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Orthogonal spline collocation methods for Schrödinger-type equations in one space variable

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Abstract. We examine the use of orthogonal spline collocation for the semi-discretization of the cubic Schrödinger equation and the two-dimensional parabolic equation of Tappert. In each case, an optimal order L^2 estimate of the error in the semidiscrete approximation is derived. For the cubic Schrödinger equation, we present the results of numerical experiments in which the integration in time is performed using a routine from a software library.

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1. Introduction

In this paper, we examine the use of orthogonal spline collocation, that is, spline collocation at Gauss points, for the semi-discretization of two problems of Schrödinger type. We first consider the initial value problem for the cubic Schrödinger equation

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + q|u|^2u = 0, \quad (x,t) \in (-\infty,\infty) \times (0,T], \\ &u(x,0) = g(x), \quad x \in (-\infty,\infty), \end{aligned} \tag{1.1}$$

where $\mathrm{i}^2 = -1$, q is a given positive constant, the given function $g(x)$ is complex-valued, and $|g(x)| \to 0$ as $|x| \to \infty$. This equation describes many physical phenomena, including the behavior of a plane stationary light beam in a medium with a nonlinear refractive index [33] and the evolution of deep water waves (see [11] and references cited therein).

For the numerical solution of (1.1), one restricts x to a finite interval $[a,b]$ which is chosen so that the modulus of the solution $u(x,t)$ is negligible for x outside $[a,b]$. By imposing homogeneous Dirichlet or homogeneous Neumann boundary conditions at $x = a$ and $x = b$, the pure initial value problem (1.1) is converted into an initial boundary-value problem (IBVP). Since the choice of Neumann or Dirichlet boundary conditions is not significant, we consider the IBVP

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(1.2a) $\mathrm{i}u_t + u_{xx} + q|u|^2u = 0, \quad (x,t) \in (a,b) \times (0,T],$

(1.2b) $u(x,0) = g(x), \quad x \in (a,b),$

(1.2c) $u(a,t) = u(b,t) = 0, \quad t \in [0,T].$

Several numerical studies have examined the use of finite difference methods and finite element methods based on the L^2-Galerkin semi-discretization for solving IBVPs of this form; see, for example, [1, 2, 6, 11, 12, 14, 23–28, 30 and 31]. In [20], both the L^2-Galerkin method and the H^{-1}-Galerkin method were considered.

The second Schrödinger-type equation which we consider is the two-dimensional parabolic equation (PE) of Tappert [29]

$$\rho^{-1}(x)u_t = \frac{\mathrm{i}}{2}k_0\rho^{-1}(x)\left[n^2(x,t) - 1 + i\nu(x,t)\right]u + \frac{\mathrm{i}}{2k_0}(\rho^{-1}(x)u_x)_x, \tag{1.3}$$

which has been used effectively in many problems involving long-range, low-frequency underwater acoustic wave propagation. In (1.3), x and t represent the depth and range, respectively, $u(x,t)$ is the pressure field, k_0 is a given positive constant, $\rho(x)$ is the density, and $n(x,t)$ and $\nu(x,t)$ are prescribed functions. Previous studies of this equation have employed various numerical schemes including split-step Fourier methods, explicit and implicit finite difference schemes, the method of lines with finite-difference semi-discretizations with respect to depth, and finite element Galerkin methods; see, for example, [15] and references therein, [10, 13, 19].

In the present study, we examine the use of orthogonal spline collocation for the discretization with respect to x of (1.2) and (1.3). Numerical methods based on spline collocation have proved to be exceedingly effective for the approximate solution of a broad class of problems, particularly boundary value problems for ordinary differential equations and in the method of lines solution of parabolic and hyperbolic initial-boundary value problems; see [9] for a comprehensive survey. The popularity of collocation methods is due in part to their conceptual simplicity, wide applicability, and ease of implementation. The obvious advantage of spline collocation methods over finite element Galerkin methods is that the calculation of the coefficients in the equations determining the approximate solution is very fast since no integrals need be evaluated or approximated. In comparison with finite difference methods, spline collocation provides approximations to the solution and its derivative with respect to x at all points of the domain of the problem. From the standpoint of solving a nonlinear equation such as the cubic Schrödinger equation, the ease of implementation of the collocation method makes it possible to achieve computational efficiency without resorting to product approximation [5] as is usually done in finite element Galerkin schemes, and enables one to obtain approximations of high order accuracy.

A brief outline of this paper is as follows. In Sect. 2, we introduce definitions, notation, and other preliminaries. In Sect. 3, we describe the continuous-time orthogonal spline collocation method for the solution of (1.2), and derive an optimal order L^2 error estimate. In Sect. 4, we present numerical results which demonstrate the efficacy of the collocation method for the solution of the cubic Schrödinger equation. In Sect. 5, an analysis similar to that of Sect. 3 is given for the collocation solution of the PE of Tappert in horizontally stratified media, in which case the density function $\rho(x)$ is piecewise constant. It is shown that the interface conditions arising in such problems can be incorporated into the collocation procedure in a very natural way.

2. Preliminaries

Unless explicitly stated otherwise, functions considered in this paper are complex-valued. For a given function f, we denote by f_1 and f_2 its real and imaginary parts, respectively. Given any space S of functions, let $\mathfrak{R}(S) = \{g \in S \mid \text{Im } g = 0\}$, that is, the set of all real-valued functions in S.

In the analysis, we assume without loss of generality that $a = 0$ and $b = 1$ in (1.2). We use $(\cdot,\cdot)$ to denote the usual L^2 inner product for complex-valued functions on the interval $I = (0,1)$, namely,

$$(f,g) = \int_0^1 f(x)g^*(x)dx,$$

where * indicates complex conjugation, and let $\|\cdot\|$ denote the corresponding L^2 norm defined by

$$\|f\| = \|f\|_{L^2(I)} = (f,f)^{\frac{1}{2}}.$$

In addition,

$$\|f\|_{L^\infty(I)} = \text{ess} \sup_{x\in I} |f|.$$

For r a nonnegative integer, we denote by

$$\|v\|_{H^r(I)} = \left(\sum_{j=0}^{r} \left\|\frac{d^j v}{dx^j}\right\|^2\right)^{\frac{1}{2}}$$

the norm on the usual Sobolev space $H^r(I)$. If X is a normed space with norm $\|\cdot\|_X$ and $v : [0,T] \to X$, then

$$\|v\|_{L^2(X)} = \left(\int_0^T \|v(t)\|_X^2 dt\right)^{\frac{1}{2}}$$

and

$$\|v\|_{L^\infty(X)} = \text{ess} \sup_{t\in[0,T]} \|v(t)\|_X.$$

Throughout, we use C to denote a generic positive constant whose value is not necessarily the same on each occurrence, and make repeated use of the inequality

$$de \le \varepsilon d^2 + \frac{1}{4\varepsilon}e^2, \quad \varepsilon > 0,$$

for $d, e \in \mathbb{R}$, without explicit mention being made each time.

Given a partition

$$\Delta : 0 = x_1 < x_2 < \dots < x_{N+1} = 1 \tag{2.1}$$

of $\bar{I}$, let $I_j = (x_j, x_{j+1})$, $h_j = x_{j+1} - x_j$, $j = 1,2,\dots,N$, and $h = \max_{1\le j\le N} h_j$. A family $\mathscr{F}$ of partitions is said to be quasi-uniform if there exists a finite positive number σ such that

$$\max_{1\le j\le N} \frac{h}{h_j} \le \sigma$$

for every partition in $\mathscr{F}$ (cf. [8]). We assume that the partition Δ is a member of a quasi-uniform family. We define

$$\mathscr{M}_p(\Delta) = \{v | v \in C^1(\bar{I}); v|_{\bar{I}_j} \in P_p, j = 1, 2, ..., N\} \cap \{v | v(0) = v(1) = 0\},$$

where $p \geq 3$ and P_p denotes the set of all polynomials of degree at most p.

Let $\{\lambda_k\}_{k=1}^{p-1}$ denote the nodes for the $(p-1)$-point Gaussian quadrature rule on the interval I with corresponding weights $\{w_k\}_{k=1}^{p-1}$, $w_k > 0$, and set

$$\lambda_{jk} = x_j + h_j \lambda_k, \ \ j = 1, 2, ..., N, \ \ k = 1, 2, ..., p-1.$$

We define the quadratic form $\{\cdot, \cdot\}$ by

$$\{f, g\} \ = \ \sum_{j=1}^{N} \{f, g\}_j,$$

where

$$\{f, g\}_j \ = \ h_j \sum_{k=1}^{p-1} w_k f(\lambda_{jk}) g^*(\lambda_{jk}),$$

and set

$$|f|_D^2 = \sum_{j=1}^{N} \{f, f\}_j.$$

Since Δ belongs to a quasi-uniform family of partitions, it can be shown using results of [7] and [18] that there exist positive constants $C_1 = C_1(p)$ and $C_2 = C_2(p, \sigma)$ such that, for any $\psi \in \mathscr{M}_p(\Delta)$,

$$C_1 |\psi|_D \leq \|\psi\| \leq C_2 |\psi|_D. \tag{2.2}$$

In the remainder of this section, we introduce notation and some basic results which are used in Sect. 5 when we consider the collocation solution of (1.3) with piecewise constant $\rho(x)$. The corresponding results required in Sect. 3 are easily obtained by setting $\rho = 1$.

We assume that each partition Δ of $\bar{I}$ under consideration contains the points defining the locations of the media interfaces. To simplify the exposition, we define $\rho(x)$ by

$$\rho(x) = \rho_j, \quad x_j < x < x_{j+1}, \quad j = 1, 2, ..., N,$$

where $\rho_j > 0$ for all j, noting, of course, that the only discontinuities in $\rho(x)$ occur at the media interfaces. Then we define

$$\begin{aligned} \mathscr{M}'_p(\Delta) = \quad & \{v | v \in C^0(\bar{I}); v|_{\bar{I}_j} \in P_p, j = 1, 2, ..., N\} \\ & \cap \{v | \rho_{j-1}^{-1} v'(x_j^-) = \rho_j^{-1} v'(x_j^+), j = 1, 2, ..., N\} \cap \{v | v(0) = v(1) = 0\}, \end{aligned}$$

where $p \geq 3$. Note that if $\rho = 1$, then $\mathscr{M}'_p(\Delta) = \mathscr{M}_p(\Delta)$. Hence, any results derived for the space $\mathscr{M}'_p(\Delta)$ also apply to $\mathscr{M}_p(\Delta)$.

Minor extensions of results of [4] and [18] show that there exist positive constants $C_1 = C_1(p)$ and $C_2 = C_2(p, \sigma, \hat{\rho})$, where

$$\hat{\rho} = \max_{j,k} \frac{\rho_j}{\rho_k},$$

such that (2.2) is satisfied for $\psi \in \mathscr{M}'_p(\Delta)$.

The following lemma is required in the analyses (cf. [4, 7]).

Lemma 2.1. *For $f, g \in \Re(\mathscr{M}_p'(\Delta))$,*

$$-\{\rho^{-1} f, g''\} = \int_0^1 \rho^{-1} f' g' dx + \mu_p \sum_{j=1}^{N} \rho_j^{-1} f_j^{(p)} g_j^{(p)} h_j^{2p-1},$$

where $f_j^{(p)}$ (resp. $g_j^{(p)}$) is the constant value of the p^{th} derivative of f (resp. g) on $\overline{I}_j$ and μ_p is a positive constant depending on p.

Proof. The argument given in [4, p.231] yields

$$-\{\rho_j^{-1} f, g''\}_j = \rho_j^{-1} \left(\int_{x_j}^{x_{j+1}} f' g' dx + \mu_p f_j^{(p)} g_j^{(p)} h_j^{2p-1} - f g' |_{x_j^+}^{x_{j+1}^-} \right),$$

from which it follows that

$$-\{\rho^{-1} f, g''\} = \int_0^1 \rho^{-1} f' g' dx + \mu_p \sum_{j=1}^{N} \rho_j^{-1} f_j^{(p)} g_j^{(p)} h_j^{2p-1} - \sum_{j=1}^{N} \rho_j^{-1} f g' |_{x_j^+}^{x_{j+1}^-}.$$

From this we obtain the desired result since the last sum on the right-hand side is zero because of the boundary conditions, the continuity of f, and the interface conditions satisfied by g, namely $\rho_{j-1}^{-1} g'(x_j^-) = \rho_j^{-1} g'(x_j^+)$. □

In our analyses, we use a variant of an interpolation operator introduced in [7], which we now describe. Let $B_p \in \Re(P_{p+1}(I))$ be defined by

$$B_p(x) = \frac{1}{(2p-2)!} \frac{d^{p-3}}{dx^{p-3}} [x^{p-1}(x-1)^{p-1}].$$

The roots of $B_p(x) = 0$ are double roots at $x = 0$ and $x = 1$ and exactly $p - 3$ simple roots at points γ_k such that

$$0 < \gamma_1 < \gamma_2 < \ldots < \gamma_{p-3} < 1.$$

For the partition Δ of (2.1), let

$$\gamma_{jk} = x_j + h_j \gamma_k, \quad j = 1, 2, \ldots, N; \quad k = 1, 2, \ldots, p-3.$$

Let

$$\begin{aligned} PC^1(I) = \quad & \{v | v \in C^0(\overline{I}); v \in C^1(\overline{I}_j),\ j = 1, 2, \ldots, N\} \\ & \cap \{v | \rho_{j-1}^{-1} v'(x_j^-) = \rho_j^{-1} v'(x_j^+),\ j = 2, 3, \ldots, N\} \cap \{v | v(0) = v(1) = 0\}. \end{aligned}$$

We define $I_{p,\Delta}$ to be the interpolation operator from $PC^1(I)$ to $\mathscr{M}_p'(\Delta)$ given by the conditions

(i) $\quad (I_{p,\Delta} v)(x_j) = v(x_j), \quad j = 1, 2, \ldots, N+1,$

(ii) $\quad (I_{p,\Delta} v)(\gamma_{jk}) = v(\gamma_{jk}), \quad j = 1, 2, \ldots, N,\ k = 1, 2, \ldots, p-3,$

(iii) $\quad (I_{p,\Delta} v)'(x_j^+) = v'(x_j^+), \quad j = 1, 2, \ldots, N,$

(iv) $\quad (I_{p,\Delta} v)'(x_j^-) = v'(x_j^-), \quad j = 2, 3, \ldots, N+1,$

for $v \in PC^1(I)$. Since locally on $\bar{I}_j$ this is the same set of conditions satisfied by the interpolation operator defined in [7, p.9], the local estimates of the interpolation error resulting from Lemma 2.2 of that reference are valid here also.

We now prove the following theorem. It should be noted that, in the remainder of the paper, when appropriate, $\|v\|_{H^r(I)}$ is meant in the piecewise sense

$$\|v\|_{H^r(I)} = \left(\sum_{j=1}^{N} \|v\|^2_{H^r(I_j)}\right)^{\frac{1}{2}}.$$

Theorem 2.2. *Let $v \in \Re(PC^1(I))$ such that $v \in H^{p+3}(I_j)$, $j = 1, 2, \ldots, N$, and suppose that $V \in \Re(\mathscr{M}'_p(\Delta))$ satisfies*

$$-(\rho^{-1}V')'(\lambda_{jk}) + \rho^{-1}V(\lambda_{jk}) = -(\rho^{-1}v')'(\lambda_{jk}) + \rho^{-1}v(\lambda_{jk}), \tag{2.3}$$

$j = 1, 2, ..., N$, $k = 1, 2, ..., p-1$. *Then*

$$\|v - V\|_{L^\infty(I)} \leq Ch^{p+1}\|v\|_{H^{p+3}(I)}. \tag{2.4}$$

Proof. From (2.3) it follows that

$$\{\rho^{-1}(v'' - V''), w\} - \{\rho^{-1}(v - V), w\} = 0, \quad w \in \Re(\mathscr{M}'_p(\Delta)). \tag{2.5}$$

If we let

$$\alpha = I_{p,\Delta}v - V, \quad \beta = I_{p,\Delta}v - v, \tag{2.6}$$

and set $w = \alpha$ in (2.5), we have

$$-\{\rho^{-1}\alpha'', \alpha\} + \{\rho^{-1}\alpha, \alpha\} = -\{\rho^{-1}\beta'', \alpha\} + \{\rho^{-1}\beta, \alpha\}. \tag{2.7}$$

Turning first to the left-hand side of (2.7), it is easily shown using Lemma 2.1 that

$$-\{\rho^{-1}\alpha'', \alpha\} \geq C(\rho)\|\alpha'\|^2.$$

Also,

$$\{\rho^{-1}\alpha, \alpha\} \geq C(\rho)|\alpha|^2_D.$$

Therefore

$$-\{\rho^{-1}\alpha'', \alpha\} + \{\rho^{-1}\alpha, \alpha\} \geq K\left[\|\alpha'\|^2 + |\alpha|^2_D\right] \geq K\|\alpha'\|^2, \tag{2.8}$$

where K is a constant depending on ρ.

Using the approach in [7, pp.16-17], it is easy to show that

$$|\{\rho^{-1}\beta'', \alpha\} - \{\rho^{-1}\beta, \alpha\}| \leq \frac{1}{2}K\|\alpha'\|^2 + Ch^{2p+2}\|v\|^2_{H^{p+3}(I)}. \tag{2.9}$$

From (2.7), (2.8) and (2.9), we have

$$K\|\alpha'\|^2 \leq \frac{1}{2}K\|\alpha'\|^2 + Ch^{2p+2}\|v\|^2_{H^{p+3}(I)}, \tag{2.10}$$

from which it follows on using Sobolev's inequality that

$$\|\alpha\|_{L^\infty(I)} \leq Ch^{p+1}\|v\|_{H^{p+3}(I)}. \tag{2.11}$$

As was shown in [7, p.18] using a corollary of the Peano Kernel Theorem, we have

$$\|\beta\|_{L^\infty(I)} \leq Ch^{p+1}\|v\|_{H^{p+2}(I)}. \tag{2.12}$$

From the triangle inequality, (2.11) and (2.12), we obtain (2.4). □

3. The continuous-time orthogonal spline collocation method for the cubic Schrödinger equation

The continuous-time orthogonal spline collocation approximation to the solution u of (1.2) is a differentiable map $U : [0,T] \to \mathscr{M}_p(\Delta)$ such that

$$\mathrm{i}U_t(\lambda_{jk},t) + U_{xx}(\lambda_{jk},t) + q|U(\lambda_{jk},t)|^2U(\lambda_{jk},t) = 0, \quad j = 1,2,...,N, \quad k = 1,2,...,p-1, \tag{3.1}$$

for $t \in (0,T]$. An appropriate initial condition is specified later. An estimate of the error in this approximation is given in the following theorem.

Theorem 3.1. *Let u be the solution to (1.2) such that $u \in L^\infty(H^{p+3})$ and $u_t \in L^2(H^{p+3})$, and let U be the solution to (3.1). Define the differentiable map $W : [0,T] \to \mathscr{M}_p(\Delta)$ by*

$$-W_{xx}(\lambda_{jk},t) + W(\lambda_{jk},t) = -u_{xx}(\lambda_{jk},t) + u(\lambda_{jk},t), \tag{3.2}$$

$j = 1,2,...,N,\ k = 1,2,...,p-1$. *If*

$$\|(U-W)(0)\|_{L^\infty(I)} \le Ch^{p+1}, \tag{3.3}$$

then, for h sufficiently small,

$$\|u-U\|_{L^\infty(L^2)} \le C\{\|(U-W)(0)\| + h^{p+1}[\|u\|_{L^\infty(H^{p+3})} + \|u_t\|_{L^2(H^{p+3})}]\}. \tag{3.4}$$

Proof. Let $\xi = U - W$ and $\eta = u - W$. We assume *a priori* that, for h sufficiently small,

$$\|\xi(t)\|_{L^\infty(I)} \le h, \quad t \in [0,T]. \tag{3.5}$$

This can be justified by a continuity argument (cf. [32]), and is discussed at the end of the proof. From the smoothness assumptions on u, it follows that

$$\|u\|_{L^\infty(L^\infty)} \le C, \tag{3.6}$$

and from (3.2) and Theorem 2.2 we have $\|\eta\|_{L^\infty(L^\infty)} \le C$. Then, from the triangle inequality and (3.5), it follows that

$$\|U\|_{L^\infty(L^\infty)} \le C. \tag{3.7}$$

It is shown in [21] that (3.1) is equivalent to the discrete Galerkin method

$$\mathrm{i}\{U_t,v\} + \{U_{xx},v\} + q\{|U|^2U,v\} = 0, \quad v \in \mathscr{M}_p(\Delta). \tag{3.8}$$

Then from (1.2a), (3.2) and (3.8), we obtain

$$\{\xi_t,\xi\} - \mathrm{i}\{\xi_{xx},\xi\} = \mathrm{i}q\{|U|^2U - |u|^2u,\xi\} + \{\eta_t,\xi\} - \mathrm{i}\{\eta,\xi\}. \tag{3.9}$$

First note that

$$\mathrm{Rl}\{\xi_t,\xi\} = \frac{1}{2}\frac{d}{dt}|\xi|_D^2. \tag{3.10}$$

Secondly,

$$-\mathrm{i}\{\xi_{xx},\xi\} = \{(\xi_2)_{xx},\xi_1\} - \{(\xi_1)_{xx},\xi_2\} - \mathrm{i}[\{(\xi_1)_{xx},\xi_1\} + \{(\xi_2)_{xx},\xi_2\}].$$

Then, from Lemma 2.1, we have

$$\{(\xi_2)_{xx},\xi_1\} = -((\xi_2)_x,(\xi_1)_x) - \mu_p \sum_{j=1}^{N} \left(\frac{\partial^p \xi_2}{\partial x^p}\right)_j \left(\frac{\partial^p \xi_1}{\partial x^p}\right)_j h_j^{2p-1} \tag{3.11}$$

and

$$\{(\xi_1)_{xx},\xi_2\} = -((\xi_1)_x,(\xi_2)_x) - \mu_p \sum_{j=1}^{N} \left(\frac{\partial^p \xi_1}{\partial x^p}\right)_j \left(\frac{\partial^p \xi_2}{\partial x^p}\right)_j h_j^{2p-1}, \tag{3.12}$$

where $\left(\frac{\partial^p \xi_l}{\partial x^p}\right)_j$, $l = 1, 2$, is the constant value of the p^{th} partial derivative of ξ_l with respect to x on $\bar{I}_j$ at time t. It follows from (3.11) and (3.12) that

$$\{(\xi_2)_{xx},\xi_1\} \;=\; \{(\xi_1)_{xx},\xi_2\},$$

and as a consequence $-\mathrm{i}\{\xi_{xx},\xi\}$ is pure imaginary. Thus, taking the real part of (3.9) and using (3.10), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\xi|_D^2 &= q[\{|u|^2u_2 - |U|^2U_2,\xi_1\} - \{|u|^2u_1 - |U|^2U_1,\xi_2\}] \\ &\quad + \{(\eta_1)_t,\xi_1\} + \{(\eta_2)_t,\xi_2\} + \{\eta_2,\xi_1\} - \{\eta_1,\xi_2\}. \end{aligned} \tag{3.13}$$

To estimate the first term on the right hand side of (3.13), we use the Schwarz inequality to obtain

$$\begin{aligned} &q[\{|u|^2u_2 - |U|^2U_2,\xi_1\} - \{|u|^2u_1 - |U|^2U_1,\xi_2\}] \\ &\le C\left[\sum_{l=1}^{2} \big|\, |u|^2u_l - |U|^2U_l \,\big|_D^2 + |\xi|_D^2\right]. \end{aligned} \tag{3.14}$$

Using the triangle inequality, (3.6) and (3.7), we have, for $l = 1, 2$,

$$\begin{aligned} \big|\, |u|^2u_l - |U|^2U_l \,\big|_D &\le \big|\, |u|^2u_l - |U|^2u_l \,\big|_D + \big||U|^2u_l - |U|^2U_l\big|_D \\ &\le \|u\|_{L^\infty(L^\infty)} \big|\, |u|^2 - |U|^2 \,\big|_D + \|U\|^2_{L^\infty(L^\infty)} |u_l - U_l|_D \\ &\le C\{\big|\, |u|^2 - |U|^2 \,\big|_D + |u_l - U_l|_D\}, \end{aligned}$$

which gives

$$\sum_{l=1}^{2} \big|\, |u|^2u_l - |U|^2U_l \,\big|_D^2 \le C\{\big|\, |u|^2 - |U|^2 \,\big|_D^2 + |u - U|_D^2\}. \tag{3.15}$$

Using (3.6) and (3.7), it is easy to show that

$$\big|\, |u|^2 - |U|^2 \,\big|_D^2 \leq C|u - U|_D^2. \tag{3.16}$$

On combining (3.14), (3.15) and (3.16), it follows that

$$\begin{aligned} q[\{|u|^2 u_2 - |U|^2 U_2, \xi_1\} - \{|u|^2 u_1 - |U|^2 U_1, \xi_2\}] &\leq C[|u - U|_D^2 + |\xi|_D^2] \\ &\leq C[|\eta|_D^2 + |\xi|_D^2]. \end{aligned} \tag{3.17}$$

On substituting (3.17) in (3.13) and using the Schwarz inequality, we have

$$\frac{d}{dt}|\xi|_D^2 \leq C\{|\eta|_D^2 + |\eta_t|_D^2 + |\xi|_D^2\}. \tag{3.18}$$

Integrating (3.18) over the interval $[0, t_1]$, where $0 < t_1 \leq T$, we obtain

$$|\xi(t_1)|_D^2 \leq |\xi(0)|_D^2 + C\left[\int_0^{t_1} [|\eta|_D^2 + |\eta_t|_D^2]dt + \int_0^{t_1} |\xi(t)|_D^2 dt\right].$$

The use of the Gronwall inequality then gives

$$|\xi(t_1)|_D^2 \leq C\left\{|\xi(0)|_D^2 + \left[\int_0^{T} [|\eta|_D^2 + |\eta_t|_D^2]dt\right]\right\}. \tag{3.19}$$

We now turn our attention to the terms $|\eta|_D^2$ and $|\eta_t|_D^2$. From (3.2) it follows that, for any $t \in [0, T]$, $W_l \in \Re(\mathscr{M}_p(\Delta))$ satisfies

$$-(W_l)_{xx}(\lambda_{jk}, t) + W_l(\lambda_{jk}, t) = -(u_l)_{xx}(\lambda_{jk}, t) + u_l(\lambda_{jk}, t),$$

$l = 1, 2$. Theorem 2.2 then implies that

$$\|\eta_l(t)\|_{L^\infty(I)} \leq Ch^{p+1}\|u_l\|_{H^{p+3}(I)},$$

from which we may conclude that

$$\|\eta(t)\|_{L^\infty(I)} \leq Ch^{p+1}\|u\|_{H^{p+3}(I)}, \tag{3.20}$$

for any $t \in [0, T]$. From (3.20), it follows that

$$\|\eta\|_{L^\infty(L^2)} \leq Ch^{p+1}\|u\|_{L^\infty(H^{p+3})} \tag{3.21}$$

and

$$|\eta(t)|_D \leq Ch^{p+1}\|u\|_{H^{p+3}(I)}. \tag{3.22}$$

Also it is easy to show in a similar fashion that

$$|\eta_t(t)|_D \leq Ch^{p+1}\|u_t\|_{H^{p+3}(I)}. \tag{3.23}$$

From (3.19), (3.22) and (3.23), we obtain

$$|\xi(t_1)|_D^2 \leq C\{|\xi(0)|_D^2 + h^{2p+2}[\|u\|_{L^2(H^{p+3})}^2 + \|u_t\|_{L^2(H^{p+3})}^2]\},$$

and using (2.2),

$$\|\xi\|_{L^\infty(L^2)}^2 \leq C\{\|\xi(0)\|^2 + h^{2p+2}[\|u\|_{L^2(H^{p+3})}^2 + \|u_t\|_{L^2(H^{p+3})}^2]\}. \tag{3.24}$$

The triangle inequality and the estimates (3.21) and (3.24) yield the desired result (3.4).

At this point we show that the *a priori* assumption (3.5) is satisfied for h sufficiently small, using the inverse property

$$\|\psi\|_{L^\infty(I)} \le Ch^{-1/2}\|\psi\|, \quad \psi \in \mathscr{M}_p(\Delta); \tag{3.25}$$

cf. [32]. If h is sufficiently small, we can assume that (3.5) is satisfied for small $t > 0$ since, from (3.3), $\|\xi(0)\|_{L^\infty(I)} \le Ch^{p+1}$ and since (3.1) is a finite system of ordinary differential equations for the coefficients of U. We now show that the failure of (3.5) for some $t \in (0, T]$ leads to a contradiction. For, suppose (3.5) does fail for some $t \in (0, T]$. Let

$$\tau = \inf\{t \in (0,T] \mid (3.5) \text{ fails to hold}\}.$$

Then by continuity, the inverse assumption (3.25), and (3.24),

$$h = \|\xi(\tau)\|_{L^\infty(I)} \le Ch^{-1/2}\|\xi(\tau)\| \le Ch^{p+1/2},$$

which, since $p \ge 3$, produces a contradiction if h is sufficiently small. Thus the assumption (3.5) is valid for $t \in [0, T]$. $\square$

In this theorem, we have assumed that the initial approximation $U(0)$ is such that $\|(U - W)(0)\|_{L^\infty(I)} \le Ch^{p+1}$. One suitable choice of $U(0)$ is the interpolant $I_{p,\Delta}g$ of the initial data g, since from (2.6) and (2.11), it follows that

$$\|(U - W)(0)\|_{L^\infty(I)} \le Ch^{p+1}\|g\|_{H^{p+3}(I)}. \tag{3.26}$$

Zakharov and Shabat [33] have shown that solutions of (1.1) satisfy an infinite number of conservation laws, the two most commonly cited implying that

$$L = \int_{-\infty}^{\infty} |u|^2 dx, \tag{3.27}$$

the squared L^2 norm of the solution, and

$$H = \int_{-\infty}^{\infty} \left(|u_x|^2 - \frac{1}{2}q|u|^4\right) dx, \tag{3.28}$$

the Hamiltonian, are conserved. The collocation solution U satisfies two conservation laws corresponding to those in (3.27) and (3.28). Taking $v = U$ in (3.8) and rearranging, we obtain

$$\{U_t, U\} = \mathrm{i}\{U_{xx}, U\} + \mathrm{i}q\{|U|^2U, U\}. \tag{3.29}$$

Taking the real part of (3.29) and using an argument similar to that employed in the proof of Theorem 3.1, we obtain

$$\frac{d}{dt}|U|_D^2 = 0. \tag{3.30}$$

This conservation of the squared discrete norm of the approximate solution is a discrete analog of the conservation of the squared L^2 norm of the exact solution, (3.27). On the other hand, taking $v = U_t$ in (3.8), we obtain

$$|U_t|_D^2 = \mathrm{i}\{U_{xx}, U_t\} + \mathrm{i}q\{|U|^2U, U_t\}. \tag{3.31}$$

Using Lemma 2.1, we see that

$$\text{Im}[\mathrm{i}\{U_{xx}, U_t\}] = -\frac{1}{2}\frac{d}{dt}\left[\|U_x\|^2 + \chi_h\right], \tag{3.32}$$

where

$$\chi_h = \mu_p \sum_{j=1}^{N} \left[\left(\frac{\partial^p U_1}{\partial x^p}\right)_j + \left(\frac{\partial^p U_2}{\partial x^p}\right)_j \right] h_j^{2p-1}.$$

Direct calculation shows that

$$\text{Im}[\mathrm{i}q\{|U|^2U, U_t\}] = \frac{1}{4}q\frac{d}{dt}\left||U|^2\right|_D^2. \tag{3.33}$$

Therefore, taking the imaginary part of (3.31) and using (3.32) and (3.33), we obtain

$$\frac{d}{dt}\left[\|U_x\|^2 - \frac{1}{2}q\left||U|^2\right|_D^2 + \chi_h\right] = 0,$$

a discrete analog of the conservation law corresponding to (3.28).

4. Numerical experiments

In this section, we present numerical results obtained when the IBVP (1.2) is solved using the orthogonal spline collocation method with $p = 3$. In this case,

$$\lambda_1 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right), \quad \lambda_2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right).$$

We define the partition

$$\Delta : a = x_1 < x_2 < \dots < x_{N+1} = b$$

of $[a, b]$, with $I_j = (x_j, x_{j+1})$ and $h_j = x_{j+1} - x_j$, $j = 1, 2, \dots, N$, and let

$$\lambda_{jk} = x_j + h_j\lambda_k, \quad j = 1, 2, \dots, N, \quad k = 1,\ 2.$$

The approximate solution $U : [0, T] \to \mathscr{M}_3(\Delta)$ is expressed using a monomial representation (cf. [3, 17]). In this formulation,

$$U(x, t) = \sum_{l=1}^{4} \frac{U_{jl}(t)(x - x_j)^{l-1}}{(l-1)!}, \quad x \in \overline{I}_j, \tag{4.1}$$

where

$$\begin{aligned} &U_{j,1}(t) = U(x_j, t), \quad U_{j,2}(t) = U_x(x_j, t), \\ &U_{j,3}(t) = U_{xx}(x_j^+, t), \quad U_{j,4}(t) = U_{xxx}(x_j^+, t), \end{aligned} \tag{4.2}$$

$j = 1, 2, \dots, N$. We also define

$$U_{N+1,1}(t) = U(x_{N+1}, t), \quad U_{N+1,2}(t) = U_x(x_{N+1}, t). \tag{4.3}$$

The coefficients $U_{jl}(t)$ are complex-valued and are written in the form

$$U_{jl}(t) = V_{jl}(t) + iW_{jl}(t).$$

For notational convenience, henceforth we do not indicate the t-dependence of the coefficients explicitly; for example, we write U_{jl} instead of $U_{jl}(t)$.

We take the Hermite cubic interpolant $I_{3,\Delta}g$ of the given function g as the initial approximation $U(0)$. The requirement that the approximate solution satisfy the partial differential equation at the collocation points yields the equations

$$\text{(4.4)}\qquad \begin{aligned} \sum_{l=1}^{4} \frac{\dot{U}_{jl}(h_j\lambda_k)^{l-1}}{(l-1)!} &= \mathrm{i}\left[U_{j,3} + h_j\lambda_k U_{j,4}\right] \\ &+\mathrm{i}q\left|\sum_{l=1}^{4} \frac{U_{jl}(h_j\lambda_k)^{l-1}}{(l-1)!}\right|^2 \sum_{l=1}^{4} \frac{U_{jl}(h_j\lambda_k)^{l-1}}{(l-1)!}, \end{aligned}$$

$j = 1, 2, \ldots, N$, $k = 1, 2$, where a dot denotes differentiation with respect to t. The C^1 continuity of U requires that

$$\text{(4.5a)}\qquad -U_{j,1} - h_j U_{j,2} - \frac{h_j^2}{2} U_{j,3} - \frac{h_j^3}{6} U_{j,4} + U_{j+1,1} = 0, \quad j = 1, 2, \ldots, N,$$

and

$$\text{(4.5b)}\qquad -U_{j,2} - h_j U_{j,3} - \frac{h_j^2}{2} U_{j,4} + U_{j+1,2} = 0, \quad j = 1, 2, \ldots, N.$$

Finally, the boundary conditions yield

$$\text{(4.5c)}\qquad U_{1,1} = 0, \quad U_{N+1,1} = 0.$$

Combining equations (4.4) and (4.5), we obtain a differential-algebraic system

$$\text{(4.6)}\qquad D\dot{\mathbf{U}}(t) = E\mathbf{U}(t) + q\mathbf{F}(\mathbf{U}(t))$$

of order $8N + 4$, with

$$\mathbf{U} = [(\mathbf{U}_1^1)^{\mathrm{T}}, (\mathbf{U}_1^2)^{\mathrm{T}}, (\mathbf{U}_2^1)^{\mathrm{T}}, (\mathbf{U}_2^2)^{\mathrm{T}}, \ldots, (\mathbf{U}_N^1)^{\mathrm{T}}, (\mathbf{U}_N^2)^{\mathrm{T}}, (\mathbf{U}_{N+1}^1)^{\mathrm{T}}]^{\mathrm{T}}$$

and

$$\mathbf{F} = [\mathbf{0}_2^{\mathrm{T}}, \mathbf{F}_1^{\mathrm{T}}, \mathbf{0}_4^{\mathrm{T}}, \mathbf{F}_2^{\mathrm{T}}, \mathbf{0}_4^{\mathrm{T}}, \ldots, \mathbf{F}_N^{\mathrm{T}}, \mathbf{0}_4^{\mathrm{T}}, \mathbf{0}_2^{\mathrm{T}}]^{\mathrm{T}},$$

where

$$\mathbf{U}_j^1 = [V_{j,1}, W_{j,1}, V_{j,2}, W_{j,2}]^{\mathrm{T}}, \quad \mathbf{U}_j^2 = [V_{j,3}, W_{j,3}, V_{j,4}, W_{j,4}]^{\mathrm{T}},$$

$$\mathbf{0}_4 = [0, 0, 0, 0]^{\mathrm{T}}, \quad \mathbf{0}_2 = [0, 0]^{\mathrm{T}},$$

and

$$\mathbf{F}_j = [-(\hat{V}_{j,1}^2 + \hat{W}_{j,1}^2)\hat{W}_{j,1}, (\hat{V}_{j,1}^2 + \hat{W}_{j,1}^2)\hat{V}_{j,1}, -(\hat{V}_{j,2}^2 + \hat{W}_{j,2}^2)\hat{W}_{j,2}, (\hat{V}_{j,2}^2 + \hat{W}_{j,2}^2)\hat{V}_{j,2}]^{\mathrm{T}},$$

with

$$\hat{V}_{jk} = \sum_{l=1}^{4} \frac{V_{jl}(h_j\lambda_k)^{l-1}}{(l-1)!}, \quad \hat{W}_{jk} = \sum_{l=1}^{4} \frac{W_{jl}(h_j\lambda_k)^{l-1}}{(l-1)!},$$

$k = 1, 2$, and D and E are almost block diagonal matrices which are described in detail in [21].

For the solution of the differential-algebraic system (4.6), we employ the code D02NNF from the NAG Library Mark 13, which has been used successfully in the solution of the cubic Schrödinger equation by Galerkin methods [20, 23]. D02NNF is a general purpose routine for integrating the initial value problem for a stiff system of implicit differential equations coupled with algebraic equations of the form

$$A(t, \mathbf{y})\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}).$$

The time stepping is done using backward differentiation formula (BDF) integrators. Prior to calling D02NNF, calls are made to the appropriate linear algebra setup routine and integrator setup routine. Since D02NNF cannot handle almost block diagonal systems directly, the coefficient matrices D and E and the Jacobian of the function **F** are considered by the code as banded with upper bandwidth 5 and lower bandwidth 6. In each case, the linear algebra setup routine D02NTF is used. The integrator setup routine used is D02NVF, which implements the family of BDF integrators. All computations were performed on the University of Kentucky's IBM 3090-600J in double precision using its vectorization facilities.

In this numerical study, we consider two examples often studied in the literature, namely, a bound state of two solitons ($q = 8$) and a bound state of three solitons ($q = 18$), in which the initial condition is

$$g(x) = \operatorname{sech}(x).$$

For these problems, the conserved quantities in (3.27) and (3.28) have the values

$$L = 2, \quad H = \frac{2}{3}(1 - q), \tag{4.7}$$

respectively. Since the solution of each of these problems is an even function of x and negligible far from $x = 0$, we allow for the use of a graded mesh in our calculations. All of our numerical experiments were carried out on the region $[-20, 20] \times [0, 2.5]$, and on the interval $[-20, 20]$ we impose the mesh defined by

$$x_0 = 0, \quad x_j = 20\left(\frac{j}{J}\right)^{\kappa} = -x_{-j}, \quad j = 1, 2, \ldots, J. \tag{4.8}$$

Here, $\kappa \geq 1$ is a parameter which determines the degree of non-uniformity of the mesh. If $\kappa = 1$, the mesh is uniform, and as κ increases the mesh becomes less uniform, being finest around $x = 0$ and becoming progressively coarser as the endpoints of the interval are approached.

In all of the experiments discussed here, the parameters RTOL and ATOL, tolerances used in a mixed relative and absolute local error test in D02NNF, are set at 10^{-5} and 10^{-8}, respectively, and the maximum order of the integrator is set at 5. In all cases, output is obtained from D02NNF at the time levels

$$t_k = 0.05k, \quad k = 1, 2, \ldots, 50.$$

In each example, we graph the modulus of the approximate solution and the quantities L' and H', approximations to the two conserved quantities L and H defined by

$$L' = \int_{-20}^{20} |U|^2 dx \quad \text{and} \quad H' = \int_{-20}^{20} \left(|U_x|^2 - \frac{1}{2} q |U|^4 \right) dx,$$

which are computed exactly using seven-point Gauss quadrature. For comparison, the exact values of L and H from (4.7) are graphed on the same set of axes as L' and H'. For the $q = 8$ case, we use Gauss quadrature and the exact solution

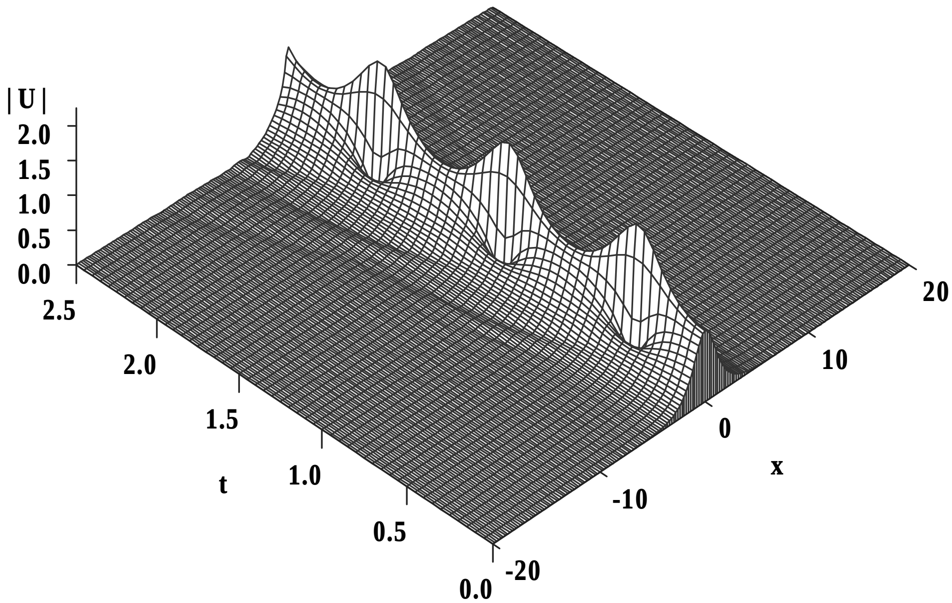

Fig. 1. Bound state of two solitons, $J = 40$, $\kappa = 1$: graph of $|U|$

$$u(x,t) = \mathrm{e}^{\mathrm{i}t}\operatorname{sech}(x) \left[\frac{1 + \frac{3}{4}\operatorname{sech}^2(x)(\mathrm{e}^{8\mathrm{i}t} - 1)}{1 - \frac{3}{4}\operatorname{sech}^4(x)\sin^2(4t)} \right],$$

given in [16], to estimate the L^2 error in our approximations.

We first consider the $q = 8$ case, a bound state of two solitons. When using (4.8) with $J = 40$ and $\kappa = 1$ (a uniform partition of 80 subintervals), we obtain the results shown in Figs. 1–3. The graph of $|U|$, Fig. 1, exhibits the correct general qualitative behavior but displays downstream ripples away from the central spine which the exact solution does not possess. In Fig. 2, while the behavior of L' is quite good, H' deviates noticeably from H in the vicinity of the periodic spikes in the graph of $|U|$. The occurrence of these spikes also coincides with significant jumps in the L^2 error (Fig. 3); this error is also quite large.

When we use the same number of grid points but with $\kappa = 1.5$ in order to concentrate the grid points close to $x = 0$, the graphs of $|U|$, L', and H' are indistinguishable from their theoretical counterparts, and are not presented here. The L^2 error, while still exhibiting jumps associated with the occurrence of the spikes in the graph of $|U|$, is reduced by two orders of magnitude (Fig. 4).

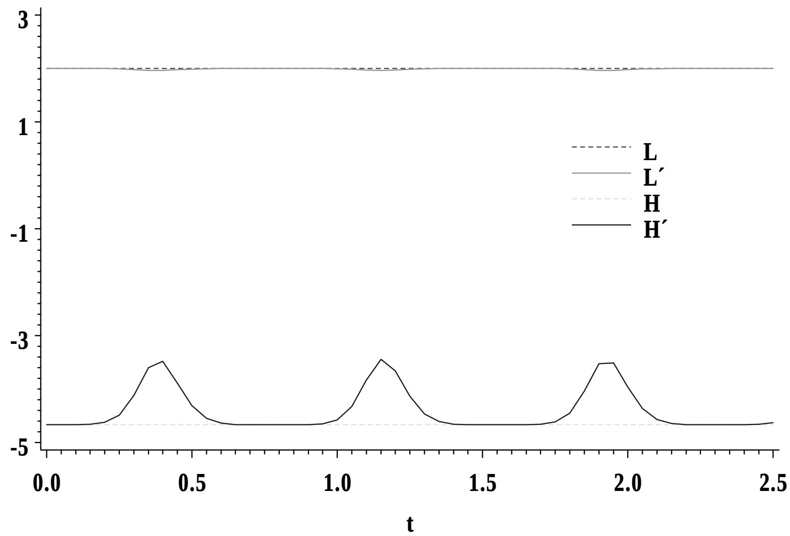

Fig. 2. Bound state of two solitons, J = 40, κ = 1: graphs of conserved quantities and approximations

We next turn to the q = 18 case, which presents a more stringent test of our numerical scheme. Using J = 80 and κ = 1 (a uniform partition with 160 subintervals), we obtain unsatisfactory results. In Fig. 5, it is shown that the time-periodic behavior of the solution has broken down by t = 2, and, in Fig. 6, the graph of H' exhibits sharp deviations from H. Using J = 80 but increasing κ to 1.2 produces a dramatic improvement. The correct time-periodic behavior is shown by the graph of $|U|$ in Fig. 7; the graphs of L' and H' are not presented as they are essentially identical to those of L and H, respectively. These results are comparable to those obtained in [20] with the L^2-Galerkin method with continuous piecewise linear basis functions and the H^{-1}-Galerkin method with discontinuous piecewise linear functions using far more mesh points. Of course, while part of this improvement may be attributed to the use of piecewise cubics, the utilization of the graded mesh feature is clearly a major factor also. Increasing κ to 1.3 produces results which are essentially indentical to those obtained using κ = 1.2.

5. Orthogonal spline collocation for the PE of Tappert

In this section, we consider (1.3), which we write in the form

$$\rho^{-1}(x)u_t - \rho^{-1}(x)\mathscr{A}(x,t)u - \frac{\mathrm{i}}{2k_0}(\rho^{-1}(x)u_x)_x = 0, \tag{5.1}$$

where

$$\mathscr{A}(x,t) = \frac{\mathrm{i}}{2}k_0\left[n^2(x,t) - 1 + \mathrm{i}\nu(x,t)\right].$$

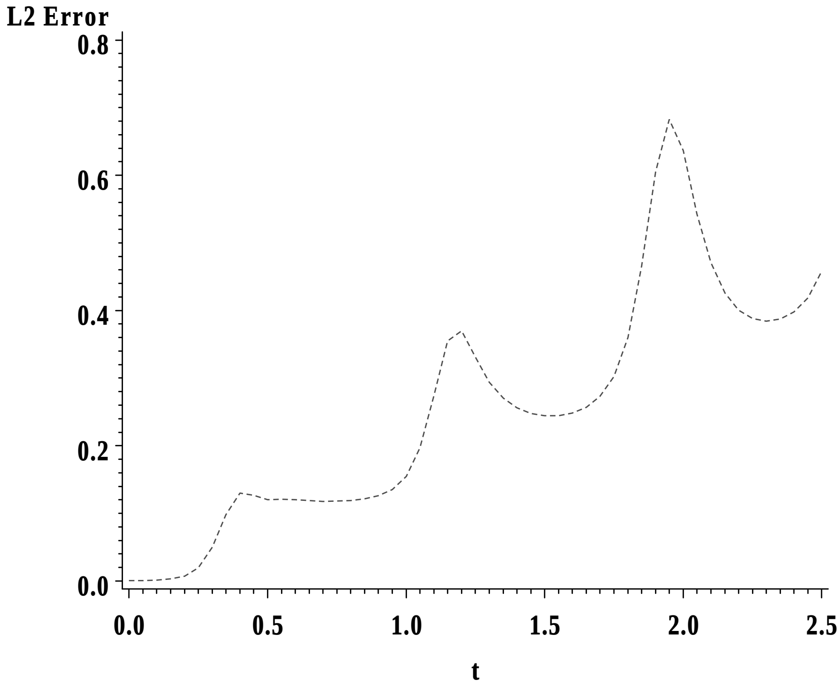

Fig. 3. Bound state of two solitons, $J = 40$, $\kappa = 1$: graph of L^2 error

The domain of the problem is the rectangular region $[0, x_B] \times [t_0, T]$ composed of M horizontally stratified layers with top boundary at depth $D_1 = 0$, bottom boundary at depth $D_{M+1} = x_B$, and interfaces at depths D_j, $2 \le j \le M$, where

$$0 = D_1 < D_2 < \ldots < D_M < D_{M+1} = x_B.$$

We assume that the density function $\rho = \rho(x)$ is piecewise constant such that

$$\rho(x) = \rho_j, \quad D_j < x < D_{j+1}, \quad j = 1, 2, \ldots, M.$$

A homogeneous Dirichlet boundary condition is customarily imposed at the top boundary, $x = 0$. Since the solution is negligible near the bottom boundary, $x = x_B$, the choice of a Dirichlet or Neumann bottom boundary condition is not critical, and we assume in this analysis that a homogeneous Dirichlet boundary condition is imposed. Also, we assume without loss of generality that $x_B = 1$ and $t_0 = 0$. Thus we consider the IBVP

(5.2a) $$\rho^{-1} u_t = \rho^{-1} \mathscr{A} u + \frac{\mathrm{i}}{2k_0} (\rho^{-1} u_x)_x, \quad x \in \bigcup_{j=1}^{M} (D_j, D_{j+1}),\ t \in (0, T],$$

(5.2b) $$u(x, 0) = g(x), \quad x \in (0, 1),$$

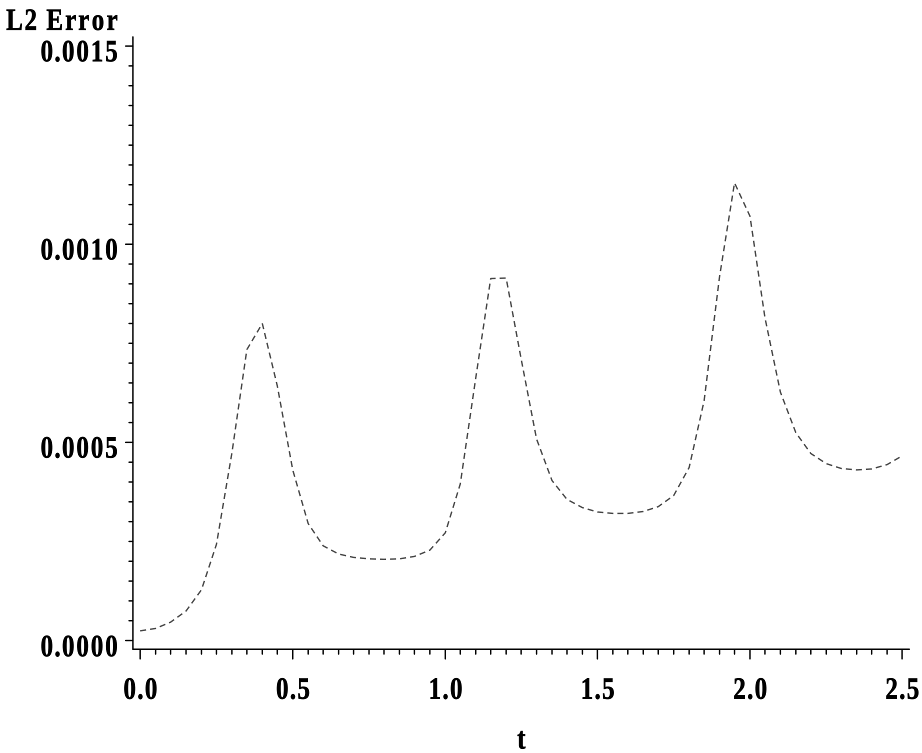

Fig. 4. Bound state of two solitons, $J = 40$, $\kappa = 1.5$: graph of L^2 error

(5.2c) $$u(0,t) = 0, \quad u(1,t) = 0, \quad t \in [0,T],$$

(5.2d) $$u(D_j^-,t) = u(D_j^+,t), \quad 2 \le j \le M, \quad t \in (0,T],$$

(5.2e) $$\frac{1}{\rho_{j-1}} u_x(D_j^-,t) = \frac{1}{\rho_j} u_x(D_j^+,t), \quad 2 \le j \le M, \quad t \in (0,T].$$

Equations (5.2d) and (5.2e) express the continuity of the solution and of the normal component of the particle velocity at the interface between two media.

To describe the collocation method, we introduce a quasi-uniform partition (2.1) of $[0,1]$ containing the points D_j, $j = 1,2,\dots,M+1$, which define the locations of the media interfaces. The continuous-time collocation approximation to the solution u of (5.2) is a differentiable map $U : [0,T] \to \mathscr{M}'_p(\Delta)$ such that

(5.3) $$\rho(\lambda_{jl})^{-1} U_t(\lambda_{jl},t) = \rho(\lambda_{jl})^{-1} \mathscr{A}(\lambda_{jl},t) U(\lambda_{jl},t) + \frac{\mathrm{i}}{2k_0} (\rho^{-1} U_x)_x(\lambda_{jl},t), \quad j = 1,2,\dots,N, \quad k = 1,2,\dots,p-1,$$

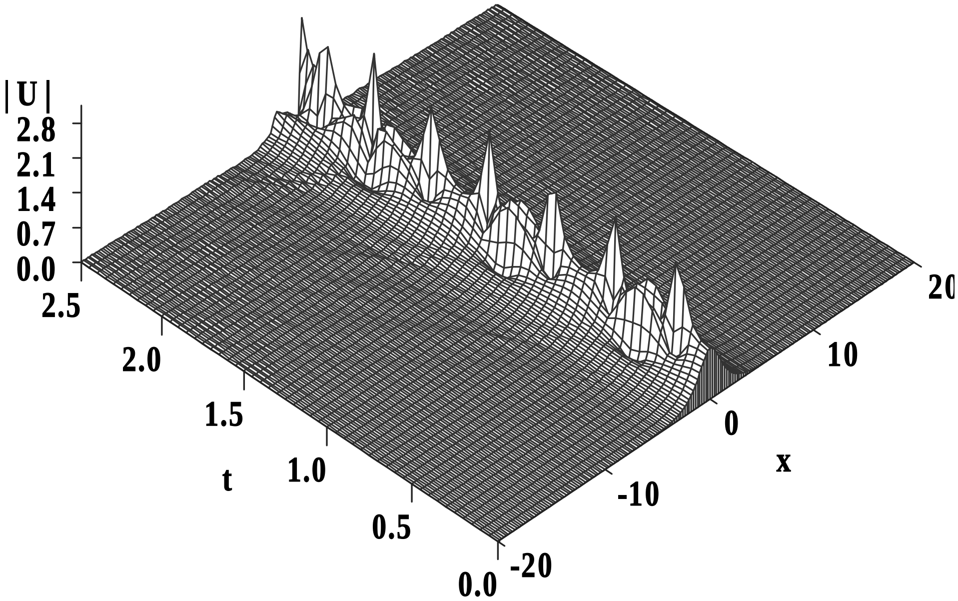

Fig. 5. Bound state of three solitons, $J = 80$, $\kappa = 1$: graph of $|U|$

for $t \in (0, T]$. An appropriate initial condition is specified later. The error estimate for the continuous-time collocation approximation defined by (5.3) is given in the following theorem.

Theorem 5.1. *Let u be the solution to (5.2) such that $u \in L^\infty(H^{p+3}(I_j))$ and $u_t \in L^2(H^{p+3}(I_j))$, $j = 1, 2, \ldots, N$, and let U be the solution to (5.3). Define the differentiable map $W : [0, T] \to \mathscr{M}'_p(\Delta)$ by*

(5.4) $-(\rho^{-1}W_x)_x(\lambda_{jk}, t) + \rho^{-1}W(\lambda_{jk}, t) = -(\rho^{-1}u_x)_x(\lambda_{jk}, t) + \rho^{-1}u(\lambda_{jk}, t),$

$j = 1, 2, ..., N,\ k = 1, 2, ..., p-1$. *Then*

(5.5) $\|u - U\|_{L^\infty(L^2)} \leq C\{\|(U - W)(0)\| + h^{p+1}[\|u\|_{L^\infty(H^{p+3})} + \|u_t\|_{L^2(H^{p+3})}]\}.$

Proof. As before, to estimate $u - U$, we bound $\xi = U - W$ in terms of $\eta = u - W$ and use the triangle inequality.

The proof parallels that of Theorem 3.1. First note that the collocation method (5.3) is equivalent to the discrete Galerkin method

(5.6) $$\{\rho^{-1}U_t, v\} = \{\rho^{-1}\mathscr{A}U, v\} + \frac{\mathrm{i}}{2k_0}\{(\rho^{-1}U_x)_x, v\}, \quad v \in \mathscr{M}'_p(\Delta).$$

Also, from (5.4),

(5.7) $$\{\rho^{-1}\eta_{xx}, v\} = \{\rho^{-1}\eta, v\}, \quad v \in \mathscr{M}'_p(\Delta).$$

Then from (5.2a), (5.6) and (5.7), we obtain, on setting $v = \xi$,

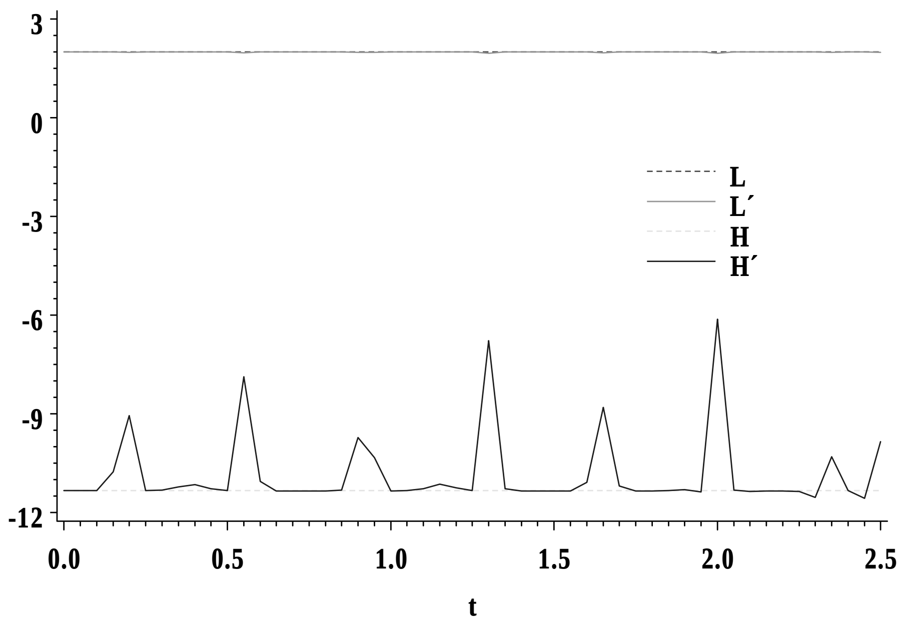

Fig. 6. Bound state of three solitons, J = 80, κ = 1: graphs of conserved quantities and approximations

(5.8)
$$\begin{aligned}&\{\rho^{-1}\xi_t,\xi\}-\{\rho^{-1}\mathscr{A}\xi,\xi\}-\frac{\mathrm{i}}{2k_0}\{\rho^{-1}\xi_{xx},\xi\}\\&\quad=\{\rho^{-1}\eta_t,\xi\}-\{\mathscr{A}\rho^{-1}\eta,\xi\}-\frac{\mathrm{i}}{2k_0}\{\rho^{-1}\eta,\xi\}.\end{aligned}$$

If we proceed as in the proof of Theorem 3.1 and take the real part of (5.8), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\rho^{-1/2}\xi|_D^2-\{\rho^{-1}\mathscr{A}_1\xi_1,\xi_1\}-\{\rho^{-1}\mathscr{A}_1\xi_2,\xi_2\}\\&\quad=\{\rho^{-1}(\eta_1)_t,\xi_1\}+\{\rho^{-1}(\eta_2)_t,\xi_2\}-\{\rho^{-1}\mathscr{A}_1\eta_1,\xi_1\}+\{\rho^{-1}\mathscr{A}_2\eta_2,\xi_1\}\\&\qquad-\{\rho^{-1}\mathscr{A}_1\eta_2,\xi_2\}-\{\rho^{-1}\mathscr{A}_2\eta_1,\xi_2\}+\frac{1}{2k_0}[\{\rho^{-1}\eta_2,\xi_1\}-\{\rho^{-1}\eta_1,\xi_2\}].\end{aligned}$$

Using the Schwarz inequality and the fact that $\|\rho^{-1}\|_{L^\infty(I)}$ and $\|\mathscr{A}\|_{L^\infty(L^\infty)}$ are bounded, it is easy to show that

$$\frac{d}{dt}|\rho^{-1/2}\xi|_D^2\le C[|\eta|_D^2+|\eta_t|_D^2+|\xi|_D^2].$$

From Theorem 2.2 it follows that

$$|\eta|_D^2\le Ch^{2p+2}\|u\|^2_{H^{p+3}(I)},\quad |\eta_t|_D^2\le Ch^{2p+2}\|u_t\|^2_{H^{p+3}(I)},$$

and

$$\|\eta\|^2_{L^\infty(L^2)}\le Ch^{2p+2}\|u\|^2_{L^\infty(H^{p+3})}.$$

Using these estimates and the Gronwall inequality, we obtain

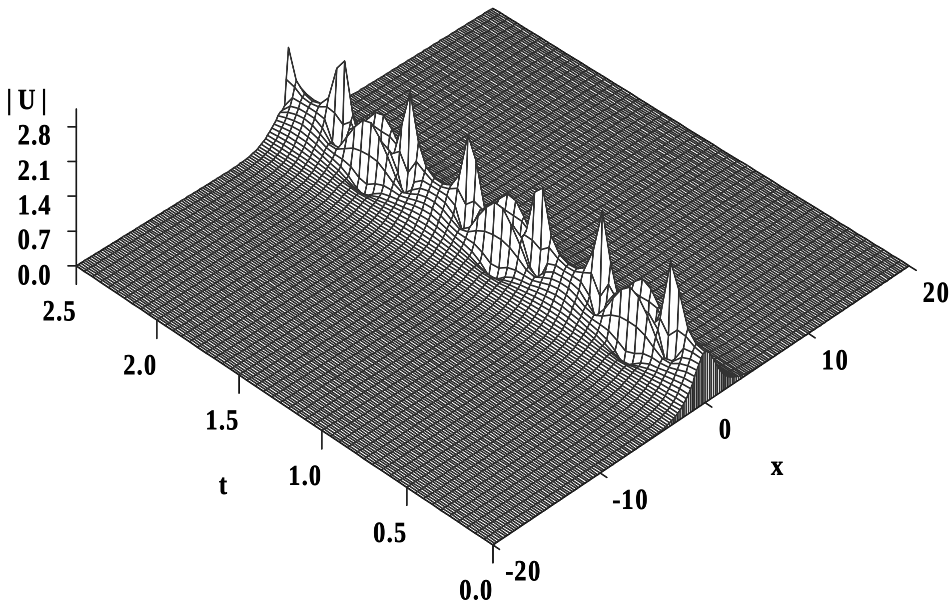

Fig. 7. Bound state of three solitons, $J = 80$, $\kappa = 1.2$: graph of $|U|$

$$\|\xi\|^2_{L^\infty(L^2)} \leq C\{\|\xi(0)\|^2 + h^{2p+2}[\|u\|^2_{L^2(H^{p+3})} + \|u_t\|^2_{L^2(H^{p+3})}]\}.$$

The final result (5.5) then follows on using the triangle inequality. □

Note that if we take $U(0) = W(0)$, Theorem 5.1 provides us with an error estimate which is $O(h^{p+1})$.

The results of numerical experiments using the orthogonal spline collocation method to solve test problems from the underwater acoustics literature are presented in [22].

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