_c}{2}(1-s^2) \equiv [\frac{\theta}{4\varepsilon} - \frac{\theta_c}{2}](s-1)^2 - \frac{\theta\varepsilon}{4} + \theta_c(1-s) \geq [\frac{\theta}{4\varepsilon} - \theta_c](s-1)^2 - \frac{\theta\varepsilon}{4} - \frac{\theta_c}{2}$, where we have applied a Young's inequality. Applying a similar bound for $s \leq -1$ yields the desired result (2.9).

We now study the corresponding regularized version of (P) as introduced by Elliott and Luckhaus (1991):

(P$_\varepsilon$) Find $\{u_\varepsilon, w_\varepsilon\}$ such that $u_\varepsilon(\cdot, 0) = u_0(\cdot)$ and for a.e. $t \in (0, T)$

$$\langle \tfrac{\partial u_\varepsilon}{\partial t}, \eta \rangle + (\nabla w_\varepsilon, \nabla \eta) = 0 \quad \forall\ \eta \in H^1(\Omega), \tag{2.10a}$$

(2.10b) $$\gamma(\nabla u_\varepsilon, \nabla\eta) + (\Psi'_\varepsilon(u_\varepsilon), \eta) = (w_\varepsilon, \eta) \quad \forall\, \eta \in H^1(\Omega).$$

It is convenient to introduce the "inverse Laplacian" operator $\mathscr{G} : \mathscr{F} \to V$ such that

(2.11) $$(\nabla \mathscr{G} v, \nabla\eta) = \langle v, \eta\rangle \quad \forall\, \eta \in H^1(\Omega)\,,$$

where $\mathscr{F} := \left\{v \in (H^1(\Omega))' : \langle v, 1\rangle = 0\right\}$ and $V := \{v \in H^1(\Omega) : (v, 1) = 0\}$. One can then define a norm on $\mathscr{F}$ by

(2.12) $$\|v\|_{-1} := |\mathscr{G} v|_1 \equiv \langle v, \mathscr{G} v\rangle^{1/2} \quad \forall\, v \in \mathscr{F}\,.$$

We note also for future reference that using a Young's inequality yields for all $\alpha > 0$ that

(2.13) $$|v|_0^2 \equiv (\nabla\mathscr{G} v, \nabla v) \le \tfrac{1}{2\alpha}\|v\|_{-1}^2 + \tfrac{\alpha}{2}|v|_1^2 \quad \forall\, v \in V\,.$$

Choosing $\eta \equiv 1$ in (2.10a) yields that $\langle \frac{\partial u_\varepsilon}{\partial t}, 1\rangle = 0$, i.e. $(u_\varepsilon(t), 1) = (u_0, 1)$ for all t. Hence it follows from the Poincaré inequality

(2.14) $$|\eta|_0 \le C_P(|\eta|_1 + |(\eta, 1)|) \quad \forall\, \eta \in H^1(\Omega)$$

that

(2.15) $$w_\varepsilon \equiv -\mathscr{G}\tfrac{\partial u_\varepsilon}{\partial t} + ⨍\, \Psi'_\varepsilon(u_\varepsilon).$$

Therefore (P_ε) can be rewritten as:
Find u_ε such that $u_\varepsilon(\cdot, 0) = u_0(\cdot)$ and for a.e. $t \in (0, T)$, $(u_\varepsilon(t), 1) = (u_0, 1)$ and

(2.16) $$\gamma(\nabla u_\varepsilon, \nabla\eta) + (\Psi'_\varepsilon(u_\varepsilon), \eta - ⨍\, \eta) + (\mathscr{G}\tfrac{\partial u_\varepsilon}{\partial t}, \eta) = 0 \quad \forall\, \eta \in H^1(\Omega).$$

Similarly, (P) can be rewritten as:
Find u such that $u(\cdot, 0) = u_0(\cdot)$ and for a.e. $t \in (0, T)$, $(u(t), 1) = (u_0, 1)$ and

(2.17) $$\gamma(\nabla u, \nabla\eta) + (\Psi'(u), \eta - ⨍\, \eta) + (\mathscr{G}\tfrac{\partial u}{\partial t}, \eta) = 0 \quad \forall\, \eta \in H^1(\Omega);$$

with

(2.18) $$w \equiv -\mathscr{G}\tfrac{\partial u}{\partial t} + ⨍\, \Psi'(u).$$

Lemma 2.1. *Let the assumptions of Theorem 1.1 hold. Then for all $\varepsilon \le \varepsilon_0$ there exists a unique solution $\{u_\varepsilon, w_\varepsilon\}$ to (P_ε) such that the following stability bounds hold independently of ε*

(2.19) $$u_\varepsilon \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))'),$$

(2.20) $$w_\varepsilon \in L^2(0, T; H^1(\Omega)) \ \textit{and}\ \ \phi_\varepsilon(u_\varepsilon) \in L^2(\Omega_T);$$

and if Ω is convex polyhedral or $\partial\Omega \in C^{1,1}$

(2.21) $$u_\varepsilon \in L^2(0, T; H^2(\Omega)).$$

Furthermore

(2.22) $$\|[u_\varepsilon - 1]_+\|_{L^\infty(0,T;L^2(\Omega))} + \|[-u_\varepsilon - 1]_+\|_{L^\infty(0,T;L^2(\Omega))} \le C\varepsilon^{1/2}.$$

Proof. Assuming that (2.16) has two solutions $u_\varepsilon^1, u_\varepsilon^2$, it follows that for a.e. $t \in (0,T)$ $d := u_\varepsilon^1 - u_\varepsilon^2 \in V$ satisfies

$$\gamma|d|_1^2 + (\phi_\varepsilon(u_\varepsilon^1) - \phi_\varepsilon(u_\varepsilon^2), d) + \tfrac{1}{2}\tfrac{d}{dt}\|d\|_{-1}^2 = \theta_c|d|_0^2. \tag{2.23}$$

Uniqueness then follows from noting (2.13), (2.6), a Grönwall inequality, (2.14) and (2.15).

Existence follows from standard arguments using Galerkin approximations and then passing to the limit. The choices of η below can be justified in a similar way.

Choosing $\eta \equiv \partial u_\varepsilon/\partial t$ in (2.16) and integrating over $(0,t)$ yields for all $t \in (0,T)$ that

$$\begin{aligned} &\frac{\gamma}{2}|u_\varepsilon(t)|_1^2 + (\Psi_\varepsilon(u_\varepsilon(t)),1) + \int_0^t \|\tfrac{\partial u_\varepsilon}{\partial s}(s)\|_{-1}^2 ds \\ &\quad = \frac{\gamma}{2}|u_0|_1^2 + (\Psi_\varepsilon(u_0),1) \le C, \end{aligned} \tag{2.24}$$

where we used the assumptions on u_0. Hence the ε independent bounds in (2.19) follow from noting (2.9) and (2.14). Noting (2.15), (2.14) and (2.19) yields that

$$\|(I - ⨍\,)w_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le C. \tag{2.25}$$

The bound (2.22) follows immediately from the bound on $(\Psi_\varepsilon(u_\varepsilon(t)),1)$ in (2.24) and (2.9).

Choosing $\eta \equiv \phi_\varepsilon(u_\varepsilon)$ in (2.16), noting that $\phi_\varepsilon'(\cdot) \ge \theta$ and (2.14) yields for a.e. $t \in (0,T)$ that

$$\begin{aligned} 2\gamma\theta|u_\varepsilon|_1^2 + |(I - ⨍\,)\phi_\varepsilon(u_\varepsilon)|_0^2 \;&\le\; |\theta_c u_\varepsilon - \mathscr{G}\frac{\partial u_\varepsilon}{\partial t}|_0^2 \\ &\le\; C\,[\,|u_\varepsilon|_0^2 + \|\frac{\partial u_\varepsilon}{\partial t}\|_{-1}^2\,]. \end{aligned} \tag{2.26}$$

Integrating the above over $t \in (0,T)$ and noting (2.19) yields that

$$\|(I - ⨍\,)\phi_\varepsilon(u_\varepsilon)\|_{L^2(\Omega_T)} \le C. \tag{2.27}$$

Choosing $\eta \equiv u_\varepsilon$ in (2.16) yields for any constant λ and for a.e. $t \in (0,T)$ that

$$\begin{aligned} (\Psi_\varepsilon'(u_\varepsilon), \lambda - ⨍\, u_\varepsilon) &= (\Psi_\varepsilon'(u_\varepsilon), \lambda - u_\varepsilon) - \gamma|u_\varepsilon|_1^2 - (\mathscr{G}\tfrac{\partial u_\varepsilon}{\partial t}, u_\varepsilon) \\ &\le (\Psi_\varepsilon(\lambda) - \Psi_\varepsilon(u_\varepsilon), 1) + \tfrac{\theta_c}{2}|u_\varepsilon - \lambda|_0^2 + C\|\tfrac{\partial u_\varepsilon}{\partial t}\|_{-1}|u_\varepsilon|_0, \end{aligned}$$

where we have noted (2.4) and (2.14). Hence it follows on choosing $\lambda = \pm 1$ and noting (2.9) that

$$\delta|\Omega||\,⨍\,\Psi_\varepsilon'(u_\varepsilon)| \le C\,[\,1 + |u_\varepsilon|_0^2 + \|\tfrac{\partial u_\varepsilon}{\partial t}\|_{-1}|u_\varepsilon|_0\,]. \tag{2.28}$$

Integrating the above over $t \in (0,T)$ and noting (2.19) yields that

$$\|\,⨍\,\Psi_\varepsilon'(u_\varepsilon)\|_{L^2(\Omega_T)} \le C. \tag{2.29}$$

Combining (2.29), (2.25) and (2.27) yields the desired result (2.20). Finally (2.21) follows from (2.10b), (2.19), (2.20) and standard elliptic regularity theory. □

Theorem 2.1. *Let the assumptions of Theorem 1.1 hold. Then there exists a unique solution* $\{u, w\}$ *to (P) such that*

$$u \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))'), \tag{2.30}$$

$$w \in L^2(0,T;H^1(\Omega)) \ \textit{and} \ \phi(u) \in L^2(\Omega_T); \tag{2.31}$$

and if Ω *is convex polyhedral or* $\partial\Omega \in C^{1,1}$

$$u \in L^2(0,T;H^2(\Omega)). \tag{2.32}$$

In addition (1.4) holds. Furthermore, we have that

$$\|u - u_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))} + \|u - u_\varepsilon\|^2_{L^\infty(0,T;(H^1(\Omega))')} \le C\varepsilon. \tag{2.33}$$

Proof. As the bounds (2.19) and (2.20) are independent of ε, it follows that there exists $u \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')$, $\phi^\star \in L^2(\Omega_T)$ and a subsequence $\{u_{\varepsilon'}\}$ such that as $\varepsilon' \to 0$

$$u_{\varepsilon'} \to u \quad \text{in } L^\infty(0,T;H^1(\Omega)) \text{ weak–star}$$
$$\text{and in } H^1(0,T;(H^1(\Omega))') \text{ weakly,} \tag{2.34}$$

$$\phi_{\varepsilon'}(u_{\varepsilon'}) \to \phi^\star \quad \text{in} \quad L^2(\Omega_T) \quad \text{weakly.} \tag{2.35}$$

Next we show that $\phi^\star \equiv \phi(u)$ by adapting an argument used in Blowey et al. (1995). It follows from (2.6) that

$$\int_{\Omega_T} [u_\varepsilon - \phi_\varepsilon^{-1}(\eta)][\phi_\varepsilon(u_\varepsilon) - \eta]dxdt \ge 0 \quad \forall\ \eta \in L^2(\Omega_T). \tag{2.36}$$

The integral being well-defined, since $|\phi_\varepsilon^{-1}(r) - \phi_\varepsilon^{-1}(s)| \le \theta^{-1}|r - s|\ \forall\ r, s$. A simple calculation, noting (2.8) and (2.3), yields that for all r

$$|\phi^{-1}(r) - \phi_\varepsilon^{-1}(r)| \le \tfrac{2\varepsilon}{\theta}\{\,[r - \phi(1-\varepsilon)]_+ + [-r + \phi(-1+\varepsilon)]_+\}. \tag{2.37}$$

Since (2.34) implies that $u_{\varepsilon'} \to u$ in $L^2(\Omega_T)$ strongly as $\varepsilon' \to 0$, see Lions (1969), noting (2.35) and (2.37) it follows from taking the limit $\varepsilon' \to 0$ in (2.36) that

$$\int_{\Omega_T} [u - \phi^{-1}(\eta)][\phi^\star - \eta]dxdt \ge 0 \quad \forall\ \eta \in L^2(\Omega_T), \tag{2.38}$$

and hence that $\phi^\star \equiv \phi(u)$. Therefore taking the limit $\varepsilon' \to 0$ in (2.16) yields that u solves (2.17). Noting (2.18) yields existence of a solution $\{u, w\}$ to (P) and the bound (2.31) for w. The bound (2.32) follows in the same way as (2.21). The bound (1.4) follows directly from the bound on $\phi(u)$ in (2.31) using the argument prior to Theorem 1.1.

Uniqueness of a solution to (P) follows as for (P_ε), see (2.23).

We now prove an error bound between the unique solutions u and u_ε of problems (P) and (P$_\varepsilon$). Define $e = u - u_\varepsilon$. Subtraction of (2.16) from (2.17) and choosing $\eta = e$ yields for a.e. $t \in (0,T)$ that

$$\gamma|e|_1^2 + (\phi(u) - \phi_\varepsilon(u_\varepsilon), e) + \tfrac{1}{2}\tfrac{d}{dt}\|e\|_{-1}^2 = \theta_c|e|_0^2 \leq \tfrac{\gamma}{2}|e|_1^2 + \tfrac{\theta_c^2}{2\gamma}\|e\|_{-1}^2, \tag{2.39}$$

where we have used (2.13) in bounding the right hand side. From the monotonicity of ϕ_ε and (2.8) it follows that for a.e. $t \in (0,T)$

$$(\phi_\varepsilon(u) - \phi_\varepsilon(u_\varepsilon), e) \geq \tfrac{\theta}{2\varepsilon}\int_{\Omega_\varepsilon^+(t)\cup\Omega_\varepsilon^-(t)} e^2 dx, \tag{2.40}$$

where

$$\Omega_\varepsilon^+(t) := \{x \in \Omega : 1-\varepsilon \leq u(x,t) \leq u_\varepsilon(x,t)\},$$
$$\Omega_\varepsilon^-(t) := \{x \in \Omega : u_\varepsilon(x,t) \leq u(x,t) \leq -1+\varepsilon\}.$$

Next we note from the definition of ϕ_ε and (2.3) that

1. If $|r| \leq 1-\varepsilon$ then $\phi_\varepsilon(r) \equiv \phi(r)$.
2. If $r \geq 1-\varepsilon$ and $s \leq r$, then $(\phi_\varepsilon(r) - \phi(r))(r-s) \leq 0$.
3. If $r \leq -1+\varepsilon$ and $r \leq s$, then $(\phi_\varepsilon(r) - \phi(r))(r-s) \leq 0$.

Hence it follows for a.e $t \in (0,T)$ that

$$\begin{aligned}(\phi_\varepsilon(u) - \phi(u), e) &\leq \int_{\Omega_\varepsilon^+(t)\cup\Omega_\varepsilon^-(t)} (\phi_\varepsilon(u) - \phi(u)) e dx \\ &\leq -\int_{\Omega_\varepsilon^+(t)\cup\Omega_\varepsilon^-(t)} \phi(u) e dx.\end{aligned} \tag{2.41}$$

Substituting (2.40) and (2.41) into (2.39) and rearranging, yields for a.e. $t \in (0,T)$

$$\begin{aligned}\tfrac{\gamma}{2}|e|_1^2 + \tfrac{1}{2}\tfrac{d}{dt}\|e\|_{-1}^2 + \tfrac{\theta}{2\varepsilon}\int_{\Omega_\varepsilon^+(t)\cup\Omega_\varepsilon^-(t)} e^2 dx &\leq (\phi_\varepsilon(u) - \phi(u), e) + \tfrac{\theta_c^2}{2\gamma}\|e\|_{-1}^2 \\ &\leq -\int_{\Omega_\varepsilon^+(t)\cup\Omega_\varepsilon^-(t)} \phi(u) e dx + \tfrac{\theta_c^2}{2\gamma}\|e\|_{-1}^2.\end{aligned}$$

Integrating over $t \in (0,T)$, using a Grönwall inequality and noting (2.14) results in

$$\begin{aligned}\|e\|_{L^2(0,T;H^1(\Omega))}^2 + \|e\|_{L^\infty(0,T;(H^1(\Omega))')}^2 + \tfrac{\theta}{2\varepsilon}\int_0^T\int_{\Omega_\varepsilon^+(t)\cup\Omega_\varepsilon^-(t)} e^2 dx dt \\ \leq -C\int_0^T\int_{\Omega_\varepsilon^+(t)\cup\Omega_\varepsilon^-(t)} \phi(u) e dx dt.\end{aligned} \tag{2.42}$$

The desired result (2.33) then follows from applying a Young's inequality and noting (2.31). □

We end this section by comparing the results of Lemma 2.1 and Theorem 2.1 with those in Elliott and Luckhaus (1991). Elliott and Luckhaus (1991) establish existence and uniqueness of a solution to (P_ε), the bounds (2.19) and a bound similar to (2.22). We have outlined these proofs above as we need similar results for the semidiscrete and fully discrete approximations in the later sections. However, in place of (2.20) they prove (2.25) and the following stability bounds which hold independently of ε:

$$\sqrt{t}\,\frac{\partial u_\varepsilon}{\partial t} \in L^2(0,T;H^1(\Omega)),\ \ \sqrt{t}\,\phi_\varepsilon(u_\varepsilon) \in L^\infty(0,T;L^2(\Omega)),$$
$$\sqrt{t}\,w_\varepsilon \in L^\infty(0,T;H^1(\Omega)). \tag{2.43}$$

Letting $\varepsilon \to 0$ they then establish Theorem 1.1. The main difference in our approach are the bounds on $\phi_\varepsilon(u_\varepsilon)$ and $\phi(u)$ in (2.20) and (2.31), which play a key role in the crucial regularization error bound (2.33) and in the regularity results (2.21) and (2.32).

3. A semidiscrete regularized approximation

Throughout the rest of the paper we assume that the assumptions (A) hold. We then define the following semidiscrete approximation to (P_ε):
($\mathbf{P}^h_\varepsilon$) Find $\{u^h_\varepsilon(t), w^h_\varepsilon(t)\} \in S^h \times S^h$ such that $u^h_\varepsilon(0) \equiv Q^h u_0$ and for a.e. $t \in (0,T)$

$$\left(\tfrac{\partial u^h_\varepsilon}{\partial t}, \chi\right) + (\nabla w^h_\varepsilon, \nabla \chi) = 0 \quad \forall\ \chi \in S^h, \tag{3.1a}$$

$$\gamma(\nabla u^h_\varepsilon, \nabla \chi) + (\Psi'_\varepsilon(u^h_\varepsilon), \chi)^h = (w^h_\varepsilon, \chi) \quad \forall\ \chi \in S^h. \tag{3.1b}$$

Here $Q^h : L^2(\Omega) \to S^h$ is defined by

$$(Q^h \eta, \chi)^h = (\eta, \chi) \quad \forall\ \chi \in S^h. \tag{3.2}$$

Similarly to (2.11), we introduce the operator $\mathscr{G}^h : \mathscr{F} \to V^h$ such that

$$(\nabla \mathscr{G}^h v, \nabla \chi) = \langle v, \chi \rangle \quad \forall\ \chi \in S^h, \tag{3.3}$$

where $V^h := \{v^h \in S^h : (v^h, 1) = 0\}$. We have the following analogues of (2.12) and (2.13). We define a norm on $\mathscr{F}$ by

$$\|v\|_{-h} := |\mathscr{G}^h v|_1 = \langle v, \mathscr{G}^h v\rangle^{1/2} \quad \forall\ v \in \mathscr{F}, \tag{3.4}$$

and for $\alpha > 0$ we have that

$$|v^h|_0^2 \equiv (\nabla \mathscr{G}^h v^h, \nabla v^h) \le \tfrac{1}{2\alpha}\|v^h\|^2_{-h} + \tfrac{\alpha}{2}|v^h|_1^2 \quad \forall\ v^h \in V^h. \tag{3.5}$$

Since $u^h_\varepsilon(0)$ is piecewise linear,

$$(u^h_\varepsilon(t), 1) = (u^h_\varepsilon(0), 1) = (u^h_\varepsilon(0), 1)^h = (u_0, 1)$$

and it follows that

(3.6) $$w_\varepsilon^h \equiv -\mathscr{G}^h \tfrac{\partial u_\varepsilon^h}{\partial t} + \tfrac{1}{|\Omega|}(\Psi_\varepsilon'(u_\varepsilon^h), 1)^h,$$

thus (P_ε^h) can be rewritten as:
Find $u_\varepsilon^h(t) \in S^h$ such that $u_\varepsilon^h(0) \equiv Q^h u_0$ and for a.e. $t \in (0,T)$, $(u_\varepsilon^h(t), 1) = (u_0, 1)$ and

(3.7) $$\gamma(\nabla u_\varepsilon^h, \nabla\chi) + (\Psi_\varepsilon'(u_\varepsilon^h), \chi - \textstyle\fint \chi)^h + (\mathscr{G}^h \tfrac{\partial u_\varepsilon^h}{\partial t}, \chi) = 0 \quad \forall\, \chi \in S^h.$$

Below we recall some well-known results concerning S^h:

(3.8) $$|(I-\pi^h)\eta|_m \le Ch^{2-m}|\eta|_2 \quad \forall\, \eta \in H^2(\Omega), \quad m = 0 \text{ or } 1.$$

(3.9) $$|\chi|_h := [(\chi,\chi)^h]^{1/2} \le |\chi|_0 \le C|\chi|_h \quad \forall\, \chi \in S^h.$$

(3.10) $$|(\chi_1,\chi_2) - (\chi_1,\chi_2)^h| \le Ch^{1+m}\|\chi_1\|_m\|\chi_2\|_1 \qquad \forall\, \chi_1,\chi_2 \in S^h, \quad m = 0 \text{ or } 1.$$

(3.11) $$|(\mathscr{G} - \mathscr{G}^h)\eta|_0 \le Ch^{2-m}\|\eta\|_{-m}, \qquad \forall\, \eta \in (H^m(\Omega))' \cap \mathscr{F}, \quad m = 0 \text{ or } 1.$$

Next we note that

(3.12) $$C_1 h^2|v^h|_1 \le C_2 h|v^h|_0 \le \|v^h\|_{-h} \le \|v^h\|_{-1} \le C_3\|v^h\|_{-h} \quad \forall\, v^h \in V^h.$$

The first inequality on the left is just an inverse inequality, recalling that the partitioning is quasi-uniform. The second follows from the first and (3.5). The third follows from noting that $|\mathscr{G}^h v^h|_1 \le |\mathscr{G} v^h|_1$. The final inequality follows from noting (3.11) with $m = 0$ and the second inequality above.

The following bounds, concerning Q^h are easily established.

(3.13) $$\|(I - Q^h)\eta\|_{-1} \le Ch|\eta|_0 \quad \forall\, \eta \in L^2(\Omega),$$

see for example Blowey and Elliott (1992).

(3.14) $$\|Q^h\eta\|_{L^\infty(\Omega)} \le \|\eta\|_{L^\infty(\Omega)} \quad \forall\, \eta \in L^\infty(\Omega);$$

since $(Q^h\eta)(x_j) \equiv (\eta,\chi_j)/(1,\chi_j)$ $\;j = 1 \to J$; where $\chi_j \in S^h$ and $\chi_j(x_i) = \delta_{ij}$. In addition, using the first inequality on the left of (3.12) and comparing with an H^1 projection, one can show that

(3.15) $$\|Q^h\eta\|_1 \le C\|\eta\|_1 \quad \forall\, \eta \in H^1(\Omega).$$

Since ϕ_ε is monotone it follows, see Elliott (1987, p. 68), that

(3.16) $$|(I-\pi^h)[\phi_\varepsilon(\chi)]|_0 \le Ch|\pi^h[\phi_\varepsilon(\chi)]|_1 \quad \forall\, \chi \in S^h.$$

Furthermore, as the partitioning is (weakly) acute, it follows from (2.7) that

(3.17) $$\tfrac{\varepsilon}{\theta}|\nabla\pi^h[\phi_\varepsilon(\chi)]|_0^2 \le (\nabla\chi, \nabla\pi^h[\phi_\varepsilon(\chi)]) \quad \forall\, \chi \in S^h,$$

see Ciavaldini (1975) and Nochetto (1991, Sect. 2.4.2).

Lemma 3.1. *Let the assumptions on u_0 of Theorem 1.1 and the assumptions (A) hold. Then for all $\varepsilon \le \varepsilon_0$ and for all $h > 0$ there exists a unique solution $\{u_\varepsilon^h, w_\varepsilon^h\}$ to (P_ε^h) such that the following stability bounds hold independently of ε and h:*

$$u_\varepsilon^h \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))'), \tag{3.18}$$

$$w_\varepsilon^h \in L^2(0,T;H^1(\Omega)) \quad \textit{and} \quad \pi^h[\phi_\varepsilon(u_\varepsilon^h)] \in L^2(\Omega_T). \tag{3.19}$$

Furthermore

$$\|\pi^h[\Psi_\varepsilon'(u_\varepsilon^h)]\|_{L^2(0,T;H^1(\Omega))} \le C\varepsilon^{-1/2}. \tag{3.20}$$

Proof. The proof is a discrete analogue of Lemma 2.1. If (3.7) has two solutions $u_\varepsilon^{h,1}, u_\varepsilon^{h,2}$, then for a.e. $t \in (0,T)$, $d^h := u_\varepsilon^{h,1} - u_\varepsilon^{h,2} \in V^h$ satisfies

$$\gamma|d^h|_1^2 + (\phi_\varepsilon(u_\varepsilon^{h,1}) - \phi_\varepsilon(u_\varepsilon^{h,2}), d^h)^h + \tfrac{1}{2}\tfrac{d}{dt}\|d^h\|_{-h}^2 = \theta_c|d^h|_h^2. \tag{3.21}$$

Uniqueness then follows from noting (3.9), (3.5), (2.6), a Grönwall inequality, (2.14) and (3.6).

Existence follows from standard ordinary differential equation theory. Choosing $\chi \equiv \partial u_\varepsilon^h/\partial t$ in (3.7) and integrating over $(0,t)$ yields for all $t \in (0,T)$ that

$$\begin{aligned}\frac{\gamma}{2}|u_\varepsilon^h(t)|_1^2 + (\Psi_\varepsilon(u_\varepsilon^h(t)),1)^h + \int_0^t \|\tfrac{\partial u_\varepsilon^h}{\partial s}(s)\|_{-h}^2 ds \\ = \frac{\gamma}{2}|Q^h u_0|_1^2 + (\Psi_\varepsilon(Q^h u_0),1)^h \le C,\end{aligned} \tag{3.22}$$

where we have noted (3.15), (3.14) and the assumptions on u_0. Hence the bounds in (3.18) follow from noting (2.9), (2.14) and (3.12).

Choosing $\chi \equiv \pi^h[\phi_\varepsilon(u_\varepsilon^h)]$ in (3.7) yields for a.e. $t \in (0,T)$ that

$$\begin{aligned}&\gamma(\nabla u_\varepsilon^h, \nabla\pi^h[\phi_\varepsilon(u_\varepsilon^h)]) + |(I - ⨍\,)\pi^h[\phi_\varepsilon(u_\varepsilon^h)]|_h^2 \\ &= -(\mathscr{G}^h \tfrac{\partial u_\varepsilon^h}{\partial t}, (I - ⨍\,)\pi^h[\phi_\varepsilon(u_\varepsilon^h)]) + \theta_c(u_\varepsilon^h, (I - ⨍\,)\pi^h[\phi_\varepsilon(u_\varepsilon^h)])^h.\end{aligned} \tag{3.23}$$

Integrating over $t \in (0,T)$, applying a Young's inequality and noting (3.9), (3.17), (2.14), (3.12) and (3.18) yields that

$$\varepsilon\int_0^T |\pi^h[\phi_\varepsilon(u_\varepsilon^h)]|_1^2 dt + \int_0^T |(I - ⨍\,)\pi^h[\phi_\varepsilon(u_\varepsilon^h)]|_h^2 dt \le C. \tag{3.24}$$

Choosing $\chi \equiv u_\varepsilon^h$ in (3.7) yields for any constant λ and for a.e. $t \in (0,T)$ that

$$\begin{aligned}(\Psi_\varepsilon'(u_\varepsilon^h), \lambda - ⨍\, u_\varepsilon^h)^h &= (\Psi_\varepsilon'(u_\varepsilon^h), \lambda - u_\varepsilon^h)^h - \gamma|u_\varepsilon^h|_1^2 - (\mathscr{G}^h \tfrac{\partial u_\varepsilon^h}{\partial t}, u_\varepsilon^h) \\ &\le (\Psi_\varepsilon(\lambda) - \Psi_\varepsilon(u_\varepsilon^h),1)^h + \tfrac{\theta_c}{2}|u_\varepsilon^h - \lambda|_h^2 + C\|\tfrac{\partial u_\varepsilon^h}{\partial t}\|_{-1}|u_\varepsilon^h|_0,\end{aligned} \tag{3.25}$$

where we have noted (2.4) and (2.14). Hence it follows on choosing $\lambda = \pm 1$, integrating over $t \in (0,T)$ and noting (2.9), (3.9), (2.14) and (3.18) that

$$\|⨍\, \pi^h[\Psi_\varepsilon'(u_\varepsilon^h)]\|_{L^2(\Omega_T)} \le C. \tag{3.26}$$

The desired results (3.19) and (3.20) then follow from combining (3.24), (3.26), (3.18), (3.6) and noting (3.9). □

Theorem 3.1. *Let the assumptions of Lemma 3.1 hold. Then for all* $\varepsilon \le \varepsilon_0$ *and* $h > 0$ *we have that*

$$\|u_\varepsilon - u_\varepsilon^h\|^2_{L^2(0,T;H^1(\Omega))} + \|u_\varepsilon - u_\varepsilon^h\|^2_{L^\infty(0,T;(H^1(\Omega))')} \le C\varepsilon^{-1}h^2. \tag{3.27}$$

Proof. We set $e_\varepsilon := u_\varepsilon - u_\varepsilon^h$, $e_\varepsilon^{\mathrm{A}} := u_\varepsilon - \pi^h u_\varepsilon$ and $e_\varepsilon^h := \pi^h u_\varepsilon - u_\varepsilon^h$. Note that $⨍\, e_\varepsilon = ⨍\, e_\varepsilon^{\mathrm{A}} + ⨍\, e_\varepsilon^h = 0$. Subtracting (3.7) from (2.16), then for a.e. $t \in (0,T)$ it follows that

$$\begin{aligned}\gamma(\nabla e_\varepsilon, \nabla\chi) &+ (\Psi_\varepsilon'(u_\varepsilon) - \Psi_\varepsilon'(u_\varepsilon^h), \chi - ⨍\, \chi) + (\mathscr{G}(\tfrac{\partial u_\varepsilon}{\partial t} - \tfrac{\partial u_\varepsilon^h}{\partial t}), \chi)\\ &= (\Psi_\varepsilon'(u_\varepsilon^h), \chi - ⨍\, \chi)^h - (\Psi_\varepsilon'(u_\varepsilon^h), \chi - ⨍\, \chi) + ((\mathscr{G}^h - \mathscr{G})\tfrac{\partial u_\varepsilon^h}{\partial t}, \chi) \quad \forall\, \chi \in S^h.\end{aligned}$$

Hence choosing $\chi \equiv e_\varepsilon^h \in S^h$ and noting (2.7), (2.13), (2.14), (3.10), (3.16) and a Young's inequality yields for a.e. $t \in (0,T)$ that

$$\begin{aligned}\gamma|e_\varepsilon|_1^2 + \tfrac{\varepsilon}{\theta}|\phi_\varepsilon(u_\varepsilon) - \phi_\varepsilon(u_\varepsilon^h)|_0^2 + \tfrac12\tfrac{d}{dt}\|e_\varepsilon\|_{-1}^2 &\le \gamma|e_\varepsilon|_1^2 + (\phi_\varepsilon(u_\varepsilon) - \phi_\varepsilon(u_\varepsilon^h), e_\varepsilon) + \tfrac12\tfrac{d}{dt}\|e_\varepsilon\|_{-1}^2\\ &= \theta_{\mathrm{c}}|e_\varepsilon|_0^2 + \gamma(\nabla e_\varepsilon, \nabla e_\varepsilon^{\mathrm{A}}) + (\Psi_\varepsilon'(u_\varepsilon) - \Psi_\varepsilon'(u_\varepsilon^h), (I - ⨍\,)e_\varepsilon^{\mathrm{A}}) + (\mathscr{G}\tfrac{\partial e_\varepsilon}{\partial t}, e_\varepsilon^{\mathrm{A}})\\ &\quad + \left[\, (\pi^h[\Psi_\varepsilon'(u_\varepsilon^h)], (I - ⨍\,)e_\varepsilon^h)^h - (\pi^h[\Psi_\varepsilon'(u_\varepsilon^h)], (I - ⨍\,)e_\varepsilon^h) \,\right]\\ &\quad + ((\pi^h - I)[\phi_\varepsilon(u_\varepsilon^h)], (I - ⨍\,)e_\varepsilon^h) + ((\mathscr{G}^h - \mathscr{G})\tfrac{\partial u_\varepsilon^h}{\partial t}, e_\varepsilon^h)\\ &\le C\Big[\, \|e_\varepsilon\|_{-1}^2 + |e_\varepsilon^{\mathrm{A}}|_1^2 + \varepsilon^{-1}|e_\varepsilon^{\mathrm{A}}|_0^2 + \|\tfrac{\partial e_\varepsilon}{\partial t}\|_{-1}|e_\varepsilon^{\mathrm{A}}|_0 + h^4\|\pi^h[\Psi_\varepsilon'(u_\varepsilon^h)]\|_1^2\\ &\quad + h^2|\pi^h[\phi_\varepsilon(u_\varepsilon^h)]|_1^2 + |(\mathscr{G} - \mathscr{G}^h)\tfrac{\partial u_\varepsilon^h}{\partial t}|_0^2\,\Big].\end{aligned} \tag{3.28}$$

Integrating over $t \in (0,T)$ and using a Grönwall inequality yields that

$$\begin{aligned}&\|e_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))} + \varepsilon\|\phi_\varepsilon(u_\varepsilon) - \phi_\varepsilon(u_\varepsilon^h)\|^2_{L^2(\Omega_T)} + \|e_\varepsilon\|^2_{L^\infty(0,T;(H^1(\Omega))')}\\ &\quad \le C\Big[\, \|(I - Q^h)u_0\|_{-1}^2 + \|e_\varepsilon^{\mathrm{A}}\|^2_{L^2(0,T;H^1(\Omega))} + \varepsilon^{-1}\|e_\varepsilon^{\mathrm{A}}\|^2_{L^2(\Omega_T)}\\ &\qquad + \|\tfrac{\partial e_\varepsilon}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')}|e_\varepsilon^{\mathrm{A}}|_{L^2(\Omega_T)} + h^4\|\pi^h[\Psi_\varepsilon'(u_\varepsilon^h)]\|^2_{L^2(0,T;H^1(\Omega))}\\ &\qquad + h^2\|\pi^h[\phi_\varepsilon(u_\varepsilon^h)]\|^2_{L^2(0,T;H^1(\Omega))} + |(\mathscr{G} - \mathscr{G}^h)\tfrac{\partial u_\varepsilon^h}{\partial t}|^2_{L^2(\Omega_T)}\Big] \le C\varepsilon^{-1}h^2;\end{aligned}$$

where we have noted (2.14), (3.13), (3.8), (2.21), (2.19), (3.18), (3.20) and (3.11) with $m = 1$. Hence the desired result (3.27) follows. □

4. A fully discrete approximation

We now consider the following fully discrete approximation to (P_ε); a regularized version of $(\mathrm{P}^{h,\Delta t})$ as studied by Copetti and Elliott (1992), see Sect. 1:
$(\mathbf{P}_\varepsilon^{h,\Delta t})$ For $n \ge 1$, find $\{U_\varepsilon^n, W_\varepsilon^n\} \in S^h \times S^h$ such that $U_\varepsilon^0 \equiv Q^h u_0$ and

$$\left(\frac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t}, \chi\right)^h + (\nabla W_\varepsilon^n, \nabla\chi) = 0 \quad \forall\, \chi \in S^h, \tag{4.1a}$$

$$\gamma(\nabla U_\varepsilon^n, \nabla\chi) + (\Psi_\varepsilon'(U_\varepsilon^n), \chi)^h = (W_\varepsilon^n, \chi)^h \quad \forall\, \chi \in S^h. \tag{4.1b}$$

Similarly to (3.7), ($\mathrm{P}_\varepsilon^{h,\Delta t}$) can be rewritten as:
For $n \geq 1$ find $U_\varepsilon^n \in S^h$ such that $U_\varepsilon^0 \equiv Q^h u_0$, $(U_\varepsilon^n, 1) = (u_0, 1)$ and

$$\gamma(\nabla U_\varepsilon^n, \nabla\chi) + (\Psi_\varepsilon'(U_\varepsilon^n), \chi - {\textstyle\fint} \chi)^h + (\hat{\mathscr{G}}^h(\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t}), \chi)^h = 0 \quad \forall\, \chi \in S^h; \tag{4.2}$$

with

$$W_\varepsilon^n \equiv -\hat{\mathscr{G}}^h(\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t}) + \tfrac{1}{|\Omega|}(\Psi_\varepsilon'(U_\varepsilon^n), 1)^h. \tag{4.3}$$

Here $\hat{\mathscr{G}}^h : \mathscr{F}^h \to V^h$ is defined by

$$(\nabla\hat{\mathscr{G}}^h v, \nabla\chi) = (v, \chi)^h \quad \forall\, \chi \in S^h \tag{4.4}$$

where $\mathscr{F}^h := \{v \in C(\overline{\Omega}) : (v, 1)^h = 0\}$. Note that $V^h \subset \mathscr{F}^h$ and the analogue of (3.5) holds: for $\alpha > 0$

$$|v^h|_h^2 \leq \tfrac{1}{2\alpha}|\hat{\mathscr{G}}^h v^h|_1^2 + \tfrac{\alpha}{2}|v^h|_1^2 \quad \forall\, v^h \in V^h. \tag{4.5}$$

It is easily deduced from (3.10), e.g. see Blowey and Elliott (1991), that

$$\|(\mathscr{G}^h - \hat{\mathscr{G}}^h)v^h\|_1 \leq Ch^2\|v^h\|_1, \qquad \forall\, v^h \in V^h. \tag{4.6}$$

In addition, we have the analogue of (3.12)

$$h^2|v^h|_1 \leq C_1 h|v^h|_h \leq C_2|\hat{\mathscr{G}}^h v^h|_1 \leq C_3|\mathscr{G}^h v^h|_1 \leq C_4|\hat{\mathscr{G}}^h v^h|_1 \quad \forall\, v^h \in V^h. \tag{4.7}$$

The first inequality on the left is just an inverse inequality on noting (3.9). The second follows from the first and (4.5). The third and fourth follow from (4.6) and noting the first two inequalities in (3.12) and (4.7), respectively.

Lemma 4.1. *Let the assumptions of Lemma 3.1 hold. Let $\Delta t \leq 4\gamma/\theta_c^2$. Then for all $\varepsilon \leq \varepsilon_0$ and for all $h > 0$ there exists a unique solution $\{U_\varepsilon^n, W_\varepsilon^n\}_{n=1}^N$ to ($P_\varepsilon^{h,\Delta t}$) such that*

$$\max_{n=1\to N}|U_\varepsilon^n|_1^2 + \sum_{n=1}^N |U_\varepsilon^n - U_\varepsilon^{n-1}|_1^2 + \Delta t\sum_{n=1}^N |\hat{\mathscr{G}}^h(\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t})|_1^2 + \Delta t\sum_{n=1}^N |W_\varepsilon^n|_1^2 + \Delta t\sum_{n=1}^N |\pi^h[\phi_\varepsilon(U_\varepsilon^n)]|_0^2 \leq C. \tag{4.8}$$

Furthermore, for $n = 1 \to N$ we have that

$$|[U_\varepsilon^n - 1]_+|_h + |[-U_\varepsilon^n - 1]_+|_h \leq C\varepsilon^{1/2}. \tag{4.9}$$

Proof. Existence and uniqueness of a solution to $(\mathrm{P}_\varepsilon^{h,\Delta t})$ follows in exactly the same way as that described in Copetti and Elliott (1992) for $(\mathrm{P}^{h,\Delta t})$. We outline the proof for completeness. Existence follows by noting that for fixed n, (4.2) is the Euler-Lagrange equation of the minimization problem

$$\min_{\chi \in K^h} \tfrac{\gamma}{2}|\chi|_1^2 + (\Psi_\varepsilon(\chi), 1)^h + \tfrac{1}{2\Delta t}|\nabla \hat{\mathscr{G}}^h(\chi - U_\varepsilon^{n-1})|_1^2;$$

where $K^h := \{\, \chi \in S^h \ :\ (\chi, 1) = (u_0, 1) \,\}$.

For fixed n, if (4.2) has two solutions $U_\varepsilon^{n,1}$ and $U_\varepsilon^{n,2}$, then $D := U_\varepsilon^{n,1} - U_\varepsilon^{n,2} \in V^h$ satisfies

$$\gamma|D|_1^2 + (\phi_\varepsilon(U_\varepsilon^{n,1}) - \phi_\varepsilon(U_\varepsilon^{n,2}), D)^h + \tfrac{1}{\Delta t}|\hat{\mathscr{G}}^h D|_1^2 = \theta_c |D|_h^2 .$$

It follows from (2.6) and (4.5) with $\alpha = \theta_c \Delta t/2$ that

$$\gamma|D|_1^2 + \theta|D|_h^2 + \tfrac{1}{\Delta t}|\hat{\mathscr{G}}^h D|_1^2 \leq \theta_c|D|_h^2 \leq \tfrac{1}{\Delta t}|\hat{\mathscr{G}}^h D|_1^2 + \tfrac{\Delta t \theta_c^2}{4}|D|_1^2$$

from which uniqueness follows under the stated condition on Δt.

The stability bound (4.8) is the analogue of (3.18, 3.19) and is proved in an analogous way. For fixed n, choosing $\chi \equiv (U_\varepsilon^n - U_\varepsilon^{n-1})/\Delta t$ in (4.2) and noting (2.4), the identity (2.5) and (4.5) yields for any $\alpha > 0$ that

$$\begin{aligned}
&\tfrac{\gamma}{2}|U_\varepsilon^n|_1^2 + \tfrac{\gamma}{2}|U_\varepsilon^n - U_\varepsilon^{n-1}|_1^2 + (\Psi_\varepsilon(U_\varepsilon^n), 1)^h + \Delta t|\hat{\mathscr{G}}^h(\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t})|_1^2 \\
&\qquad \leq \tfrac{\gamma}{2}|U_\varepsilon^{n-1}|_1^2 + (\Psi_\varepsilon(U_\varepsilon^{n-1}), 1)^h + \tfrac{\theta_c(\Delta t)^2}{2}|\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t}|_h^2 \\
&\leq \tfrac{\gamma}{2}|U_\varepsilon^{n-1}|_1^2 + (\Psi_\varepsilon(U_\varepsilon^{n-1}), 1)^h + \tfrac{\theta_c}{2}\left[\, \tfrac{\alpha}{2}|U_\varepsilon^n - U_\varepsilon^{n-1}|_1^2 + \tfrac{(\Delta t)^2}{2\alpha}|\hat{\mathscr{G}}^h(\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t})|_1^2 \right].
\end{aligned}$$

Summing the above from $n = 1 \to m$ yields for all $m \leq N$ that

$$\begin{aligned}
&\tfrac{\gamma}{2}|U_\varepsilon^m|_1^2 + (\tfrac{\gamma}{2} - \tfrac{\alpha\theta_c}{4})\sum_{n=1}^m |U_\varepsilon^n - U_\varepsilon^{n-1}|_1^2 \\
&\quad + (1 - \frac{\theta_c \Delta t}{4\alpha})\Delta t \sum_{n=1}^m |\hat{\mathscr{G}}^h(\tfrac{U_\varepsilon^n - U_\varepsilon^{n-1}}{\Delta t})|_1^2 + (\Psi_\varepsilon(U_\varepsilon^m), 1)^h \\
&\quad \leq \tfrac{\gamma}{2}|U_\varepsilon^0|_1^2 + (\Psi_\varepsilon(U_\varepsilon^0), 1)^h \equiv \tfrac{\gamma}{2}|Q^h u_0|_1^2 + (\Psi_\varepsilon(Q^h u_0), 1)^h \leq C;
\end{aligned}$$

where we have noted (3.15), (3.14) and the assumptions on u_0. Hence the first four bounds of (4.8) hold on choosing $\alpha = \theta_c \Delta t/3$, noting the stated condition for Δt and (4.3). Furthermore, the bound (4.9) follows immediately from the bound on $(\Psi_\varepsilon(U_\varepsilon^m), 1)^h$ above.

Choosing $\chi \equiv \pi^h[\phi_\varepsilon(U_\varepsilon^n)]$ in (4.2), summing from $n = 1 \to N$ and noting the bounds above yields, similarly to (3.23) and (3.24), that

$$\Delta t \sum_{n=1}^N |(I - ⨍\,)\pi^h[\phi_\varepsilon(U_\varepsilon^n)]|_h^2 \leq C. \tag{4.10}$$

Choosing $\chi \equiv U_\varepsilon^n$ in (4.2) and rearranging, similarly to (3.25) with $\lambda = \pm 1$, and summing $n = 1 \to N$ yields, similarly to (3.26), that

$$\Delta t \sum_{n=1}^{N} | ⨍ \; \pi^h[\Psi_\varepsilon'(U_\varepsilon^n)]|_h^2 \leq C . \tag{4.11}$$

Combining (4.10), (4.11) and the first bound in (4.8) and noting (3.9) yields the remaining bound in (4.8). □

The first four stability bounds of (4.8) are established in Copetti and Elliott (1992). In addition they prove discrete analogues of the bounds (2.43), established by Elliott and Luckhaus (1991); see Theorem 1.2 in the limit $\varepsilon \to 0$. The main difference in our approach is the bound on $\pi^h[\phi_\varepsilon(U_\varepsilon^n)]$, which leads to an identical bound on $\pi^h[\phi(U^n)]$; this plays a key role in the discrete regularization error bound, see Theorem 4.2.

We now prove an error estimate between the problems $(\mathrm{P}_\varepsilon^{h,\Delta t})$ and $(\mathrm{P}_\varepsilon^h)$.

Theorem 4.1. *Let the assumptions of Lemma 4.1 hold. Then we have that*

$$\|u_\varepsilon^h - \hat{U}_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))} + \|u_\varepsilon^h - U_\varepsilon\|^2_{L^\infty(0,T;(H^1(\Omega))')} \leq C \left[\Delta t + \frac{h^4}{\Delta t}\right]; \tag{4.12}$$

where

$$U_\varepsilon(t) := \tfrac{t-t_{n-1}}{\Delta t} U_\varepsilon^n + \tfrac{t_n - t}{\Delta t} U_\varepsilon^{n-1} \qquad t \in [t_{n-1}, t_n] \; n \geq 1$$

and

$$\hat{U}_\varepsilon(t) := U_\varepsilon^n \qquad t \in (t_{n-1}, t_n) \; n \geq 1.$$

Proof. Using the above notation, (4.2) can be restated as:
Find $U_\varepsilon \in H^1(0,T;S^h)$ such that $U_\varepsilon(0) \equiv Q^h u_0$ and for a.e. $t \in (0,T)$, $(U_\varepsilon(t), 1) = (u_0, 1)$ and

$$\gamma(\nabla \hat{U}_\varepsilon, \nabla \chi) + (\Psi_\varepsilon'(\hat{U}_\varepsilon), \chi - ⨍ \; \chi)^h + (\hat{\mathcal{G}}^h \tfrac{\partial U_\varepsilon}{\partial t}, \chi)^h = 0 \quad \forall \; \chi \in S^h . \tag{4.13}$$

It follows from (4.7) and (4.8) that

$$\int_0^T \|U_\varepsilon - \hat{U}_\varepsilon\|^2_{-h} dt \equiv \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (t_n - t)^2 \| \tfrac{\partial U_\varepsilon}{\partial t} \|^2_{-h} dt \leq (\Delta t)^2 \int_0^T \| \tfrac{\partial U_\varepsilon}{\partial t} \|^2_{-h} dt$$

$$\leq C(\Delta t)^2 \int_0^T |\hat{\mathcal{G}}^h \tfrac{\partial U_\varepsilon}{\partial t}|_1^2 dt \equiv C \, \Delta t \sum_{n=1}^{N} |\hat{\mathcal{G}}^h (U_\varepsilon^n - U_\varepsilon^{n-1})|_1^2 \leq C(\Delta t)^2. \tag{4.14}$$

In addition, we have from (4.8) that

$$\int_0^T | \tfrac{\partial U_\varepsilon}{\partial t} |_1^2 dt \equiv \tfrac{1}{\Delta t} \sum_{n=1}^{N} |U_\varepsilon^n - U_\varepsilon^{n-1}|_1^2 \leq \tfrac{C}{\Delta t} . \tag{4.15}$$

We set $E_\varepsilon := u_\varepsilon^h - U_\varepsilon \in V^h$ and $\hat{E}_\varepsilon := u_\varepsilon^h - \hat{U}_\varepsilon \in V^h$ for a.e. $t \in (0,T)$. Then subtracting (4.13) from (3.7) and choosing $\chi \equiv \hat{E}_\varepsilon$, and noting (3.9), (3.5), (4.6), (3.10) and a Young's inequality yields, similarly to (3.28), for a.e. $t \in (0,T)$ that

$$
\begin{aligned}
&\gamma|\hat{E}_\varepsilon|_1^2 + (\phi_\varepsilon(u_\varepsilon^h) - \phi_\varepsilon(\hat{U}_\varepsilon), \hat{E}_\varepsilon)^h + \tfrac{1}{2}\tfrac{d}{dt}\|E_\varepsilon\|_{-h}^2 \\
&= \theta_{\rm c}|\hat{E}_\varepsilon|_h^2 + (\mathscr{G}^h \tfrac{\partial E_\varepsilon}{\partial t}, E_\varepsilon - \hat{E}_\varepsilon) + ((\hat{\mathscr{G}}^h - \mathscr{G}^h)\tfrac{\partial U_\varepsilon}{\partial t}, \hat{E}_\varepsilon) \\
&\qquad\qquad\qquad + \Big[(\hat{\mathscr{G}}^h \tfrac{\partial U_\varepsilon}{\partial t}, \hat{E}_\varepsilon)^h - (\hat{\mathscr{G}}^h \tfrac{\partial U_\varepsilon}{\partial t}, \hat{E}_\varepsilon) \Big] \\
&\leq C\Big[\|\hat{E}_\varepsilon\|_{-h}^2 + \|\tfrac{\partial E_\varepsilon}{\partial t}\|_{-h}\|U_\varepsilon - \hat{U}_\varepsilon\|_{-h} + |(\mathscr{G}^h - \hat{\mathscr{G}}^h)\tfrac{\partial U_\varepsilon}{\partial t}|_0^2 + h^4\|\hat{\mathscr{G}}^h \tfrac{\partial U_\varepsilon}{\partial t}\|_1^2 \Big] \\
&\leq C\Big[\|E_\varepsilon\|_{-h}^2 + \|U_\varepsilon - \hat{U}_\varepsilon\|_{-h}^2 + \|\tfrac{\partial E_\varepsilon}{\partial t}\|_{-h}\|U_\varepsilon - \hat{U}_\varepsilon\|_{-h} \\
&\qquad + h^4\|\tfrac{\partial U_\varepsilon}{\partial t}\|_1^2 + h^4\|\hat{\mathscr{G}}^h \tfrac{\partial U_\varepsilon}{\partial t}\|_1^2 \Big]. \qquad (4.16)
\end{aligned}
$$

Noting (2.6), (2.14), (3.12), integrating over $t \in (0,T)$ and a using a Grönwall inequality yields that

$$
\begin{aligned}
&\|\hat{E}_\varepsilon\|_{L^2(0,T;H^1(\Omega))}^2 + \|E_\varepsilon\|_{L^\infty(0,T;(H^1(\Omega))')}^2 \\
&\leq C\Big[\|U_\varepsilon - \hat{U}_\varepsilon\|_{L^2(0,T;(H^1(\Omega))')}^2 + \|\tfrac{\partial E_\varepsilon}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')}\|U_\varepsilon - \hat{U}_\varepsilon\|_{L^2(0,T;(H^1(\Omega))')} \\
&\qquad + h^4\|\tfrac{\partial U_\varepsilon}{\partial t}\|_{L^2(0,T;H^1(\Omega))}^2 + h^4\|\hat{\mathscr{G}}^h \tfrac{\partial U_\varepsilon}{\partial t}\|_{L^2(0,T;H^1(\Omega))}^2 \Big].
\end{aligned}
$$

Hence noting (2.14), (3.12), (4.14), (3.18) and (4.15) yields the desired result (4.12). □

Theorem 4.2. *Let the assumptions of Lemma 4.1 hold. Then for all $\varepsilon \leq \varepsilon_0$ and $h > 0$ there exists a unique solution $\{U^n, W^n\}_{n=1}^N$ to ($\mathrm{P}^{h,\Delta t}$) such that*

$$
\begin{aligned}
&\max_{n=1\to N} |U^n|_1^2 + \sum_{n=1}^N |U^n - U^{n-1}|_1^2 + \Delta t \sum_{n=1}^N |\hat{\mathscr{G}}^h(\tfrac{U^n - U^{n-1}}{\Delta t})|_1^2 \\
&\qquad + \Delta t \sum_{n=1}^N |W^n|_1^2 + \Delta t \sum_{n=1}^N |\pi^h[\phi(U^n)]|_0^2 \leq C. \qquad (4.17)
\end{aligned}
$$

In addition (1.9) holds. Furthermore, we have that

$$
\|\hat{U} - \hat{U}_\varepsilon\|_{L^2(0,T;H^1(\Omega))}^2 + \|U - U_\varepsilon\|_{L^\infty(0,T;(H^1(\Omega))')}^2 \leq C(\varepsilon + (\Delta t)^2), \qquad (4.18)
$$

where U and $\hat{U}$ are defined similarly to their regularized counterparts, see Theorem 1.3.

Proof. The proof is a discrete analogue of Theorem 2.1. Uniqueness of a solution to ($\mathrm{P}^{h,\Delta t}$) follows as for ($\mathrm{P}_\varepsilon^{h,\Delta t}$), see also Copetti and Elliott (1992). Note that one can allow $\Delta t = 4\gamma/\theta_{\rm c}^2$. Existence of a solution follows by letting $\varepsilon \to 0$, noting the uniform bounds (4.8) and applying a discrete analogue of (2.34) $\to$ (2.38). Hence the bounds (4.17) hold. The bound (1.9) follows immediately from the bound on $\pi^h[\phi(U^n)]$ in (4.17).

We set $E := U - U_\varepsilon \in V^h$ and $\hat{E} := \hat{U} - \hat{U}_\varepsilon \in V^h$ a.e. $t \in (0,T)$. Subtracting (4.2) from its non-regularized counterpart and choosing $\chi \equiv \hat{E}$ yields for a.e. $t \in (0,T)$ that

$$\gamma|\hat{E}|_1^2 + (\phi(\hat{U}) - \phi_\varepsilon(\hat{U}_\varepsilon), \hat{E})^h + (\hat{\mathscr{G}}^h \tfrac{\partial E}{\partial t}, E)^h + (\hat{\mathscr{G}}^h \tfrac{\partial E}{\partial t}, \hat{E} - E)^h = \theta_c|\hat{E}|_h^2.$$

Noting that for $t \in (t_{n-1}, t_n)$, $\hat{E} - E \equiv (t_n - t)\frac{\partial E}{\partial t}$, (2.8) and the analogue of (2.41) yields for a.e. $t \in (0, T)$ that

$$\begin{aligned}
&\gamma|\hat{E}|_1^2 + \tfrac{\theta}{2\varepsilon}(\hat{E}, \hat{E})^h_{\omega_\varepsilon^+(t)\cup\omega_\varepsilon^-(t)} + \tfrac{1}{2}\tfrac{d}{dt}|\hat{\mathscr{G}}^h E|_1^2 \\
&\quad\le \gamma|\hat{E}|_1^2 + (\phi_\varepsilon(\hat{U}) - \phi_\varepsilon(\hat{U}_\varepsilon), \hat{E})^h + \tfrac{1}{2}\tfrac{d}{dt}|\hat{\mathscr{G}}^h E|_1^2 \\
&\quad\le \theta_c|\hat{E}|_h^2 + (\phi_\varepsilon(\hat{U}) - \phi(\hat{U}), \hat{E})^h \le \theta_c|\hat{E}|_h^2 - (\phi(\hat{U}), \hat{E})^h_{\omega_\varepsilon^+(t)\cup\omega_\varepsilon^-(t)},
\end{aligned} \tag{4.19}$$

where, recalling the notation of (1.5),

$$(\eta_1, \eta_2)^h_{\omega_\varepsilon^+(t)\cup\omega_\varepsilon^-(t)} := \sum_{j\in\omega_\varepsilon^+(t)\cup\omega_\varepsilon^-(t)} M_j \eta_1(x_j)\eta_2(x_j), \quad \forall\ \eta_1, \eta_2 \in C(\overline{\Omega});$$

and

$$\begin{aligned}
\omega_\varepsilon^+(t) &:= \{j : 1 - \varepsilon \le \hat{U}(x_j, t) \le \hat{U}_\varepsilon(x_j, t)\}, \\
\omega_\varepsilon^-(t) &:= \{j : \hat{U}_\varepsilon(x_j, t) \le \hat{U}(x_j, t) \le -1 + \varepsilon\}.
\end{aligned}$$

After noting (4.5) and applying a Young's inequality, it follows from (4.19) that for $a.e\ t \in (0, T)$

$$\begin{aligned}
&\gamma|\hat{E}|_1^2 + \tfrac{\theta}{2\varepsilon}(\hat{E}, \hat{E})^h_{\omega_\varepsilon^+(t)\cup\omega_\varepsilon^-(t)} + \tfrac{1}{2}\tfrac{d}{dt}|\hat{\mathscr{G}}^h E|_1^2 \\
&\quad\le C\left[|\hat{\mathscr{G}}^h E|_1^2 + |\hat{\mathscr{G}}^h(\hat{E} - E)|_1^2 + \varepsilon|\phi(\hat{U})|_h^2\right].
\end{aligned}$$

Integrating the above for $t \in (0, T)$, applying a Grönwall inequality and noting (2.14), (4.7), (4.14), (3.9) and (4.17) yields the desired result (4.18). □

Therefore combining (2.33), (3.27), (4.12) and (4.18) yields that

$$\begin{aligned}
&\|u - \hat{U}\|^2_{L^2(0,T;H^1(\Omega))} + \|u - U\|^2_{L^\infty(0,T;(H^1(\Omega))')} \\
&\quad\le C\left[\varepsilon + \varepsilon^{-1}h^2 + \Delta t + \frac{h^4}{\Delta t}\right].
\end{aligned} \tag{4.20}$$

Hence choosing $\varepsilon := C_1 h \le \varepsilon_0$, for some constant C_1 proves Theorem 1.3.

Finally, we note that it is not necessary to introduce the semidiscrete regularized problem (P_ε^h). One could analyse directly the error between u_ε and U_ε. However, such an analysis is rather cumbersome. We introduced (P_ε^h) in order to split the error analysis into two (more amenable) parts and in an attempt to isolate the errors due to (i) spatial discretization by finite elements and (ii) time discretization. This is very desirable as one may be interested in alternative time stepping procedures. We were not totally successful in this aim, since in order to isolate the spatial error of the fully practical scheme ($\mathrm{P}_\varepsilon^{h,\Delta t}$): $(\frac{\partial u_\varepsilon^h}{\partial t}, \chi)$ in (3.1a) and (w_ε^h, χ) in (3.1b) should be replaced by $(\frac{\partial u_\varepsilon^h}{\partial t}, \chi)^h$ and $(w_\varepsilon^h, \chi)^h$; that is, $(\mathscr{G}^h \frac{\partial u_\varepsilon^h}{\partial t}, \chi)$ in (3.7) should be replaced by $(\hat{\mathscr{G}}^h \frac{\partial u_\varepsilon^h}{\partial t}, \chi)^h$. However, we were not able to prove an error bound in this case; since an analogue of the key bound (4.15) in the fully discrete case is not available.

5. A numerical experiment

As no exact solution to (P) is known, a comparison between the solutions of $(\mathrm{P}^{h,\Delta t})$ on a coarse mesh, $\hat{U}$, with that on a fine mesh, $\hat{u}$, was made. The data used in each experiment on the coarse meshes were $\Omega = (0,1)$, $\gamma = 5 \times 10^{-3}$, $\theta = 0.15$, $\theta_c = 1.0$, $T = 0.36$, $\Delta t = 0.48h$, $h = 1/(J-1)$ where $J = 2^k + 1$ ($k = 6, 7, 8$), tol $= 1 \times 10^{-7}$ and $\mu = 1$; the last two quantities were parameters used to vary the degree and speed of convergence in the iterative method (method II of Copetti and Elliott (1992)) to solve for U^n at each time level in $(\mathrm{P}^{h,\Delta t})$. The data were the same for the fine mesh except $J = 2^{11} + 1$. The initial data u_0 was taken to be a smooth function, see Fig. 1. In fact we took $u_0 \equiv U^N$, where U^N is the final solution of problem $(\mathrm{P}^{h,\Delta t})$ with the parameters stated above except $T = 0.18$, $h = 2^{-6}$ and random initial data perturbed about the mean value of -0.6; this simulates two components A and B being quenched into an unstable state. This choice of initial data u_0 ensured that the logarithm played a role. In each of the experiments after one time step there existed a point where $|U_i^n| > 0.99$. In Fig. 1 the solution on the fine mesh is plotted at time intervals of 0.12.

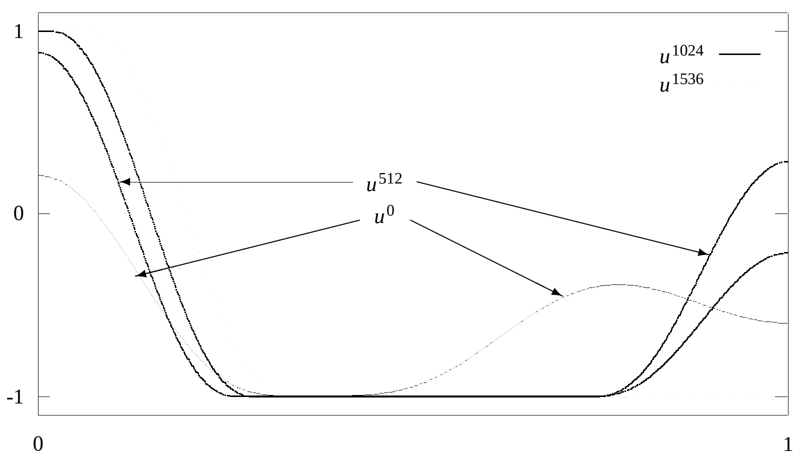

Fig. 1. u^n plotted at intervals of 0.12

We computed the quantity

$$\xi := \left[\sum_{n=1}^{N} \Delta t |\pi^h u^n - U^n|_1^2 \right]^{1/2},$$

and obtained the following table:

J	65	129	257
ξ^2	4.31×10^{-3}	1.05×10^{-3}	2.37×10^{-4}

We see that the ratio of consecutive ξ^2 is approximately 4.1 and 4.5 which is better than 2, the rate of convergence proved in Theorem 1.3.

References

Blowey, J.F., Elliott, C.M. (1991): The Cahn-Hilliard gradient theory for phase separation with non-smooth free energy, part i: Mathematical analysis. Eur. J. Appl. Math. **2**, 233–279

Blowey, J.F., Elliott, C.M. (1992): The Cahn-Hilliard gradient theory for phase separation with non-smooth free energy part ii: Numerical analysis. Eur. J. Appl. Math. **3**, 147–179

Blowey, J.F., Copetti, M.I.M., Elliott, C.M. (1995): The numerical analysis of a model for phase separation of a multi-component alloy. IMAJNA (to appear)

Cahn, J.W. (1961): On spinodal decomposition. Acta Metall. **9**, 795–801

Cahn, J.W., Hilliard, J.E. (1958): Free energy of a non-uniform system. I. Interfacial free energy. J. Chem. Phys. **28**, 258–267

Cialvaldini, J.F. (1975): Analyse numérique d'un problème de Stefan à deux phases par une méthode d'elements finis. SIAM J. Numer. Anal. **12**, 464–487

Copetti, M.I.M., Elliott, C.M. (1992): Numerical analysis of the Cahn-Hilliard equation with logarithmic free energy. Numer. Math. **63**, 39–65

Elliott, C.M. (1987): Error analysis of the enthalpy method. IMAJNA **7**, 61–71

Elliott, C.M., Luckhaus, S. (1991): A generalised diffusion equation for phase separation of a multi-component mixture with interfacial free energy. IMA, University of Minnesota, Preprint 887

Lions, J.L. (1969): Quelques méthodes de résolution des problèmes aux limites nonlinéaires. Dunod, Paris

Nochetto, R.H. (1991): Finite element methods for parabolic free boundary problems. In: W. Light, ed., Advances in numerical analysis, vol. 1, O.U.P. pp. 34–95

This article was processed by the author using the LaTeX style file *pljour1m* from Springer-Verlag.