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The boundary integro-differential equations of three-dimensional Neumann problem in linear elasticity

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Summary. In this paper, we mainly consider the three dimensional Neumann problem in linear elasticity, which is reduced to a system of integro-differential equations on the boundary based on a new representation of the derivatives of the double-layer potential. Furthermore a new boundary finite element method for this Neumann problem is presented.

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1. Introduction

We mainly consider the three dimensional Neumann problems in linear elasticity in this paper.

Let Ω denote a bounded domain in $\mathbb{R}^3$ with boundary Σ, which is a sufficiently smooth closed surface. Ω_c denotes the unbounded domain with boundary Σ. We now recall a few of notions of linear elasticity [4]. Let $u(x) = (u_1(x), u_2(x), u_3(x))^T$ be the displacement vector u and $\sigma_{ij}(u)$ be the stress field corresponding to the displacement vector u. The stress-displacement relationships for an isotropic elastic material are

$$\sigma_{ij}(u) = \lambda\delta_{ij}\mathrm{div}u + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad 1 \le i, j \le 3, \tag{1.1}$$

where λ and μ are Lame constants, δ_{ij} is the Kronecker delta. The strain tensor ε_{ij} in terms of displacements is given by

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad 1 \le i, j \le 3, \tag{1.2}$$

For $x \in \mathbb{R}^3$ and an arbitrary unit vector $n(x) = (n_1(x), n_2(x), n_3(x))^T$ the matrix differential operator

$$T(\partial_x, n(x)) = (T_{ij}(\partial_x, n(x))_{3\times 3} \tag{1.3}$$

is defined by

$$T_{ij}(\partial_x, n(x)) = \lambda n_i(x)\frac{\partial}{\partial x_j} + \mu n_j(x)\frac{\partial}{\partial x_i} + \mu\delta_{ij}\frac{\partial}{\partial n(x)}, \quad 1 \le i, j \le 3. \tag{1.4}$$

$T(\partial_x, n(x))$ is called the stress operator. $T(\partial_x, n(x))u(x)$ denotes the stress at the point x along the direction $n(x)$ in terms of the displacement $u(x)$. Furthermore we introduce the differential operator

$$\mathrm{U}(\partial_x, n(x)) = (\mathrm{U}_{ij}(\partial_x, n(x))_{3\times 3} \tag{1.5}$$

with

$$\mathrm{U}_{ij}(\partial_x, n(x)) = n_j(x)\frac{\partial}{\partial x_i} - n_i(x)\frac{\partial}{\partial x_j}, \qquad 1 \le i, j \le 3. \tag{1.6}$$

Now giving i and j in (1.6) the values 1,2 and 3, we can write out the following operators

$$\begin{aligned}
&\mathrm{U}_{11}(\partial_x, n(x)) = \mathrm{U}_{22}(\partial_x, n(x)) = \mathrm{U}_{33}(\partial_x, n(x)) \equiv 0,\\
&\mathrm{U}_{32}(\partial_x, n(x)) = -\mathrm{U}_{23}(\partial_x, n(x)) = n_2(x)\frac{\partial}{\partial x_3} - n_3(x)\frac{\partial}{\partial x_2} \equiv \frac{\partial}{\partial S_1(x)},\\
&\mathrm{U}_{13}(\partial_x, n(x)) = -\mathrm{U}_{31}(\partial_x, n(x)) = n_3(x)\frac{\partial}{\partial x_1} - n_1(x)\frac{\partial}{\partial x_3} \equiv \frac{\partial}{\partial S_2(x)},\\
&\mathrm{U}_{21}(\partial_x, n(x)) = -\mathrm{U}_{12}(\partial_x, n(x)) = n_1(x)\frac{\partial}{\partial x_2} - n_2(x)\frac{\partial}{\partial x_1} \equiv \frac{\partial}{\partial S_3(x)}.
\end{aligned}$$

Here $S_1(x) = (0, -n_3(x), n_2(x))^{\mathrm{T}}$, $S_2(x) = (n_3(x), 0, -n_1(x))^{\mathrm{T}}$, and $S_3(x) = (-n_2(x), n_1(x), 0)^{\mathrm{T}}$, which are perpendicular to the vector $n(x)$. $\mathrm{U}(\partial_x, n(x))$ is an antisymmetrical matrix differential operator.

The equilibrium equations without the body force in elastostatics are

$$\sum_{j=1}^{3}\frac{\partial\sigma_{ij}(u)}{\partial x_j} = 0, \qquad i = 1, 2, 3 \tag{1.7}$$

or

$$\mu\Delta u + (\lambda + \mu)\ \mathrm{grad\ div}\ u = 0 \tag{1.8}$$

in terms of displacements.

We now consider the following Neumann problems:

$$\begin{aligned}
&\mu\Delta u + (\lambda + \mu)\ \mathrm{grad\ div}\ u = 0, \qquad \text{in} \quad \Omega,\\
&T(\partial_x, n(x))u = g, \qquad \text{on} \quad \Sigma
\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}
&\mu\Delta u + (\lambda + \mu)\ \mathrm{grad\ div}\ u = 0, \qquad \text{in} \quad \Omega_{\mathrm{c}},\\
&T(\partial_x, n(x))u = g, \qquad \text{on} \quad \Sigma
\end{aligned}$$

$$(1.10) \qquad u(x) \to 0, \qquad \text{when} \quad |x| \to +\infty,$$

where $n(x)$ denotes the unit outward normal vector on Σ for the domain Ω and $g = (g_1, g_2, g_3)^{\mathrm{T}}$ is a given vector valued function satisfying

$$(1.11) \qquad \int_\Sigma g(x)dS = 0 \qquad \text{and} \qquad \int_\Sigma x \times g(x)dS = 0.$$

Under the condition (1.11), the external problem (1.10) has a unique solution and the internal problem (1.9) has a unique solution apart from a difference of a rigid displacement [4, P.312].

Let $\Gamma(x-y)$ denote the fundamental solution matrix of the Navier system (1.8), then we have

$$\Gamma(x-y) \equiv (\Gamma^1(x-y), \Gamma^2(x-y), \Gamma^3(x-y)) \equiv (\Gamma_{ij}(x-y))_{3\times 3}$$

with

$$\Gamma_{ij}(x-y) = \frac{1}{8\pi\mu(\lambda+2\mu)} \left\{ (\lambda+3\mu)\frac{\delta_{ij}}{|x-y|} + (\lambda+\mu)\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3} \right\}.$$

The matrix $\Gamma(x)$ is symmetric and every column and row of $\Gamma(x)$ satisfy the equations (1.8) at every point $x \in \mathbb{R}^3$, except the origin. Furthermore we note that

$$(1.12) \qquad \operatorname{div} \Gamma^i(x) = \frac{1}{4(\lambda+\mu)} \frac{\partial}{\partial x_i} \frac{1}{|x|}, \qquad i = 1, 2, 3,$$

$$(1.13) \qquad \frac{\partial \Gamma(x-y)}{\partial x_i} = -\frac{\partial \Gamma(x-y)}{\partial y_i}, \qquad i = 1, 2, 3.$$

Consider the double-layer potential

$$(1.14) \qquad u(x, \varphi) = \int_\Sigma (T(\partial_y, n(y))\Gamma(x-y))^{\mathrm{T}} \varphi(y) dS_y,$$

where $n(y)$ denotes the unit outward normal vector at $y \in \Sigma$ for the domain Ω and $\varphi = (\varphi_1, \varphi_2, \varphi_3)^{\mathrm{T}}$ is a vector valued function on Σ to be determined. We know that for a given $\varphi(x) \in C^{1,\beta}(\Sigma), u(x)$ is a solution of (1.8) [4]. For every $x \notin \Sigma$ and an arbitrary unit vector $n(x)$, we have

$$(1.15) \quad T(\partial_x, n(x))u = \int_\Sigma T(\partial_x, n(x))(T(\partial_y, n(y))\Gamma(x-y))^{\mathrm{T}} \varphi(y) dS_y, \qquad \forall x \notin \Sigma.$$

For $x \in \Sigma$, the kernel in the integral of (2.2) has a singularity which is of order $|x-y|^{-3}$ when x and y are close. Thus the integral in the right-hand side of (2.2) is defined as a finite part in the sense of Hadamard. The double-layer potential (2.1) has been used to reduce the Neumann problems (1.9) and (1.10) to the boundary integral equation with non integrable kernels [6],

$$(1.16) \qquad \int_\Sigma T(\partial_x, n(x))(T(\partial_y, n(y))\Gamma(x-y))^{\mathrm{T}} \varphi(y) dS_y = g(x), \quad \forall x \in \Sigma$$

assuming that the following formulas hold:

$$\lim_{x_0\in\Omega\to x\in\Sigma}\int_\Sigma T(\partial_x,n(x))(T(\partial_y,n(y))\Gamma(x_0-y))^{\mathrm{T}}\varphi(y)dS_y$$
$$=\int_\Sigma T(\partial_x,n(x))(T(\partial_y,n(y))\Gamma(x-y))^{\mathrm{T}}\varphi(y)dS_y,\quad \forall x\in\Sigma$$

and

$$\lim_{x_0\in\Omega_c\to x\in\Sigma}\int_\Sigma T(\partial_x,n(x))(T(\partial_y,n(y))\Gamma(x_0-y))^{\mathrm{T}}\varphi(y)dS_y$$
$$=\int_\Sigma T(\partial_x,n(x))(T(\partial_y,n(y))\Gamma(x-y))^{\mathrm{T}}\varphi(y)dS_y,$$
$$\forall x\in\Sigma\,. \tag{1.17}$$

From the mathematical point of view it is not obvious that equalities (1.17) hold. Also, the integral equation (1.16) is quite difficult to approximate because of the hypersingular kernel. Hence a modified formulation of the hypersingular integral (1.16) using weakly singular integrals was given by Nedelec [6], based on equalities (1.17) again. But his formula is extremely complicated. It involve a tensor of order 4 which is a sum of 9 terms and up to 6 derivative have to be applied to these terms. It therefore remains an open problem to derive an integral equation for the equations of three-dimensional elasticity with Neumann boundary conditions which is convenient for numercial wish. These problems are important, for example, in crack propagation.

In this paper, a new representation of the derivative of the double-layer potential will be presented. Based on this new representation, Neumann problems (1.9) and (1.10) will be reduced to a system of boundary integro-differential equations, which is equivalent to the hypersingular integral equation (1.16) given by Nedelec [6], under the assumption that equalities (1.17) hold. This system of boundary integro-differential equations is a much simpler formula for solving Neumann problems (1.9) and (1.10) that uses a weakly singular kernal and tangential derivatives of the test and trial functions. The derivatives of the test and trial functions can be easily expressed using a given parametrization of the surface. This new representation of the derivative of double layer potential is of high practical importance, especially for crack problems and coupling of finite element method and boundary element method. Similar representations of the derivative of double layer potentials in two dimensional case can be found in [2,3]. Finally, we mention that as a consequence of our new representation, the limits that occur on the left side (1.17) exist.

2. A new representation of the derivative of double-layer potential

In this section, a new representation of (1.15) will be presented, which will play an important role throughout this paper. A computation shows [4, p. 282]

Lemma 2.1.

$$(T(\partial_y,n(y))\Gamma(x-y))^{\mathrm{T}}=2\mu(\mathrm{U}(\partial_y,n(y))\Gamma(x-y))^{\mathrm{T}}$$
$$+\frac{1}{4\pi}\left(I\frac{\partial}{\partial n(y)}\frac{1}{|x-y|}+\mathrm{U}(\partial_y,n(y))\frac{1}{|x-y|}\right), \tag{2.1}$$

where I is the unit matrix.

Substituting (2.1) into (1.15) and applying Stokes Theorem, we obtain [4, p. 283]:

Lemma 2.2. *The double-layer potential (1.14) may be expressed in the following form:*

$$u(x,\varphi) = \frac{1}{4\pi}\int_{\Sigma}\varphi\frac{\partial}{\partial n(y)}\frac{1}{|x-y|}dS_y - \frac{1}{4\pi}\int_{\Sigma}\frac{1}{|x-y|}(\mathrm{U}(\partial_y, n(y))\varphi(y))dS_y$$
$$+ 2\mu\int_{\Sigma}\Gamma(x-y)(\mathrm{U}(\partial_y, n(y))\varphi(y))dS_y. \tag{2.2}$$

In (2.2) the double-layer potential $u(x,\varphi)$ is expressed as a sum of the harmonic double-layer potential, the harmonic single-layer potential and the single-layer potential. Based on the representation (2.2), we calculate $T(\partial_x, n(x))u(x,\varphi)$ for $x \notin \Sigma$. We have

$$T(\partial_x, n(x))u(x,\varphi) = \frac{1}{4\pi}\int_{\Sigma}\left(T(\partial_x, n(x))I\frac{\partial}{\partial n(y)}\frac{1}{|x-y|}\right)\varphi(y)dS_y$$
$$+ \int_{\Sigma}T(\partial_x, n(x))\left(2\mu\Gamma(x-y) - \frac{1}{4\pi}\frac{I}{|x-y|}\right)\mathrm{U}(\partial_y, n(y))\varphi(y)dS_y$$
$$\equiv I_0 + J. \qquad \forall x \notin \Sigma \tag{2.3}$$

By the definition of $T(\partial_x, n(x))$, we get

$$I_0 = \frac{\mu}{4\pi}\int_{\Sigma}\left(\frac{\partial^2}{\partial n(x)\partial n(y)}\frac{1}{|x-y|}\right)\varphi(y)dS_y$$
$$+ \frac{\lambda+\mu}{4\pi}\int_{\Sigma}n(x)\left(\left(\mathrm{grad}_x\frac{\partial}{\partial n(y)}\frac{1}{|x-y|}\right)\cdot\varphi(y)\right)dS_y$$
$$+ \frac{\mu}{4\pi}\int_{\Sigma}\mathrm{U}(\partial_x, n(x))\frac{\partial}{\partial n(y)}\frac{1}{|x-y|}\varphi(y)dS_y. \tag{2.4}$$

By Lemma 2.1 we have

$$T(\partial_x, n(x))\Gamma(x-y) = 2\mu\mathrm{U}(\partial_x, n(x))\Gamma(x-y)$$
$$+ \frac{1}{4\pi}\left(I\frac{\partial}{\partial n(x)}\frac{1}{|x-y|} - \mathrm{U}(\partial_x, n(x))\frac{1}{|x-y|}\right). \tag{2.5}$$

Substituting (2.5) into J and using the definition of $T(\partial_x, n(x))$ we obtain

$$J = \int_{\Sigma}\mathrm{U}(\partial_x, n(x))\left(4\mu^2\Gamma(x-y) - \frac{3\mu}{4\pi}\frac{I}{|x-y|}\right)\mathrm{U}(\partial_y, n(y))\varphi(y)dS_y$$
$$- \frac{\lambda+\mu}{4\pi}\int_{\Sigma}n(x)\left(\mathrm{grad}_x\frac{1}{|x-y|}\cdot\mathrm{U}(\partial_y, n(y))\varphi(y)\right)dS_y$$
$$+ \frac{\mu}{4\pi}\int_{\Sigma}\frac{\partial}{\partial n(x)}\frac{1}{|x-y|}\mathrm{U}(\partial_y, n(y))\varphi(y)dS_y. \tag{2.6}$$

Thus we have

$$T(\partial_x, n(x))u(x,\varphi) = \frac{\mu}{4\pi}\int_\Sigma \frac{\partial^2}{\partial n(x)\partial n(y)}\frac{1}{|x-y|}\varphi(y)dS_y$$
$$+\int_\Sigma \mathrm{U}(\partial_x, n(x))\left(4\mu^2\Gamma(x-y) - \frac{3\mu}{4\pi}\frac{I}{|x-y|}\right)\mathrm{U}(\partial_y, n(y))\varphi(y)dS_y$$
$$+ E_1 + E_2, \tag{2.7}$$

with

$$E_1 = \frac{\lambda+\mu}{4\pi}\int_\Sigma n(x)\left\{(\mathrm{grad}_x\frac{\partial}{\partial n(y)}\frac{1}{|x-y|}\cdot\varphi(y) - \mathrm{grad}_x\frac{1}{|x-y|}\cdot\mathrm{U}(\partial_y, n(y))\varphi(y)\right\}dS_y,$$
$$E_2 = \frac{\mu}{4\pi}\int_\Sigma\left\{(\mathrm{U}(\partial_x, n(x))\frac{\partial}{\partial n(y)}\frac{1}{|x-y|}\varphi(y) + \frac{\partial}{\partial n(x)}\frac{1}{|x-y|}\mathrm{U}(\partial_y, n(y))\varphi(y)\right\}dS_y.$$

For the further simplication of (2.7), we have the following lemma

Lemma 2.3. *The following equalities hold for $x \neq y$*

$$\frac{\partial^2}{\partial n(x)\partial n(y)}\frac{1}{|x-y|} = -\sum_{k=1}^{3}\frac{\partial^2}{\partial S_k(x)\partial S_k(y)}\frac{1}{|x-y|} \tag{2.8}$$

$$\mathrm{grad}_x\frac{\partial}{\partial n(y)}\frac{1}{|x-y|} = \mathrm{U}(\partial_y, n(y))\mathrm{grad}_x\frac{1}{|x-y|} \tag{2.9}$$

$$\mathrm{U}(\partial_x, n(x))\frac{\partial}{\partial n(y)}\frac{1}{|x-y|} - \mathrm{U}(\partial_y, n(x))\frac{\partial}{\partial n(x)}\frac{1}{|x-y|}$$
$$= \left\{\mathrm{U}(\partial_y, n(y))\mathrm{U}(\partial_x, n(x)) - \mathrm{U}(\partial_x, n(x))\mathrm{U}(\partial_y, n(y))\right\}\frac{1}{|x-y|} \tag{2.10}$$

Proof. The proof of (2.8) can be found in [4]. To prove (2.9), let

$$\mathrm{U}(\partial_y, n(y))\mathrm{grad}_x\frac{1}{|x-y|} = (H_1, H_2, H_3)^{\mathrm{T}},$$

then we have

$$\begin{aligned} H_i &= \sum_{l=1}^{3}\mathrm{U}_{il}(\partial_y, n(y))\frac{\partial}{\partial x_l}\frac{1}{|x-y|} \\ &= \sum_{l=1}^{3}\left(n_l(y)\frac{\partial}{\partial y_i} - n_i(y)\frac{\partial}{\partial y_l}\right)\frac{\partial}{\partial x_l}\frac{1}{|x-y|} \\ &= \sum_{l=1}^{3}n_l(y)\frac{\partial^2}{\partial x_i\partial y_l}\frac{1}{|x-y|} - n_i(y)\sum_{l=1}^{3}\frac{\partial^2}{\partial x_l\partial y_l}\frac{1}{|x-y|} \\ &= \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial n(y)}\frac{1}{|x-y|}\right), \qquad i = 1,2,3. \end{aligned}$$

The equality (2.9) is proved completely. To prove (2.10), let

$$-\{\mathrm{U}(\partial_y, n(y))\mathrm{U}(\partial_x, n(x)) - \mathrm{U}(\partial_x, n(x))\mathrm{U}(\partial_y, n(y))\} \frac{1}{|x-y|} = (H_{ij})_{3\times 3},$$

then we obtain

$$\begin{aligned}
H_{ij} &= \left\{ \sum_{l=1}^{3} \mathrm{U}_{il}(\partial_x, n(x))\mathrm{U}_{lj}(\partial_y, n(y)) - \sum_{l=1}^{3} \mathrm{U}_{il}(\partial_y, n(y))\mathrm{U}_{lj}(\partial_x, n(x)) \right\} \frac{1}{|x-y|} \\
&= \sum_{l=1}^{3} \left(n_l(x) \frac{\partial}{\partial x_i} - n_i(x) \frac{\partial}{\partial x_l} \right) \left(n_j(y) \frac{\partial}{\partial y_l} - n_l(y) \frac{\partial}{\partial y_j} \right) \frac{1}{|x-y|} \\
&- \sum_{l=1}^{3} \left(n_l(y) \frac{\partial}{\partial y_i} - n_i(y) \frac{\partial}{\partial y_l} \right) \left(n_j(x) \frac{\partial}{\partial x_l} - n_l(x) \frac{\partial}{\partial x_j} \right) \frac{1}{|x-y|} \\
&= \sum_{l=1}^{3} n_l(x) n_j(y) \frac{\partial^2}{\partial x_i \partial y_l} \frac{1}{|x-y|} + \sum_{l=1}^{3} n_i(x) n_l(y) \frac{\partial^2}{\partial x_l \partial y_j} \frac{1}{|x-y|} \\
&- \sum_{l=1}^{3} n_l(y) n_j(x) \frac{\partial^2}{\partial y_i \partial x_l} \frac{1}{|x-y|} - \sum_{l=1}^{3} n_i(y) n_l(x) \frac{\partial^2}{\partial y_l \partial x_j} \frac{1}{|x-y|} \\
&= n_j(y) \frac{\partial}{\partial y_i} \frac{\partial}{\partial n(x)} \frac{1}{|x-y|} + n_i(x) \frac{\partial}{\partial x_j} \frac{\partial}{\partial n(y)} \frac{1}{|x-y|} \\
&- n_j(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial n(y)} \frac{1}{|x-y|} - n_i(y) \frac{\partial}{\partial y_j} \frac{\partial}{\partial n(x)} \frac{1}{|x-y|} \\
&= \mathrm{U}_{ij}(\partial_y, n(y)) \frac{\partial}{\partial n(x)} \frac{1}{|x-y|} - \mathrm{U}_{ij}(\partial_x, n(x)) \frac{\partial}{\partial n(y)} \frac{1}{|x-y|}, \qquad 1 \le i, j \le 3.
\end{aligned}$$

The equality (2.10) follows immediately. □

Now we return to simplify (2.7). From the equality (2.9) we have

$$\begin{aligned}
&\int_{\Sigma} \mathrm{grad}_x \frac{\partial}{\partial n(y)} \frac{1}{|x-y|} \cdot \varphi(y) dS_y \\
&= \int_{\Sigma} \mathrm{U}(\partial_y, n(y)) \mathrm{grad}_x \frac{1}{|x-y|} \cdot \varphi(y) dS_y \\
&= \int_{\Sigma} \sum_{i,j=1}^{3} \left(\mathrm{U}_{ij}(\partial_y, n(y)) \frac{\partial}{\partial x_j} \frac{1}{|x-y|} \right) \varphi_i(y) dS_y \\
&= - \int_{\Sigma} \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \frac{1}{|x-y|} \left(\sum_{i=1}^{3} \mathrm{U}_{ij}(\partial_y, n(y)) \varphi_i(y) \right) dS_y \\
&= \int_{\Sigma} \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \frac{1}{|x-y|} \left(\sum_{i=1}^{3} \mathrm{U}_{ji}(\partial_y, n(y)) \varphi_i(y) \right) dS_y \\
&= \int_{\Sigma} \mathrm{grad}_x \frac{1}{|x-y|} \cdot \mathrm{U}(\partial_y, n(y)) \varphi(y) dS_y
\end{aligned}$$

Hence we have $E_1 = 0$. An application of (2.10) yields

$$E_2 = \frac{\mu}{4\pi}\int_\Sigma \left\{ (\mathrm{U}\,(\partial_x, n(x))\,\frac{\partial}{\partial n(y)}\frac{1}{|x-y|})\varphi(y) - (\mathrm{U}\,\big(\partial_y, n(y)\big)\,\frac{\partial}{\partial n(x)}\frac{1}{|x-y|})\varphi(y)\right\} dS_y$$
$$= \frac{\mu}{4\pi}\int_\Sigma \left\{ [\mathrm{U}(\partial_y, n(y))\mathrm{U}(\partial_x, n(x)) - \mathrm{U}(\partial_x, n(x))\mathrm{U}(\partial_y, n(y))]\,\frac{1}{|x-y|}\right\}\varphi(y) dS_y$$
$$= -\frac{\mu}{4\pi}\int_\Sigma \left[\mathrm{U}(\partial_x, n(x))\frac{1}{|x-y|}\mathrm{U}(\partial_y, n(y))\right]^{\mathrm{T}}\varphi(y)dS_y$$
$$+ \frac{\mu}{4\pi}\mathrm{U}(\partial_x, n(x))\int_\Sigma \frac{1}{|x-y|}\mathrm{U}(\partial_y, n(y))\varphi(y)dS_y\ .$$

Finally we obtain

$$T(\partial_x, n(x))u(x,\varphi) = \sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}\int_\Sigma \frac{\mu}{4\pi}\frac{1}{|x-y|}\frac{\partial\varphi(y)}{\partial S_k(y)}dS_y$$
$$+ \mathrm{U}(\partial_x, n(x))\int_\Sigma \left(4\mu^2\Gamma(x-y) - \frac{\mu}{2\pi}\frac{I}{|x-y|}\right)\mathrm{U}(\partial_y, n(y))\varphi(y)dS_y$$
$$- \frac{\mu}{4\pi}\int_\Sigma \left[\mathrm{U}(\partial_x, n(x))\frac{1}{|x-y|}\mathrm{U}(\partial_y, n(y))\right]^{\mathrm{T}}\varphi(y)dS_y$$

(2.11) $\forall x \notin \Sigma$.

Hence $T(\partial_x, n(x))u(x,\varphi)$ is presented as a sum of the derivative of single-layer potentials with respect to $S_k(x)(k = 1, 2, 3)$. By the continuity of the tangential derivatives of the single-layer potential on the boundary Σ [4, p. 312], we obtain

$$T(\partial_x, n(x))u(x,\varphi) = \sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}\int_\Sigma \frac{\mu}{4\pi}\frac{1}{|x-y|}\frac{\partial\varphi(y)}{\partial S_k(y)}dS_y$$
$$+ \mathrm{U}(\partial_x, n(x))\int_\Sigma \left(4\mu^2\Gamma(x-y) - \frac{\mu}{2\pi}I\frac{1}{|x-y|}\right)\mathrm{U}(\partial_y, n(y))\varphi(y)dS_y$$
$$- \frac{\mu}{4\pi}\int_\Sigma \left[\mathrm{U}(\partial_x, n(x))\frac{1}{|x-y|}\mathrm{U}(\partial_y, n(y))\right]^{\mathrm{T}}\varphi(y)dS_y$$

(2.12) $\forall x \in \Sigma$,

provided $\varphi(y) \in C^{1,\beta}(\Sigma)$, where $n(x)$ denote the unit outward normal vector at $x \in \Sigma$ for domain Ω. In particular, we conclude that the limits on the left side of (1.17) exist and are equal to the right side of (2.12).

Let $u(x,\varphi)$ given by (1.14) be the solution of problem (1.9), then the unknown function $\varphi(y)$ satisfies the following system of boundary integro-differential equations

$$-\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}\int_\Sigma \frac{\mu}{4\pi}\frac{1}{|x-y|}\frac{\partial\varphi(y)}{\partial S_k(y)}dS_y$$
$$- \mathrm{U}(\partial_x, n(x))\int_\Sigma \left(4\mu^2\Gamma(x-y) - \frac{\mu}{2\pi}\frac{I}{|x-y|}\right)\mathrm{U}(\partial_y, n(y))\varphi(y)dS_y$$

$$\frac{\mu}{4\pi}\int_{\Sigma}\left[\mathrm{U}(\partial_x, n(x))\frac{1}{|x-y|}\mathrm{U}(\partial_y, n(y))\right]^{\mathrm{T}}\varphi(y)dS_y$$

$$= -g(x), \qquad \forall x \in \Sigma. \tag{2.13}$$

Similarly let $u(x,\varphi)$ given by (1.14) be the solution of (1.10), then the unknown function $\varphi(y)$ satisfies (2.13) also. We have therefore obtained a system of boundary integro-differential equations under the assumption $\varphi \in C^{1,\beta}(\Sigma)$. The kernels contain weak singularities. In the next section we weaken the assumption on φ, and we show how to find φ by solving (2.13).

3. The system of boundary integro-differential equations (2.13)

Let $H^m(\Omega), H^\alpha(\Sigma)$ be the usual Sobolev spaces with norm $\|\cdot\|_{m,\Omega}$ and $\|\cdot\|_{\alpha,\Sigma}$ and $W^m(\Omega) = (H^m(\Omega))^3, W^\alpha(\Sigma) = (H^\alpha(\Sigma))^3$, where m and α are two real numbers [5]. Furthermore let [7]

$$H^1_*(\Omega_{\mathrm{c}}) = \left\{v(x) \mid \frac{v(x)}{(1+|x|^2)^{\frac{1}{2}}} \in L^2(\Omega_{\mathrm{c}}); \quad \nabla v \in (L^2(\Omega_{\mathrm{c}}))^3\right\},$$

$$W^1_*(\Omega_{\mathrm{c}}) = (H^1_*(\Omega_{\mathrm{c}}))^3,$$

$$W_*^{-\frac{1}{2}}(\Sigma) = \left\{\varphi \in W^{-\frac{1}{2}}(\Sigma) \quad \text{and} \quad \int_\Sigma \varphi ds = 0, \quad \int_\Sigma x \times \varphi ds = 0, \right\},$$

$$W_*^{\frac{1}{2}}(\Sigma) = \left\{\varphi \in W^{\frac{1}{2}}(\Sigma) \quad \text{and} \quad \int_\Sigma \varphi ds = 0, \quad \int_\Sigma x \times \varphi ds = 0\right\}.$$

Suppose that a function v defined in Ω (Ω_{c}) is continuously extendible to a point $x \in \Sigma$, we let $v^+(x)(v^-(x))$ denote the limit and we set $[v] = v^+(x) - v^-(x)$ on Σ.

For any given $\varphi(x) \in C^{1,\beta}(\Sigma)$, the double-layer potential $u(x,\varphi)$ given by (1.14) is the solution of the following problem:

$$\sum_{j=1}^{3}\frac{\partial\sigma_{ij}(u)}{\partial x_j} = 0 \qquad \text{in} \quad \Omega \quad \text{and} \quad \Omega_{\mathrm{c}}, \qquad i = 1,2,3,$$

$$[u] = -\varphi \qquad \text{on} \quad \Sigma$$

$$[T(\partial_x, n(x))u] = 0 \qquad \text{on} \quad \Sigma$$

$$u(x) = o(1), \qquad \frac{\partial u}{\partial x_i} = o\left(\frac{1}{|x|}\right), \quad i = 1,2,3, \qquad \text{when} \quad |x| \to +\infty. \tag{3.1}$$

Furthermore, for given $\varphi \in W^{\frac{1}{2}}(\Sigma)$ we consider the problem (3.1). Let $u_0(x)$ denote the weak solution of the following problem with given $\varphi \in W^{\frac{1}{2}}(\Sigma)$:

$$\sum_{j=0}^{3}\frac{\partial\sigma_{ij}(u_0)}{\partial x_j} = 0, \quad \text{in } \Omega,$$

$$u_0(x) = \varphi(x) \quad \text{on } \Sigma. \tag{3.2}$$

We know [5] that problem (3.2) has a unique solution $u_0(x) \in W^1(\Omega)$ and

$$\|u_0\|_{1,\Omega} \leq C\|\varphi\|_{\frac{1}{2},\Sigma}, \tag{3.3}$$

where C is a constant and C will also denote a constant in the following, which may have different values at different places. Moreover from trace theorem we have

$$\|\varphi\|_{\frac{1}{2},\Sigma} \leq C\|u_0\|_{1,\Omega}, \tag{3.4}$$

$$\|T(\partial_x, n(x))u_0(x)\|_{-\frac{1}{2},\Sigma} \leq C\|u_0\|_{1,\Omega}. \tag{3.5}$$

Setting

$$\tilde{u}(x) = \begin{cases} u(x) + u_0(x) & \text{in } \Omega, \\ u(x) & \text{in } \Omega_c, \end{cases}$$

Then $\tilde{u}(x)$ satisfies the following problem:

$$\begin{aligned}
&\sum_{j=1}^{3} \frac{\partial \sigma_{ij}(\tilde{u})}{\partial x_j} = 0, \quad \text{in} \quad \Omega \quad \text{and} \quad \Omega_c, \quad i = 1,2,3, \\
&[\tilde{u}] = 0, \quad \text{on} \quad \Sigma \\
&[T(\partial_x, n(x))\tilde{u}] = h(x), \quad \text{on} \quad \Sigma \\
&\tilde{u}(x) = o(1), \quad \frac{\partial \tilde{u}}{\partial x_i} = o\left(\frac{1}{|x|}\right), \quad i = 1,2,3, \quad \text{when} \quad |x| \to +\infty.
\end{aligned} \tag{3.6}$$

with $h(x) = T(\partial_x, n(x))u_0(x)|_\Sigma$. Introduce spaces

$$X = \{v \mid v|_\Omega \in W^1(\Omega), \ \ v|_{\Omega_c} \in W^1_*(\Omega_c) \ \text{ and } \ [v]_\Sigma \in W_*^{\frac{1}{2}}(\Sigma)\},$$

$$X_0 = \{v \in X, \ \text{ and } \ [v] = 0 \ \text{ on } \Sigma\}.$$

with norm $\|v\|_X^2 = \|v\|^2_{W^1(\Omega)} + \|v\|^2_{W^1_*(\Omega_c)}$.

Then the problem (3.6) is reduced to the following variational problem:

$$\text{Find } \ \tilde{u} \in X_0, \ \text{ such that}$$

$$a(\tilde{u}, v) = \int_\Sigma h \ v ds, \quad \forall v \in X_0, \tag{3.7}$$

with

$$a(u, v) = \int_{\Omega \cup \Omega_c} \sum_{i,j=1}^{3} \sigma_{ij}(u)\varepsilon_{ij}(v) dx. \tag{3.8}$$

It is straightforward to check that $a(v,v)^{\frac{1}{2}}$ is a norm on the space X and it can be shown, using the Korn inequalities, that $a(v,v)^{\frac{1}{2}}$ is an equivalent norm on X. From the Lax-Milgram lemma [1], the variational problem (3.7) has a unique solution $\tilde{u} \in X_0$ for given $h(x) \in W_*^{-\frac{1}{2}}(\Sigma)$ and

$$\|\tilde{u}(x)\|_X \leq C\|h\|_{-\frac{1}{2},\Sigma}. \tag{3.9}$$

Let

$$u(x) = \begin{cases} \tilde{u}(x) - u_0(x) & \text{in } \Omega, \\ \tilde{u}(x) & \text{in } \Omega_{\text{c}}. \end{cases}$$

Then u is the unique weak solution of problem (3.1) and

$$\|u\|_X \leq C\|\varphi\|_{\frac{1}{2},\Sigma}. \tag{3.10}$$

Thus for given $\varphi \in W^{\frac{1}{2}}(\Sigma)$ the equality (1.14) is meaningful and the equality (2.13) holds in the weak sense for given $g \in W_*^{-\frac{1}{2}}(\Sigma)$. Furthermore from Green's formula,

$$-\int_{\Sigma}(T(\partial_x, n(x))u(x)) \cdot \psi(x)dS_x = a(u, v), \tag{3.11}$$

where $u(x)$ and $v(x)$ are the solutions of problem (3.1) corresponding to the given functions $\varphi(x)$ and $\psi(x) \in W^{\frac{1}{2}}(\Sigma)$. Substituting (2.11) into (3.11) and integrating by parts we obtain

$$\begin{aligned} a(u,v) = b(\varphi,\psi) \equiv & \int_{\Sigma}\int_{\Sigma} \frac{\mu}{4\pi}\frac{1}{|x-y|}\left(\sum_{k=1}^{3}\frac{\partial\psi(x)}{\partial S_k(x)} \cdot \frac{\partial\varphi(y)}{\partial S_k(y)}\right)dS_x dS_y \\ & + \int_{\Sigma}\int_{\Sigma}(\mathrm{U}(\partial_x, n(x))\psi(x))^{\mathrm{T}}\left(\frac{\mu}{2\pi}\frac{I}{|x-y|} - 4\mu^2\Gamma(x-y)\right)\mathrm{U}(\partial_y, n(y))\varphi(y)dS_x dS_y \\ & + \int_{\Sigma}\int_{\Sigma}\sum_{i,j,k=1}^{3}(\mathrm{U}_{kj}(\partial_x, n(x))\psi_i(x))\frac{1}{|x-y|}(\mathrm{U}_{ki}(\partial_y, n(y))\varphi_j(y)dS_x dS_y \end{aligned} \tag{3.12}$$

For the bilinear form b defined by (3.12), we have

Lemma 3.1. *(i)* $b(\varphi, \psi)$ *is a bounded symmetric bilinear form on* $W^{\frac{1}{2}}(\Sigma) \times W^{\frac{1}{2}}(\Sigma)$, *namely there is a constant* $M > 0$, *such that*

$$|b(\varphi,\psi)| \leq M\|\varphi\|_{\frac{1}{2},\Sigma}\|\psi\|_{\frac{1}{2},\Sigma}, \qquad \forall\varphi,\psi \in W^{\frac{1}{2}}(\Sigma). \tag{3.13}$$

(ii) There exists a constant $\alpha > 0$, *such that*

$$b(\varphi,\varphi) \geq \alpha\|\varphi\|^2_{\frac{1}{2},\Sigma}, \qquad \forall\varphi \in W_*^{\frac{1}{2}}(\Sigma). \tag{3.14}$$

Proof. (i) Since $a(u, v) = b(\varphi, \psi)$, we have

$$|b(\varphi,\psi)| = |a(u,v)| \leq C\|u\|_X\|v\|_X,$$

where u, v are the solutions of problem (3.1) corresponding to the functions φ and ψ. From the estimate (3.10), the inequality (3.13) follows immediately.

(ii) For given $\varphi \in W_*^{\frac{1}{2}}(\Sigma)$, the solution u of (3.1), is in X. Since $a(u, u)^{\frac{1}{2}}$ is an equivalent norm of X, we have

$$b(\varphi,\varphi) = a(u,u) \geq \alpha_0\|u\|^2_X. \tag{3.15}$$

On the other hand,

$$\begin{aligned}\|\varphi\|_{\frac{1}{2},\Gamma} &= \|u^+ - u^-\|_{\frac{1}{2},\Sigma} \\ &\leq \|u^+\|_{\frac{1}{2},\Sigma} + \|u^-\|_{\frac{1}{2},\Sigma} \\ &\leq C\|u\|_X . \end{aligned} \tag{3.16}$$

The last inequality is from the trace theorem. Combining (3.15) and (3.16), the inequality (3.14) is proved. □

We now return to the system of the boundary integro-differential equations (2.13). For any given $g \in W_*^{-\frac{1}{2}}(\Sigma)$, the problem (2.13) is equivalent to the following variational problem:

$$\text{Find} \quad \varphi(y) \in W_*^{\frac{1}{2}}(\Sigma), \text{ such that}$$

$$b(\varphi, \psi) = -\int_\Sigma g \cdot \psi dS, \qquad \forall \psi \in W_*^{-\frac{1}{2}}(\Sigma) \tag{3.17}$$

An application of the Lax-Milgram lemma yeilds the following theorem.

Theorem 3.1. *For any $g \in W_*^{-\frac{1}{2}}(\Sigma)$, the variational problem (3.7) has a unique solution $\varphi \in W_*^{\frac{1}{2}}(\Sigma)$.*

As a consequence of Theorem 3.1, Eq. (2.13) has a unique solution for any $g \in W_*^{-\frac{1}{2}}(\Sigma)$, so (1.9) has a unique weak solution $u \in W_*^1(\Omega_c)$, and (1.10) has a weak solution $u \in W^1(\Omega)$ that unique up to rigid displacements, for any $g \in W_*^{-\frac{1}{2}}(\Sigma)$.

Suppose that V_h is a finite dimensional subspace of $W_*^{\frac{1}{2}}(\Sigma)$, then we consider the discrete problem of (3.17):

$$\text{Find } \varphi_h \in V_h, \quad \text{such that}$$

$$b(\varphi_h, \psi_h) = -\int_\Sigma g \cdot \psi_h dS, \qquad \forall \varphi_h \in V_h, \tag{3.18}$$

We obtain

Theorem 3.2. *The discrete problem (3.18) has a unique solution $\varphi_h \in V_h$ and*

$$\|\varphi - \varphi_h\|_{\frac{1}{2},\Sigma} \leq \frac{M}{\alpha} \inf_{\psi_h \in V_h} \|\varphi - \psi_h\|_{\frac{1}{2},\Sigma},$$

where $\varphi \in W_*^{\frac{1}{2}}(\Sigma)$ is the solution of (3.18).

This conclusion follows immediately from Lemma 3.1, the Lax-Milgram lemma and tha Cea Lemma [1].

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