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An asymptotically optimal finite element scheme for the arch problem

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1. Introducing the problem

Numerical locking has received much attention recently, the most comprehensive study being probably the article of Babuška and Suri [1] where different examples are considered.

We consider here the Kirchhoff-Love approximation to the clamped arch problem. This appears not to be a very difficult one, as we are dealing with a system of ordinary differential equations. However it is more complicated than the beam problem which is studied in [1] and its study may shed some light on more general and complex problems. On the other hand, arch optimization requires pretty robust methods.

Chenais and Paumier [2] have considered a class of problems of which the arch problem is a member. Namely:

$$(\mathrm{P_g})\quad \text{Find } u \in V \quad \text{such that} \quad \forall v \in V,\ \ t^{-1}(Au^t, Av)_X + a_1(u^t, v) = \langle f, v\rangle_{V',V}\ .$$

Here and in what follows, V is a Banach space, V' its dual as usual, X some Hilbert space, A a linear continuous application from V into X and a_1 a continuous bilinear symmetric form on V. Setting:

$$a^t(u, v) = t^{-1}(Au, Av)_X + a_1(u, v)\ ,$$

it is assumed that a^t is uniformly coercive on V for $t \in]0, 1]$. We say that the sequence of linear subspaces V_N approximates V iff:

$$\forall u \in V \lim_{N\to\infty} d_V(u, V_N) = 0\ .$$

Chenais and Paumier give a necessary and sufficient condition for the Ritz-Galerkine procedure associated with such a sequence to converge to u^t uniformly in t for any second member $f \in V'$. Let us denote by G the kernel of A and by G_N its intersection with V_N. The condition is that G_N approximate G.

Let us now specialize this to the arch problem. The arch profile is parametrized by the linear ascissa. Let e denote the thickness of the arch, L its length, $c(s)$ the curvature of the arch profile at abscissa s, the function c is assumed $\mathscr{C}^4[0, L]$. After proper scaling, we set:

$$\begin{aligned}
&t = e^2 \\
&V = H_0^1(0, L) \times H_0^2(0, L) \\
&X = L^2(0, L) \\
&Av = K_0(v_1' + cv_2) \\
&a_1(u, v) = K_1 \int_0^L (u_2' - cu_1)'(v_2' - cv_1)' ds \ ,
\end{aligned}$$

where K_0 and K_1 are positive constants, depending only on the material. Here, the kernel of A is easily described:

$$G = \left\{ (v_1, v_2); v_2 \in H_0^2, \int_0^L c(s)v_2(s)ds = 0 \ , \right.$$
$$\left. v_1(s) = -\int_0^s c(x)v_2(x)dx \right\} \ .$$

We assume from now on that N is the dimension of V_N and that f belongs to some fixed ball of $W = L^2 \times H^{-1}$. Denote by u_N^t the solution of the discrete problem:

$$(P_N) \qquad u_N^t \in V_N, \ \forall v \in V_N at(u_N^t, v) = \langle f, v \rangle_{V',V} \ .$$

Under these assumptions, Chenais and the author [3] have proved that the best order of convergence in both N and t which can be obtained for every $f \in W$ is in $1/N$. Schemes which give uniform convergence in t are known, at least for constant curvature, but one does not know whether they are asymptotically optimal. In this article a finite element scheme will be proposed for more general arch profiles and it will be proved to satisfy the asymptotically optimal estimate:

$$||u^t - u_N^t|| \leq C/N$$

where C is independant of f and t.

2. The scheme

This scheme is directly suggested by the condition of Chenais and Paumier. We first reformulate the problem, taking as our unknown function $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ where:

$$\begin{aligned}
&\tilde{u}_1 = u_1 \\
&\tilde{u}_2 = cu_2 \ .
\end{aligned}$$

Setting:

$$\tilde{a}^t(\tilde{u}, v) = t^{-1}K_0 \int_0^L (\tilde{u}_1' + \tilde{u}_2)(v_1' + v_2)ds) + K_1 \int_0^L ((\tilde{u}_2/c)' - c\tilde{u}_1)'((v_2/c)' - cv_1)'ds \ ,$$

our problem becomes:

$(\tilde{\text{P}}_a)$ Find $\tilde{u} \in V$ such that $\forall v \in V$:

$$\tilde{a}^t(\tilde{u}, v) = \langle f_1, v_1 \rangle_{H^{-1}, H_0^1} + \langle f_2/c, v_2 \rangle_{H^{-2}, H_0^2} \ ..$$

Of course, u is easily recovered from $\tilde{u}$. We shall assume c positive; roughly speaking, this means that the arch profile has no inflexion point. So, for $0 \leq j \leq 4$, both multiplication and division by c are linear continuous applications of H_0^j into itself and this is true for H^j as well. u and $\tilde{u}$ have exactly the same regularity properties and the new problem is as coercive as the initial one.

We divide the interval $[0, L]$ into N subintervals with end points $s_0 = 0 < s_1 < ... < s_N = L$. We assume this sequence of subdivisions quasi-regular in the sense that the product of $h_N = \max(s_j - s_{j-1})$ by N remains bounded.

It remains to define the approximation space $\tilde{V}_N$ for $\tilde{u}$. We set:

$$\tilde{V}_N = V_N^1 \times V_N^2 \ .$$

We take for V_N^2 the usual approximation space: continuously differentiable functions which are polynomials of degree at most three on each subinterval and vanish with their derivative at the end points. For V_N^1, we have to take polynomials of degree at most four on each subinterval in order to satisfy the condition of Chenais and Paumier. We take them continuously differentiable too. This leads us to determine a function v in $\tilde{V}_N$ by $v_1(s_j)$, $v_2(s_j)$, $v_2'(s_j)$ for $j = 1$ to $N-1$, $v_1'(s_j)$ for $j = 0$ to N and $v_1(\frac{s_{j-1}+s_j}{2})$ for $j = 1$ to N. All told, we have $5N - 2$ degrees of freedom.

3. The error estimate

Theorem. *With the notations and under the assumptions of the preceding sections, define* $u_N^t = (\tilde{u}_{N,1}^t, \tilde{u}_{N,2}^t/c)$*, where* $\tilde{u}_N^t$ *is the solution of the problem:*

$(\tilde{\text{P}}_N)$ Find $\tilde{u}_N^t \in \tilde{V}_N$ such that $\forall v \in \tilde{V}_N$,

$$\tilde{a}^t(\tilde{u}_N^t, v) = \langle f_1, v_1 \rangle_{H^{-1}, H_0^1} + \langle f_2/c, v_2 \rangle_{H^{-2}, H_0^2} \ .$$

Then:

$$||u^t - u_N^t||_V \leq C/N \ ,$$

where C *does not depend on* t *nor* f.

Proof. The relation between the u's and the $\tilde{u}$'s shows that it is as good to prove:

$$||\tilde{u}^t - \tilde{u}_N^t||_V \leq C/N \ .$$

It is well known (see e.g. Ciarlet [4]) that, in view of the uniform coerciveness, this amounts to:

$$d_V(\tilde{u}^t, \tilde{V}_N) \le C/N\ .$$

Let us define the following functions and subspaces of V:

$$\tilde{G} = \left\{ (v_1, v_2); v_2 \in H_0^2, \int_0^L v_2(s)ds = 0\ , \quad v_1(s) = -\int_0^s v_2(x)dx \right\},$$

$$\tilde{G}_N = \tilde{G} \cap V_N\ ,$$

$$u^0 \in G \quad \text{such that} \quad \forall v \in G \quad a_1(u^0, v) = \langle f, v\rangle_{V',V}\ ,$$

$$\tilde{u}^0 = (u_1^0, cu_2^0)\ ,$$

$$\tilde{u}_N^0 \in \tilde{G}_N \quad \text{such that} \quad \forall v \in \tilde{G}_N :$$

$$K_1 \int_0^L ((\tilde{u}_2/c)' - c\tilde{u}_1)'((v_2/c)' - cv_1)'ds$$

$$= \langle f_1, v_1\rangle_{H^{-1},H_0^1} + \langle f_2/c, v_2\rangle_{H^{-2},H_0^2}\ .$$

Let us agree that the letter C will denote constants in the sense that they do not depend on either t or f (as long as this last is restricted to a fixed ball of W); these constants may differ from one relation to the next.

The following sections will prove the inequalities:

(1) $$||u^t - u^0|| \le Ct$$

(2) $$d_V(\tilde{u}^0, \tilde{G}_N) \le C/N$$

(3_1) $$||u_1||_{H^2} \le C$$

(3_2) $$u_2 = w_1 + w_2 \text{ with } ||w_1||_{H^3} \le C \text{ and } ||w_2||_{H^4} \le C/t\ .$$

Notice that inequalities (1) and (3) imply the same with u replaced by $\tilde{u}$ everywhere.

We get by (1), this remark and (2):

$$d_V(\tilde{u}^t, \tilde{V}_N) \le C(t + N^{-1})\ .$$

From the $\tilde{u}$ version of the inequalities (3) and standard approximation properties (again see e.g. Ciarlet [4]):

$$d_V(\tilde{u}^t, \tilde{V}_N) \le C(N^{-1} + N^{-2}t^{-1})\ .$$

We complete the proof using the first inequality for $t \le 1/N$ and the last one for $t > 1/N$. □

4. Proof of inequality (1)

u^0 is defined as the Ritz-Galerkine approximation of u^t corresponding to the approximating subspace G (the fact that it is infinite dimensional is immaterial here). From this and the uniform coerciveness of a^t, it follows that:

$$||u^t - u^0||_V \leq C d_V(u^t, G) \ .$$

We now prove that A has a right inverse A^-. Choose a fixed $\phi \in H_0^2$ such that $\int_0^L c(x)\phi(x)dx = 1$. Then for any $\psi \in L^2$, set:

$$A^-\psi = (v_1, v_2) \quad \text{with}$$
$$v_2 = \phi \int_0^L \psi(s)ds \quad \text{and} \quad v_1(s) = \int_0^s [\psi(x) - c(x)v_2(x)]dx \ .$$

You are done.

Let $\psi \in L^2$. We have:

$$(Au^t, \psi)_{L^2} = t[\langle f, A^-\psi\rangle_{V',V} - a_1(u^t, A^-\psi)]$$

from which follow:

$$(Au^t, \psi)_{L^2} \leq Ct||\psi||_{L^2} \quad \text{and} \quad ||Au^t||_{L^2} \leq Ct \ .$$

$d_V(u^t, G)$ is the norm of the image $\overline{u}^t$ of u^t in the quotient space V/G and the operator $\overline{A}$ from this space to L^2 such that $\overline{A}\overline{v} = Av$ is continuously invertible so that:

$$d_V(u^t, V) \leq C||Au^t||_{L^2} \ .$$

Putting together these inequalities, we get (1).

5. Proof of inequality (2)

Define:

$$H = \left\{ w \in H_0^2; \int_0^L w(s)ds = 0 \right\} ,$$
$$H_N = V_N^2 \cap H \ .$$

It is enough to approximate $\tilde{u}_2^0$ by H_N. Indeed, if $w \in H_N$, set $W(s) = \int_0^s w(x)dx$. Then W belongs to V_N^1 and $||\tilde{u}_1^N - W||_{H_0^1} \leq C||\tilde{u}_2^N - w||_{H_0^2}$.

As the restriction of a_1 to G is coercive, u_2^0 is bounded in H^2. As a consequence, u_1^0 is bounded in H^3. As:

$$u_2^{0(4)} = f_2 + (cu_1^0)'''$$

and u_2^0 satisfies Dirichlet boundary conditions, it is bounded in H^3, and so is $\tilde{u}_2^0$. This implies:

(4) $$d_{H_0^2}(\tilde{u}_2^0, V_N^2) \leq C/N \ .$$

Let now w be the orthogonal projection of $\tilde{u}_2^0$ on V_N^2, i the vector in H_0^2 such that:

$$(\phi, i)_{H_0^2} = \int_0^L \phi(s)ds$$

and i_N its orthogonal projection on V_N^2. As V_N^2 approximates H_0^2:

$$\lim_{N\to\infty} i_N = i \quad \text{and} \quad \lim_{N\to\infty} ||i_N||_{H_0^2} = ||i||_{H_0^2} .$$

By (4):

$$\left| \int_0^L w(s)ds \right| \le C/N ,$$

but the distance from w to H_N is

$$\frac{\left|\int_0^L w(s)ds\right|}{||i_N||_{H_0^2}} ;$$

whence finally:

$$d_{H_0^2}(\tilde{u}_2^0, H_N) \le d_{H_0^2}(\tilde{u}_2^0, V_N^2) + d_{H_0^2}(w, H_N) \le C/N$$

which implies (2) as we have seen.

6. Proof of the inequalities (3)

Introduce the auxiliary function

$$y = (u_2^{t\prime} - cu_1^t)' ,$$

we write the differential system satisfied by u^t:

$$-\frac{(Au^t)'}{t} + cy' = f_1 \tag{5}$$

$$\frac{1}{t} cAu^t + y'' = f_2 . \tag{6}$$

In view of the uniform coerciveness, y and Au^t are bounded in L^2, and, to take advantage of (6), we can use the following lemma, to be proved at the end:

Lemma. *There exists a constant c such that if z is bounded in $H^{-1}(]0, L[)$ and $z' = f_0 + f_1$ where $f_0 \in L^2$ and $f_1 \in H^{-1}$ then:*

$$z = z_0 + z_1$$

with $||z_0||_{L^2} \le c(||z||_{H^{-1}} + ||f_1||_{H^{-1}})$ *and* $||z_1||_{H^1} \le c||f_0||_{L^2}$.

So we get:

$$y' = g_0 + g_1 \quad \text{with} \quad ||g_0||_{L^2} \le C \quad \text{and} \quad ||g_1||_{H^1} \le C/t .$$

From the definition of y and the fact that u_1 and u_2' vanish at the end points, it follows:

$$u_2^{t\prime} - cu_1^t = h_1 + h_2 \quad \text{with} \quad ||h_1||_{H^2} \le C \quad \text{and} \quad ||h_2||_{H^3} \le C/t .$$

This and (5) prove (3_1). Finally (3_2) follows by the formula:

$$u_2^t(s) = \int_0^s c(x)u_1^t(x)dx + \int_0^s [u_2^{t\prime}(x) - cu_1^t(x)]dx \ .$$

Proof of the lemma. Define $z_1(x) = \int_0^x f_0(y)dy$ and $z_0 = z - z_1$. We have the inequalities:

$$||z_0||_{H^{-1}} \leq ||z||_{H^{-1}} + c_1||f_0||_{L^2}$$
$$||z_0'||_{H^{-1}} \leq ||z'||_{H^{-1}} + ||f_0||_{L^2} \ .$$

Let $\phi_0 \in \mathscr{D}$ be such that $\int_0^L \phi_0(x)dx = 1$. For any $\phi \in \mathscr{D}$, set:

$$\phi_1 = \phi - \phi_0 \int_0^L \phi(x)dx \ .$$

We have:

$$\int_0^L z_0(x)\phi(x)dx = \int_0^L z_0(x)\phi_0(x)dx \int_0^L \phi(x)dx + \int_0^L z_0(x)\phi_1(x)dx \ .$$

The first term is no larger than $||\phi_0||_{H_0^1} \ ||z_0||_{H^{-1}} \ ||\phi||_{L^2}$; call the second one A. There is a unique $\psi \in H_0^1$ such that $\psi' = \phi_1$ and we have:

$$A = \langle z_0, \psi' \rangle = -\langle z_0', \psi \rangle$$
$$|A| \leq ||\psi||_{H_0^1}||z_0'||_{H^{-1}} \leq c_2||\phi_1||_{L^2}||z_0'||_{H^{-1}} \leq c_3||\phi||_{L^2}||z_0'||_{H^{-1}} \ ;$$

together with the inequalities which we already proved about z_0, this ends the proof. □

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