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An analysis of bilinear transform polynomial methods of inversion of Laplace transforms

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Summary. Methods for the numerical inversion of a Laplace transform $F(s)$ which use a special bilinear transformation of s are particularly effective in many cases and are widely used. The main purpose of this paper is to analyze the convergence and conditioning properties of a special class of such methods, characterized by the use of Lagrange interpolation. The results derived apply both to complex and real inversion, and show that some known inversion methods are in fact in this class.

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1. Introduction

Among the numerical methods for the inversion of a *Laplace transform* F, i.e., for solving with respect to f the equation

$$F(s) = \pounds\{f\}(s) = \int_0^{\infty} \mathrm{e}^{-sx} f(x)\, dx \ ,$$

particularly relevant are those which use a bilinear mapping like

$$w = \frac{s+\alpha}{s+\beta} \qquad (\beta > \alpha)$$

and then approximate $\psi(w) = F(s(w))$ [3].
Such methods, including both *Laguerre polynomial methods* [10, 11, 15, 18] and *Piessens' methods* [12], have led to reliable and efficient algorithms and software [4, 5, 13].

The use of Lagrange interpolation at two special sets of knots for Laguerre polynomial methods is discussed in [15].

The approach of the present paper allows us to deal in a unified way with several polynomial-type methods of inversion of the Laplace transform. We generalize and formally characterize the approach in [15], referring to such collocation methods as *C-methods*. We study their convergence and conditioning properties. Furthemore, we show that both the methods in [10] and [12] are in fact C-methods, which converge on any class $\mathscr{S}_\gamma$ of functions whose analytic continuation is given by

$$F(s) = s^{-\gamma} G(s) \ , \tag{1}$$

for some fixed $\gamma > 0$ and G analytic at infinity. The previous result accomplishes the analysis in [2], where in fact Hardy's theorem [9] has been generalized to orthogonal polynomial expansions of $F \in \mathscr{S}_\gamma$ to prove the convergence of Piessens' methods.

The background of the problem is summarized in Sect. 2. Section 3 is devoted to defining C-methods and deriving their explicit representation. In Sect. 4 a sufficient condition for the (exponentially fast) convergence of C-methods is given, while their conditioning properties are discussed in Sect. 5. Finally, Sect. 6 provides various examples of such methods; both the methods in [10] and [12] are analyzed.

2. Preliminaries

This section provides some results and terminology on the inversion of a Laplace transform (Sect. 2.1) and on conformal mappings (Sect. 2.2), which we will need later.

2.1.

In the sequel we shall refer to the usual definition of *Laplace absolutely transformable* functions and of the *Laplace absolute convergence abscissa* σ_0 [9, vol.II, p.200]. Moreover, we shall denote by R the radius of the smallest circle centered at the origin and enclosing the singularities of G.

Lemma 1. (Riemann) [9]**.** *Let f be an absolutely transformable function and F its Laplace transform. If* $\sigma > \sigma_0$ *and if* $x > 0$ *is a point of continuity of* f*, then*

$$f(x) = \frac{1}{2\pi \mathrm{i}} \,\mathrm{vp} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} \mathrm{e}^{sx} F(s) \, ds \ .$$

Lemma 2. *Let F be in* $\mathscr{S}_\gamma$ *for some* $\gamma > 0$*. Then F has a continuous, absolutely transformable original function f and* $\sigma_0 \le R$ *. Moreover, if C is any piecewise smooth, closed curve intersecting the real axis exactly twice and lying in the domain* $|s| > R$*, then*

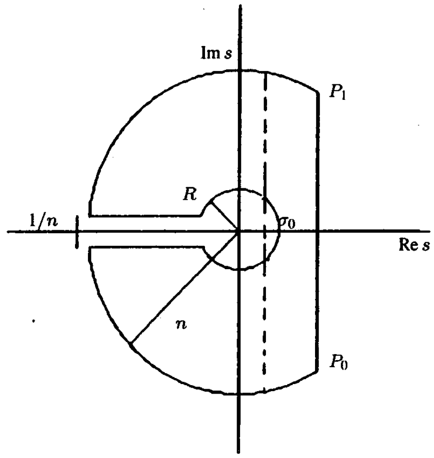
$$f(x) = \frac{1}{2\pi \mathrm{i}} \oint_C \mathrm{e}^{sx} s^{-\gamma} G(s) \, ds + \frac{\sin(\pi\gamma)}{\pi} \int_c^\infty \mathrm{e}^{-sx} s^{-\gamma} G(-s) \, ds \ , \quad x > 0. \tag{2}$$

In Eq. (2) the integration along C is carried out in the anticlockwise direction, and $-c$ *is the abscissa of the point where* C *intersects the real negative axis.*

Sketch of the proof. The first part of the lemma easily follows from Hardy's theorem [9, vol.II, p.248]. To prove Eq. (2), consider a family $\{C_n\}$ of integration paths as shown in the figure.

According to Cauchy's theorem and Lemma 1,

$$2\pi \mathrm{i} f(x) = \lim_n \int_{\overline{P_0 P_1}} \mathrm{e}^{sx} F(s) \, ds = \lim_n \int_{\widehat{P_0 P_1}} \mathrm{e}^{sx} s^{-\gamma} G(s) \, ds \ , \quad x > 0.$$



Thus Eq. (2) follows from Jordan's lemma [9, vol.I, p.255] and from the properties of the function $s^{-\gamma}$. □

Remark. For any natural number n, $F \in \mathscr{S}_n$ implies that f is a function of exponential type, and Eq. (2) becomes

$$f(x) = \frac{1}{2\pi \mathrm{i}} \oint_C \mathrm{e}^{sx} s^{-n} G(s)\, ds \ , \quad x > 0.$$

In this case the conclusions of Lemma 2 follow from Pincherle's theorem [9, vol.II, p.308].

2.2.

Given a closed Jordan curve J, we denote by $\overset{\circ}{J}$ the bounded open region having J as its boundary, by $\overline{J}$ the simply connected, closed set $J \cup \overset{\circ}{J}$ and by J^{c} the complement of $\overline{J}$ with respect to the complex plane. Moreover, U is the unit circle $|s| = 1$.

According to the Riemann mapping theorem, corresponding to every curve J there exists a unique function ϕ which conformally maps J^{c} onto U^{c} such that $\phi(J) = U$, $\phi(\infty) = \infty$ and such that $\phi'(\infty) > 0$. The quantity $\kappa_J = [\phi'(\infty)]^{-1}$ is the *capacity* of the set $\overline{J}$ [16, p.13]. Lastly, we denote by $J(r)$ $(r > 1)$ the curve lying in J^{c} whose image through ϕ is the circle U_r of radius r centered at the origin.

Consider now the transformation

$$w = \frac{s - \sigma - b}{s - \sigma + b} \ , \qquad \sigma \in (-\infty, \infty) \ , \ b > 0 \ . \tag{3}$$

Eq. (3) defines a one-to-one, conformal map, which transforms circles into circles[1]. In particular, (3) maps any circle C for which the points $s = \sigma \pm b$ are symmetrical [2] onto some U_r, so that to the axis Re $s = \sigma$ corresponds U and to the positive direction on C corresponds the negative one on U_r.

[1] Straight lines are generalized circles

[2] Let D be a circle in the complex plane. The points p, q are symmetrical with respect to D if there exists a constant k such that $|s - p| = k|s - q|$ for any $s \in D$

It follows that C^c is mapped into $\overset{\circ}{U}_r$ and $r > 1$ or $r < 1$ depending as $\sigma - b \in \overset{\circ}{C}$ or $\sigma + b \in \overset{\circ}{C}$, respectively [3]. Moreover, (3) maps straight lines parallel to the real and imaginary axes into circles which are tangent in $w = 1$ to the real axis, and to the axis $\mathrm{Re}\, w = 1$, respectively. In the foregoing correspondence, the real axis and the lower and upper half-planes are mapped into their analogues in the w-plane. The line $\mathrm{Re}\, s = \sigma - b$ is transformed into $\mathrm{Re}\, w = 1$, and the half-plane $\mathrm{Re}\, s < \sigma - b$ is mapped onto $\mathrm{Re}\, w > 1$.

Lastly, it is easy to prove:

Lemma 3. *If J is a closed Jordan curve in the s-plane for which $\sigma - b \in \overset{\circ}{J}$, then its image J' through (3) is a closed Jordan curve in the w-plane such that $1 \in \overset{\circ}{J'}$. Moreover, J is piecewise smooth if and only if J' is.*

3. C-methods

Let $F \in \mathscr{S}_\gamma$ for some $\gamma > 0$. If $\sigma > \sigma_0$, $b > 0$ and w is given by (3), then the assumption on F implies that (see Sect. 2.2)

$$\psi(w) = \left(\frac{2b}{1-w}\right)^\gamma F\left(\frac{2b}{1-w} + \sigma - b\right) \tag{4}$$

is an analytic function on $\overline{U}$. Given the sequence

$$\underline{w} : h = 0, 1, 2, \ldots \to \underline{w}_h \equiv [w_{h0}, \ldots, w_{hh}] \subset \overline{U} \ , \tag{5}$$

where $w_{hm} \neq w_{hn}$ if $m \neq n$, consider the Lagrange interpolation polynomial ℓ_N which interpolates ψ at the $N + 1$ coordinates of $\underline{w}_N$. We can choose, as an approximation of f, the continuous function f_N such that

$$\pounds\{f_N\}(s) = (s - \sigma + b)^{-\gamma}\, \ell_N\left(\frac{s - \sigma - b}{s - \sigma + b}\right) \ .$$

Let $\{p_n\}$ $(n = 0, 1, 2, \ldots)$ be any basis for the space of polynomials, and let $\{\varphi_n\}$ be the sequence of the continuous functions whose Laplace transforms are

$$(s - \sigma + b)^{-\gamma}\, p_n\left(\frac{s - \sigma - b}{s - \sigma + b}\right) \ .$$

If

$$\ell_N(w) = \sum_{n=0}^{N} c_n p_n(w) \ , \tag{6}$$

then f_N can be obtained by:

I. evaluating ψ at the coordinates w_{Nk} of $\underline{w}_N$;

[3] If C is the line $\mathrm{Re}\, s = \sigma$, we must take as C^c the right half-plane $\mathrm{Re}\, s > \sigma$

II. solving the linear system in the unknowns c_n

$$[c_0, \ldots, c_N]\tilde{V}(\underline{w}_N) = [\psi_0, \ldots, \psi_N] \ ,$$

where $\psi_k = \psi(w_{Nk})$ and

$$\tilde{V}(\underline{w}_N) = \begin{bmatrix} 1 & \ldots & 1 \\ p_1(w_{N0}) & \ldots & p_1(w_{NN}) \\ \vdots & & \vdots \\ p_N(w_{N0}) & \ldots & p_N(w_{NN}) \end{bmatrix}$$

is a *Vandermonde-like* matrix on $\underline{w}_N$ [6];

III. evaluating

$$f_N(x) = \sum_{n=0}^{N} c_n \varphi_n(x; \gamma, \sigma, b).$$

Definition 1. A C-method of order γ on the system of nodes $\underline{w}$, briefly denoted by $CM(\underline{w}, \gamma)$, is a numerical method for Laplace transform inversion consisting of steps I)-III) above.

The results of Sect. 2.2 show that C-methods involve the evaluation of F at prescribed sets of points in the half-plane Re $s \geq \sigma$. They include both complex and real inversion methods. The latter are characterized by the condition

$$\underline{w}_h \subset [-1, 1] \ , \qquad h = 0, 1, 2, \ldots \ .$$

Let L_n^α be the generalized Laguerre polynomial of degree n and order α ($\alpha > -1$); then

$$\pounds\{x^\alpha L_n^\alpha(x)\}(s) = \frac{\Gamma(\alpha + n + 1)(s - 1)^n}{n! \ s^{\alpha+n+1}} \ .$$

If in (6) we choose $p_n(w) = w^n$, it follows that

$$\varphi_n(x; \gamma, \sigma, b) = \mathrm{e}^{(\sigma-b)x} x^{\gamma-1} \frac{n!}{\Gamma(\gamma + n)} L_n^{\gamma-1}(2bx) \ ,$$

and $f_N(x)$ is computed by step I) and replacing steps II) and III) by:

II'. solving the linear system

$$[c_0, \ldots, c_N]V(\underline{w}_N) = [\psi_0, \ldots, \psi_N] \ ,$$

where $V(\underline{w}_N) = [\underline{w}_N^n]^{\mathrm{T}}$ ($n = 0, .., N$) is the Vandermonde matrix on $\underline{w}_N$;

III'. evaluating

$$f_N(x) = \mathrm{e}^{(\sigma-b)x} x^{\gamma-1} \sum_{n=0}^{N} \frac{n! \, c_n}{\Gamma(\gamma + n)} L_n^{\gamma-1}(2bx) \ .$$

We refer to such C-methods, which are special Laguerre polynomial methods, as *LC-methods*.

If

$$l_m(w) = \sum_{n=0}^{N} l_{mn} w^n$$

is the *m*th elementary Lagrange interpolation polynomial of degree N, then it is easy to prove (see for example [6]) that

$$V^{-1}(\underline{w}_N) = [l_{mn}] \qquad (m, n = 0, \ldots, N) \ .$$

Thus,

$$c_n = \sum_{m=0}^{N} l_{mn} \psi_m \qquad (n = 0, \ldots, N) \tag{7}$$

and

$$f_N(x) = x^{\gamma-1} \mathrm{e}^{(\sigma-b)x} \sum_{m=0}^{N} \psi_m \lambda_m(x) \ , \tag{8}$$

where

$$\lambda_m(x) = \sum_{n=0}^{N} l_{mn} \frac{n!}{\Gamma(\gamma+n)} L_n^{\gamma-1}(2bx) \ . \tag{9}$$

For any fixed $\gamma, b > 0$ and any real number σ, let $\Psi_{\sigma,b,\gamma}$ be the set of functions ψ defined by Eq. (4), where $F \in \mathscr{S}_\gamma$ has (absolute) convergence abscissa $\sigma_0 < \sigma$. Let $R_N^{(\sigma,b,\gamma)}$ be the operator

$$R_N^{(\sigma,b,\gamma)} : \psi \in \Psi_{\sigma,b,\gamma} \to f_N \ , \tag{10}$$

where f_N is the function defined by steps I)-III). Obviously, $R_N^{(\sigma,b,\gamma)}$ is a linear operator on the space of holomorphic functions on $\overline{U}$, whose explicit representation is given by (8).

4. A convergence criterion

Henceforth we will denote by $\|\cdot\|_S$ the uniform norm on the set S.

Let f_N be obtained by steps I)-III).

Definition 2. The method $CM(\underline{w}, \gamma)$ converges on the set $\mathscr{T}$ of Laplace transforms if and only if any $F \in \mathscr{T}$ has an original function f such that

$$f(x) = \lim_{N\to\infty} f_N(x) \ , \qquad x \geq 0 \ ,$$

whenever $\sigma > \sigma_0$ and $b > 0$.

Given a closed Jordan curve J, let $\underline{w}$ be any sequence

$$\underline{w} : h = 0, 1, 2, \ldots \rightarrow \underline{w}_h = [w_{h0}, \ldots, w_{hh}] \subset \overline{J} \ ,$$

where $w_{hm} \neq w_{hn}$ if $m \neq n$. Consider any function ψ on $\overline{J}$ and construct for it the Lagrange interpolating polynomial ℓ_h at the coordinates of $\underline{w}_h$. We denote by ω_{h+1} the monic polynomial

$$\omega_{h+1}(w) = \prod_{k=0}^{h}(w - w_{hk}) \ .$$

The next lemma follows from the Fekete–Walsh convergence principle [16, p.17].

Lemma 4. *Let ψ be any function regular on $\overline{J}$. The sequence $\{\ell_h\}$ converges uniformly to ψ on $\overline{J}$ if and only if*

$$\lim_{h\to\infty} \sqrt[h+1]{\|\omega_{h+1}\|_J} = \kappa_J$$

or, equivalently,

$$\lim_{h\to\infty} \sqrt[h+1]{|\omega_{h+1}(w)|} = \kappa_J|\phi(w)| \ ,$$

uniformly on every compact set in J^{c}, where κ_J is the capacity of $\overline{J}$ and ϕ is the conformal map defined in Sect. 2.2. Moreover, if ϕ is regular on $\overline{J}(r)$ $(r > 1)$, then

$$\|\phi - \ell_h\|_J = O(r^{-(h+1)}) \ .$$

Theorem 1. *The method $CM(\underline{w}, \gamma)$ converges exponentially on $\mathscr{S}_\gamma$ $(\gamma > 0)$ if there exists a sequence $\{J_n\}$ of piecewise smooth, closed Jordan curves intersecting the real axis exactly twice and satisfying*

i) $\forall r > 1 \quad \exists \nu : \ \{w_{hm}\} \subseteq \overline{J}_\nu \subseteq \overline{U}_r$ *and* $1 \in \mathring{J}_\nu$ *;*
ii) $\lim_{h\to\infty} \sqrt[h+1]{\|\omega_{h+1}\|_{J_n}} = \kappa_{J_n}$.

Condition (ii) can be replaced by

ii') $\lim_{h\to\infty} \sqrt[h+1]{|\omega_{h+1}|} = \kappa_{J_n}|\phi_n|$ *uniformly for any compact set in* J_n^{c}*,*

where ϕ_n is the conformal map corresponding to J_n.

Proof. Given any $F \in \mathscr{S}_\gamma$, $\sigma > \sigma_0$ and $b > 0$, let $C_{\sigma,b}$ be the smallest circle in $\mathrm{Re}\, s > \sigma$ for which the points $s = \sigma \pm b$ are symmetrical such that $\overline{C}_{\sigma,b}$ contains all the singularities of F. According to the results of Sect. 2.2, the image through (3) of $C_{\sigma,b}$ is a circle U_R of radius $R = R_{\sigma,b} > 1$, and the function ψ, defined by Eq. (4), is regular in $\mathring{U}_R$. Condition (i) implies that for any $1 < r < R$ it is always possible to choose n so that $\{w_{hm}\} \subseteq \overline{J}_n \subseteq \overline{U}_r$ and $1 \in \mathring{J}_n$. Let J'_n be the inverse image under (3) of J_n. Lemma 3 implies that J'_n is a piecewise smooth, closed Jordan curve such that $\sigma - b \in \mathring{J'}_n$ and $J'_n \subseteq C^{\mathrm{c}}_{\sigma,b}$. Thus, we must have $C_{\sigma,b} \subseteq \mathring{J'}_n$.

On the other hand, if G_N and G are the two functions defined by

$$G_N(s) = \ell_N\left(\frac{s - \sigma - b}{s - \sigma + b}\right) \quad , \quad G(s) = \psi\left(\frac{s - \sigma - b}{s - \sigma + b}\right) \ ,$$

then from Lemma 2 it follows that F has its continuous original function f satisfying for $x > 0$ the equality

$$\begin{aligned} f(x) - f_N(x) &= \frac{1}{2\pi \mathrm{i}} \oint_{J_n'} \mathrm{e}^{sx} (s - \sigma + b)^{-\gamma} [G(s) - G_N(s)]\, ds \\ &+ \frac{\sin(\pi\gamma)}{\pi} \int_{-s_n}^{\infty} \mathrm{e}^{-sx} (s + \sigma - b)^{-\gamma} [G(-s) - G_N(-s)]\, ds \ , \end{aligned} \tag{11}$$

where $s_n \in J_n'$ satisfies $s_n < \sigma - b$. Since $(-\infty, s_n) \subset J_n'^{\mathrm{c}}$, Eq.(11) and the maximum modulus principle imply that

$$f(x) - f_N(x) = O(\|\psi - \ell_N\|_{J_n}) \quad , \quad x \geq 0 \ .$$

Let $\rho > 1$ be any number satisfying $J_n(\rho) \subset \overset{\circ}{U}_R$; according to condition (ii) and Lemma 4,

$$\|\psi - \ell_N\|_{J_n} = O(\rho^{-(N+1)}) \ ,$$

so that the theorem is proved. □

Corollary 1. *The method* $CM(\underline{w}, \gamma)$ *converges exponentially on* $\mathscr{S}_\gamma$ *if there exists a sequence* $\{\theta_h\} \subset [0, 2\pi]$ *and a number* $\rho \in (0, 1]$ *such that*

$$w_{hk} = \rho \mathrm{e}^{\mathrm{i}(\theta_h + k \frac{2\pi}{h+1})} \ , \qquad h = 0, 1, 2, \ldots; \ k = 0, \ldots, h.$$

Proof. Since $\underline{w}_h$ is the vector of the roots of order $h + 1$ of $(\rho \mathrm{e}^{\mathrm{i}\theta_h})^{h+1}$, we have

$$\omega_{h+1}(w) = w^{h+1} - (\rho \mathrm{e}^{\mathrm{i}\theta_h})^{h+1}.$$

If $w \in U_r$, then we find that

$$|\omega_{h+1}(w)|^2 = r^{2(h+1)} - \rho^{2(h+1)} - 2(r\rho)^{h+1} \cos[(h+1)(\theta - \theta_h)], \qquad \theta \in [0, 2\pi],$$

so that, for any fixed $r > 1$,

$$\lim_{h \to \infty} \sqrt[h+1]{\|\omega_{h+1}\|_{U_r}} = r.$$

The function $\phi(w) = w/r$ conformally maps U_r^{c} onto U^{c} and, moreover, $\phi(U_r) = U$, $\phi(\infty) = \infty$, $\phi'(\infty) = 1/r$. Thus, $K_{U_r} = r$, and the assertion follows by applying Theorem 1 with $J_n = U_{1+1/n}$. □

Corollary 2. *Let* $\{p_n\}$ *be a sequence of polynomials orthogonal with respect to the measure* $d\mu$ *on the segment* $[-1, 1]$. *If* $\mu' > 0$ *almost everywhere in* $[-1, 1]$ *and* $\underline{w}_h$ $(h = 0, 1, 2, \ldots)$ *is the vector whose coordinates are the zeros of* p_{h+1}, *then* $CM(\underline{w}, \gamma)$ *converges exponentially on* $\mathscr{S}_\gamma$. [4]

[4] A weaker version of this corollary follows from a theorem of Van Assche [17, p.18]. We omit it because it is of no practical interest

Proof. Let E_r $(r > 1)$ be the ellipse defined by the parametric equation

$$w = \frac{1}{2}\left(r\mathrm{e}^{\mathrm{i}\theta} + \frac{1}{r\mathrm{e}^{\mathrm{i}\theta}}\right) , \qquad 0 \le \theta \le 2\pi .$$

Since E_r has foci at $w = \pm 1$ and semi axes

$$a = \frac{1}{2}(r + 1/r), \quad b = \frac{1}{2}(r - 1/r),$$

there exists a sequence of E_r satisfying condition (i) of Theorem 1.
E_r^{c} is conformally mapped onto U^{c} by the function

$$\phi(w) = \frac{1}{r}(w + \sqrt{w^2 - 1})$$

and $\phi(E_r) = U$, $\phi(\infty) = \infty$, $\phi'(\infty) = 2/r$. On the other hand, if ω_{h+1} is the monic polynomial whose zeros are the coordinates of $\underline{w}_h$, then [8, p.151]

$$\lim_{h\to\infty} \sqrt[h+1]{|\omega_{h+1}|} = \frac{|w + \sqrt{w^2 - 1}|}{2}$$

uniformly on any compact subset of the domain $|w+\sqrt{w^2 - 1}| > 1$. Thus (ii') follows, and the corollary is proved. □

5. Conditioning

According to the results of Sect. 3 , it is natural to consider the quantity

$$k_N(\gamma, \sigma, b) = \sup_{\psi\in\Psi} \frac{\|R_N^{(\gamma,\sigma,b)}(\psi)\|_I}{\|\psi\|_U} \tag{12}$$

to represent an (absolute) condition number of the problem of evaluating f_N on the interval $I \subset [0, \infty)$, starting with ψ.

Theorem 2. *Let $I = [x_1, x_2]$ be any interval such that $I \subset (0, \infty)$. If $R_N^{(\gamma,\sigma,b)}$ is the linear operator in (10) then*

$$k_N(\gamma, \sigma, b) \le \frac{\mathrm{e}^{\sigma x_2} B}{\Gamma(\gamma)} \sum_{m,n=0}^{N} |l_{mn}| , \tag{13}$$

where

$$B = \begin{cases} x_2^{\gamma-1} & \text{if } \gamma \ge 1 \\ x_1^{\gamma-1}\left(\frac{2\,N!}{\prod_{k=0}^{N-1}(\gamma+k)} - 1\right) & \text{if } 0 < \gamma < 1. \end{cases}$$

Proof. Since $R_N^{(\gamma,\sigma,b)}$ is linear and ψ belongs to the set $C(U)$ of continuous functions on $\overline{U}$, then

$$k_N(\gamma,\sigma,b) \le \sup_{\psi\in C(\overline{U}),\|\psi\|=1} \sup_{x\in I} |\sum_{m=0}^{N} \psi_m \lambda_m| = \sup_{x\in I} \sum_{m=0}^{N} |\lambda_m| \ . \tag{14}$$

Moreover, the following inequalities hold [1, p.786]:

$$|L_h^{\gamma-1}(2bx)| \le \frac{\Gamma(\gamma+h)}{\Gamma(\gamma)h!} \mathrm{e}^{bx} \quad (\gamma \ge 1) \ ,$$

$$|L_h^{\gamma-1}(2bx)| \le \left(2 - \frac{\Gamma(\gamma+h)}{\Gamma(\gamma)h!}\right) \mathrm{e}^{bx} \quad (0 < \gamma \le 1) \ .$$

Thus, the statement of Theorem 2 follows from (9), (14) and the fact that

$$\frac{\Gamma(\gamma)h!}{\Gamma(\gamma+h)} = \frac{h!}{\prod_{k=0}^{h-1}(\gamma+k)}$$

is an increasing function of h for $0 < \gamma < 1$. □

Equation (13) illustrates the terms that affect the conditioning. The factor $\mathrm{e}^{\sigma x_2} B / \Gamma(\gamma)$ is related to the parameters of the methods, while the sum $\sum_{m,n=0}^{N} |l_{mn}|$ depends only on the set of knots. Since $B \to \infty$ as $\gamma \to 0$, particularly relevant is the influence of γ, according to the fact that the slower $F(s)$ decreases to zero as $s \to \infty$, the more difficult is the inversion of F.

Equation (13) suggests choosing the parameter σ as small as possible in the interval $\sigma > \sigma_0$. The factor $\mathrm{e}^{\sigma_0 x}$ is inherent in the inversion of *all* Laplace transforms and is not special to C-methods.

As regards the choice of the knots, we emphasize that, according to step III), their effects can be analyzed by means of the conditioning of Vandermonde matrices, extensively studied by Gautschi (see [6] for a general discussion and further references).

6. Some remarkable C-methods

In the following examples we assume some typical configurations of interpolation points and study both the convergence of the related C-methods and the conditioning to evaluate f_N from ψ. We consider both complex (Sect. 6.1) and real (Sect. 6.2) interpolation points. We show that the methods in [10, 12] are in fact convergent collocation methods.

Henceforth, ω_{h+1} shall denote, as in Sect. 4, the monic polynomial whose zeros are the coordinates of $\underline{w}_h$.

6.1. C-methods at the roots of unity

An important class of C-methods at complex knots is obtained by choosing the sequence $\underline{w}$ whose term $\underline{w}_h$ is ρ times $(0 < \rho \leq 1)$ the vector of the $(h+1)$st roots of unity,

$$w_{hk} = \rho \mathrm{e}^{\frac{2\pi \mathrm{i} k}{h+1}} \ , \qquad h = 0, 1, 2, \dots \ ; \ k = 0, \dots, h \ .$$

We refer to these methods simply as *C-methods at the roots of unity*.
Obviously, $\underline{w}$ satisfies the conditions stated in Corollary 1, so that any C-method of order γ at the roots of unity converges exponentially on $\mathscr{S}_\gamma$.
Since

$$l_m(w) = \frac{\omega_{N+1}(w)}{(w - w_{Nm})\omega'_{N+1}(w_{Nm})}$$

and $\omega_{N+1}(w) = w^{N+1} - \rho^{N+1}$, elementary calculations show that

$$l_{mn} = \frac{\rho^{-n}}{N+1} \mathrm{e}^{\frac{-2\pi \mathrm{i} mn}{N+1}} \ . \tag{15}$$

From (15) and Theorem 2 it follows that

$$k_N(\gamma, \sigma, b) \leq \begin{cases} \frac{\mathrm{e}^{\sigma x_2} B}{\rho^N \Gamma(\gamma)} \frac{1-\rho^{N+1}}{1-\rho} & \text{if } \rho < 1 \ , \\ \frac{\mathrm{e}^{\sigma x_2} B}{\Gamma(\gamma)} (N+1) & \text{if } \rho = 1 \ , \end{cases} \tag{16}$$

where B is the bound in (13). Eq. (16) shows that in this case the problem of evaluating f_N starting from ψ is well conditioned if and only if $\gamma \geq 1$ and $\rho \approx 1$.

The approximation formulae relating to the LC-methods at the roots of unity are easily obtained by Eqs. (7) and (15). We have

$$c_n = \frac{\rho^{-n}}{N+1} \sum_{m=0}^{N} \psi(\rho \mathrm{e}^{\frac{2\pi \mathrm{i} m}{N+1}}) \mathrm{e}^{\frac{-2\pi \mathrm{i} mn}{N+1}} \ , \qquad n = 0, \dots, N \ , \tag{17}$$

which proves that the methods for inverting a Laplace transform in [10] are precisely LC-methods of order one at the roots of unity [5].

Similar results are valid if $\underline{w}_h$ is ρ times the vector of the $(h+1)$st roots of -1,

$$w_{hk} = \rho \mathrm{e}^{\frac{k+1/2}{h+1} \pi \mathrm{i}} \qquad h = 0, 1, 2, \dots \ ; \ k = 0, \dots, h \ .$$

Any C-method of order $\gamma > 0$ on $\underline{w}$ converges exponentially on $\mathscr{S}_\gamma$, and $k_N(\gamma, \sigma, b)$ satisfies the same inequalities as before.

[5] In [10], Eq.(17) is obtained by approximating the Cauchy integral representation of the derivatives of ψ at the origin, as in the Weeks method. The authors refer to their methods as *modified Weeks' methods*. In spite of these analogies, Weeks' method is not a C-method

6.2. C-methods at the roots of Jacobi polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}$ $(\alpha, \beta > -1;\ n = 0, 1, 2, \ldots)$ satisfy the conditions stated in Corollary 2; thus, with any fixed $\alpha, \beta > -1$, the method $CM(\underline{w}, \gamma)$ converges exponentially on $\mathscr{S}_\gamma$ if $\underline{w}_h$ $(h = 0, 1, 2, \ldots)$ is the vector of the roots of $P_{h+1}^{(\alpha,\beta)}$. Among these real inversion methods are the Piessens' methods [12], which are obtained by expressing the polynomial ℓ_N in terms of Jacobi polynomials. To prove this, let

$$\ell_N(w) = \sum_{n=0}^{N} c_n P_n^{(\alpha,\beta)}(w)$$

be the polynomial which interpolates ψ in (4) at the zeros of $P_{N+1}^{(\alpha,\beta)}$. Then

$$c_n = \frac{1}{H_n} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta \ell_N(x) P_n^{(\alpha,\beta)}(x)\, dx = \frac{1}{H_n} \sum_{k=0}^{N} W_k \psi_k P_n^{(\alpha,\beta)}(w_{Nk}) \ , \tag{18}$$

where

$$H_n = \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta [P_n^{(\alpha,\beta)}(x)]^2\, dx \ ,$$

and the sum in the right-hand side of (18) is the Gauss-Jacobi quadrature formula on $\underline{w}_{N+1}$. Substitution of (18) in III) leads to Piessens' approximation [12, Eq.(9)].

Particularly relevant are the two choices $\alpha = \beta = \mp 1/2$, corresponding to the Chebyshev polynomials of the first and second kind, respectively. In comparison with other numerical real inversion methods, these methods are very accurate on a wide range of functions [3, 13].

We conclude this section by studying the conditioning in the case $\alpha = \beta = -1/2$; similar results are valid for $\alpha = \beta = 1/2$. Therefore, let $\underline{w}_h$ be the vector whose coordinates are the zeros of T_{h+1}, the Chebyshev polynomial of the first kind. The orthogonality properties of the T_h imply that [14]

$$l_m(w) = \frac{2}{N+1} \sum_{h=0}^{N}{}' T_h(w_{Nm}) T_h(w) \ ,$$

so that, if

$$T_h(w) = \sum_{k=0}^{h} t_{hk} w^k \ ,$$

then[6]

$$l_{mn} = \frac{2}{N+1} \sum_{h=n}^{N}{}^{\star} t_{hn} T_h(w_{Nm}) \ . \tag{19}$$

Thus,

$$\sum_{m,n=0}^{N} |l_{mn}| \le \frac{2}{N+1} \sum_{m,n=0}^{N} \sum_{h=n}^{N} |t_{hn}| = 2 \sum_{h=0}^{N} \sum_{n=0}^{h} |t_{hn}| \ .$$

Since [14, p.63]

[6] $\sum_{h=n}^{\star}$ means $\sum_{h=n}'$ if $n = 0$ and $\sum_{h=n}$ otherwise

$$\sum_{n=0}^{h} |t_{hn}| = |T_h(i)| = \frac{1}{2}((1+\sqrt{2})^h + (1-\sqrt{2})^h) \ ,$$

it follows from Theorem 2 that

$$k_N(\gamma,\sigma,b) \leq \frac{\mathrm{e}^{\sigma x_2} B}{\Gamma(\gamma)} \frac{(1+\sqrt{2})^{N+1} - (1-\sqrt{2})^{N+1} - \sqrt{2}}{\sqrt{2}} \ .$$

On the other hand, if Q, R are nonsingular matrices, and $\operatorname{cond} Q$ denotes the *condition number* of Q respect to its *row norm*, then

$$\left|\frac{\varepsilon_R^u}{\varepsilon_R^l}\right| \frac{1}{\operatorname{cond} Q} \leq \operatorname{cond} QR \ ,$$

where ε_R^l and ε_R^u are the lower and upper (in modulus) eigenvalues of R, respectively.

Let V_N be the Vandermonde matrix on $\underline{w}_N$. From (19) it follows that

$$V_N^{-1} = T_N D_N L_N \ ,$$

where $D_N = \frac{1}{N+1}\operatorname{diag}[1,2,\ldots,2]$, $T_N = [T_j(w_{Ni})]$ and $L_N = [t_{ij}]$ $(i,j = 0,\ldots,N)$. The matrix L_N is lower triangular and, again by the orthogonality properties of Chebyshev polynomials, we have

$$(T_N D_N)^{-1} = T_N^{\mathrm{T}} \ .$$

Thus,

$$\operatorname{cond} T_N D_N \leq 2(N+1) \quad , \quad \varepsilon_{L_N}^u = t_{NN} = 2^{N-1} \quad , \quad \varepsilon_{L_N}^l = t_{00} = 1 \ ,$$

and

$$\frac{2^{N-1}}{2(N+1)} \leq \operatorname{cond} V_N \ .$$

In [7] it is proved that the growth of $\operatorname{cond} V(\underline{w}_N)$ is at least $O(2^{(N+1)/2})$ for *any* real vector $\underline{w}_N$ whose coordinates are located symmetrically with respect to the origin. Assuming Gautschi's conjecture [6, 7] about the optimality of symmetric knots for the conditioning of Vandermonde matrices, we may conclude that C-methods of approximating f using only real values of the Laplace transform F are always exponentially ill-conditioned.

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