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On the locking phenomenon for a class of elliptic problems

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Summary. In this paper we study the numerical behaviour of elliptic problems in which a small parameter is involved and an example concerning the computation of elastic arches is analyzed using this mathematical framework. At first, the statements of the problem and its Galerkin approximations are defined and an asymptotic analysis is performed. Then we give general conditions ensuring that a numerical scheme will converge uniformly with respect to the small parameter. Finally we study an example in computation of arches working in linear elasticity conditions. We build one finite element scheme giving a locking behaviour, and another one which does not.

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1. Introduction

The locking phenomenon is well known in the world of engineering. It generally consists in a loss of meaning of numerical results when the computations are made for some values of the parameters, while the results of the computations remain valid for a wide range of other values (we are refering to the *meaning of the numerical results* with regard to a supposed suitable model representing a physically unknown quantity).

This phenomenon appears in elastic structures such as plates and shells where the parameter is the thickness of the structure and the unknown is its displacement. It happens that common numerical schemes modeling the elastic behavior of the structure give wrong displacements if the thickness is too small.

For example, when we consider the so called Mindlin-Reissner model for moderately thin plates which is often used by engineers, it is well known that many numerical schemes for this model are satisfactory only when the thickness parameter ε is not too small. But, for a very small ε, some bad behavior (such as the ”locking” phenomenon) can occur. The problem is to present methods which are uniformly good as ε approaches zero and to show how error estimates remain independent of

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ε. This problem is solved in the following papers: a thorough analysis is provided by Arnold in [2] for the one -dimensional case (Timoshenko model) and in [3] for the two -dimensional case; Brezzi and Fortin [8] give a mixed formulation of the Mindlin-Reissner plate model and derive uniform error estimates from a F. E. M.

The locking effects may appear in the finite element approximation of elasticity problems when the Poisson ratio ν is close to 0.5 : it is well-known that the performance of certain commonly used finite elements deteriorates as ν goes to 0.5 (see [5]).

The locking phenomenon also might occur in three-dimensional elasticity when we compute the displacement of a thick plate (thickness parameter ε). As a matter of fact, M. Vidrascu [20] uses prismatic finite elements Q2 of height ε and obtains, for small values of ε, a very poor behaviour of the displacement compared to the associated two-dimensional thin plate displacement given by the asymptotic theory developped by Ciarlet [11] (Kirchhoff Love model). An analysis of this locking phenomenon is carried out in [15] for the Galerkin method.

J.-L. Akian and E. Sanchez-Palencia give in [1] a phenomenon of membrane locking in the approximation of thin elastic shells by flat elements. In this work the *mean surface* of the shell (with kinematic boundary conditions) admits inextensional displacements and is approximated by a polyhedral surface where the edges enjoy stiffness properties causing a phenomenon of membrane locking.

Another example concerns arches studied under the Kichhoff-Love model. It is standard to compute the displacement field using an approximation of the arch by straight beam elements of small length. For a fixed thickness, the convergence of this scheme is studied in [6] : *if h denotes the length of a beam, the displacement converges in $O(h)$ but when the thickness is very small, the numerical results are wrong*. This phenomenon has been specialy studied in [13] : *examples are given where the numerical results deteriorate when the mesh size is fixed and the thickness becomes small* and it is proved that *if h is taken in the order of the square of the thickness, the convergence is insured*. But this is numerically prohibitive. For this arch problem, F. Kikuchi [14] gives an answer, exhibiting a uniformly converging scheme. The question of finding the best possible behaviour with respect to the thickness for a scheme which has a chosen order with respect to the mesh size becomes essential and is studied in [9].

At last, in a recent paper, Ivo Babuška and Manil Suri [4] develop precise mathematical definitions for locking and robustness, give their quantitative characterization, and prove some general theorems. Moreover a model problem involving heat transfer is analyzed using this mathematical framework, and various related computational results are described.

Let us briefly outline the content of this paper.

In Sect. 2 we give the statements of the class of considered problems and we perform an asymptotic analysis. In Sect. 3 we define a family of Galerkin approximations in which there might happen a loss of convergence when the small parameter reaches zero. In Sect. 4 we give general conditions ensuring that a numerical scheme will converge uniformly with respect to the small parameter without needing the use of a mixed formulation.

In Sect. 5 we study the example of the computation of arches working in linear elasticity conditions, under the Kirchhoff-Love model. The uniform error estimates was already considered by Kikuchi [14] using direct computations. Our present analysis is quite different because we apply the result of the previous sections and we give some sufficient conditions to get uniform convergence. We build one finite element scheme giving a locking behaviour (Subsect. 5.2), and one which does not (Subsect. 5.3). Both do not need the use of mixed formulations.

2. The continuous problem. General setting and limit when ε goes to zero

Let V be a real Hilbert space where the inner product is denoted $< u, v >$ and the associated norm $||u||$. We are given two bilinear continuous symmetric forms defined on V, say:

$$a_0 : V \times V \to \mathbb{R}, \qquad a_1 : V \times V \to \mathbb{R}.$$

Let $\varepsilon \in]0, 1]$ be a small real parameter. We define $a^\varepsilon : V \times V \to \mathbb{R}$ by:

$$a^\varepsilon(u, v) = \frac{1}{\varepsilon}\ a_0(u, v) + a_1(u, v).$$

The *fundamental hypothesis* in this paper are the following:

(H1) For each $\varepsilon \in]0, 1]$, $a^\varepsilon(u, v)$ is coercive:

$$\forall\, \varepsilon\ \in\ ]0, 1], \qquad \exists\, \alpha(\varepsilon) > 0 \text{ such that } \qquad \forall v\ \in V,\ a^\varepsilon(v, v) \geq \alpha(\varepsilon)\ ||v||^2.$$

(H2) The bilinear form a_0 is positive and its kernel:

$$G = \{w \in V;\ a_0(w, v) = 0, \forall v \in V\}$$

is not reduced to the origin.

We remark that the kernel G is also equal to the set $\{w \in V; a_0(w, w) = 0\}$ since we have the inequality $|a_0(w, v)|^2 \leq a_0(v, v)\, a_0(w, w)$ which holds for any $v \in V$ and any $w \in G$.

Under these hypotheses it is clear that:

– all the forms a^ε, for $\varepsilon \in]0, 1]$ are uniformly α - coercive where we denote α for $\alpha(1)$:

$$\forall\, \varepsilon\ \in\ ]0, 1], \qquad \forall\, v \in V, \qquad a^\varepsilon(v, v)\ \geq\ a^1(v, v)\ \geq \alpha ||v||^2,$$

– the bilinear form a_1 is G - elliptic since we have:

$$\forall\, w \in G, \qquad a^1(w, w) = a_1(w, w)\ \geq \alpha ||w||^2.$$

Let $l : V \to \mathbb{R}$ be a given linear form defined on V. From the Lax-Milgram theorem, for each $\varepsilon \in]0, 1]$, the equation:

(1) $$u^\varepsilon \in V, \qquad a^\varepsilon(u^\varepsilon, v) = l(v), \qquad \forall v \in V,$$

has one and only one solution as well as the equation:

(2) $$u^0 \in G, \qquad a_1(u^0, w) = l(w), \qquad \forall w \in G.$$

We are interested in the limit of u^ε when ε goes to zero. We have the following result:

Theorem 1. *When $\varepsilon \to 0$, the solution u^ε of the Eq. (1) strongly converges in V to the unique solution of the Eq. (2). The limit u^0 is equal to 0 if and only if $l \in G^\perp$.*

Proof. It is classical. It is very similar, for instance to the proof of Theorem 3.3.1 in [11]. It is done in several steps:

1. *A priori estimate and weak limit:* from the inequalities

$$\forall \varepsilon \in]0,1], \qquad \alpha ||u^\varepsilon||^2 \leq a^\varepsilon(u^\varepsilon, u^\varepsilon) \leq ||l||_{V'} ||u^\varepsilon||$$

we deduce that the family $(u^\varepsilon)_{\varepsilon>0}$ is bounded. There exists a subsequence that we still denote by $(u^\varepsilon)_{\varepsilon>0}$ which converges weakly in the space V. We denote by w^0 its limit. We will prove that it is the strong limit of u^ε when $\varepsilon \to 0$.

2. *Properties of* w^0: we first multiply the Eq. (1) by ε. Using the bounded linear operators A_0 and A_1 defined on V by:

$$< A_0.u, v >= a_0(u, v), \qquad < A_1.u, v >= a_1(u, v),$$

we get:

$$< u^\varepsilon, A_0.v > +\varepsilon < u^\varepsilon, A_1.v >= \varepsilon\ l(v), \qquad \forall v \in V.$$

Passing to the limit when $\varepsilon \to 0$, we get:

$$< w^0, A_0.v >= a_0(w^0, v) = 0, \qquad \forall v \in V,$$

which says that w^0 is in the kernel G of A_0. Now, in Eq. (1), we choose $w \in G$ as test functions. We get:

$$u^\varepsilon \in V, \qquad a_1(u^\varepsilon, w) = l(w), \qquad \forall w \in G,$$

and passing to the limit as previously:

$$w^0 \in G \qquad a_1(w^0, w) = l(w), \qquad \forall w \in G,$$

which is Eq. (2). Since this equation has one and only one solution we have $w^0 = u^0$. Moreover, $u^0 = 0$ if and only if $l(w)$ is equal to zero for all $w \in G$. This means that $l \in G^\perp$.

3. *Strong convergence:* Considering the previous subsequence $(u^\varepsilon)_{\varepsilon>0}$ which converges weakly to u^0 we have:

$$\alpha ||u^\varepsilon - u^0||^2 \leq a^\varepsilon(u^\varepsilon - u^0, u^\varepsilon - u^0) = a^\varepsilon(u^\varepsilon, u^\varepsilon) - 2a^\varepsilon(u^\varepsilon, u^0) + a^\varepsilon(u^0, u^0).$$

As we have :

$$a^\varepsilon(u^0, u^0) = a_1(u^0, u^0) = l(u^0) \qquad a^\varepsilon(u^\varepsilon, u^0) = l(u^0),$$

passing to the limit, we obtain:

$$\lim_{\varepsilon \to 0} a^\varepsilon(u^\varepsilon, u^\varepsilon) = \lim_{\varepsilon \to 0} l(u^\varepsilon) = l(u^0),$$

so:

$$\lim_{\varepsilon \to 0} ||u^\varepsilon - u^0||^2 = 0.$$

Hence the subsequence $(u^\varepsilon)_{\varepsilon>0}$ converges strongly to u^0. Then, as any cluster point of the family u^ε has to satisfy Eq. (2) which has u^0 as a unique solution, the whole family converges to u^0 in V. This ends the proof of the Theorem 1. □

Remark 1. We also have $\frac{1}{\varepsilon} a_0(u^\varepsilon, u^\varepsilon) \to 0$ because $\frac{1}{\varepsilon} a_0(u^\varepsilon, u^\varepsilon) = l(u^\varepsilon) - a_1(u^\varepsilon, u^\varepsilon)$ which tends to 0 with ε.

3. A family of discrete problems

We are now interested in the following problem: we want to find a way to compute an approximate solution of Eq. (1). As explained in the introduction, one of our aims is to prove that there exists a *Galerkin method* such that u^ε be approximated by u_h^ε with:

$$||u_h^\varepsilon - u^\varepsilon|| \ \to 0 \qquad \text{when } h \to 0, \qquad \text{uniformly in } \varepsilon \in]0,1].$$

So, let $(V_h)_{h>0}$ be a family of finite dimensional subspaces of V. For every $\varepsilon \in]0,1]$ we use the Galerkin method for the Eq. (1). For a fixed ε, we have to solve the following equations:

$$(3) \qquad \forall h > 0, \qquad u_h^\varepsilon \in V_h, \qquad a^\varepsilon(u_h^\varepsilon, v_h) = l(v_h), \qquad \forall v_h \in V_h.$$

Obviously each one has a unique solution. Let $G_h := G \cap V_h$. We have:

$$G_h = \{w_h \in V_h;\ a_0(w_h, v_h) = 0,\ \forall v_h \in V_h\} = \{w_h \in V_h;\ a_0(w_h, w_h) = 0\}.$$

We also consider the Galerkin method for the Eq. (2), which requires to solve :

$$(4) \qquad \forall h > 0, \qquad u_h^0 \in G_h, \qquad a_1(u_h^0, v_h) = l(v_h), \qquad \forall v_h \in G_h.$$

Each of these equations has one and only one solution. We study first the two following limits:

$$\lim_{\varepsilon \to 0} u_h^\varepsilon \qquad \text{for a fixed } h,$$

$$\lim_{h \to 0} u_h^\varepsilon \qquad \text{for a fixed } \varepsilon.$$

We have:

Theorem 2. *For a given $h > 0$, when $\varepsilon \to 0$, the solution u_h^ε of (3) converges in V_h to the unique solution u_h^0 of Eq. (4).*

Proof. This is exactly the same proof as Theorem 1. It is just a little simplified because weak and strong convergence are the same in finite dimensional spaces. □

Now, in order to pass to the limit when h goes to zero we recall that *a family of finite dimensional subspaces $(X_h)_{h>0}$ of a Hilbert space X is said to approach X if and only if:*

$$\forall u \in X, \qquad \lim_{h \to 0} \inf_{v_h \in X_h} ||u - v_h|| = 0.$$

The following result is classical [10]:

Proposition 1. *i. If the family $(V_h)_{h>0}$ approaches V, then for any given $\varepsilon > 0$, the solution u_h^ε of the Eq. (3) converges to the solution u^ε of (1) when $h \to 0$,*

ii. If the family $(G_h)_{h>0}$ approaches G, then the solution u_h^0 of the Eq. (4) converges to the solution u^0 of (2) when $h \to 0$.

4. Uniform convergence when h goes to zero

The aim of this section is the proof of the following result: *if the family* $(V_h)_{h>0}$ *approaches* V *and the associated family* $(G_h)_{h>0}$ *approaches* G*, then* u_h^ε *converges to* u^ε *when* h *goes to zero, uniformly in* $\varepsilon \in [0,1]$.

This will be proved in several steps. In particular, we separate the study of a neigbourhood of $\varepsilon = O$, and the study on intervals of the form $[\varepsilon_0, 1]$ with $\varepsilon_0 > 0$.

Proposition 2. *If* $(V_h)_{h>0}$ *approaches* V*, then:*

$$\forall \varepsilon_0 > 0, \qquad \lim_{h\to 0} \sup_{\varepsilon_0 \le \varepsilon \le 1} ||u_h^\varepsilon - u^\varepsilon|| = 0.$$

Proof. Let $C^0([\varepsilon_0, 1])$ be the space of continuous real valued functions defined on $[\varepsilon_0, 1]$, equipped with the norm of the uniform convergence. We have to prove that for any sequence (h_n) converging to zero:

$$\phi_n(\varepsilon) \to 0 \text{ in } C^0([\varepsilon_0, 1]), \qquad \text{where } \phi_n(\varepsilon) = ||u^\varepsilon - u_{h_n}^\varepsilon||.$$

Considering that this sequence converges ponctually to 0 when $n \to \infty$ (see Proposition 1), in order to get a uniform convergence, it is sufficient to show that this family is equicontinuous (Ascoli Theorem). This can be done for instance showing that the mapping $\varepsilon \mapsto u^\varepsilon - u_h^\varepsilon$ is differentiable and that its differential is uniformly bounded in ε and h.

Let $A^\varepsilon = \frac{1}{\varepsilon} A_0 + A_1$ (see the proof of Theorem 1) and $A_h^\varepsilon = \frac{1}{\varepsilon} A_{0,h} + A_{1,h}$ be the bounded linear operators respectively defined by :

$$< A^\varepsilon.u, v >= a^\varepsilon(u, v), \quad \forall u, v \in V \qquad < A_h^\varepsilon.u_h, v_h >= a^\varepsilon(u_h, v_h), \quad \forall u_h, v_h \in V_h,$$

and $L \in V$, $L_h \in V_h$ be defined by:

$$< L, v >= l(v), \quad \forall v \in V, \qquad < L_h, v_h >= l(v_h), \qquad \forall\ v_h \in V_h.$$

Considering the Eqs. (1) and (3), we can see that $A^\varepsilon\ u^\varepsilon = L$ and $A_h^\varepsilon\ u_h^\varepsilon = L_h$. Clearly, the mappings $\varepsilon \mapsto A^\varepsilon$ and $\varepsilon \mapsto A_h^\varepsilon$ are indefinitely differentiable on the interval $[\varepsilon_0, 1]$ with values in the Banach space of bounded linear operators respectively on V and V_h. So by the implicit functions theorem, $\varepsilon \mapsto u^\varepsilon$ and $\varepsilon \mapsto u_h^\varepsilon$ are also indefinitely differentiable on the interval $[\varepsilon_0, 1]$ with values respectively in V and V_h, and we have :

$$a^\varepsilon(\frac{du^\varepsilon}{d\varepsilon}, v) = \frac{1}{\varepsilon^2} a_0(u^\varepsilon, v), \quad \forall v \in V, \qquad a^\varepsilon(\frac{du_h^\varepsilon}{d\varepsilon}, v_h) = \frac{1}{\varepsilon^2} a_0(u_h^\varepsilon, v_h), \qquad \forall v_h \in V_h.$$

Choosing $v = \frac{du^\varepsilon}{d\varepsilon}$ and $v_h = \frac{du_h^\varepsilon}{d\varepsilon}$, we get:

$$\alpha||\frac{du^\varepsilon}{d\varepsilon}|| \le \frac{1}{\varepsilon^2}||a_0||\ ||u^\varepsilon|| \le \frac{1}{\varepsilon_0^2}||a_0||\frac{||f||_{V'}}{\alpha}$$

$$\alpha||\frac{du_h^\varepsilon}{d\varepsilon}|| \le \frac{1}{\varepsilon^2}||a_0||\ ||u_h^\varepsilon|| \le \frac{1}{\varepsilon_0^2}||a_0||\frac{||f||_{V'}}{\alpha}.$$

Thus, $||\frac{du^\varepsilon}{d\varepsilon}||$ and $||\frac{du_h^\varepsilon}{d\varepsilon}||$ are bounded, uniformly in h and $\varepsilon \in [\varepsilon_0, 1]$.The family $\phi_n(\varepsilon)$ is then equicontinuous, therefore it converges uniformly to zero on the compact set $[\varepsilon_0, 1]$. $\square$

Let us now study the neighbourhood of $\varepsilon = 0$. This is where the fundamental hypothesis requiring that $(G_h)_{h>0}$ approaches G interferes. We have :

Proposition 3. *If $(G_h)_{h>0}$ approaches G, then:*
for all $\eta > 0$ there exist $\varepsilon_0 = \varepsilon_0(\eta) > 0$ and $h_0 = h_0(\eta) > 0$ such that:

$$\forall h \in]0, h_0[, \qquad \forall \varepsilon \in]0, \varepsilon_0[, \qquad ||u^\varepsilon - u_h^\varepsilon|| < \eta.$$

Proof. we know that u_h^ε is the orthogonal projection of u^ε on V_h for the inner product associated to the bilinear form a^ε. Therefore :

$$a^\varepsilon(u^\varepsilon - u_h^\varepsilon, u^\varepsilon - u_h^\varepsilon) = \inf_{v_h \in V_h} a^\varepsilon(u^\varepsilon - v_h, u^\varepsilon - v_h).$$

Using the uniform coerciveness of a^ε and choosing $v_h = w_h \in G_h$, we get:

$$\alpha\ ||u^\varepsilon - u_h^\varepsilon||^2 \le \frac{1}{\varepsilon}\ a_0(u^\varepsilon, u^\varepsilon) + a_1(u^\varepsilon - w_h, u^\varepsilon - w_h), \qquad \forall w_h \in G_h.$$

Let us estimate the first term. As we have seen in the Remark 1, we know that $\frac{1}{\varepsilon}\ a_0(u^\varepsilon, u^\varepsilon)$ tends to 0 with ε. So:

$$\forall \eta > 0, \qquad \exists \varepsilon_1(\eta) > 0 \ \text{ such that } \qquad 0 < \varepsilon < \varepsilon_1(\eta) \Rightarrow \frac{1}{\varepsilon}\ a_0(u^\varepsilon, u^\varepsilon) < \frac{\eta}{2}.$$

Let us now consider the second term. As the bilinear form a_1 is continuous, there exists a constant C which does not depend on ε, h and $w_h \in G_h$ such that

$$\forall w_h \in G_h, \qquad a_1(u^\varepsilon - w_h, u^\varepsilon - w_h) \le C||u^\varepsilon - w_h||^2 \le C(||u^\varepsilon - u^0|| + ||u^0 - w_h||)^2.$$

Moreover, we have the following estimates:

- as $u^\varepsilon \to u^0$ when $\varepsilon \to 0$:

$$\forall \eta > 0 \qquad \exists \varepsilon_2(\eta) > 0 \text{ such that } \qquad 0 < \varepsilon < \varepsilon_2(\eta) \ \Rightarrow ||u^\varepsilon - u^0|| < \sqrt{\frac{\eta}{8C}}$$

- as G_h approaches G:

$$\forall \eta > 0 \qquad \exists h_0(\eta) > 0 \ s.t.$$

$$0 < h < h_0 \Rightarrow \left\{ \exists\ w_h \in G_h \text{ such that } ||u^0 - w_h|| < \sqrt{\frac{\eta}{8C}} \right\}.$$

Putting these two conditions together, we get the result with $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. □

Now, with Propositions 3 and 2, it is staightforward to get:

Theorem 3. *If $(V_h)_{h>0}$ approaches V and $(G_h)_{h>0}$ approaches G, then:*

$$\lim_{h \to 0} \sup_{0 < \varepsilon < 1} ||u_h^\varepsilon - u^\varepsilon|| = 0.$$

5. Example, the arch problem

In this paragraph, we show that an arch working in linear elasticity conditions is an example of the general framework studied in this paper. The small parameter is the thickness of the arch. It is well-known that classical numerical methods often give a locking behaviour when the thickness is too small (see [14], [17], [2], [12], [13]).

In this section, we are going to show a standard finite element procedure, and see that it is likely to lock for a small thickness. Then, we will modify it to get another procedure which converges uniformly when the thickness is small.

In the first subsection, we give the description of the continuous model, and show that it is an example of the general framework described in the previous sections. Then, both finite element procedures will be described.

5.1. The continuous problem

An arch is a solid three dimensional body, built around a midcurve. It is supposed to be invariant in one direction, and it has a thickness $\sqrt{\varepsilon}$, which is supposed to be small compared to its other dimensions. The midcurve is generated by a plane curve that we parametrize with the curvilinear abscissa s. As in classical litterature, we suppose that it is a C^3 curve. The arch is submitted to a loading that we also suppose invariant in one direction. So, this becomes a plane problem. Moreover, considering the small thickness, it is standard to reduce the linear elasticity equations to a system of differential equations written on the midcurve. The problem is finally reduced to a one dimensional variational equation. Several models can be considered, coresponding to different types of loadings, for a given geometry. We consider here the standard Kirchhoff-Love model, in which normal and shear stresses are neglected.

When it is loaded, a displacement field $\mathbf{u}(s)$ appears on the arch. In the Kirchhoff-Love model, it is decomposed on the local basis $(\mathbf{t}(s), \mathbf{n}(s))$ made of the tangent and normal unit vectors at the point of local coordinate s. So the unknowns are two scalar functions $u_1(s), u_2(s)$ such that:

$$\mathbf{u}(s) = u_1(s)\mathbf{t}(s) + u_2(s)\mathbf{n}(s).$$

We suppose that the arch length is 1. We also suppose that it is clamped at its ends. The notations are the following:

- $V = H_0^1(]0,1[) \times H_0^2(]0,1[)$ where the Sobolev spaces are defined as follows:

  $$H_0^1(]0,1[) = \{v \in L^2(]0,1[); v' \in L^2(]0,1[) \text{ and } v(0) = v(1) = 0\},$$

  $$H_0^2(]0,1[) = \{v \in L^2(]0,1[); v', v'' \in L^2(]0,1[) \\ \text{and } v(0) = v(1) = v'(0) = v'(1) = 0\}$$

  (by v' and v" we mean the first and second derivatives of v in the sense of distributions *see* [7]) .
- For $v = (v_1, v_2) \in V$ we denote:

  $$\gamma(v) = v_1' + cv_2 \qquad \textit{and} \qquad \rho(v) = (cv_1 - v_2')'$$

  where $c \in C^1([0,1])$ is the curvature of the midcurve.

– We define the bilinear form:

$$\forall u, v \in V, \qquad a^\varepsilon(u,v) = \frac{C}{\varepsilon}\ a_0(u,v) + D\ a_1(u,v)$$

with:

$$a_0(u,v) = \int_0^1 \gamma(u)\gamma(v)\ ds, \qquad\qquad a_1(u,v) = \int_0^1 \rho(u)\rho(v)\ ds.$$

C and D are mechanical constants that we will suppose to be equal to 1 in the sequel.

Let f^ε be a given loading to which the arch is submitted. It has to belong to V'. The displacement field which appears in the arch is the solution of the following problem:

$$u^\varepsilon \in V, \qquad \varepsilon^{3/2}\ a^\varepsilon(u^\varepsilon, v) = f^\varepsilon(v), \qquad \forall v \in V.$$

It is known (see [10]) that this equation has one and only one solution. Actually a^ε is coercive, uniformly in $\varepsilon \in]0,1]$, and it satisfies the hypothesis (H1,H2) required in the general setting (Sect. 2) of this article. Let us make some comments about the loading f^ε. Usually it is of the form:

$$f^\varepsilon = f_0 + \sqrt{\varepsilon} f_1.$$

The term f_0 comes from surface loadings, like a pressure produced by wind, the other term comes from volume loadings, like the self-weight of the arch. Anyway, as the equation is linear, these two types of loadings can be studied separately. Moreover, still using arguments of linearity, the study of the equation:

$$u^\varepsilon \in V, \qquad a^\varepsilon(u^\varepsilon, v) = l(v), \qquad \forall v \in V,$$

where l does not depend on ε is sufficient to get a full information for realistic loadings, because the real displacement field $\mathbf{u}^\varepsilon$ in the arch can be deduced from u^ε multiplying this one by a proper power of ε. Now the last thing we have to look at, in order to show that we are in the general setting of this paper, is the kernel of the bilinear form $a_0(u,v)$. As before, we denote it by $G = \{w \in V;\ a_0(w,v) = 0,\ \forall v \in V\}$. We have :

Proposition 4. *Assume that the curvature c is not identical to 0. We have:*

$$G = \{w = (w_1, w_2);\ w_2 \in H_0^2(]0,1[),\ \int_0^1 (cw_2)(s)\ ds = 0,\ w_1(s) = -\int_0^s (cw_2)(t)\ dt\},$$

and this space is isomorphic to a closed hyperplan of $H_0^2(]0,1[)$.

Proof. A function w is in G if and only if $\gamma(w) = 0$. As $\gamma(w) = w_1' + cw_2$, we have:

$$w \in G \iff \{w_1 \in H_0^1(]0,1[),\ w_2 \in H_0^2(]0,1[), w_1' = -cw_2\}.$$

But, as $c \in L^\infty$, if w_2 is in H^2 and $w_1' = -cw_2$, w_1 will be in H^1. So, if w_1 has been defined by:

$$w_1(s) = -\int_0^s (cw_2)(t)\ dt$$

the pair (w_1, w_2) will belong to G if and only if:

$$w_1(1) = -\int_0^1 (cw_2)(t)\ dt = 0.$$

This achieves the description of G.

Moreover, the mapping $w_2 \mapsto (-\int_0^s (cw_2)(t)dt,\ w_2(s))$ is clearly linear continuous from the space $\{w_2 \in H_0^2(]0,1[);\ \int_0^1 (cw_2)(t)\ dt = 0\}$ into the space G. This achieves the proof of the proposition. □

Remark 2. If the curvature c of the arch is identical to 0 (which means that the arch is a beam),

$$G = \{0\} \times H_0^2(]0,1[).$$

5.2. A locking discrete model

Let us first discretize the interval $[0,1]$. We choose an integer $N > 0$, define $h = 1/N$, and:

$$x_0 = 0, \qquad x_i = ih, \quad i = 1, ..., N, \qquad x_{N+1} = 1.$$

We denote by P_k the space of *piecewise polynomials* defined on this discretization, of degree not greater than k.

We define the space $V_h = V_{h1} \times V_{h2}$ by:

$$V_{h1} = P_1 \cap H_0^1(]0,1[), \qquad V_{h2} = P_3 \cap H_0^2(]0,1[).$$

It is wellknown (see [10]) that $(V_h)_{h>0}$ $approaches$ V. So we get:

$$\forall \varepsilon > 0, \qquad u_h^\varepsilon \to u^\varepsilon, \qquad \text{when } h \to 0.$$

Let us now look at the spaces G and $(G_h)_{h>0}$. If the curvature is identical to 0, which means that the arch is a beam, then:

$$G = \{0\} \times H_0^2 \qquad G_h = \{0\} \times V_{h2}$$

so that $(G_h)_{h>0}$ approaches G, and the convergence is uniform for small thicknesses. Actually, in this very particular case, the components u_1^ε and u_2^ε are decoupled, which gives very special properties.

On the other hand, let us consider an arch which does have curvature. We have seen in proposition 4 that G is isomorphic to a closed hyperplan of H_0^2. Moreover:

$$G_h = \{w_h \in V_h; \gamma(w_h) = 0\}, \qquad \text{with} \qquad \gamma(w_h) = w'_{h1} + cw_{h2}.$$

As w_{h1} is piecewise P_1, its first derivative is piecewise constant, and if w_h belongs to G_h, then cw_{h2} is piecewise constant. Moreover, as c is continuous, we also have:

$$cw_{h2} \in C^0([0,1]), \qquad cw_{h2}(0) = cw_{h2}(1) = 0$$

and if c is nonzero except at isolated points:

$$w_{h2} = 0.$$

Then, still using the relation $w'_{h1} + cw_{h2} = 0$, we deduce that:

$$w'_{h1} = 0, \qquad w_{h1} \in C^0([0,1]), \qquad w_{h1}(0) = w_{h1}(1) = 0.$$

Therefore we have:

$$w_{h1} = 0,$$

and consequently, for all $h > 0$ the space G_h is reduced to the origin:

$$G_h = \{0\}.$$

As the space G obviously contains nonzero elements, it is clear that:

$$(G_h)_{h>0} \text{ does not approach } G.$$

We even know that generally locking does occur. Actually, we know that to avoid it, u_h^0 has to converge to u^0 when $h \to 0$. But here $u_h^0 = 0$ because it belongs to G_h, and u^0 is not equal to 0, unless the loading functional $l \in G^\perp$ (see Theorem 1). So we have the following result:

Proposition 5. *If the loading functional l does not belong to $G^\perp$ and if the arch is not a beam (the curvature is not identical to 0) then the discretization scheme described in this section does not converge when $h \to 0$ uniformly in $\varepsilon \in]0,1]$. A locking phenomenon does occur and:*

$$\sup_{0<\varepsilon<1} ||u_h^\varepsilon - u^\varepsilon|| \geq ||u^0|| > 0.$$

5.3. A nonlocking conforming model for a circular arch

The interval $[0,1]$ is discretized just like in the previous section. We still denote by P_k the space of piecewise polynomials of degree not greater than k. We are going to choose:

$$V_{h1} \subset P_k \cap H_0^1(]0,1[), \qquad V_{h2} = P_3 \cap H_0^2(]0,1[)$$

with $k \geq 1$. We want to prove that it is possible to adjust k and to choose the subset V_{h1} so that $(G_h)_{h>0}$ *approaches* G.

The idea consists in finding spaces V_{h1} and V_{h2} giving less restrictions than in the previous case, so that G_h be bigger. Supposing that the curvature is constant, we use the relation defining G_h saying that as w_{h2} is P_3 polynomial, we choose k such that w'_{h1} be of the same degree. We get:

Theorem 4. *The curvature is supposed to be a nonzero constant. We define:*

$$V_{h1} = P_4 \cap H_0^1(]0,1[), \qquad V_{h2} = P_3 \cap H_0^2(]0,1[).$$

Then $(V_h)_{h>0}$ approaches V, and $(G_h)_{h>0}$ approaches G. Therefore we have:

$$\lim_{h\to 0} \sup_{\varepsilon\in]0,1]} ||u_h^\varepsilon - u^\varepsilon|| = 0.$$

Remark 3. The approximation of arch equations studied in [14] fits in the family described here.

Proof of the Theorem. It is wellknown that the family $(V_h)_{h>0}$ approaches the space V (see [10]). Let us prove that $(G_h)_{h>0}$ approaches G.

We have seen (Proposition 4) that:

$$G = \{(w_1, w_2); w_2 \in H_0^2([0,1]),\ \int_0^1 w_2 = 0,\ w_1(s) = -c \int_0^s w_2(t)\ dt\}.$$

Here the curvature is constant, we suppose that it is equal to 1. This space is isomorphic to:

$$F = \{w_2 \in H_0^2([0,1]); \int_0^1 w_2 = 0\}.$$

In the same way:

$$G_h = \{(w_{h1}, w_{h2}); w_{h2} \in H_0^2([0,1]),\ \int_0^1 w_{h2} = 0,\ w_{h1}(s) = -c \int_0^s w_{h2}(t)\ dt\}$$

is isomorphic to the following subspace of $H_0^2(]0,1[)$:

$$F_h = \{w_{h2} \in V_{h2}; \int_0^1 w_{h2} = 0\}.$$

Since the mapping $v \mapsto \int_0^s v(t)\ dt$ is linear and bounded from F into $H_0^1(]0,1[)$, it is sufficient to prove that $(F_h)_{h>0}$ approaches F in order to prove that $(G_h)_{h>0}$ approaches G.

As it is known that $V_{h2} = P_3 \cap H_0^2(]0,1[)$ *approaches* $H_0^2(]0,1[)$ we only have to prove the mean value compatibility. Let $\phi \in F$ be given. We have:

$$\forall \eta > 0,\ \exists\, h_0 > 0,\ \textit{such that} \qquad \{\ 0 < h < h_0\ \Rightarrow \exists \psi_h \in V_{h2}\ :\ ||\phi - \psi_h||_2 \leq \eta\}$$

where $||\psi||_2$ denotes the norm in H^2. Let us then choose $\theta_h \in V_{h2}$ such that $\int_0^1 \theta_h = 1$, and consider:

$$\phi_h = \psi_h - \theta_h \int_0^1 \psi_h$$

which belongs to F_h. It is easily checked that:

$$||\phi - \phi_h||_2 \leq ||\phi - \psi_h||_2 [1 + ||\theta_h||_2].$$

So the Theorem will be proved if it is possible to choose θ_h such that its norm (in H^2) remains bounded when $h \to 0$. Let us build such a function.

We take $\eta \in H_0^2$ such that $\int_0^1 \eta = 1$. Then, there exists $h_1 > 0$ such that, for all h in $]0, h_1[$, there exists $\lambda_h \in V_{h2}$ such that $||\eta - \lambda_h||_2 < \frac{1}{2}$. If we choose:

$$\theta_h = \frac{\lambda_h}{\int_0^1 \lambda_h},$$

it is in V_{h2} with meanvalue equal to 1, and:

$$||\theta_h||_2 = \frac{1}{|\int_0^1 \lambda_h|} ||\lambda_h||_2.$$

Moreover, as $|\ |\int_0^1 \lambda_h| - 1\ | \leq |\ \int_0^1 (\lambda_h - \eta)\ | \leq ||\lambda_h - \eta||_2$, we have:

$$0 < h < h_1 \qquad \Rightarrow \qquad |\int_0^1 \lambda_h| > \frac{1}{2}.$$

Thus there exists $C > 0$ such that:

$$0 < h < h_1 \qquad \Rightarrow \qquad ||\theta_h||_2 \ \leq \ C.$$

This ends the proof of the Theorem. □

Remark 4. Generally the curvature is not constant and the space G_h strongly depends on the function c. In this case and generally speaking, this space will not be big enough to approach G. But if the curvature c is approximated by a piecewise constant function the above $\{P_4,\ P_3\}$-method may be valid.

6. Conclusion and perspective

We have studied the numerical behaviour of elliptic problems in which a small parameter is involved. We have given general conditions ensuring that a numerical scheme will converge uniformly with respect to the small parameter, without needing the use of mixed method.
An example is studied, concerning the computation of arches working in linear elasticity conditions, studied under the Kirchhoff-Love model. The uniform error estimates was already considered by Kikuchi [14] using direct computations. Our present analysis is different because we apply an asymptotic analysis on a finite dimensional abstract model and we give some sufficient conditions to get uniform convergence (this analysis has been already used in the case of a three dimensional elastic thin plate [15]). In the case of the arch we have shown one finite element scheme giving a locking behaviour, and one which does not. This one behaves as well as possible, as shown in [9]: it converges uniformly with respect to the small parameter, which is the thickness of the arch.
Another possibility is to use a mixed formulation. But this addition of a new field complicates the implementation and its generalization to nonlinear problems.
It is known that a reduced integration is a good method to avoid a locking phenomenon. In this way one computes the terms of the stiffness matrix arising from the bilinear form a_0 with a low-order quadrature (see [2] for the one dimensional Midlin-Reissner model, [14] in the case of arches and [16] in the case of shells, ...). Our present analysis could be extended to cover the case of the reduced integration by redefining the finite dimensional problem (3) as:

$$(5) \qquad u_h^\varepsilon \in V_h, \qquad a_h^\varepsilon(u_h^\varepsilon, v_h) = l_h(v_h), \qquad \forall v_h \in V_h,$$

where a_h^ε and l_h are approximations of the forms a^ε and l, respectively. In this case $(G_h)_{h>0}$ would become an *external* approximation of the space G.

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