Numer. Math. 68: 149–167 (1994)

Numerische Mathematik

A polyhedral approach to multicommodity survivable network design

Mechthild Stoer[1], **Geir Dahl**[2]

[1] Konrad-Zuse-Zentrum für Informationstechnik Berlin, Heilbronner Strasse 10, D-10711 Berlin, Germany
E-mail: stoer@zib-berlin.de
[2] Norwegian Telecom Research, P.O.Box 83, N-2007 Kjeller, Norway
E-mail: geir.dahl@tf.tele.no

Received August 26, 1993 / Revised version received February 1994

Dedicated to Professor Josef Stoer on the occasion of his 60th birthday

Summary. The design of cost-efficient networks satisfying certain survivability constraints is of major concern to the telecommunications industry. In this paper we study a problem of extending the capacity of a network by discrete steps as cheaply as possible, such that the given traffic demand can be accommodated even when a single edge or node in the network fails. We derive valid and nonredundant inequalities for the polyhedron of capacity design variables, by exploiting its relationship to connectivity network design and knapsack-like subproblems. A cutting plane algorithm and heuristics for the problem are described, and preliminary computational results are reported.

Mathematics Subject Classification (1991): 65K05, 90C35

1. Introduction

It is of major concern to telecommunication operators to design networks with a suitable degree of survivability towards component failures, cable cuts etc. Motivated by this, there has been a large amount of recent research on the design of networks satisfying specified connectivity constraints. Most of this research concentrates on *uncapacitated networks*, i.e., where each link can support all the traffic at once. However, for many important present and future telecommunication networks, capacities play a fundamental role.

In this paper we study an integer programming model for the following integrated planning problem: decide which links to install in the network and which capacities to install on these links such that the network allows routing of point-to-point traffic even under single node or edge failures. Thus the model addresses MULTIcommodity SUrvivable Network design, for short: MULTISUN.

More specifically, we have given a set V of nodes and traffic demands between certain pairs of these nodes. Each demand represents a certain amount of point-to-point traffic to be routed in the network between origin and destination nodes. In addition, a set of edges joining pairs of nodes in V are given; these represent direct physical links (e.g., a fiber cable or a radio relay system). For each edge one wants to decide which capacity to install, chosen among a discrete set of alternatives, each with an associated building cost. The number of such alternatives depends on the application and the desired level of detail. We are interested in capacity extensions such that all demands can be routed simultaneously in the resulting network. Such a routing is called a multicommodity flow. Especially, we may require that the network allows a multicommodity flow also in certain failure situations, e.g., when a single edge or node fails. In our model, we allow traffic demands to be split up and routed on several paths, i.e., we consider continuous flows. The discreteness of the model lies in deciding the design/capacity extension. The optimization problem in MULTISUN is to find such a feasible network extension of minimum total building cost.

The purpose of this paper is to present and analyze an integer linear programming model for the MULTISUN problem using a polyhedral approach. We study properties of polytopes that are naturally associated with the model. Specifically, we present classes of nonredundant inequalities that strengthen the original formulation and may be (in fact, are) used in a cutting plane algorithm for solving real-world planning problems. In deriving these inequalities, we exploit relations to the knapsack problem and also the design of (uncapacitated) networks with connectivity constraints.

A large amount of work has been done by Minoux and others on the related model with a *continuous* cost function, see [15] and the references given there. Minoux [15] introduced a general survivability framework for multicommodity flow network design, based on telecommunication studies. Gavish et al. [8] considered an even more general model arising in fiber optic networks involving selection of different cable types and a discrete cost function. They developed bounding procedures using Lagrangian relaxation. Balakrishnan and Graves [2] considered the problem of designing a (directed) network supporting multicommodity flow without survivability constraints and with a continuous piecewise linear cost function on each arc. A special case of the MULTISUN problem is the well known fixed charge network flow problem (assuming that only design costs are present), see e.g., [16]. For work on the design of uncapacitated networks satisfying connectivity constraints, see the work of Grötschel, Monma, and Stoer [9, 11, 19], and for directed networks, see Dahl [6].

This paper is organized as follows. The integer linear programming formulation of the MULTISUN problem is given in Sect. 2. Two models are introduced, one with and one without survivability requirements, and associated 0/1-polytopes are defined. We discuss basic properties (dimension and trivial facets) of these polytopes in Sect. 3. The remaining part of the paper discusses stronger formulations of the problem. In Sect. 4 classes of facet defining inequalities are derived from underlying knapsack structures of the original model. We then, in Sect. 5, exploit the mentioned relation of MULTISUN to connectivity design problems, and derive facet defining inequalities, so-called partition and lifted two-cover inequalities, from this. In Sect. 6 a cutting plane algorithm using some of these inequalities is described together with a few computational results. Some suggestions for further work are given in the concluding section.

We use fairly standard notation from graph theory and polyhedral theory, see [4, 18], but a few notions need to be explained. $\mathbb{R}^E$ denotes the set of real vectors indexed by E, where E is a finite set. Let $G = (V, E)$ be an undirected graph without loops and multiple edges. If w is a node in G, we let $G-w$ denote the graph obtained from G by removing w and its incident edges from G. Similarly, $G-e$ is the graph obtained by removing the edge e. The *cut* $\delta_G(W)$ induced by a subset W of V is the set of edges with one end node in W and the other outside W; W and $V \setminus W$ are called *shores* of the cut. By $G[W] = (W, E(W))$ we denote the graph induced by node set W. For two nodes u and v, a $[u,v]$-path P is a sequence of nodes and edges $(v_0, e_1, v_1, e_2, \ldots, v_{l-1}, e_l, v_l)$, where each edge e_i is incident to the nodes v_{i-1} and v_i ($i = 1, \ldots, l$), where $v_0 = u$ and $v_l = v$, and where no node or edge appears more than once in P. A graph G is said to be *2-edge* (or *2-node*) *connected* with respect to some given node set R, if between any two nodes $u, v \in R$ there exist at least two edge- (or node-)disjoint $[u,v]$-paths. Similarly, we say that G is *connected* with respect to R, if G contains a $[u,v]$-path for each pair of nodes $u, v \in R$. If G is moreover an edge-minimal connected graph w.r.t. R, then G is a *Steiner tree* with terminal set R.

A *network* $\mathcal{N} = (G, c)$ is a graph G with weights (capacities or demands) $c_e \geq 0$ associated with the edges e. Finally, given a *supply* network (G, c) and a *demand* network (H, d), where G and H have the same node set, a *multicommodity flow* (w.r.t. (H, d)) is defined as a collection of $[u,v]$-paths P^i_{uv} of G together with numbers $\lambda^i_{uv} \geq 0$ (for all $uv \in E(H)$, $i = 1, \ldots$), such that $\sum_i \lambda^i_{uv} = d_{uv}$, for each $uv \in E(H)$. The network (G, c) or the capacity vector c is said to *allow* a multicommodity flow w.r.t. (H, d), if, moreover, for each edge $e \in E(G)$ the sum of λ^i_{uv} over all paths containing e is at most its capacity c_e.

Given a vector $x \in \mathbb{R}^I$ indexed by some set I and given a subset S of I, we write $x(S)$ instead of $\sum_{i \in S} x_i$. By $\chi^S \in \mathbb{R}^I$ we denote the incidence vector of S.

2. Mathematical model

We present a mathematical model for the MULTISUN problem. In fact, we will study two models; Model 1, which does not take survivability into account, and Model 2, which does.

2.1. Model 1

The existing lines (for example transmission links), plus the lines that can be established, are given as a *supply* graph $G = (V, E)$ with node set V and edge set E. The traffic demands are given as a *demand* graph $H = (V, D)$ whose edges represent the different demands, and the amount of traffic $d_{uv} > 0$ for each demand edge $uv \in D$.

For each edge $e \in E$ one has to choose a capacity y_e from among a small set of discrete capacity extensions with associated costs, such that the so constructed network satisfies all traffic demands and is of minimum total cost. Below we describe in detail how the cost function and the multicommodity flow constraints are modeled.

The discrete capacities for each edge $e \in E$ are given by numbers $0 < M_e^1 < M_e^2 < \cdots < M_e^{T_e}$ (where $T_e \geq 1$), denoting, for instance, sizes of cables and/or of terminal equipment. The capacity $M_e^{T_e}$ is supposed to be at least as large as the sum of all demands. This can always be achieved with a sufficiently high cost, if necessary. Define M_e^0 to be 0. Let m_e^t denote the extension steps $M_e^t - M_e^{t-1}$, for $t = 1, \ldots, T_e$. The cost of extending the capacity from M_e^{t-1} to M_e^t for $t = 1, \ldots, T_e$ is given by $c_e^t \geq 0$. So the cost of installing capacity M_e^t on edge e is $\sum_{\tau=1}^{t} c_e^\tau m_e^\tau$.

One can view the capacity/cost function on each edge e as a step function with step lengths m_e^t and step heights c_e^t. In [7] we introduced a more general model with a piecewise linear cost function, where sloped steps are allowed. But in this paper we only consider step functions.

The cost function is modeled with binary *design variables* x_e^t (sometimes written $x(e,t)$) for all edges $e \in E$, indicating the incremental capacity installations. The values of x_e^t are required to be nonincreasing with increasing t. We define $I := \{ (e,t) \mid t = 1, \dots, T_e, e \in E \}$ as the index set of all the design variables x_e^t, and let $x \in \mathbb{R}^I$ be the vector of all these variables. For a design vector $x \in \mathbb{R}^I$ the corresponding cost is $c^{\mathrm{T}}x$ and the *associated capacity vector* y is given by $y_e = \sum_{t=1}^{T_e} m_e^t x_e^t$.

Let $\bar{y}$ be the capacity vector associated with some design vector x. The network $(G, \bar{y})$ is supposed to allow a multicommodity flow carrying all traffic. The feasibility of a capacity vector $\bar{y}$ can be expressed in terms of linear inequalities as follows. For some given nonnegative vector $\mu \in \mathbb{R}^E$ and demand edge $f \in D$ let π_f^μ denote the shortest path length in G between the two end nodes of f with respect to edge lengths μ_e. It can be shown (see [12]) that $\bar{y}$ is feasible if and only if

$$(1) \qquad \sum_{e \in E} \mu_e \bar{y}_e \geq \sum_{f \in D} \pi_f^\mu d_f \quad \text{for all } \mu \geq 0.$$

The necessity of these inequalities stems from the fact that the cheapest way to route a multicommodity flow, if no capacity constraints but edge costs μ_e are given, is to route each flow individually on its shortest path w.r. to μ. Thus a lower bound for $\mu^{\mathrm{T}}\bar{y}$ is the sum of the shortest-path-lengths multiplied with the demand value. The sufficiency of the system (1) comes from linear programming duality. This characterization of feasible capacities is known as the “Japanese theorem”, first stated in [12, 17]. The inequalities (1) are called *metric inequalities*. This name is motivated by the fact that, for complete input and demand graphs, any vector $\mu \in \mathbb{R}^E$ defining a nonredundant inequality in (1) induces a (pseudo-)metric on G, that is, it is nonnegative, symmetric and satisfies the triangle inequality $\mu_{uv} + \mu_{vw} \geq \mu_{uw}$ for any three nodes u, v, w. (Here, μ_{uu} is supposed to be 0).

In the inequality system (1) we can restrict ourselves to the inequalities defined by vectors (μ, π) in the set Π of extreme rays of the cone

$$\{ \mu \in \mathbb{R}^E, \pi \in \mathbb{R}^D \mid \mu \geq 0,\ \pi_f = \pi_f^\mu \text{ for all } f \in D \}.$$

The extreme rays of this cone were investigated in [1, 13].

A special type of metric inequality is the so-called *cut inequality* defined by a node set $W \neq \emptyset$, $W \neq V$

$$(2) \qquad \bar{y}(\delta_G(W)) \geq d(\delta_H(W)).$$

This can be seen to be a metric inequality when μ is set as the incidence vector of $\delta_G(W)$, and when we assume that $G[W]$ and $G[V \setminus W]$ are connected. Cut inequalities express the fact that the total demand crossing a cut should not exceed its capacity.

Model 1 with cost function and multicommodity flow constraints can now be stated by inserting $\bar{y}_e := \sum_{t=1}^{T_e} m_e^t x_e^t$ into the equation system (1):

Model 1

min $c^{\mathrm{T}}x$

subject to

(3)

(i)	$1 \geq x_e^1 \geq x_e^2 \geq \cdots \geq x_e^{T_e} \geq 0$	for all $e \in E$
(ii)	$\sum_{e \in E} \mu_e \sum_{t=1}^{T_e} m_e^t x_e^t \ \geq\ \sum_{f \in D} \pi_f^\mu d_f$	for all $(\mu, \pi) \in \Pi$
(iii)	x_e^t integer	for all $(e,t) \in I$.

The constraints (i) are called *ordering constraints*, and the constraints (ii) are called *metric inequalities*, as inequalities (1). If $S \subseteq I$ is such that its incidence vector χ^S satisfies all the constraints in Model 1, we say that χ^S and S are *feasible*.

This concludes the discussion of the validity of Model 1 as an integer linear programming formulation of the MULTISUN problem when no survivability is taken into account.

2.2. Model 2

The second model includes survivability constraints, which means that if any single line or node of the network fails all the traffic should still be accommodated in the remaining operating network. More precisely, we require that the installed capacity vector $\overline{y}$ satisfies the following constraints:

(i) for each $e \in E$ the capacity vector $\overline{y}$ restricted to $E\setminus\{e\}$ allows a multicommodity flow w.r.t. (H, d),

(ii) for each $v \in V$ the capacity vector $\overline{y}$ restricted to $E \setminus \delta_G(v)$ allows a multicommodity flow w.r.t. $(H - v, d)$.

The constraints (i), called *edge failure constraints*, assure that the network $(G, \overline{y})$ has sufficient reserve capacity to protect against any single edge failure. Similarly, the *node failure constraints* (ii) protect the network against any single node failure. Note that if a node fails, then all the demands originating in this node are deleted from the demand graph. In some applications one may have more complex changes of the demand graph in case of node failures (see [7]), but this is not considered here.

It is convenient to introduce index sets representing the *operating states* of the network. Let S^E (S^V) have one element for each $e \in E$ ($v \in V$) corresponding to the failure of e (v, resp.), and let S^0 represent the case without any node or edge failure. In the following, we are interested in the following subsets of operating states, namely $S := S^0$, $S := S^0 \cup S^E$, and $S := S^0 \cup S^E \cup S^V$. Furthermore, for an operating state $s \in S^0 \cup S^E \cup S^V$ we define sets $E(s)$ and $D(s)$ of operating supply and demand edges:

- For $s \in S^0$, let $E(s) := E$ and $D(s) := D$.
- For $s \in S^E$ representing the failure of edge e, define $E(s) := E \setminus \{e\}$, and $D(s) := D$.
- For $s \in S^V$ representing the failure of edge v, define $E(s) := E \setminus \delta_G(v)$, and $D(s) := D \setminus \delta_H(v)$.

For any operating state s, the shortest-path values π_f^μ are defined with respect to the graph of operating supply edges $(V, E(s))$, and the set $\Pi(s)$ is defined analogous to Π in Model 1.

The integer linear programming formulation of the MULTISUN problem with survivability constraints now becomes:

Model 2

min $c^{\mathrm{T}}x$

subject to

(4)

$$\begin{array}{lll}
\text{(i)} & 1 \geq x_e^1 \geq x_e^2 \geq \cdots \geq x_e^{T_e} \geq 0 & \text{for all } e \in E \\
\text{(ii)} & \sum_{e \in E(s)} \mu_e \sum_{t=1}^{T_e} m_e^t x_e^t \geq \sum_{f \in D(s)} \pi_f^{\mu} d_f & \text{for all } \pi \in \Pi(s),\ s \in S \\
\text{(iii)} & x_e^t \text{ integer} & \text{for all } (e,t) \in I.
\end{array}$$

We remark that this model could be used for more general failure situations as well, for example the simultaneous failure of two or more nodes. This would be reflected in other choices of $E(s)$ and $D(s)$. For $S := S^0$, Model 2 is equivalent with Model 1.

2.3. Polytopes associated with the models

We introduce the polytopes associated with the MULTISUN models.

$$\mathrm{MSUN}_S(G, m, H, d) := \mathrm{conv}\{\, x \in \mathbb{R}^I \mid x \text{ satisfies (4)(i)–(iii)} \,\}$$

If no misunderstanding is possible, we drop the parameter list and write MSUN$_S$. When we consider only edge failures $S := S^0 \cup S^E$, we also write MSUN$_E$; when $S := S^0 \cup S^E \cup S^V$, we write MSUN$_{E\cup V}$, and for $S := S^0$, we denote the polytope by MSUN$_\emptyset$.

Since the vertices of the polytope MSUN$_S$ are exactly the feasible x with integer components, the MULTISUN problem can now also be written as

$$\min c^{\mathrm{T}}x \quad \text{subject to } x \in \mathrm{MSUN}_S.$$

To optimize over a polyhedron using a linear programming code, one needs to know all, or at least “sufficiently” many, of its defining inequalities. The inequalities in (4) are generally not enough to achieve good lower bounds. So, a main purpose of the subsequent study is to find more classes of valid and nonredundant inequalities for MSUN$_S$.

3. Basic properties of MULTISUN polyhedra

We begin the polyhedral investigations of MULTISUN polyhedra by a study of their dimension. Let a supply graph $G = (V, E)$ and a demand graph $H = (V, D)$ with demand vector d be given.

Proposition 5. *Let MSUN$_S$ be one of the polytopes MSUN$_\emptyset$, MSUN$_E$ or MSUN$_{E\cup V}$.*

Then MSUN$_S(G, m, H, d)$ is full-dimensional if and only if MSUN$_S(G-e, m, H, d)$ is nonempty for all $e \in E$.

Proof. Assume that MSUN$_S(G - e, m, H, d)$ is empty for some $e \in E$. Then either MSUN$_S(G, m, H, d)$ is empty, or each feasible $x \in \mathbb{R}^I$ satisfies $x_e^1 = 1$, hence MSUN$_S$ is not full-dimensional.

To prove the sufficiency, we assume that MSUN$_S$ is not full-dimensional, i.e., there is a linear equation $a^{\mathrm{T}}x = \alpha$ with nonzero a satisfied by each point in MSUN$_S$. Let $f \in E$. By hypothesis, there exists a solution $x \in$ MSUN$_S$ with $x_f^t = 0$ for $t = 1, \ldots, T_f$. By monotonicity, the solutions x^k obtained from x by changing x_f^t to 1 for $t = 1, \ldots, k$ are feasible. Thus we have $a^{\mathrm{T}}x = a^{\mathrm{T}}x^1 = a^{\mathrm{T}}x^2 = \cdots = \alpha$, and by subtraction, we get $a_f^t = 0$ for all $t = 1, \ldots, T_f$. Since f was chosen arbitrarily in E, we get $a = 0$, a contradiction. Thus the equality system of MSUN$_S$ is empty, and the polyhedron is full-dimensional. $\square$

We hereafter assume that each MSUN-polytope under consideration is full-dimensional; this simplifies polyhedral arguments. In most applications the supply graph will be sufficiently connected anyway.

Proposition 6. *Let MSUN$_S$ be one of the polytopes MSUN$_\emptyset$, MSUN$_E$ or MSUN$_{E\cup V}$.*

(i) The inequalities $1 \geq x_e^1 \geq x_e^2 \geq \cdots \geq x_e^{T_e}$ define facets of MSUN$_S$ for all $e \in E$.
(ii) The inequality $x(e, T_e) \geq 0$ defines a facet of MSUN$_S$ if and only if for all $f \neq e$ the polytope MSUN$_S(G - f, m, H, d) \cap \{x \in \mathbb{R}^I \mid x(e, T_e) = 0\}$ is nonempty.

Proof. (i): For each $(f, k) \in I$, define the vector $x^{f,k} \in \{0, 1\}^I$ by setting $x^{f,k}(e, t) :=$ 1 for all $e \neq f$ and all t, $x^{f,k}(f, t) := 1$ if $k < t$, and $x^{f,k}(f, t) := 0$ if $k \geq t$. Define furthermore $x' \in \mathbb{R}^I$ to be the vector of all 1s. These $|I| + 1$ points are feasible and affinely independent. Each inequality in (i) holds with equality for exactly $|I|$ of the points. Thus the face induced by this inequality is a facet.

(ii): If MSUN$_S(G - f, m, H, d) \cap \{x \mid x(e, T_e) = 0\}$ is empty for some f, then $x(e, T_e) = 0$ implies $x_f^1 = 1$, therefore the nonnegativity inequality cannot define a facet. If, on the other hand, the given condition holds, then the $|I|$ vectors $y^{f,k}$ derived from $x^{f,k}$ for all $(f, k) \in I$ by setting the component $x(e, T_e)$ to 0 are feasible and affinely independent. $\square$

The inequalities (4)(ii) do not define facets of MSUN$_S$ except in very special cases. This indicates the need of stronger formulations than the LP relaxation given by (4). In the remaining part of the paper we therefore give different classes of inequalities that lead to improved LP formulations of the MULTISUN problem.

4. Improved formulations based on Knapsack substructures

4.1. Band inequalities

Several ideas may lead to improved, i.e., stronger, formulations of the MULTISUN problem. We describe a class of valid inequalities for MSUN$_\emptyset$ called *band inequalities*. They are derived as facet-defining inequalities for a relaxation of MSUN$_\emptyset$, the so-called ICOV-polytope.

Let $\sum_{e\in E} \sum_{t=1}^{T_e} g_e^t x_e^t \geq b$ be a metric inequality (3)(ii), where g_e^t is defined as $\mu_e m_e^t$. We will assume that $g_e^{T_e} \geq b$ for all $e \in E$. This can be done without loss of generality, because we assume that the highest capacity of each edge is "large". Let F

be the support of a given metric inequality, that is, the set of edges with positive g_e^1, and consider the following polytope

$$
\begin{aligned}
\text{ICOV}(g,b) := \text{conv}\{\, (\, x_e^t :\ t = 1, \ldots, T_e,\, e \in F) \mid \\
\textstyle\sum_{e\in F} \sum_{t=1}^{T_e} g_e^t x_e^t \geq b, \\
1 \geq x_e^1 \geq x_e^2 \geq \cdots \geq x_e^{T_e} \geq 0 \quad \text{for all } e \in F, \\
x \text{ integral}\,\}.
\end{aligned} \tag{7}
$$

The polytope ICOV(g,b) can be viewed as a knapsack polytope with additional ordering constraints. Facial properties of knapsack polytopes have been studied in recent years, see e.g., [3, 5]. Knapsack polytopes with the additional constraints $\sum_t x_e^t \leq 1$ (for all $e \in F$), and $x \geq 0$, have been studied in [14, 20] under the names "multiple-choice knapsack problem" or "knapsack problem with generalized upper bounds" (which is actually a larger class of problems). The polytope ICOV(g,b) can be linearly transformed into a knapsack polytope with generalized upper bounds, so all results pertaining to that polytope apply also to the ICOV-polytope.

Any inequality that is valid for ICOV(g,b) is clearly also valid for MSUN$_\emptyset$, if the missing coefficients (for (e,t) with $e \notin F$) are filled up with zeros.

To describe the band inequalities we introduce some notation. Define the index set $I(A) := \{\, (e,t) \in I \mid t = 1,\ldots,T_e, e \in A\,\}$ for each $A \subseteq E$. For simplicity, we write $I(e)$ in stead of $I(\{e\})$. A *band* B of F is a subset of $I(F)$ containing exactly one element $(e, t_e^{k_e})$ in each $I(e)$, $e \in F$. Given a band $B \subseteq I(F)$, we will from now on write t_e^B instead of $t_e^{k_e}$. Let $B^< := \{(e,t) \in I(F) \mid t < t_e^B\}$, and define similarly $B^>$. A band B is called *valid* if $g(B^<) < b$. We say that a band B' of F is *above* a band B of F if $t_e^B \leq t_e^{B'}$ for all $e \in F$, and $t_e^B < t_e^{B'}$ for at least one $e \in F$.

Whenever B is a valid band, the inequality

$$x(B) := \sum_{(e,t)\in B} x_e^t \geq 1, \tag{8}$$

is valid for ICOV(g,b). It is called a *band inequality*. The band inequalities are, after transformation, equivalent to the GUB cover inequalities (in [20]) for the knapsack problem with generalized upper bounds, but their nonredundancy is not proved there.

Figure 1 illustrates a band inequality with $F = \{e_1,\ldots,e_4\}$ and $b = 4$. Each column depicts a different edge. The (e,t) are represented by a box of width 1 and height g_e^t. For each e, the boxes $(e,1)$, $(e,2)$, $\ldots$, (e,T_e) are stacked on top of each other, with $(e,1)$ being lowest, and (e,T_e) being highest. In our example, $g_e^1 = 1$ for each e, and $b = 4$. The 1-coefficients of the band inequality are depicted inside the boxes. The value $g(B^<)$ equals the area below the 1's.

We next characterize when a band inequality defines a facet of ICOV(g,b), which is of some interest for the strength of the similar MSUN inequality. First, however, we remark that ICOV(g,b) is full-dimensional. This follows from the assumption $g_e^{T_e} \geq b$ and $|F| \geq 2$, using the same construction as in the proof of Proposition 5.

Proposition 9. *Let B be a band in F, where $|F| \geq 2$. Then the band inequality $x(B) \geq 1$ defines a facet of ICOV(g,b) if and only if there is no valid band above B.*

Proof. Assume that B' is a valid band above B. It follows from the ordering constraints that $x(B) \geq x(B')$ for each $x \in$ ICOV(g,b), so $x(B) \geq 1$ is implied by other valid inequalities and therefore redundant.

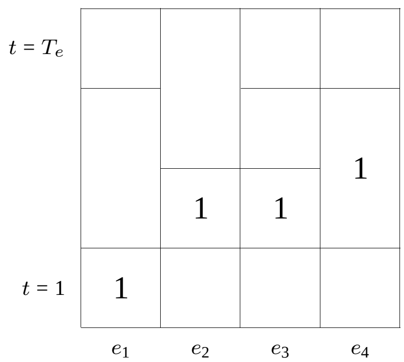

Fig. 1. Band inequality

To prove the converse, assume that there is no valid band above B. This implies that, for all $e \in F$ and all $t \geq t_e^B$, the incidence vectors of

$$B^< \cup \{(e,1),(e,2),\ldots,(e,t)\}$$

lie in the face defined by the present band inequality. Moreover, by the assumption that $g_e^{T_e} \geq b$, we have that, for all $e, f \in F$ and all $t < t_e^B$, the incidence vectors of

$$I(f) \cup \{(e,1),(e,2),\ldots,(e,t)\}$$

lie in the face. These are sufficiently many affinely independent vectors to prove (by standard polyhedral methods) that the band inequality defines a facet of ICOV(g,b). □

A special case are the band inequalities derived from cut inequalities (2). They are a subclass of the partition inequalities for Model 1 studied in Subsect. 5.1, many of which define facets also for MSUN$_\emptyset$.

The separation problem for band inequalities for ICOV(g,b) or MSUN$_\emptyset$ is easily seen to be equivalent to the NP-complete multiple-choice knapsack problem. Thus it is not easy to determine, for a given vector $\overline{x} \in \mathbb{R}^{I(F)}$, whether there exists a band inequality that is violated by $\overline{x}$ but valid for ICOV(g,b).

Besides the band inequalities there are other nonredundant inequalities for ICOV (g,b). Some properties of their coefficient structure are stated in the next proposition.

Proposition 10. *(i) All nonredundant inequalities for ICOV(g,b) that are not equivalent to any of the ordering constraints $1 \geq x_e^1 \geq x_e^2 \geq \ldots \geq x_e^{T_e}$ have nonnegative coefficients. The same holds for nonredundant inequalities of MSUN$_\emptyset$.*

(ii) For all nonredundant inequalities of ICOV(g,b) that are not equivalent to any ordering constraint, the sum of coefficients over $I(e)$ is equal to its right-hand side, for all $e \in F$.

Proof. (i): If the inequality has a negative coefficient for some (e,t), and if $t > 1$, then any vector satisfying the given nonredundant inequality with equality must also satisfy $x_e^{t-1} = x_e^t$. Otherwise we could construct a contradiction to the validity of the inequality by increasing x_e^t. Since the given inequality is nonredundant, it is equivalent to the ordering constraint $x_e^{t-1} \geq x_e^t$. By a similar argument, if the coefficient of some $(e, 1)$ is negative then the inequality is equivalent to $x_e^1 \leq 1$.

(ii): Since the incidence vector of $I(e)$ is feasible for all $e \in F$, and the given inequality is valid, the sum of coefficients for each fixed edge is not less than the right-hand side. Suppose that for some $f \in F$ it is strictly larger than the right-hand side. Since the face P defined by the given inequality is not contained in the face defined by $x(f, T_f) \geq 0$ there exists a set $Z \subseteq I$ whose incidence vector is in P and satisfies $x(f, T_f) > 0$, which implies that $I(f)$ is contained in Z. By (i), the inequality has nonnegative coefficients, so the incidence vector of Z cannot satisfy the inequality with equality. This contradicts our choice of Z. Therefore the sum of coefficients over $I(e)$ is equal to the right-hand side of the inequality, for all $e \in E$. □

The band inequalities (8) are exactly those nonredundant inequalities with integral coefficients whose coefficients in $I(e)$ sum up to 1 for each edge $e \in F$.

The band inequalities can be generalized to *k-band inequalities* whose integer coefficients sum up to k for each fixed edge, but we do not discuss this generalization further in this paper.

4.2. Strengthened band inequalities (model 2)

Unfortunately, the band inequalities for Model 1 are not facet-defining for MSUN$_E$ (with edge survivability constraints). But it is possible, as can be seen in the next lemma, to raise the right-hand side of a band inequality for MSUN$_E$ to 2 to achieve a valid inequality (under some further assumptions).

Lemma 11. *Let $g^{\mathrm{T}} x \geq b > 0$ be a metric inequality for MSUN$_\emptyset(G, m, H, d)$, let F be the set of edges with positive g_e^1, and let B be a band of F. We suppose that $|F| \geq 2$. If $g(B^< \setminus I(e)) < b$ for all $e \in F$ then*

$$x(B) \geq 2 \tag{12}$$

is a valid inequality for MSUN$_E$.

Proof. First note that $g^{\mathrm{T}} x \geq b$ is also a valid inequality for MSUN$_\emptyset(G - e, m, H, d)$, for all $e \in F$. Since $g(B^< \setminus I(e)) < g(B^<) < b$, the inequality $x(B) \geq 1$ is valid for MSUN$_\emptyset(G - e, m, H, d)$, for all $e \in F$. Suppose there is a vertex χ^Z of MSUN$_E(G, m, H, d)$ satisfying $\chi^Z(B) = 1$, say $B \cap Z = \{f\}$. Then $\sum_{e \neq f} \sum_{t=1}^{T_e} g_e^t \chi^Z(e, t) \leq g(B^< \setminus I(f)) < b$, which contradicts the survivability condition when edge f is deleted from the supply graph. So $\chi^Z(B) \geq 2$ for all vertices χ^Z of MSUN$_E$. □

We call inequality (12) *strengthened band inequality*.

If the given band was derived from a cut inequality, then the strengthened band inequality can be shown to define a facet of MSUN$_E$ in many cases. This is done in Subsect. 5.2.

5. Improved formulations based on relations to connectivity network design

In this section we exploit the relation of the MULTISUN problem to the design of uncapacitated networks satisfying connectivity requirements. This leads to several classes of facet defining inequalities for the MSUN-polytopes.

First, we give a simple but useful lemma. It describes the main relations between the two classes of problems and will be used in later facet proofs. To simplify the presentation, we assume throughout this section that the set of demand edges and its adjacent nodes induce a connected graph with node set R, the *terminal set*. We assume that the supply graph G is connected as well. Let the polytope CON in $\mathbb{R}^E$ be the convex hull of incidence vectors of subgraphs of G that are connected with respect to R, that is, those subgraphs containing a Steiner tree with terminal set R. Similarly, let 2ECON (2NCON) be the polytope associated with 2-edge connected (2-node connected) subgraphs with respect to R. For $F \subseteq E$ and a band B of F, let $I(F)$, $B^<$ and $B^>$ be defined as in Sect. 4.

Lemma 13. *Let $F \subseteq E$. Then the following statements hold.*

(i) If (V, F) is connected w.r.t. R, then $I(F)$ is feasible for MSUN$_\emptyset$.
(ii) If (V, F) is 2-edge connected (2-node connected) w.r.t. R, then $I(F)$ is feasible for MSUN$_E$ (MSUN$_{E\cup V}$).
(iii) If $a^{\mathrm{T}}y \geq \alpha$ is a valid inequality for CON (or 2ECON, 2NCON, resp.), then $\sum_e a_e x_e^1 \geq \alpha$ is a valid inequality for MSUN$_\emptyset$ (MSUN$_E$, MSUN$_{E\cup V}$ resp.)

Proof. Straightforward, using the fact that we can install a sufficiently "large" capacity on each edge. □

In the following we will exploit fact (iii) of this lemma to derive valid and nonredundant inequalities for MSUN$_S$ polyhedra from inequalities for CON and 2ECON polyhedra. We will concentrate on some nonredundant inequalities listed in [9, 11]: partition, cut, and lifted two-cover inequalities.

Partition inequalities for CON are defined as

$$x(\cup_{i=1}^{p}\delta(V_i)) \geq p-1 \tag{14}$$

for all partitions $\mathscr{P} = \{V_1, \ldots, V_p\}$ of V (here $p \geq 2$) where each V_i contains some node of R. Such partitions are called *proper partitions*. Associated with $\mathscr{P}$ we have the graph $\hat{G} = (\hat{V}, \hat{E})$ obtained by shrinking each node set V_i into one node and then deleting parallel edges.

Remark 15. By [9], partition inequalities define facets of CON if and only if (i) $G[V_i]$ is connected for all i, (ii) $G[V_i]$ is 2-edge connected w.r.t. R for all i, and (iii) $\hat{G}$ is 2-node connected.

Cut inequalities for 2ECON are defined as

$$x(\delta(W)) \geq 2 \tag{16}$$

for all subsets W of V such that both W and $V \setminus W$ contain nodes of R.

Remark 17. By [11, 19], the cut inequality defines a facet of 2ECON if $G[W]$ and $G[V \setminus W]$ are 2-edge connected and G is 3-edge connected. (Actually, somewhat weaker conditions are already sufficient and at the same time necessary).

Lifted two-cover inequalities for 2ECON are defined by pairwise disjoint nonempty node sets $W_1, \ldots, W_p$, each containing some node of R, and edges $e_1, \ldots, e_k$ ($3 \leq k \leq p$) with $e_i \in \delta(W_i) \cap \delta(W)$ for $i = 1, \ldots, k$. Here $W \neq V$ denotes the union of all W_i, and k is assumed to be odd. The lifted two-cover inequalities have the form

$$x((\cup_{i=1}^{p} \delta(W_i)) \setminus \{e_1, \ldots, e_k\}) \geq p - \lfloor k/2 \rfloor. \tag{18}$$

Remark 19. In [19] it was shown that the lifted two-cover inequality defines a facet of 2ECON if G is complete, and the graphs induced by $V \setminus W$ and by all W_i are three-edge connected.

By Lemma 13 (iii) any valid inequality $a^{\mathrm{T}} y \geq \alpha$ for CON (2ECON, 2NCON) can be transformed to a valid inequality for the respective MSUN-polyhedron. When a is a 0/1-vector, the set $B := \{ (e, 1) \in I \mid a_e > 0 \}$ can be interpreted as a band of the support of $a^{\mathrm{T}} y \geq \alpha$. In the next subsections we will see how the inequality $\sum_e a_e x_e^1 \geq \alpha$ can be made facet-inducing for the respective MSUN-polyhedron by finding a band above B such that the inequality stays valid. Certainly this technique can be applied as well to other valid inequalities for CON (or 2ECON, 2NCON) not listed here. Especially we observe that it can be applied to the so-called node-partition inequalities (in [11]) valid for 2NCON, but not 2ECON. They are a generalization of partition inequalities (14).

5.1. Partition inequalities

Consider a proper partition $\mathscr{P}$ of V into node sets V_i, $i = 1, \ldots, p$, called *shores*. Let $\hat{G} = (\hat{V}, \hat{E})$ be defined as above, and similarly, let $\hat{H} = (\hat{V}, \hat{D})$ be obtained from H by shrinking each node set V_i into one node. Note that the edge set $\hat{E}$ consists of those edges in E with end nodes belonging to different shores.

Definition 20. *A band B in $\hat{E}$ is called a valid $\mathscr{P}$-band if for each nonempty $W \subset \hat{V}$ with connected shores (i.e., $\hat{G}[W]$ and $\hat{G}[\hat{V} \setminus W]$ are connected) we have that*

$$m(\{ (e,t) \in B^{<} \mid e \in \delta_{\hat{G}}(W) \}) \; < \; d(\delta_{\hat{H}}(W)). \tag{21}$$

A valid $\mathscr{P}$-band B is called maximal if no valid $\mathscr{P}$-band above B exists.

Proposition 22. *Let $\mathscr{P}$ be a proper partition and assume that B is a valid $\mathscr{P}$-band. Then the partition inequality*

$$x(B) \geq p - 1 \tag{22}$$

is valid for MSUN$_\emptyset(G, m, H, d)$.

Proof. Let u and v be two distinct nodes in $\hat{V}$, and let W' be a node subset of $\hat{V}$ such that $u \in W'$ and $v \notin W'$. We observe that if, say, $\hat{G}[W']$ is not connected, then we can find $W \subseteq W'$ or $W \supseteq W'$ such that the cut $\delta_{\hat{G}}(W)$ is a subset of $\delta_{\hat{G}}(W')$, has connected shores and separates u and v (recall that $\hat{G}$ is connected). Since B is a valid $\mathscr{P}$-band, each feasible $Z \subseteq I$ satisfies $(e, t_e^B) \in Z$ for at least one edge $e \in \delta_{\hat{G}}(W) \subseteq \delta_{\hat{G}}(W')$, otherwise the capacity across this cut would be too small. Thus, by Menger's theorem, the subgraph of $\hat{G}$ consisting of those edges e with $(e, t_e^B) \in Z$ contains a $[u, v]$-path, and, since u and v were arbitrary, this graph contains a spanning tree of $p-1$ edges. This proves that the partition inequality holds for χ^Z, and, by convexity, the inequality is valid for MSUN$_\emptyset$. $\square$

We next discuss those partition inequalities that define facets of $\mathrm{MSUN}_\emptyset$.

Theorem 24. *Let $\mathscr{P}$ be a proper partition and assume that B is a valid $\mathscr{P}$-band. Then the partition inequality (22) defines a facet of $\mathrm{MSUN}_\emptyset$ if the following conditions hold:*

(i) The partition inequality (14) defined by partition $\mathscr{P}$ induces a facet of CON (see Remark 15), and
(ii) B is maximal.

Moreover, condition (ii) is necessary for (22) to define a facet of $\mathrm{MSUN}_\emptyset$.

Proof. Necessity of (ii) can be seen by adding ordering constraints to a partition inequality defined by a maximal band.

To prove sufficiency, assume that both conditions hold, and let $a^{\mathrm{T}}x \geq \alpha$ be the inequality (22), with $\alpha = p - 1$. Let $F(a, \alpha) = \{x \in \mathrm{MSUN}_\emptyset \mid a^{\mathrm{T}}x = \alpha\}$ be the face of $\mathrm{MSUN}_\emptyset$ induced by this inequality. We may assume that $F(a, \alpha) \subseteq F(b, \beta)$, where $F(b, \beta)$ is the facet of $\mathrm{MSUN}_\emptyset$ induced by some inequality $b^{\mathrm{T}}x \geq \beta$. We prove now that (b, β) is a scalar multiple of (a, α).

First, consider an edge $e \notin \hat{E}$, which is contained in, say, shore V_1. There is a Steiner tree $F \subseteq E \setminus \{e\}$ such that χ^F satisfies (14) with equality, because this inequality defines a facet of CON different from a nonnegativity constraint. Thus we see from Lemma 13 (i) that $\chi^{I(F)} \in F(a, \alpha)$. For $t = 1, \ldots, T_e$, the incidence vector of $I(F)_{(t)} := I(F) \cup \{(e, 1), \ldots, (e, t)\}$ also lies in $F(a, \alpha)$, hence in $F(b, \beta)$. Then b_e^t must be zero for $t = 1, \ldots, T_e$. By similar arguments one can show that $b_e^t = 0$ for each $(e, t) \in B^<$.

Consider an edge $e \in \hat{E}$ for which $t_e^B < T_e$. Since B is assumed maximal, the band obtained from B by replacing (e, t_e^B) by $(e, t_e^B + 1)$ is *not* a valid $\mathscr{P}$-band. Thus there exists a nonempty node set $W \subset \hat{V}$ with connected shores and an edge $e \in \delta_{\hat{G}}(W)$, such that

$$m(\{\,(g,t) \in B^< \mid g \in \delta_{\hat{G}}(W)\,\}) \;+\; m(e, t_e^B) \;\geq\; d(\delta_{\hat{H}}(W)). \tag{25}$$

Now, choose a spanning tree T in $\hat{G}$ that contains e but no other edge of $\delta_{\hat{G}}(W)$. This is possible since the shores of this cut both are connected. Define Z as the union of the four index sets $I(E \setminus \hat{E})$, $I(T \setminus \{e\})$, $\{\,(g, t) \in B^< \mid g \in \delta_{\hat{G}}(W)\,\}$, and $\{(e, t_e^B)\}$. We claim that Z is feasible for $\mathrm{MSUN}_\emptyset$. To prove this, we describe a feasible multicommodity flow in the network $\mathscr{N} = (G, y)$, where y is the capacity vector associated with the design Z. By connectedness of each $G[V_i]$ and $\hat{G}[W]$ (and the "large" capacities there), we can clearly route within W (or $V \setminus W$) all demands with both end nodes belonging to W (or $V \setminus W$). All demands in $\delta_H(W)$ can be routed across the cut, since the capacity of the cut is at least as large as the sum of crossing demands, by (25). Thus Z is feasible. Furthermore, χ^Z satisfies $a^{\mathrm{T}}x \geq \alpha$ with equality. By monotonicity, $Z' := Z \cup \{(e, t_e^B + 1)\}$ is also feasible and its incidence vector also lies in $F(a, \alpha)$, from which we obtain that $b(e, t_e^B + 1) = 0$, as above. Continuing in this manner, we conclude that $b_e^t = 0$ for $t > t_e^B$ and $e \in \hat{E}$.

Thus, so far we have shown that $b_e^t = 0$ for each $(e, t) \in I \setminus B$. Finally we prove that all b_e^t with $(e, t) \in B$ must be equal. In fact, since (14) defines a facet of CON, whose affine hull does not contain the origin, there are $|E|$ linearly independent incidence vectors of edge sets containing Steiner trees such that each of these vectors satisfies (14) with equality. Let A be a nonsingular $|\hat{E}| \times |\hat{E}|$-dimensional submatrix whose columns correspond to spanning trees of $\hat{G}$. Also let b' (resp. a') be the $|\hat{E}|$-dimensional vector obtained from b (resp. a) by removing the components for $(e,t) \in I \setminus B$; all shown above to be zero. Thus we have $(b')^{\mathrm{T}} A = \beta\mathbf{1}$ and $(a')^{\mathrm{T}} A = (p-1)\mathbf{1}$, which implies that (a, α) is a scalar multiple of (b, β). Therefore $F(a, \alpha)$ is a facet. $\square$

Unfortunately, it may not be easy to check algorithmically whether a band is a valid $\mathscr{P}$-band. However, for small p it can be done by simply checking all cuts explicitly. Furthermore, we remark that a band is valid provided that the following multicommodity flow problem is feasible. First, we define a capacity vector y by increasing with (a suitably small) ϵ the total capacity below the band B on each edge, and let d be the demand vector as usual. The multicommodity flow problem of interest is then obtained by reversing the roles of y and d, and of G and H (that is, viewing y as demands and d as capacities).

5.2. Cut inequalities

In the following sections we consider the polytope $\mathrm{MSUN}_E(G,H,m,d)$ associated with Model 2. Let R be the terminal set, and let $y(\delta(W)) \geq 2$ be a cut inequality (16) for 2ECON, defined by node set W such that both W and $V \setminus W$ contain nodes of R (the terminal set). Let B be a band of $\delta_G(W)$ with the property

(26) $$m(B^< \setminus I(e)) < d(\delta_H(W)) \quad \text{for all } e \in \delta_G(W).$$

Then the *cut inequality* for MSUN_E defined by W and B is

(27) $$x(B) \geq 2.$$

Since this inequality can also be seen as a strengthened band inequality derived from the metric inequality $y(\delta_G(W)) \geq d(\delta_H(W))$, it is valid for MSUN_E (by Lemma 11) and therefore also for $\mathrm{MSUN}_{E\cup V}$.

The next theorem gives a sufficient condition for a cut inequality (27) to define a facet of MSUN_E.

Theorem 28. *The cut inequality defined by W and B defines a facet of $\mathrm{MSUN}_E(G,H,m,d)$ if the following conditions are satisfied:*

(i) The cut inequality $y(\delta(W)) \geq 2$ defines a facet for 2ECON (see Remark 17).
(ii) For $g \in \delta(W)$ minimizing $m(B^< \cap I(g))$ and for all $f \in \delta(W) \setminus \{g\}$, the edge set $E(W) \cup E(V \setminus W) \cup \{f, g\}$ defines a 2-edge connected graph with respect to R.
(iii) There is no band above B satisfying condition (26).

Proof. Let $b^{\mathrm{T}} x \geq \beta$ be a valid inequality defining a facet of MSUN_E that contains the face defined by $x(B) \geq 2$.

With condition (i) and Lemma 13 (ii) one can prove $b_e^t = 0$ for all $(e,t) \in I(E(W) \cup E(V \setminus W))$ and for all $(e,t) \in B^<$ in much the same way as demonstrated in the proof of the previous theorem.

Let f be some edge in $\delta_G(W)$ with $t_f^B < T_f$ and let g minimize $m(B^< \cap I(g))$. If $g = f$, choose a $g \neq f$ minimizing $m((B^< \setminus I(f)) \cap I(g))$. Since, by condition (iii), it is not possible to replace t_f^B by $t_f^B + 1$ without violating (26), we have $m(B^< \setminus I(g)) + m(f, t_f^B) \geq d(\delta_H(W))$. Condition (ii) ensures that the incidence vector of

$$I(E(W) \cup E(V \setminus W)) \;\cup\; B^{<} \;\cup\; \{(f, t_f^B)\} \;\cup\; I(g)$$

is feasible and satisfies $x(B) = 2$. Especially, if edge g is failing, the cut $\delta_G(W)$ has still sufficiently high capacity to route all demands between W and $V \setminus W$. By increasing the capacity of edge f, we can prove $b_f^t = 0$ for all $t > t_f^B$. This shows that $b_e^t = 0$ for all $(e,t) \in B^{>}$.

With condition (i) one can now prove that all b_e^t with $(e,t) \in B$ have the same value, as in the last part of the proof of Theorem 24. Therefore all coefficients of the band must be the same. This implies that $b^{\mathrm{T}}x \geq \beta$ is a positive scalar multiple of $x(B) \geq 2$ as desired. □

5.3. Lifted two-cover inequalities

The lifted two-cover inequalities (18) for the 2ECON polytope can be generalized in much the same way as the cut and partition inequalities.

Let $W_1, \ldots, W_p, e_1, \ldots, e_k$ define a lifted two-cover inequality (18). We assume that $\delta(W_i) \cap D \neq \emptyset$, for $i = 1, \ldots, p$ (recall that D is the set of demand edges). Now define, for each $i \leq k$, a band B_i of $\delta(W_i) \setminus \{e_i\}$ such that $x(B_i) \geq 1$ is valid for $\mathrm{MSUN}_{\emptyset}(G - e_i, m, H, d)$. Define also, for $i = k+1, \ldots, p$, a band B_i of $\delta(W_i)$, such that $x(B_i) \geq 2$ is valid for $\mathrm{MSUN}_E(G, m, H, d)$.

Moreover, we assume that these bands are "consistent" in the sense that if an edge e lies in two bands B_i and B_j, then $t_e^{B_i} = t_e^{B_j}$. Denote the union of all B_i by B. Then the *lifted two-cover inequality* defined by all W_i, e_i and B is

$$x(B) \geq p - \lfloor k/2 \rfloor. \tag{29}$$

Lemma 30. *The lifted two-cover inequality (29) is valid for MSUN_E.*

Proof. Note that $x(B_i) \geq 1$ for $i = 1, \ldots, k$, is valid for MSUN_E. Let W be the union of all W_i. Add the valid inequalities

$$\begin{aligned} x(B_i) &\geq 1 \quad \text{for } i = 1, \ldots, k, \\ x(B_i) &\geq 2 \quad \text{for } i = k+1, \ldots, p, \\ x(e, t_e^B) &\geq 0 \quad \text{for all } e \in \delta(W) \setminus \{e_1, \ldots, e_k\}, \end{aligned}$$

divide the result by 2, and round up the coefficients of the right-hand side. The resulting valid inequality is the lifted two-cover inequality (29). □

From the validity proof it can be seen that the lifted two-cover inequality is valid, but redundant, if k is even. Note that, for odd k, an integer vector $x \in \mathrm{MSUN}_E$ satisfies (29) with equality if and only if it satisfies with equality all except one of the inequalities used in the validity proof. The next theorem gives a sufficient facet condition for the lifted two-cover inequalities.

Theorem 31. *The lifted two-cover inequality (29) defines a facet of MSUN_E if*

- *(i) the lifted two-cover inequality (18) defined by the W_i and e_i induces a facet of 2ECON (see Remark 19);*
- *(ii) for any $i = k+1, \ldots, p$ and for any two edges $f, g \in \delta(W_i)$, there exists a vertex y of 2ECON with $y_f = y_g = 1$, satisfying (18) and $y(\delta(W_i)) \geq 2$ with equality; and*
- *(iii) the band B is maximal in the sense that it is not possible to increase any t_e^B without violating the conditions of the definition of B.*

Proof. Let $\{ x \in \text{MSUN}_E \mid x(B) = p - \lfloor k/2 \rfloor \} \subseteq \{ x \in \text{MSUN}_E \mid b^{\mathrm{T}}x = \beta \}$, where the last set is a facet of MSUN_E. Since (18) defines a facet of 2ECON, it is possible to prove $b_e^t = 0$ for all $e \in E$ and $t < t_e^B$ (or $t \leq T_e$ if t_e^B does not exist).

To prove $b_f^t = 0$ for all $(f,t) \in B^>$, pick some edge $f \in \cup_i \delta(W_i)$. By the maximality of B it is not possible to increase t_f^B by 1, so there exists W_i adjacent to f such that $x(B_i) \geq 1$ or 2 is not valid any more. Suppose $i = p > k$. Then B_p must violate (26), that is, there must exist an edge $g \neq f$ such that $m(B_p^< \setminus I(g)) + m(f, t_f^B) \geq d(\delta_H(W_p)) > 0$. Now, construct a feasible solution x with $b^{\mathrm{T}}x = \beta$ by equipping edge f with exactly capacity $m(f, t_f^B)$, edge g with highest capacity, and all other edges in $\delta(W_p)$ with capacity $m(e, t_e^B - 1)$. (So far, exactly two nonzero coefficients of the lifted two-cover inequality have been "used".) To complete the present solution with high-capacity edges, we use condition (ii) for $i = p$. Furthermore all edges with $b(I(e)) = 0$ can be equipped with high capacity. The so constructed vector x is feasible for MSUN_E, because when edge g fails, only the demands originating in W_p need to be routed on low-capacity edges, and the capacity of $\delta(W_p)$ is high enough to make this possible. Moreover, $G[W_p]$ is connected, otherwise (18) would not define a facet of 2ECON. Since any further capacity extension of edge f also leads to a feasible solution with $b^{\mathrm{T}}x = \beta$ we have $b_f^t = 0$ for all $t > t_f^B$. Similar constructions can be done for all other edges f in $\cup\delta(W_i) \setminus \{e_1, \ldots, e_k\}$. So we have proved that all coefficients of value 0 in the lifted two-cover inequality have also value 0 in $b^{\mathrm{T}}x \geq \beta$. To prove that both inequalities are in fact equivalent, we can use the $|E|$ affinely independent integer vectors in 2ECON multiplied with the high capacity (and proceed as in the last part of the proof of Theorem 24). □

Concerning the separation problems for the inequalities listed here, it was shown in [10] that the problem of determining a violated partition inequality (14) for CON and a violated lifted two-cover inequality (18) for 2ECON, given some nonnegative vector $y \in \mathbb{R}^E$, is NP-complete. Therefore the separation problems for inequalities (22) and (29) are NP-complete, too. The NP-complete separation problem of band inequalities (8) for $\text{ICOV}(g,b)$ can be reduced to separation of cut inequalities (27) for $\text{MSUN}_E(G,m,H,d)$ with suitably defined (G,m,H,d), hence the latter problem is also NP-complete.

Despite these negative results, it is possible to design heuristics to determine violated inequalities for MSUN-polyhedra. Since partition, cut, and lifted two-cover inequalities played a major role in solving connectivity problems to optimality, see [10], the corresponding inequalities will probably be useful in finding good lower bounds for MULTISUN-problems. Our computational results for MSUN_E using only cut inequalities (27) and band inequalities (8) were already quite encouraging, as is reported in the next section.

6. Computational results

We describe in this section the main features of a cutting plane algorithm and heuristics for Model 2, where the failure of any single node or link is considered. Furthermore, some preliminary computational results are given.

The basic idea of the cutting plane algorithm is to solve a sequence of ever tighter LP-relaxations of the problem at hand, by generating linear inequalities as they are needed to ensure feasibility. The algorithm is as follows:

1. solve an initial LP containing a few valid inequalities for MSUN_S, using the cost function c;
2. if the optimal solution $\overline{x}$ of the current LP is integral and satisfies all inequalities of (4), output $\overline{x}$ as an optimal solution to Model 2; otherwise $\overline{x}$ is not feasible for MSUN_S, so try to find strengthened band inequalities and other inequalities valid for MSUN_S that are violated by $\overline{x}$;
3. if violated inequalities were found, add them to the current LP, solve this new LP, and go to Step 2; otherwise stop and output $c^{\mathrm{T}}\overline{x}$ as the best lower bound found for the problem.

This approach is called a cutting plane algorithm, because geometrically, in Step 3 the infeasible $\overline{x}$ is cut off by hyperplanes, so called cutting planes, from the region of feasible solutions MSUN_S. Unfortunately, we do not have a complete linear description of MSUN_S, so we have to settle with a partial linear description of this polytope given by the inequalities in this paper. But the hope is that this is sufficient to produce good lower bounds for the optimal value of the problem.

We next explain how the subproblems are solved. In Step 1 the inequalities of the initial LP are chosen by approximately "solving" in greedy fashion the dual LP of min $c^{\mathrm{T}}x$, where x satisfies all cut inequalities (27). Whatever inequality is "used" in the greedy solution is placed into the initial LP.

In Step 2, the feasibility check of $\overline{x}$ is done as follows: if $\overline{x}$ defines a network that is not two-node connected, violated cut inequalities (27) can be derived from the two-connected components of the network. These inequalities are added to the current LP. If this simple connectivity test fails to produce inequalities, we compute the capacity vector $\overline{y} \in R^E$ from $\overline{x}$ by $\overline{y}_e := \sum_{t=1}^{T_e} m_e^t \overline{x}_e^t$ for all $e \in E$. Now it needs to be decided whether the network G with capacity $\overline{y}$ allows a multicommodity flow in each failure situation. This can be done by solving, for each failure situation, a certain LP, which gives back either the routings for each traffic demand, or a violated metric inequality (1), if a multicommodity flow does not exist. Since all these LPs are very similar, they can be efficiently solved by using the knowledge of previously found optimal solutions. If the violated metric inequality is a cut inequality (2) (most of them are), we rewrite it in terms of x as $g^{\mathrm{T}}x \geq b$ and try to heuristically identify a band inequality (8) of ICOV(g, b) that is violated by $\overline{x}$. After this, we attempt to strengthen the found band inequality of right-hand side 1 to a band inequality (12) valid for MSUN_E with right-hand side 2. This is then a cut inequality of type (16), which is often facet-defining.

Lifted two-cover inequalities are so far not identified by our program.

Since the cutting plane approach is not guaranteed to produce a feasible and optimal solution, we implemented several heuristics to find feasible and hopefully good solutions. Basically these heuristics first "blow up" an (infeasible) $\overline{x}$ obtained at some step of the algorithm until it becomes feasible, and afterwards attempt to sequentially reduce overflow capacities.

Unfortunately, we have not yet got hold of a real-world test example, but it was possible to determine a quite realistic test example with the following characteristics. The underlying graph has 27 nodes (denoting some cities in southern Norway), 51 edges, and its demand graph contains 19 demands, three of them of size 100, the others of size 6. For each line one has the choice between four different capacities: the already existing (free) capacity, 63, 252, and 1008, whose costs depend on the physical length of the cable. By varying the free capacity between 0 and 6, we obtained different versions of the problem. The relative gaps between the best heuristic and

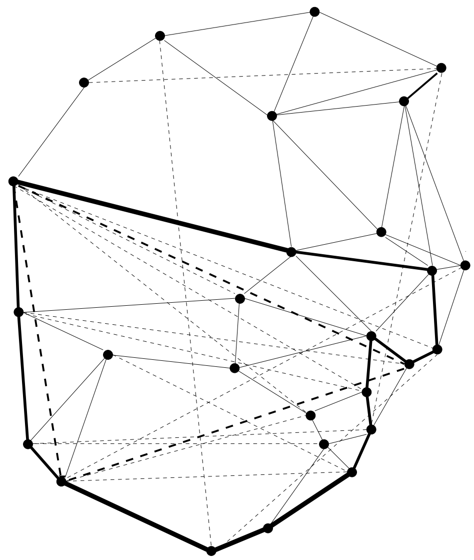

Fig. 2. An optimal solution

and the best lower bound ranged between 0 and 5 percent (of the lower bound value). The running times of the cutting plane stage are between 15 seconds and 2 minutes on a Solbourne work station with a Sparc 10 processor using CPLEX as the LP solver. The time for the heuristics was between one and five minutes.

When the free capacity was set to 6, the algorithm terminated with an optimal solution, depicted in Fig. 2. There, the broken lines denote demands, with thick lines for high demands, and the unbroken lines define the network. The ring of thick lines is of capacity 252, except for four lines that have capacity 1008, the single thick line on the upper right is of capacity 63, and all other lines only use the already existing capacity of 6.

7. Conclusions and future work

We have studied an integer linear programming model for the multicommodity survivable network design problem (MULTISUN): find a minimum cost capacitated network that allows certain multicommodity flows under single edge and node failures. By exploiting connections to both knapsack-like problems and uncapacitated connectivity design problems, we found several classes of valid inequalities (band-, partition and lifted two-cover inequalities) for MULTISUN polytopes. Furthermore we discussed facet conditions for these inequalities. We presented some preliminary computational results with a cutting plane algorithm for solving MULTISUN problems, based on the polyhedral results in this paper. Further tests will be reported in a forthcoming paper.

Another interesting area is to develop fast heuristics for the MULTISUN problem accompanied with worst-case analysis. (For connectivity design problems, empirical good heuristic methods have been found).

The models presented here can be extended or varied in various ways. For instance, one may include flow (routing) costs in the objective function. Also, from an algorithmic point of view, it may be useful to exploit possible further structure on the cost function in certain applications. We would also like to study additional constraints on the routing, like regional constraints and diversification (saying that, for each demand, only a certain fraction of the flow may go through each node or edge, see [7]). These modifications are interesting from an applied point of view and are worthwhile to be investigated theoretically.

References

1. Avis, D. (1980): On the extreme rays of the metric cone. Can. J. Math. **32**, 126–144
2. Balakrishnan, A., Graves, S.C. (1989): A composite algorithm for a concave-cost network flow problem. Networks **19**, 175–202
3. Balas, E., Zemel, E. (1984): Lifting and complementing yields all the facets of positive zero-one programming polytopes. In: Cottle, R.W., Kelmanson, M.L., Korte, B., eds., Mathematical Programming, pp. 13–24. Elsevier Science Publisher, North-Holland
4. Berge, C. (1973): Graphs and Hypergraphs. North-Holland, Amsterdam
5. Crowder, H., Johnson, E.L., Padberg, M. (1983): Solving large-scale zero-one linear programming problems. Oper. Res. **31**(5), 803–834
6. Dahl, G. (1991): Contributions to the Design of Survivable Directed Networks. PhD thesis, University of Oslo. Technical Report TF R 48/91, Norwegian Telecom Research, Kjeller, Norway
7. Dahl, G., Stoer, M. (1992): MULTISUN – mathematical model and algorithms. Technical Report TF R 46/92, Norwegian Telecom Research, Kjeller, Norway
8. Gavish, G., Trudeau, G., Dror, M., Gendreau, M., Mason, L. (1989): Fiberoptic circuit network design under reliability constraints. IEEE J. Selected Areas Commun. **7**(8), 1181–1187
9. Grötschel, M., Monma, C.L. (1990): Integer polyhedra associated with certain network design problems with connectivity constraints. SIAM J. Discrete Math. **3**, 502–523
10. Grötschel, M., Monma, C.L., Stoer, M. (1992): Computational results with a cutting plane algorithm for designing communication networks with low-connectivity constraints. Oper. Res. **40**(2), 309–330
11. Grötschel, M., Monma, C.L., Stoer, M. (1992): Facets for polyhedra arising in the design of communication networks with low-connectivity constraints. SIAM J. Optimization **2**(3), 474–504
12. Iri, M. (1971): On an extension of the maximum-flow minimum-cut theorem to multicommodity flows. J. Oper. Res. Society Japan **13**, 129–135
13. Lomonosov, M.V. (1985): Combinatorial approaches to multiflow problems. Discrete Appl. Math. **11**, 1–93
14. Martello, S., Toth, P. (1990): Knapsack Problems, Algorithms and Computer Implementations, chapter The Multiple-Choice Knapsack Problem, pp. 77–80. Wiley, Chichester
15. Minoux, M. (1971: Optimum synthesis of a network with non-simultaneous multicommodity flow requirements. In: Hansen, P., ed., Studies on Graphs and Discrete Programming, pp. 269–277. North-Holland Publishing Company
16. Nemhauser, G.L., Wolsey, L.A. (1988): Integer and Combinatorial Optimization. Wiley, Chichester
17. Onaga, K., Kakusho, O. (1971): On feasibility conditions of multicommodity flows in networks. Trans. Circuit Theory **CT-18**(4), 425–429
18. Schrijver, A. (1986): Theory of Linear and Integer Programming. Wiley, Chichester
19. Stoer, M. (1992): Design of Survivable Networks, volume 1531 of Lect. Notes Math. Springer, Berlin
20. Wolsey, L.A. (1990): Valid inequalities for 0-1 knapsacks and MIPs with generalised upper bound constraints. Discrete Appl. Math. **29**, 251–261

This article was processed by the author using the LaTeX style file *pljour1* from Springer-Verlag.