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On the multistep time discretization of linear initial-boundary value problems and their boundary integral equations

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Summary. Convergence estimates in terms of the data are shown for multistep methods applied to non-homogeneous linear initial-boundary value problems. Similar error bounds are derived for a new class of time-discrete and fully discrete approximation schemes for boundary integral equations of such problems, e.g., for the single-layer potential equation of the wave equation. In both cases, the results are obtained from convergence and stability estimates for operational quadrature approximations of convolutions. These estimates, which are also proved here, depend on bounds of the Laplace transform of the (distributional) convolution kernel outside the stability region scaled by the time stepsize, and on the smoothness of the data.

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1. Introduction

The present article is about time discretization methods for linear time-invariant non-homogeneous evolution equations. These include initial-boundary value problems for partial differential equations of hyperbolic and parabolic type, and boundary integral equations for such problems. A common feature is that the solution operator is a temporal convolution $k * g$ with the data g. Here, the (distributional) convolution kernel k is not known explicitly, but various concepts of well-posedness can be phrased in terms of bounds for its Laplace transform $K(s)$, for s varying in a half-plane Re $s > \sigma_0$. Even $K(s)$, which is the solution operator of the Laplace transformed problem, is usually not known, but it is modeled implicitly in the time discretization of partial differential equations by linear multistep methods (in fact, also by Runge-Kutta methods), and it is a composition of boundary integral operators and inverses of such operators in the case of time-dependent boundary integral equations. We thus would like to have stable and convergent numerical methods when all the information available is an implicit definition of $K(s)$, bounds of $K(s)$ for s varying in suitable regions of the complex plane, and the time-dependent data $g(t)$.

In Sect. 2 we present in more detail the analytical background of the situation sketched above. In Sect. 2.1 we consider one-sided convolutions. We find it convenient to use the operational notation $K(\partial_t)g := k * g$. This emphasizes the role of $K(s)$ instead of $k(t)$, and suggests by itself algebraic manipulations and discretization schemes that would otherwise appear unmotivated. Section 2.2 deals with linear initial-boundary value problems which are "strongly well-posed in the generalized sense" according to the definition of Kreiss and Lorenz [KrL]. They fit perfectly into the framework of Sect. 2.1. In Sect. 2.3 we discuss boundary integral equations for the wave equation as a specific example, building upon the substantial work of Bamberger and Ha Duong [BaH1].

In Sect. 3 we study "operational quadrature methods" which approximate the convolution $K(\partial_t)g$ by a discrete convolution, using only $K(s)$ and the data $g(t)$. The basic idea is intriguingly simple and can be traced back at least as far as Liouville's work on fractional derivatives: Replace the time derivative ∂_t by a finite difference quotient ∂_t^h and approximate $K(\partial_t)g$ by $K(\partial_t^h)g$ (defined appropriately!). In [Lu1] the author discussed the implementation of such methods, and proved optimal-order convergence in the (parabolic) case where $K(s)$ is analytic and polynomially bounded outside a sector of the complex plane with an acute angle to the negative real axis. Here, we consider the (hyperbolic) case where bounds of $K(s)$ are available only in a half-plane Re $s > \sigma_0$, and obtain sharp convergence estimates in terms of the data g for A-stable discretizations ∂_t^h. If in addition $K(s)$ is analytic and bounded as before outside a large circle (a situation typical for stable space discretizations), then we get conditional convergence estimates of the same type for methods which are not necessarily A-stable, but contain in their stability region a half-disk in the left half-plane with center at the origin.

In Sect. 4 we use the results of Sect. 3 for standard linear multistep methods applied to linear initial-boundary value problems, or to their (stable) space discretizations in the method of lines. Here, the numerical solution is just $K(\partial_t^h)g$ in the notation of Sect. 3, with $K(s)$ denoting the solution operator of the Laplace transformed problem. Therefore the error bounds of Sect. 3 apply, and so we obtain conditional and unconditional convergence estimates in terms of the data, i.e., of the inhomogeneities in the differential equation and the boundary conditions.

In Sect. 5 we study full discretization of a boundary integral equation for the Dirichlet problem of the (possibly dissipative) wave equation, as an illustration of techniques which apply more generally. We use a Galerkin boundary element discretization in space, and operational quadrature in time. This leads to a method where the time-dependent fundamental solution is not required at all (fortunately so, because it is a very complicated expression for the dissipative wave equation and completely unknown for other problems of interest to engineers). Only its Laplace transform and the time-dependent boundary data are evaluated. The computational complexity is nearly linear in the number of time steps. Such a method has been proposed and analyzed in [LuS] for boundary integral equations of the heat equation (the extension to more general parabolic problems being straightforward), but the stability and convergence properties for hyperbolic problems remained open. Here, this is studied in detail for the wave equation. The algorithms and results extend without additional difficulty to transient boundary integral equations of other well-posed problems, such as those of elasticity (cf. [Be]), as soon as a-priori estimates for the solutions of the Laplace transformed problem are available. Extensive numerical experiments with the numerical methods discussed here are reported in [LuS] for a boundary integral equation of the heat equation, in [Lu2] for an integral equation for a time-dependent Schrödinger equation, and in [EgL] for a Wiener-Hopf integral equation.

2. Analytical background

2.1. One-sided convolution

In this subsection we consider one-sided convolutions $k * g$ when only bounds for the Laplace transform $K(s)$ of the (distributional) convolution kernel $k(t)$ are available, which itself may be unknown. Instances of such situations will be discussed in the following two subsections.

Let $K(s)$ be an analytic function in a half-plane Re $s > \sigma_0$, which (for large arguments s) is bounded by

$$|K(s)| \le M \cdot |s|^\mu \ . \tag{2.1}$$

If we write $K(s) = s^m K_m(s)$ with $m > \mu + 1$, then the Laplace inversion formula

$$k_m(t) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma + i\mathbb{R}} \mathrm{e}^{st} K_m(s)\, ds \ , \quad t \in \mathbb{R} \quad (\sigma > \sigma_0)$$

defines a continuous and exponentially bounded function $k_m(t)$ which by Cauchy's integral theorem vanishes for $t < 0$. To emphasize the dependence on $K(s)$ rather than the possibly distributional kernel $k = (d/dt)^m k_m$, we denote the convolution $k * g$ by

$$(K(\partial_t)g)(t) := \left(\frac{d}{dt}\right)^m \int_{-\infty}^{t} k_m(t-\tau)\, g(\tau)\, d\tau = \int_0^\infty k_m(\tau)\, g^{(m)}(t-\tau)\, d\tau \ . \tag{2.2}$$

This defines a smooth function on the real line for smooth data g whose support is bounded to the left, and the definition is independent of the choice of m and σ. An important observation is that $K(\partial_t)g(t)$ does not depend on future values $g(t')$ with $t' > t$. A rationale for the notation (2.2) (which essentially dates back to O. Heaviside a century ago) comes from the fact that for $K(s) = s$ we have $\partial_t g = g'$, and from the composition rule

$$K_2(\partial_t)K_1(\partial_t)g = (K_2 \cdot K_1)(\partial_t)g \ , \tag{2.3}$$

which expresses the associativity of convolution.

In the following we restrict our attention to functions g which vanish on the negative half-axis $t < 0$. Then also $K(\partial_t)g(t) = 0$ for $t < 0$, and the Laplace transforms are related by

$$\mathscr{L}(K(\partial_t)g)(s) = K(s) \cdot (\mathscr{L} g)(s) \ , \quad \text{Re } s > \sigma_0 \ , \tag{2.4}$$

for smooth functions g whose derivatives do not grow stronger than exponentially with rate σ_0. If $\mu = 0$ in (2.1), then (2.4) and Parseval's formula show the L^2 bound

$$\|\mathrm{e}^{-\sigma t} K(\partial_t) g\|_{L^2(\mathbb{R}_+)} \le M \cdot \|\mathrm{e}^{-\sigma t} g\|_{L^2(\mathbb{R}_+)} \ , \quad \sigma > \sigma_0 \ .$$

For general μ, we get similar bounds in Sobolev norms: For real r, we let $H^r(\mathbb{R}) = \{g : (1+|\omega|)^r \cdot \mathscr{F} g(\omega) \in L^2(\mathbb{R})\}$ (with $\mathscr{F}$ denoting the Fourier transform on $\mathbb{R}$) be the Sobolev space of order r, and on finite intervals $(0,T)$ we denote[1]

$$H_0^r(0,T) = \{g|_{(0,T)} : g \in H^r(\mathbb{R}) \text{ with } g \equiv 0 \text{ on } (-\infty, 0)\} \ . \tag{2.5}$$

An equivalent norm on $H_0^r(0,T)$ is $\|\partial_t^r g\|_{L^2(0,T)}$. For integer r, the space $H_0^r(0,T)$ consists of those functions g whose r-th distributional derivative is in $L^2(0,T)$ and which have $g(0) = \ldots = g^{(r-1)}(0) = 0$. Parseval's formula gives us the following.

Lemma 2.1. *If $K(s)$ is bounded by (2.1) in a half-plane* Re $s > \sigma_0$, *then $K(\partial_t)$ extends by density to a bounded linear operator*

$$K(\partial_t) : H_0^{r+\mu}(0,T) \to H_0^r(0,T) \tag{2.6}$$

for arbitrary real r.

In order to get pointwise estimates of $K(\partial_t)g(t)$, using only the bound (2.1), we can apply the embedding $H^r(0,T) \subset C[0,T]$ for $r > \frac{1}{2}$. A simple alternative, which we will later use to derive pointwise error bounds for the numerical methods, is to insert (2.4) directly in the Laplace inversion formula:

$$K(\partial_t)g(t) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma + \mathrm{i}\mathbb{R}} e^{st} \frac{K(s)}{s^m} \cdot s^m (\mathscr{L} g)(s)\, ds \ ,$$

to estimate

$$\sup_{s \in \sigma + \mathrm{i}\mathbb{R}} |s^m \cdot \mathscr{L} g(s)| = \sup_{s \in \sigma + \mathrm{i}\mathbb{R}} |\mathscr{L}(\partial_t^m g)(s)| \leq \int_0^\infty \mathrm{e}^{-\sigma t} |\partial_t^m g(t)|\, dt \ , \tag{2.7}$$

and to use the fact that $K(s)/s^m$ is integrable along $\sigma + \mathrm{i}\mathbb{R}$ for $m > \mu + 1$ by (2.1).

For simplicity, let now m be a non-negative integer. We let $W_0^{m,1}(0,T)$ denote the space of functions g on $(0,T)$ with the m-th distributional derivative in $L^1(0,T)$ and with $g(0) = \ldots = g^{(m-1)}(0) = 0$, equipped with the norm $\|g^{(m)}\|_{L^1(0,T)}$. We have shown

Lemma 2.2. *If $K(s)$ is bounded by (2.1) in a half-plane* Re $s > \sigma_0$, *then $K(\partial_t)$ extends for $m > \mu + 1$ to a bounded linear operator*

$$K(\partial_t) : W_0^{m,1}(0,T) \to C[0,T] \ . \tag{2.8}$$

Remark. Similarly as above, we would get a bounded extension $K(\partial_t) : A_0^\mu(0,T) \to C[0,T]$, with $A^\mu(\mathbb{R}) = \{g : (1+|\omega|)^\mu \cdot \widehat{g}(\omega) \in L^1(\mathbb{R})\}$ and $A_0^\mu(0,T)$ defined analogously as in (2.5). By embeddings of suitable smoothness spaces into $A_0^\mu(0,T)$ (e.g., Hölder-type spaces using Bernstein's or Zygmund's theorem [Ka], Theorems I.6.3 and 4) further pointwise bounds can thus be obtained. We remark that stronger results

[1] Compared to standard notation, this differs in that the subscript 0 in H_0^r only refers to the left end-point of the interval instead of both end-points. Moreover, if $r-1/2$ is integer, then the above space becomes what is usually denoted $H_{00}^r(0,T)$, cf. [LiM]

can be shown if $K(s)$ is analytic and bounded by (2.1) not only in a half-plane, but in a larger sector $|\arg(s-\sigma_0)| < \pi - \vartheta$ with $\vartheta < \pi/2$.

An obvious, but important generalization of the preceding material concerns the situation where $K(s)$, Re $s > \sigma_0$, is an analytic family of linear operators between two *Hilbert spaces*, whose operator norms are bounded by (2.1). As Parseval's formula is not used in Lemma 2.2, this lemma remains valid even in a *Banach space* setting. Such generalizations will be used freely in the sequel. The corresponding spaces will be denoted by $H_0^r(0,T;X)$ etc. to indicate the underlying space X when appropriate.

2.2. Initial-boundary value problems which are strongly well-posed in the generalized sense (cf. [KrL])

On a smooth domain $\Omega \subset \mathbb{R}^d$ with boundary Γ we consider the linear initial-boundary value problem for $u(x,t) \in \mathbb{R}^n$, given by the system of differential equations

$$\partial_t u = P(x,\partial_x)u + f(x,t)\ , \qquad x \in \Omega\ ,\ t > 0\ , \tag{2.9a}$$

with boundary conditions

$$L(x,\partial_x,\partial_t)u = g(x,t)\ , \qquad x \in \Gamma\ ,\ t > 0\ , \tag{2.9b}$$

and the special initial condition

$$u(x,0) = 0\ , \qquad x \in \Omega\ . \tag{2.9c}$$

(More general initial conditions $u(x,0) = u_0(x)$ can be reduced to this case by considering the equation for $u(x,t) - u_0(x)$.)

According to the definition in [KrL], p.227, the problem is called *strongly well-posed in the generalized sense*, if for all smooth compatible data f and g there is a unique smooth solution u, and for every finite time interval $0 \le t \le T$ there is a constant C_T such that

$$\begin{aligned} &\int_0^t \|u(\cdot,\tau)\|_\Omega^2\,d\tau + \int_0^t \|u(\cdot,\tau)\|_\Gamma^2\,d\tau \\ &\le C_T \cdot \left\{ \int_0^t \|f(\cdot,\tau)\|_\Omega^2\,d\tau + \int_0^t \|g(\cdot,\tau)\|_\Gamma^2\,d\tau \right\} \qquad \text{in}\ \ 0 \le t \le T. \end{aligned} \tag{2.10}$$

Here the norms are L^2 norms over Ω or Γ, as indicated by the subscripts. Examples of hyperbolic, parabolic, and mixed systems satisfying these conditions can also be found in [KrL].

If the constants C_T are allowed to grow only exponentially with T, then it follows via Parseval's formula that strong well-posedness in the generalized sense is equivalent – or nearly so[2] – to the following condition: The boundary value problem obtained by formal Laplace transformation of (2.9),

$$\begin{aligned} sU &= P(x,\partial_x)U + F(x)\ , \qquad x \in \Omega \\ L(x,\partial_x,s)U &= G(x)\ , \qquad x \in \Gamma \end{aligned} \tag{2.11}$$

[2] To infer the existence of smooth solutions from one problem (2.9) or (2.11) to the other, one needs in addition bounds for the spatial derivatives of solutions: exponential growth at a fixed rate in t, polynomial growth conditions in s

has a unique smooth solution $U(x,s)$ for complex s with sufficiently large real part (Re $s > \sigma_0$, say) and smooth data F and G, and there is a constant M such that

$$\|U(\cdot,s)\|_\Omega^2 + \|U(\cdot,s)\|_\Gamma^2 \le M^2 \cdot \left\{ \|F\|_\Omega^2 + \|G\|_\Gamma^2 \right\} \ , \quad \text{Re } s > \sigma_0 \ . \tag{2.12}$$

Hence, the solution operator of (2.11) extends by density to a bounded linear operator

$$K(s) : L^2(\Omega)^n \times L^2(\Gamma)^m \to L^2(\Omega)^n \times L^2(\Gamma)^n : (F,G) \mapsto (U, U_\Gamma)$$

with operator norms bounded independently of s:

$$\|K(s)\| \le M \ , \qquad \text{Re } s > \sigma_0 \ . \tag{2.13}$$

Transforming back, it follows that the solution of (2.9) is given for smooth compatible data f and g as

$$\begin{pmatrix} u \\ u|_\Gamma \end{pmatrix} = K(\partial_t) \begin{pmatrix} f \\ g \end{pmatrix} \ , \tag{2.14}$$

in the notation of formula (2.2). Via Parseval's formula, the bound (2.13) yields the well-posedness estimate (2.10). We are thus back in the framework of Subsect. 2.1, with $\mu = 0$ in (2.1).

2.3. Boundary integral equations for the wave equation (cf. [BaH1],[BaH2])

To determine the outgoing wave scattered by an acoustically soft obstacle occupying a smooth bounded domain $\Omega_0 \subset \mathbb{R}^3$, one requires the solution of the exterior Dirichlet problem for the wave equation (here $\Omega = \mathbb{R}^3 \setminus \overline{\Omega}_0$, $\Gamma = \partial\Omega$)

$$\begin{aligned} \partial_t^2 u &= \Delta u \ , \qquad x \in \Omega \ , \ 0 < t < T \ , \\ u &= g(x,t) \ , \quad x \in \Gamma \ , \ 0 < t < T \ , \end{aligned} \tag{2.15}$$

with initial conditions $u(x,0) = \partial_t u(x,0) = 0$ for $x \in \Omega$. For smooth compatible boundary data g there exists a unique smooth solution u with $u(\cdot,t) \in H^1(\Omega)$ for all t, which can be represented as a single-layer wave potential

$$u(x,t) = \int_0^t \int_\Gamma k(x-\xi, t-\tau)\, \varphi(\xi,\tau)\, d\xi\, d\tau \ , \qquad x \in \Omega \ , \ 0 < t < T \ . \tag{2.16}$$

Here $k(x,t)$ is the fundamental solution of the wave equation, which is the weighted and shifted delta-function $k(x,t) = \frac{1}{4\pi|x|}\delta(t-|x|)$. It has the Laplace transform $K(x,s) = \frac{1}{4\pi|x|}\mathrm{e}^{-|x|\cdot s}$ which will be seen to play a more important role than k itself. Letting in (2.16) x tend to the boundary, we see that the density φ has to solve the integral equation

$$\int_0^t \int_\Gamma k(x-\xi, t-\tau)\, \varphi(\xi,\tau)\, d\xi\, d\tau = g(x,t) \ , \qquad x \in \Gamma \ , \ 0 < t < T \ , \tag{2.17}$$

to which we refer as the *single-layer potential equation of the wave equation*. For smooth compatible Dirichlet data g this equation has a smooth solution: Green's formula shows that the difference between the normal derivatives of the solutions of the exterior and the interior Dirichlet problem of the wave equation satisfies (2.17).

The further solution theory for equation (2.17) follows from that of the family of equations

$$\int_\Gamma K(x-\xi,s)\,\Phi(\xi)\,d\xi = G(x)\ , \qquad x\in\Gamma\ ,\ \text{Re}\ s>\sigma_0>0\ . \tag{2.18}$$

This is the single-layer potential equation of the Helmholtz equation $\Delta U - s^2 U = 0$, and we abbreviate it as

$$V(s)\Phi = G\ . \tag{2.19}$$

Mapping properties of $V(s)$ between suitable Sobolev spaces over Γ are quoted in the following theorem.

Proposition 2.3. [BaH1, Prop.3] *For* Re $s=\sigma>0$, *the single-layer potential operator* $V(s)$ *extends by density to an isomorphism*

$$V(s) : H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)$$

which for all $\psi\in H^{-1/2}(\Gamma)$ *satisfies*

$$\text{Re}\ \langle sV(s)\psi,\psi\rangle \geq c\cdot\frac{\min(1,\sigma)}{|s|}\|\psi\|^2_{H^{-1/2}(\Gamma)}$$

$$\|V(s)\psi\|_{H^{1/2}(\Gamma)} \leq C\cdot|s|\cdot\frac{\max(1,\sigma^{-2})}{\sigma}\cdot\|\psi\|_{H^{-1/2}(\Gamma)}$$

Here $\langle\cdot,\cdot\rangle$ *denotes the anti-duality between* $H^{1/2}(\Gamma)$ *and* $H^{-1/2}(\Gamma)$, *and* c *and* C *are positive constants which depend only on* Γ.

In particular, it follows that the operator norm of $V^{-1}(s) : H^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)$ is bounded by

$$\|V^{-1}(s)\| \leq M(\sigma_0)\cdot|s|^2\ , \qquad \text{Re}\ s>\sigma_0>0\ . \tag{2.20}$$

Since the single-layer potential equation (2.17) can be written in the notation of formula (2.2) as

$$V(\partial_t)\,\varphi = g\ , \tag{2.21}$$

we get from the composition rule (2.3) that for temporally smooth data $g(\cdot,t)\in H^{1/2}(\Gamma)$ which vanish near $t=0$, there exists a unique smooth solution $\varphi(\cdot,t)\in H^{-1/2}(\Gamma)$ which is given by

$$\varphi = V^{-1}(\partial_t)g\ . \tag{2.22}$$

Using the bound (2.20) in Lemma 2.1 (with $V^{-1}(s)$ in the role of $K(s)$ and $\mu=2$), we obtain the a-priori estimate

$$\|\varphi\|_{H^r_0(0,T;H^{-1/2}(\Gamma))} \leq C_T\cdot\|g\|_{H^{r+2}_0(0,T;H^{1/2}(\Gamma))} \qquad (r\in\mathbb{R}) \tag{2.23}$$

and thus also pointwise bounds in $0\leq t\leq T$:

$$\|\varphi(\cdot,t)\|_{H^{-1/2}(\Gamma)} \leq C_T\cdot\|g\|_{H^r_0(0,T;H^{1/2}(\Gamma))} \qquad (r>\tfrac{5}{2}) \tag{2.24}$$

The solution operator $V^{-1}(\partial_t)$ can clearly be extended to an operator between spaces as indicated by the norms above. This defines generalized solutions of (2.17) for nonsmooth data g. With the single-layer potential operator $S(s)$ on Ω defined by

$$(S(s)\Phi)(x) = \int_\Gamma K(x-\xi, s)\, \Phi(\xi)\, d\xi \ , \qquad x \in \Omega \tag{2.25}$$

(like $V(s)$, but for x off the boundary), the representation formula (2.16) becomes

$$u = S(\partial_t)\, \varphi = (SV^{-1})(\partial_t) g \ , \tag{2.26}$$

using (2.22) and the composition rule (2.3) in the second equality. The composed operator $(SV^{-1})(s) : G \mapsto U$, which maps Dirichlet data to the solution of the Helmholtz equation $\Delta U - s^2 U = 0$ on Ω, extends to an operator from $H^{1/2}(\Gamma)$ to $H^1(\Omega)$ which for Re $s = \sigma > \sigma_0 > 0$ is bounded by [BaH1, Prop.1]:

$$\|U\|_{H^1(\Omega)} + \|sU\|_{L^2(\Omega)} \le C(\sigma_0)/\sigma \cdot |s|^{3/2} \cdot \|G\|_{H^{1/2}(\Gamma)} \ . \tag{2.27}$$

As before, this leads to estimates of $u(x,t)$ of (2.26) in terms of the data $g(x,t)$.

This analysis can be considerably extended. For example, a nearly identical treatment applies to the Dirichlet problem of the *dissipative* wave equation

$$\partial_t^2 u + \alpha\, \partial_t u = \Delta u \quad \text{with} \quad \alpha > 0 \ .$$

Here the time-dependent fundamental solution $k_\alpha(x,t)$ is of a very complicated nature, but its Laplace transform is nearly as simple as before: $K_\alpha(x,s) = \frac{1}{4\pi|x|} e^{-|x|\cdot\sqrt{s^2+\alpha s}} = K(x, \sqrt{s^2+\alpha s})$. With $V_\alpha(s) = V(\sqrt{s^2+\alpha s})$, the single-layer potential equation reads $V_\alpha(\partial_t)\, \varphi = g$, and since again $\|V_\alpha^{-1}(s)\| \le M(\sigma_0)\cdot|s|^2$ for Re $s > \sigma_0 > 0$, its solution $\varphi = V_\alpha^{-1}(\partial_t) g$ is bounded as in (2.23) and (2.24) above. The two-dimensional case can be treated in the same way.

To mention further extensions: All the boundary integral equations of classical potential theory have analogues for the (dissipative or pure) wave equation and the heat equation, and their solution theory follows from the analysis of the corresponding integral equations for complex Helmholtz equations[3]. See [BaH2] for the Neumann problem of the wave equation, and [LuS] for the heat equation. Nonstationary boundary integral equations of elasticity have been studied in [Be], again by translating properties of the Laplace transformed problems back into the time domain. Concerning practical applications of time-dependent boundary integral equations in engineering, we refer to [An],[Br], and the many references therein.

[3] This is not to say, however, that they are all equally useful: Second-kind boundary integral equations appear to have little to recommend for the wave equation, quite in contrast to the heat equation

3. Convolution quadrature

3.1. Operational quadrature methods (cf. [Lu1])

In the situation of Sect. 2.1, we approximate the convolution $K(\partial_t)g = k * g$ by a discrete convolution which we denote

$$\left(K(\partial_t^h)g\right)(t) := \sum_{j\geq 0} \omega_j g(t-jh) \ . \tag{3.1}$$

Here $h > 0$ is a time stepsize, and the convolution quadrature weights $\omega_j \equiv \omega_j(h, K)$ are defined as the coefficients of the generating power series

$$\sum_{j=0}^{\infty} \omega_j \zeta^j := K\left(\frac{\delta(\zeta)}{h}\right) \ , \qquad |\zeta| \text{ small.} \tag{3.2}$$

Here $\delta(\zeta) = \sum_0^\infty \delta_j \zeta^j$ is the quotient of the generating polynomials of a linear multistep method $\sum_{j=0}^k \alpha_j y_{n-j} = h \sum_{j=0}^k \beta_j f_{n-j}$ for differential equations $y' = f(y)$:

$$\delta(\zeta) = \frac{\alpha_0 + \alpha_1\zeta + \ldots + \alpha_k \zeta^k}{\beta_0 + \beta_1\zeta + \ldots + \beta_k\zeta^k} \tag{3.3}$$

We always assume that $\alpha_0/\beta_0 > 0$, so that (3.2) is well-defined at least for sufficiently small h.

The notation (3.1) is used because in analogy to (2.2),

$$\partial_t^h g(t) = \frac{1}{h} \sum_{j\geq 0} \delta_j \, g(t-jh) \tag{3.4}$$

is a backward difference approximation of $\partial_t g = g'$, and there is again the composition rule

$$K_2(\partial_t^h) K_1(\partial_t^h) g = (K_2 \cdot K_1)(\partial_t^h) g \ , \tag{3.5}$$

which follows from $\omega_n(h, K_2K_1) = \sum_{j=0}^n \omega_{n-j}(h, K_2)\, \omega_j(h, K_1)$, a direct consequence of the definition (3.2).

As before, we restrict our attention to functions g which vanish on the negative half-axis $t < 0$ (and, for convenience, also for very large positive arguments t). On the grid $t_n = n \cdot h$ ($n = 0, 1, 2, \ldots$) we then have

$$y_n := K(\partial_t^h) g\,(nh) = \sum_{j=0}^{n} \omega_{n-j} \, g(jh) \ , \tag{3.6}$$

and the generating functions $Y(\zeta) = \sum_{n=0}^\infty y_n \zeta^n$ and $G(\zeta) = \sum_{n=0}^\infty g(nh)\zeta^n$ are related by

$$Y(\zeta) = K(\delta(\zeta)/h) \cdot G(\zeta) \ , \tag{3.7}$$

in analogy to (2.4).

Our interest is in deriving convergence estimates for $K(\partial_t^h)g(t)$ when $K(s)$, Re $s > \sigma_0$, is bounded by (2.1), and g is a fairly smooth function. This will be possible under the assumption that the multistep method is *A-stable*, i.e.,

$$\text{Re } \delta(\zeta) > 0 \quad \text{for} \quad |\zeta| < 1 \ . \tag{3.8}$$

The multistep method is of *order* p if

$$\frac{1}{h}\delta(\mathrm{e}^{-h}) = 1 + O(h^p) \quad \text{as} \quad h \to 0 \ . \tag{3.9}$$

We recall that the order p cannot exceed 2 for A-stable multistep methods [Da]. Well-known examples of second-order A-stable methods are the backward difference formula $\delta(\zeta) = \frac{3}{2} - 2\zeta + \frac{1}{2}\zeta^2$, and the trapezoidal rule which corresponds to $\delta(\zeta) = 2\frac{1-\zeta}{1+\zeta}$.

If $K(s)$ is analytic and bounded by (2.1) in a larger sector $|\arg(s - \sigma_0)| < \pi - \vartheta$ with $\vartheta < \frac{\pi}{2}$, then $A(\alpha)$-stability with $\alpha > \vartheta$ is sufficient, and stronger convergence results than those below can be derived, see [Lu1],[Eg].

Computationally, the quadrature weights ω_n are obtained by approximating the Cauchy integral

$$\omega_n = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=\rho} K(\delta(\zeta)/h) \cdot \zeta^{-n-1}\, d\zeta$$

by the trapezoidal rule

$$\omega_n \doteq \frac{\rho^{-n}}{L} \sum_{\ell=0}^{L-1} K(\delta(\zeta_\ell)/h) \cdot \mathrm{e}^{-2\pi \mathrm{i} \cdot n\ell/L} \ , \qquad n = 0, 1, \ldots, N \ , \tag{3.10}$$

with $\zeta_\ell = \rho \cdot \mathrm{e}^{2\pi \mathrm{i}\ell/L}$.

Let us first consider the case $\mu \le 0$ in (2.1). If we assume that the values of K in (3.10) are computed with an error bounded by ϵ, then the choice $L = 2N$ and $\rho^n = \sqrt{\epsilon}$ yields an error in ω_n of size $O(\sqrt{\epsilon})$, see [Lu1, Sect.7]. The sums in (3.10) are computed by FFT, and thus one obtains $\omega_0, \ldots, \omega_N$ using $O(N \log N)$ arithmetical operations.

For $\mu > 0$, it appears preferable to rewrite $K(s) = K_m(s)s^m$ with $m \ge \mu$, to compute the backward difference quotients $(\partial_t^h)^m g$, and then $K(\partial_t^h)g = K_m(\partial_t^h)\left((\partial_t^h)^m g\right)$, using the weights $\omega_n(h, K_m)$ computed by (3.10).

3.2. Pointwise error bounds

We shall now give pointwise error bounds of $K(\partial_t^h)g(t)$, uniformly over bounded intervals. We are mainly interested in the case where g is a smooth function on $[0, T]$, whose extension by 0 to the negative half-axis need not necessarily be smooth at $t = 0$. We may then split g into its Taylor polynomial at 0 and the remainder whose extension by 0 is sufficiently smooth. By linearity, we may study the error of the parts separately.

Depending on the sign of the exponent μ in the bound (2.1) of $K(s)$, we will make additional assumptions about the discretization method:

$$\delta(\zeta) \text{ has no poles on the unit circle.} \tag{3.11}$$

(3.12) $\delta(\zeta)$ has no zeros on the unit circle, with the exception of $\zeta = 1$.

We note that condition (3.11) excludes the trapezoidal rule.

Theorem 3.1. *Let* $K(s)$, Re $s > \sigma_0$, *be analytic and bounded by (2.1). Let the discretization method be A-stable, (3.8), and of order* p, *(3.9). If* $\mu > 0$ *(resp.* $\mu < 0$*) in (2.1), then condition (3.11) (resp. (3.12)) is to be satisfied.*

(i) Let $m \geq \max(p + 2 + \mu, p)$. *For smooth data* g *on* $[0, T]$ *with* $g(0) = \ldots = g^{(m-1)}(0) = 0$, *we have in* $0 \leq t \leq T$

$$|K(\partial_t^h)g(t) - K(\partial_t)g(t)| \leq C \cdot h^p \cdot \int_0^t |g^{(m)}(\tau)|\, d\tau \ . \tag{3.13}$$

(ii) For $g(t) = t^r$ *(extended by 0 to negative* t*) with real* $r > \mu$ *and* $r \geq 0$, *we have in* $0 \leq t \leq T$

$$|K(\partial_t^h)g(t) - K(\partial_t)g(t)| \leq C \cdot h^\alpha \tag{3.14}$$

with $\alpha = \min((r - \mu)\frac{p}{p+1}, r + 1, p)$.

These error bounds are valid for $0 < h \leq h_0$, *where* h_0 *depends only on* σ_0 *and the discretization method* $\delta(\zeta)$. *The constants* C *are proportional to* M *and depend on* μ, σ_0, T, h_0, *and the method, and in (ii) additionally on* r.

Proof. (a) In case (i), we first extend $g(t)$ to $t > T$ as the $(m - 1)$-st degree Taylor polynomial of g at T. The error function

$$e_h(t) = K(\partial_t^h)g(t) - K(\partial_t)g(t) \tag{3.15}$$

has the Laplace transform

$$\mathscr{L} e_h(s) = \left\{K\left(\delta(\mathrm{e}^{-sh})/h\right) - K(s)\right\} \cdot \mathscr{L} g(s) \ . \tag{3.16}$$

The A-stability condition (3.8) implies that there exist $\sigma_1 \geq \sigma_0$ and $h_0 > 0$ such that

$$\mathrm{Re}\ \delta(\mathrm{e}^{-sh})/h > \sigma_0 \quad \text{for } \ \mathrm{Re}\ s > \sigma_1\ ,\ 0 < h < h_0\ . \tag{3.17}$$

Consequently, $\mathscr{L} e_h(s)$ exists for Re $s = \sigma > \max(\sigma_1, 0)$. Provided that this is integrable along $\sigma + \mathrm{i}\mathbb{R}$, the Laplace inversion formula gives

$$e_h(t) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma+\mathrm{i}\mathbb{R}} \mathrm{e}^{st} \left\{K\left(\delta(\mathrm{e}^{-sh})/h\right) - K(s)\right\} \cdot \mathscr{L} g(s)\, ds \ . \tag{3.18}$$

We will show in part (b) of the proof that for $r > \max(\mu, 0)$ and for fixed $\sigma > \max(\sigma_1, 0)$ we have

$$\int_{\sigma+\mathrm{i}\mathbb{R}} \left|\left\{K\left(\delta(\mathrm{e}^{-sh})/h\right) - K(s)\right\} \cdot s^{-r-1}\right| \cdot |ds| = O(h^\alpha) \tag{3.19}$$

with α as in the theorem. Since $\Gamma(r + 1) \cdot s^{-r-1}$ (with Euler's Γ-function) is the Laplace transform of t^r, this yields statement (ii) of the theorem for $r > 0$. The case $r = 0 > \mu$ will be studied in part (c) of the proof. The error bound in case (i) is obtained from (3.19) with $r + 1 = m$, and from the estimate

$$\sup_{\text{Re } s=\sigma} |s^m \mathscr{L} g(s)| = \sup_{\text{Re } s=\sigma} |\mathscr{L} g^{(m)}(s)| \le \int_0^\infty e^{-\sigma t} |g^{(m)}(t)|\, dt \ .$$

By our extension of g beyond $[0,T]$, this is actually the integral from 0 to T, and the result follows.

(b) We now prove (3.19). We write $s \equiv \sigma + \mathrm{i}\omega$, with fixed $\sigma > \max(\sigma_1, 0)$. The integral in question is bounded by

$$\begin{aligned} I + II + III = & \int_{|\omega h| \le \pi} \left| \left\{ K\left(\delta(e^{-sh})/h\right) - K(s) \right\} \cdot s^{-r-1} \right| \, d\omega \\ & + \int_{|\omega h| \le \pi} \left| K\left(\delta(e^{-sh})/h\right) \right| \cdot \sum_{n \ne 0} |s + 2\pi \mathrm{i} \cdot n/h|^{-r-1} \, d\omega \\ & + \int_{|\omega h| \ge \pi} |K(s) \cdot s^{-r-1}| \, d\omega \ . \end{aligned}$$

By (2.1) we have $III = O(h^{r-\mu})$. For $r > 0$ the series in II converges, and so we get

$$II \le C\, h^{r+1} \cdot \int_{-\pi/h}^{\pi/h} \left| K\left(\delta(e^{-sh})/h\right) \right| \, d\omega \le C\, h^{r+1} \cdot \int_{-\pi/h}^{\pi/h} M \cdot |\delta(e^{-sh})/h|^\mu \, d\omega$$

Now the consistency condition (3.9) and condition (3.11) or (3.12) (depending on the sign of μ) imply

(3.20) $$\left|\delta(e^{-sh})/h\right|^\mu \le C \cdot |s|^\mu \qquad \text{for} \quad s = \sigma + \mathrm{i}\omega \ , \quad |\omega h| \le \pi \ .$$

It follows that

$$II = \begin{cases} O(h^{r+1}) & \mu < -1 \\ O(h^{r+1} \log h) & \mu = -1 \\ O(h^{r-\mu}) & \mu > -1 \ . \end{cases}$$

We still have to bound I. First we note that by Cauchy's integral formula for the derivative, also $K'(s)$ is bounded in the same way as $K(s)$ over a half-plane:

(3.21) $$|K'(s)| \le \frac{M}{\sigma - \sigma_0} \cdot |s|^\mu \ , \quad \text{Re } s = \sigma > \sigma_0 \ .$$

By (3.20), we therefore have for $s = \sigma + \mathrm{i}\omega$ with $|\omega h| \le \pi$

(3.22) $$\left| K\left(\delta(e^{-sh})/h\right) - K(s) \right| \le C \cdot |s|^\mu \cdot \left|\delta(e^{-sh})/h - s\right| \le C' \cdot h^p \cdot |s|^{p+1+\mu} \ ,$$

using (3.9) in the last inequality. By (2.1) and (3.20) we also have

(3.23) $$\left| K\left(\delta(e^{-sh})/h\right) - K(s) \right| \le C \cdot |s|^\mu \ ,$$

which is a tighter bound for $|s| \ge h^{-p/(p+1)}$. We thus get

$$\begin{aligned} I \le & \ C \cdot \int_{\sigma \le |s| \le h^{-p/(p+1)}} h^p \cdot |s|^{p+1+\mu} \cdot |s|^{-r-1} \cdot |ds| \\ & + C \cdot \int_{|s| \ge h^{-p/(p+1)}} |s|^\mu \cdot |s|^{-r-1} \cdot |ds| = O(h^p) + O(h^{(r-\mu) \cdot p/(p+1)}) \ . \end{aligned}$$

This yields (3.19).

(c) The case $r = 0 > \mu$ requires a separate treatment. We use (3.7) with $G(\zeta) = \sum_0^\infty 1 \cdot \zeta^n = 1/(1-\zeta)$ and apply Cauchy's integral formula to obtain for $nh \le t < (n+1)h$

$$K(\partial_t^h)1(t) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=\rho} \zeta^{-n-1} K\left(\frac{\delta(\zeta)}{h}\right) \frac{1}{1-\zeta}\, d\zeta \ .$$

Substituting $\zeta = \mathrm{e}^{-sh}$ with $s \equiv \sigma + \mathrm{i}\omega$, we thus have

$$K(\partial_t^h)1(t) = \frac{1}{2\pi} \int_{|\omega h| \le \pi} \mathrm{e}^{st}\, K\left(\frac{\delta(\mathrm{e}^{-sh})}{h}\right) \frac{h}{1-\mathrm{e}^{-sh}}\, d\omega \ .$$

Subtracting $K(\partial_t)1(t)$ expressed by the Laplace inversion formula, we get for the error

$$|e_h(t)| \le \frac{\mathrm{e}^{\sigma t}}{2\pi} \left(I + III + \int_{|\omega h| \le \pi} \left| K\left(\frac{\delta(\mathrm{e}^{-sh})}{h}\right) \left\{ \frac{h}{1-\mathrm{e}^{-sh}} - s^{-1} \right\} \right| d\omega \right) \ ,$$

where I and III are the same expressions as in part (b) of the proof. By (2.1) and (3.20), the integrand in the last term is bounded by $Ch|s|^\mu$, and hence the integral is bounded like II above, with $r = 0$. So we get the desired result also for the case $r = 0$. □

Theorem 3.2. *Under the conditions of Theorem 3.1, we have for data $g \in H_0^{r+1/2}(0,T)$ (see (2.5)) with $r > \max(\mu, 0)$ the following error bound in $0 \le t \le T$:*

(3.24) $$|K(\partial_t^h)g(t) - K(\partial_t)g(t)| \le C \cdot h^\beta |\log h| \cdot \|g\|_{H^{r+1/2}}$$

with $\beta = \min((r-\mu)\frac{p}{p+1}, r, p)$. If the first two terms in the definition of β are strictly greater than p, then the factor $|\log h|$ can be omitted. The constant C depends on the same quantities as in Theorem 3.1.

Proof. The Sobolev inequality gives $|e_h(t)| \le (C/\epsilon) \cdot \|e_h\|_{H^{1/2+\epsilon}}$ for $\epsilon > 0$. We use Parseval's formula in (3.16) to get the bound

$$\|e_h\|_{H^{1/2+\epsilon}} \le C_\sigma \cdot \sup_{\mathrm{Re}\ s=\sigma} \left| \left\{ K(\delta(\mathrm{e}^{-sh})/h) - K(s) \right\} \cdot s^{-r+\epsilon} \right| \cdot \|g\|_{H^{r+1/2}} \ .$$

Estimating the supremum as in part (b) of the preceding proof and setting $\epsilon = 1/|\log h|$ yields the result. □

Remarks. (a) If $\sigma_0 < 0$ in (2.1), then a slight modification of the proof shows that the bounds in Theorem 3.1(i) and Theorem 3.2 are valid uniformly over the whole half-line $t \geq 0$. The same is true for Theorem 3.1(ii) with t^r replaced by the exponentially decaying function $t^r \cdot \mathrm{e}^{-t}$.

(b) If the derivative of $K(s)$ satisfies in Re $s > \sigma_0'$ a bound $|K'(s)| \leq M' \cdot |s|^{\mu'}$ with $\mu' < \mu$, then the error bounds hold with $p/(p+1)$ replaced by $p/(p+1+\mu'-\mu)$ in the definition of α and β (and γ below.) This follows easily from the estimation of the term I in part (b) of the proof of Theorem 3.1. Note that always $\mu - 1 \leq \mu' \leq \mu$.

3.3. ℓ^2 error bounds

For brevity, we denote the error at the grid points by

$$e_n = y_n - y(nh) = K(\partial_t^h)g(nh) - K(\partial_t)g(nh)\ , \quad n = 0, 1, 2, \dots \tag{3.25}$$

The following ℓ^2 error bounds are slightly more favorable than their pointwise counterparts.

Theorem 3.3. *Let the assumptions of Theorem 3.1 hold.*
(i) For $g \in H_0^r(0,T)$ with $r > \frac{1}{2} + \max(\mu, 0)$, we have with $\beta = \min\big((r-\mu)\frac{p}{p+1}, r, p\big)$

$$\Big(h \textstyle\sum_{n=0}^N |e_n|^2\Big)^{1/2} \leq C \cdot h^\beta \cdot \|g\|_{H^r}\ . \tag{3.26}$$

(ii) For $g(t) = t^r$ with $r > \mu$ and $r \geq 0$, we have with $\gamma = \min\big((r-\mu+\frac{1}{2})\frac{p}{p+1}, r+1, p\big)$

$$\Big(h \textstyle\sum_{n=0}^N |e_n|^2\Big)^{1/2} \leq C \cdot h^\gamma\ . \tag{3.27}$$

The bounds are valid uniformly for $0 < h \leq h_0$ and $Nh \leq T$. The constants C depend on the same quantities as in Theorem 3.1.

Proof. The proof uses Poisson's summation formula, Parseval's formula, and estimates from the proof of Theorem 3.1.

After extending g to $g \in H^r(\mathbb{R})$ with compact support in $t \geq 0$, we start from the relation (3.7). By Poisson's summation formula, the generating function of the grid-values of g at $\zeta = \mathrm{e}^{-sh}$ equals

$$h \sum_{n=0}^\infty g(nh)\, \mathrm{e}^{-shn} = \sum_{m=-\infty}^\infty \mathscr{L} g(s + 2\pi \mathrm{i} \tfrac{m}{h})\ .$$

By (3.7) and (3.17), this gives for $0 < h \leq h_0$

$$h \sum_{n=0}^\infty y_n\, \mathrm{e}^{-shn} = K\big(\delta(\mathrm{e}^{-sh})/h\big) \cdot \sum_{m=-\infty}^\infty \mathscr{L} g(s + 2\pi \mathrm{i} \tfrac{m}{h})\ , \quad \text{Re } s > \sigma_1\ . \tag{3.28}$$

We use Poisson's summation formula once more for $y = K(\partial_t)g$ and insert (2.4) to get

$$h \sum_{n=0}^{\infty} y(nh)\, \mathrm{e}^{-shn} = \sum_{m=-\infty}^{\infty} K(s + 2\pi \mathrm{i} \tfrac{m}{h}) \cdot \mathscr{L} g(s + 2\pi \mathrm{i} \tfrac{m}{h}) \,, \quad \mathrm{Re}\ s > \sigma_0 \,. \tag{3.29}$$

The lower bounds on r required in the theorem guarantee the absolute convergence of the right-hand series in (3.28) and (3.29) and the validity of Poisson's summation formula. (An exception is again the case $r = 0 > \mu$ of (ii). Here the infinite sum in (3.28) has to be interpreted as the Abel mean.)

Parseval's formula now shows that for $s \equiv \sigma + i\omega$ with fixed $\sigma > \sigma_1 (\geq \sigma_0)$ we have

$$h \sum_{n=0}^{\infty} \mathrm{e}^{-2\sigma nh} |y_n - y(nh)|^2 = \frac{1}{2\pi} \int_{|\omega h| \leq \pi} \left| \sum_{m=-\infty}^{\infty} \left\{ K(\delta(\mathrm{e}^{-sh})/h) - K(s + 2\pi \mathrm{i} \tfrac{m}{h}) \right\} \cdot \mathscr{L} g(s + 2\pi \mathrm{i} \tfrac{m}{h}) \right|^2 d\omega \,. \tag{3.30}$$

We extract the term for $m = 0$ from the sum (according to $|a + b|^2 \leq 2(|a|^2 + |b|^2)$) and estimate its integral by

$$\left(\sup_{\mathrm{Re}\ s = \sigma} \left| \left\{ K(\delta(\mathrm{e}^{-sh})/h) - K(s) \right\} \cdot s^{-r} \right| \right)^2 \cdot \|\mathrm{e}^{-\sigma t} g\|^2_{H^r(\mathbb{R})} \,.$$

The supremum has already been seen to be $O(h^\beta)$ in the proof of Theorem 3.2. We apply the Cauchy-Schwarz inequality to the remaining term in (3.30) to get it bounded by

$$\int_{|\omega h| \leq \pi} \sum_{m \neq 0} \left| \left\{ K(\delta(\mathrm{e}^{-sh})/h) - K(s + 2\pi \mathrm{i} \tfrac{m}{h}) \right\} \cdot (s + 2\pi \mathrm{i} \tfrac{m}{h})^{-r} \right|^2 d\omega \cdot \|\mathrm{e}^{-\sigma t} g\|^2_{H^r(\mathbb{R})} \,.$$

The above integral is bounded by

$$\int_{|\omega h| \leq \pi} \left| K\left(\delta(\mathrm{e}^{-sh})/h \right) \right|^2 \cdot \sum_{m \neq 0} \left| s + 2\pi \mathrm{i} \tfrac{m}{h} \right|^{-2r} d\omega + \int_{|\omega h| \geq \pi} |K(s) \cdot s^{-r}|^2 \, d\omega \,.$$

By the same arguments as in the proof of Theorem 3.1, we obtain that both these terms are $O(h^{2\beta})$. This gives (i).

In (ii), $\mathscr{L} g(s)$ is a multiple of s^{-r-1}. Here we bound the terms for $m \neq 0$ in the same way as the above integrals, with r replaced by $r + 1$. The term coming from $m = 0$ is now

$$\int_{|\omega h| \leq \pi} \left| \left\{ K(\delta(\mathrm{e}^{-sh})/h) - K(s) \right\} \cdot s^{-r-1} \right|^2 d\omega \,,$$

and the inequalities (3.22) and (3.23) imply that this is $O(h^{2\gamma})$. □

3.4. Conditional error bounds

The following situation typically arises from stable space discretizations: $K(s) = K_\Delta(s)$ is parameterized by a small parameter $\Delta > 0$ and is analytic and bounded by

(2.1) not only in a half-plane Re $s > \sigma_0$, but additionally outside a circle with radius $1/\Delta$ and center at the origin. In this case, we can weaken the A-stability assumption to the following condition: There exists $R > 0$ such that

$$\text{(3.31)} \qquad \text{Re } \delta(\zeta) > 0 \quad \text{or} \quad |\delta(\zeta)| > R \;, \qquad \text{for all} \quad |\zeta| < 1 \;.$$

Equivalently, this states that the half-disk $\{\text{Re } z < 0\} \cap \{|z| < R\}$ is contained in the stability region of the multistep method. Such methods are called *locally stable* in [KrS], see also Sect. 5 of [ReT]. This class of methods includes explicit and implicit multistep methods of arbitrary order. The arguments of the foregoing proofs apply unchanged to give us the following result.

Theorem 3.4. *Let $K(s)$ be analytic and bounded by (2.1) on $\{\text{Re } s > \sigma_0\} \cup \{|s\Delta| > 1\}$. Under the stepsize restriction*

$$h \le R\Delta$$

the error bounds of Theorems 3.1–3.3 remain valid when the A-stability assumption is replaced by condition (3.31). The constants depend on the same quantities as previously, and in particular are independent of Δ.

4. Multistep time discretization of linear initial-boundary value problems

We consider again the initial-boundary value problem (2.9) which we assume throughout to be strongly well-posed in the generalized sense. We discretize in time by a linear multistep method, with the special starting values $u_n = 0$ for $n \le 0$, compatible with (2.9c). (More general starting values might be put into right-hand terms in equation (4.1) below.) In the notation of (3.1), this can be equivalently written as a system of the form (2.9) with ∂_t formally replaced by ∂_t^h:

$$\text{(4.1)} \qquad \begin{aligned} \partial_t^h u^h &= P(x, \partial_x) u^h + f(x,t) \;, \qquad && x \in \Omega \;, \; t = nh \ge 0 \\ L(x, \partial_x, \partial_t^h) u^h &= g(x,t) \;, && x \in \Gamma \;, \; t = nh \ge 0 \;. \end{aligned}$$

By considering the generating function $U^h(x,\zeta) = \sum_0^\infty u^h(x, nh)\zeta^n$ and similarly $F(x,\zeta)$, $G(x,\zeta)$ (to avoid convergence problems we may assume that f and g vanish for sufficiently large t), we get from (3.7) that U^h has to solve the boundary-value problem

$$\text{(4.2)} \qquad \begin{aligned} \frac{\delta(\zeta)}{h} \cdot U^h &= P(x, \partial_x)\, U^h + F \qquad \text{in } \Omega \\ L(x, \partial_x, \frac{\delta(\zeta)}{h})\, U^h &= G \qquad \text{on } \Gamma \end{aligned}$$

which is (2.11) with s replaced by $\delta(\zeta)/h$. For spatially smooth data f and g, we obtain a unique solution in terms of the solution operator $K(s)$ of (2.11):

$$\begin{pmatrix} U^h \\ U^h|_\Gamma \end{pmatrix} = K\left(\frac{\delta(\zeta)}{h}\right) \begin{pmatrix} F \\ G \end{pmatrix} \;.$$

Transforming back via (3.7), we thus get

$$\begin{pmatrix} u^h \\ u^h|_\Gamma \end{pmatrix} = K(\partial_t^h) \begin{pmatrix} f \\ g \end{pmatrix} . \tag{4.3}$$

For nonsmooth data, this defines generalized solutions (like in the time-continuous case). As a first consequence of (4.3) and the bound (2.13) we can state the following stability lemma.

Lemma 4.1. *Consider the semi-discretization (4.1) by an A-stable multistep method. For spatially smooth data f and g, there exists a unique solution $u^h = (u_n)_{n\geq 0}$ which is bounded by*

$$h \sum_{n=0}^{N} \|u_n\|_\Omega^2 + h \sum_{n=0}^{N} \|u_n\|_\Gamma^2 \leq C \cdot \left\{ h \sum_{n=0}^{N} \|f(\cdot, nh)\|_\Omega^2 + h \sum_{n=0}^{N} \|g(\cdot, nh)\|_\Gamma^2 \right\} .$$

This holds uniformly for $0 \leq Nh \leq T$ and $0 \leq h \leq h_0$, where h_0 depends on σ_0 of (2.13) and the multistep method (h_0 is arbitrary if $\sigma_0 \leq 0$.) The constant C can be chosen to depend only on M of (2.13), on σ_0, T, and h_0.

Proof. For ease of exposition, let us first consider the case $\sigma_0 \leq 0$ in (2.13). Then Parseval's formula implies that the ℓ^2 operator norm of $K(\partial_t^h)$ is bounded by

$$\sup_{|\zeta|<1} \|K(\delta(\zeta)/h)\| \leq \sup_{\mathrm{Re}\ s>0} \|K(s)\| \leq M .$$

Here we have used A-stability (3.8) in the first inequality, and (2.13) in the second inequality. This gives the stated result with $C = M$ for $\sigma_0 \leq 0$.

For the general case we observe that Re $\delta(\zeta) > h\sigma_0$ for $|\zeta| < \mathrm{e}^{-\gamma h}$, with $\gamma = c\sigma_0 + O(h)$ and a positive constant c. (Actually, $c = 1$ if $\zeta = 1$ is the only zero of $\delta(\zeta)$ on the unit circle, and if $\delta(\zeta)$ has no poles on the unit circle.) Again, we get via Parseval's formula

$$\sum_{n=0}^{\infty} \mathrm{e}^{-2\gamma nh} \left\{ \|u_n\|_\Omega^2 + \|u_n\|_\Gamma^2 \right\} \leq M^2 \sum_{n=0}^{\infty} \mathrm{e}^{-2\gamma nh} \left\{ \|f(\cdot, nh)\|_\Omega^2 + \|g(\cdot, nh)\|_\Gamma^2 \right\}$$

which yields the stated bound. □

Remark. For algebraically stable Runge-Kutta methods, a corresponding stability estimate follows with the proof of Proposition 10 in [LuO].

In the usual way, the stability bound leads to ℓ^2 convergence estimates in terms of the solution. On the other hand, in view of (2.14) and (4.3), the convergence results of the foregoing section yield error bounds in terms of the data. For example, we get from Theorem 3.1 the following pointwise estimates. Further error estimates are given by Theorems 3.2 and 3.3.

Corollary 4.2. *For smooth data (f, g) on $[0, T]$ with $f(\cdot, 0) = 0$ in Ω and $g(\cdot, 0) = 0$ on Γ, the error of the temporal semi-discretization (4.1) by an A-stable multistep method of order 2 is bounded at $t = nh$ by*

$$\|u_n - u(\cdot,t)\|_\Omega + \|u_n - u(\cdot,t)\|_\Gamma \le C \cdot \sum_{m=1}^{3} h^{2m/3} \left\{ \|f^{(m)}(\cdot,0)\|_\Omega + \|g^{(m)}(\cdot,0)\|_\Gamma \right\}$$
$$+ C \cdot h^2 \int_0^t \left\{ \|f^{(4)}(\cdot,\tau)\|_\Omega + \|g^{(4)}(\cdot,\tau)\|_\Gamma \right\} d\tau$$

uniformly for bounded t and $h \le h_0$.

Here, non-vanishing time derivatives of f and g at 0 should be interpreted as violations of compatibility conditions. Arbitrary starting values can be accounted for by adding a polynomial of degree not exceeding the order of the multistep method to both the exact and the time-discrete solution. These can then be interpreted as solutions of an equation of the form (2.9) (with modified f and g) with arbitrary initial value and starting values.

Stronger results are valid for parabolic problems where (2.13) (or equivalently (2.12)) holds in a sector $|\arg(s - \sigma_0)| < \pi - \vartheta$ with $\vartheta < \frac{\pi}{2}$. For p-th order strongly $A(\alpha)$-stable methods with $\alpha > \vartheta$, the full order of convergence p can be restored on time intervals bounded away from 0 in the case of smooth data which are incompatible at $t = 0$, cf. [LuS].

As the constants in Lemma 4.1 and Corollary 4.2 depend only on bounds for the solution operator of (2.11), the results extend to the fully discrete situation where the multistep method is applied to stable space discretizations of the initial-boundary value problem. Here, Theorem 3.4 yields in addition conditional convergence for methods which are not necessarily A-stable.

5. Time discretization of boundary integral equations

In this section we study temporal and to some extent spatial discretization of the single-layer potential equation of the (possibly dissipative) wave equation, as an instructive example of time-dependent boundary integral equations.

5.1. Semi-discretization in time

We recall the single-layer potential equation of the wave equation (2.17) in the notation of (2.2):

(5.1) $$V(\partial_t)\,\varphi = g \quad \text{on } \Gamma \times (0,T)\ ,$$

where $V(s) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ denotes the single-layer potential operator of the Helmholtz equation $\Delta U - s^2 U = 0$, see (2.18) and Prop. 2.3. Given a time stepsize $h > 0$, we obtain a semi-discretization in time by setting for $t = nh$ $(n = 0, 1, 2, \ldots)$

(5.2) $$V(\partial_t^h)\,\varphi^h = g \quad \text{on } \Gamma \times (0,T)\ ,$$

in the notation of (3.1). With $\varphi_j = \varphi^h(\cdot, jh)$, this is written more explicitly as the temporally discrete convolution equation

$$\sum_{j=0}^{n} \omega_{n-j}(h,V)\,\varphi_j = g(\cdot, nh) \quad \text{on } \Gamma\ , \quad \text{for } n = 0,1,2,\dots \tag{5.3}$$

Here the "quadrature weights" are linear operators $\omega_n(h,V) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ defined by (3.2), viz.,

$$\textstyle\sum_{n=0}^{\infty} \omega_n(h,V)\,\zeta^n = V(\delta(\zeta)/h)\ , \tag{5.4}$$

with an A-stable discretization method $\delta(\zeta)$. Provided that $g(\cdot, nh) \in H^{1/2}(\Gamma)$ for all n, equation (5.3) defines recursively $\varphi_n \in H^{-1/2}(\Gamma)$, since $\omega_0(h,V) = V(\delta(0)/h)$ is an isomorphism between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ by Proposition 2.3. The composition rule (3.5) gives us that

$$\varphi^h = V^{-1}(\partial_t^h) g\ . \tag{5.5}$$

Convergence of the semi-discretization (5.2) is therefore obtained by using the bound (2.20) for $V^{-1}(s)$ in Theorems 3.1–3.3. In particular, Theorem 3.3 gives us the following ℓ^2 error bound. Here and in the sequel we denote for brevity the norm on $H_0^r(0,T;H^q(\Gamma))$ by $\|\cdot\|_{q,r}$.

Theorem 5.1. *Let the discretization method be A-stable and of order 2 (see (3.8) and (3.9)), and satisfy (3.11). For smooth compatible data g on $\Gamma \times [0,T]$ (thus sufficiently many time derivatives of g vanish at $t = 0$!) the error of the semi-discretization (5.2) is bounded by*

$$\Big(h \textstyle\sum_{n=0}^{N} \|\varphi_n - \varphi(\cdot, nh)\|^2_{H^{-1/2}(\Gamma)}\Big)^{1/2} \le C \cdot h^2 \cdot \|g\|_{\frac{1}{2},5}\ , \tag{5.6}$$

uniformly for bounded $T = Nh$ and $h \le h_0$. The constant C is independent of h and g.

For data g which are less smooth or less compatible than indicated by the norm on the right-hand side of (5.6), Theorems 3.1–3.3 predict an order reduction.

Once an approximation φ^h of the density has been obtained, the solution of the wave equation can be approximated as a semi-discrete single-layer potential, cf. (2.26),

$$u^h = S(\partial_t^h)\,\varphi^h \quad \text{on } \Omega \times (0,T)\ , \tag{5.7}$$

where $S(s) : H^{-1/2}(\Gamma) \to H^1(\Omega)$ is defined by (2.25). This approximate solution is now compared to direct multistep semi-discretization of the wave equation.

Theorem 5.2. *For smooth compatible Dirichlet data g, the approximate solution $u^h = (u_n)_{n\geq 0}$ given by (5.7) and (5.2) is identical to the multistep semi-discretization of the wave equation (2.15):*

$$\begin{aligned} \left(\partial_t^h\right)^2 u^h &= \Delta u^h \quad \text{in } \Omega \times (0,T) \\ u^h &= g \quad \text{on } \Gamma \times (0,T) \end{aligned} \tag{5.8}$$

(with starting values $u^h(\cdot,t) = 0$ for $t < 0$ and $u^h(\cdot,t) \in H^1(\Omega)$ for all t.) Under the method assumptions of Theorem 5.1, the error is bounded by

$$\left(h\sum_{n=0}^N \|u_n - u(\cdot,nh)\|^2_{H^1(\Omega)} + \|\partial_t^h u_n - \partial_t u(\cdot,nh)\|^2_{L^2(\Omega)}\right)^{1/2} \leq C\cdot h^2 \cdot \|g\|_{\frac{1}{2},\frac{9}{2}} , \tag{5.9}$$

and pointwise in time by

$$\|u_n - u(\cdot,nh)\|_{H^1(\Omega)} + \|\partial_t^h u_n - \partial_t u(\cdot,nh)\|_{L^2(\Omega)} \leq C \cdot h^2\,|\log h| \cdot \|g\|_{\frac{1}{2},5} , \tag{5.10}$$

uniformly on bounded intervals $0 \leq nh \leq Nh = T$. The constants are again independent of $h \leq h_0$ and g.

Proof. We start from the multistep semi-discretization (5.8). Since every time step requires the solution of a Dirichlet problem for a non-homogeneous Helmholtz equation $\Delta u_n - (\delta(0)/h)^2 \cdot u_n = f_n$ with f_n a linear combination of previous solution values, equation (5.8) has a unique, spatially smooth solution $u_n \in H^1(\Omega)$ ($n \geq 0$). The generating functions $U^h(x,\zeta) = \sum_0^\infty u_n(x)\,\zeta^n$ and $G(x,\zeta) = \sum_0^\infty g(x,nh)\,\zeta^n$ (where again we may assume for convenience that g vanishes for large t) are related via

$$\begin{aligned} (\delta(\zeta)/h)^2 \cdot U^h &= \Delta U^h \quad \text{in } \Omega \\ U^h &= G \quad \text{on } \Gamma \ . \end{aligned}$$

This Dirichlet problem for the Helmholtz equation has a unique smooth solution in $H^1(\Omega)$ which can be represented as a single-layer potential (2.25):

$$U^h = S(\delta(\zeta)/h)\,\Phi^h \ ,$$

where Φ^h satisfies the single-layer potential equation (2.19):

$$V(\delta(\zeta)/h)\,\Phi^h = G \ .$$

Now (3.7) shows that u^h of (5.8) satisfies (5.7) with (5.2). The error estimates follow from the bound (2.27) used in Theorems 3.2 and 3.3. □

The above results extend immediately to the dissipative wave equation, cf. Sect. 2.3. Similar results for the Neumann problem follow by combining the estimates given in formulas (2.6) and (2.7) of [BaH2] with Theorems 3.1–3.3. For transient boundary integral equations of elasticity, convergence of temporal semi-discretizations is obtained by using the a-priori bounds of [Be] for the Laplace transformed problem in our Theorems 3.1–3.3. Stronger results, in particular for incompatible data, are available for boundary integral equations of parabolic problems [LuS].

5.2. Galerkin semi-discretization in space

To prepare for the fully discrete scheme, we study in this subsection space discretization by a Galerkin boundary element method. We write Δx for a small spatial discretization parameter, and let $X_{\Delta x} \subset L^2(\Gamma)$ denote a family of finite-dimensional approximation spaces of order m, in the sense that

$$\inf_{\psi_{\Delta x} \in X_{\Delta x}} \|\psi_{\Delta x} - \psi\|_{H^{-1/2}(\Gamma)} \leq C \cdot \Delta x^{m+\frac{1}{2}} \cdot \|\psi\|_{H^m(\Gamma)} , \qquad \text{for all } \psi \in H^m(\Gamma) . \tag{5.11}$$

As shown in Sect. 1.1 of [Ne], this is satisfied with $m = 1$ or $m = 2$, respectively, for approximation by piecewise constant or continuous piecewise linear functions over non-degenerate triangulations of maximum meshwidth Δx.

A spatial semi-discretization of the single-layer potential equation (5.1) can be done by a Galerkin method: Given smooth compatible data g on $\Gamma \times [0, T]$, find $\varphi_{\Delta x}(\cdot, t) \in X_{\Delta x}$ depending smoothly on t, such that in $0 \leq t \leq T$

$$\langle \big(V(\partial_t)\, \varphi_{\Delta x}\big)(\cdot, t), \psi_{\Delta x}\rangle = \langle g(\cdot, t), \psi_{\Delta x}\rangle \qquad \text{for all } \psi_{\Delta x} \in X_{\Delta x} , \tag{5.12}$$

with the $L^2(\Gamma)$ scalar product. (One verifies that this spatial semi-discretization is identical to that of formula (4.1) in [BaH1].) Stability and convergence are shown next.

Lemma 5.3. *For smooth compatible data g on $\Gamma \times [0, T]$, the above time-continuous Galerkin scheme has a unique solution $\varphi_{\Delta x}$, which is bounded like (2.23) for arbitrary $r \in \mathbb{R}$ (even with the same constant C_T):*

$$\|\varphi_{\Delta x}\|_{-\frac{1}{2}, r} \leq C_T \cdot \|g\|_{\frac{1}{2}, r+2} . \tag{5.13}$$

The error is bounded by

$$\|\varphi_{\Delta x} - \varphi\|_{-\frac{1}{2}, r} \leq C \cdot \Delta x^{m+\frac{1}{2}} \cdot \|\varphi\|_{m, r+3} . \tag{5.14}$$

Proof. We denote by $P_{\Delta x} : L^2(\Gamma) \to X_{\Delta x}$ and $\Pi_{\Delta x} : H^{-1/2}(\Gamma) \to X_{\Delta x}$ the orthogonal projections onto $X_{\Delta x}$.

(a) With

$$V_{\Delta x}(s) = P_{\Delta x} V(s) P_{\Delta x}^* : X_{\Delta x} \to X_{\Delta x} \tag{5.15}$$

the scheme (5.12) is written equivalently as

$$V_{\Delta x}(\partial_t)\, \varphi_{\Delta x} = P_{\Delta x} g \ . \tag{5.16}$$

Since by definition,

$$\langle s V_{\Delta x}(s) \psi_{\Delta x}, \psi_{\Delta x} \rangle = \langle s V(s) \psi_{\Delta x}, \psi_{\Delta x} \rangle \qquad \text{for all} \quad \psi_{\Delta x} \in X_{\Delta x} \ ,$$

the coercivity estimate of Proposition 2.3 implies the invertibility of $V_{\Delta x}(s)$ for Re $s > 0$ and the same bound as in (2.20):

$$\| V_{\Delta x}^{-1}(s) P_{\Delta x} \| \le c^{-1} \cdot |s|^2 \qquad (\text{Re } s > \sigma_0 > 0) \tag{5.17}$$

for the $H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ operator norm. The composition rule (2.3) now shows that

$$\varphi_{\Delta x} = V_{\Delta x}^{-1}(\partial_t) P_{\Delta x} g$$

is the unique temporally smooth solution of (5.16), and (5.17) together with Lemma 2.1 shows that this is bounded by (5.13).

(b) To prove the error bound, we form the difference of (5.16) and equation (5.1) premultiplied by $P_{\Delta x}$ to get

$$V_{\Delta x}(\partial_t)\,(\varphi_{\Delta x} - \Pi_{\Delta x}\varphi) = P_{\Delta x} V(\partial_t)\,(I - \Pi_{\Delta x})\,\varphi \ .$$

The stability estimate (5.13) thus gives us

$$\| \varphi_{\Delta x} - \Pi_{\Delta x}\varphi \|_{-\frac{1}{2},r} \le C_T \cdot \| V(\partial_t)(I - \Pi_{\Delta x})\,\varphi \|_{-\frac{1}{2},r+2} \ .$$

The bound of $V(s)$ in Proposition 2.3 together with Lemma 2.1 then shows that the right-hand side is bounded by a constant multiple of $\|(I - \Pi_{\Delta x})\,\varphi\|_{-\frac{1}{2},r+3}$, which in turn is bounded by the right-hand side of (5.14) because of the approximation property (5.11). □

5.3. Full discretization

We now combine the Galerkin boundary element method in space with operational quadrature in time and show unconditional convergence of the resulting fully discrete scheme. For clarity, we denote in this subsection the time step size by Δt (instead of h). Superscripts refer to time, subscripts to space.

We get a fully discrete scheme by formally replacing ∂_t by $\partial_t^{\Delta t}$ in (5.12): Find $\varphi_{\Delta x}^{\Delta t} = (\varphi^n)_{n \ge 0}$ in $X_{\Delta x}$ such that for $0 \le t = n\Delta t \le T$

$$\big\langle \big(V(\partial_t^{\Delta t})\, \varphi_{\Delta x}^{\Delta t} \big)(\cdot, t), \psi_{\Delta x} \big\rangle = \langle g(\cdot, t), \psi_{\Delta x} \rangle \quad \text{for all } \psi_{\Delta x} \in X_{\Delta x} \ . \tag{5.18}$$

With all the preparations done, convergence follows without more ado.

Theorem 5.4. *Let the time discretization method be A-stable and of order p and satisfy (3.11), and let the space discretization be of order m as specified by (5.11). For smooth compatible data g, the fully discrete method (5.18) (Galerkin in space, operational quadrature in time) is unconditionally convergent of optimal order:*

$$\|\varphi^n - \varphi(\cdot, n\Delta t)\|_{H^{-1/2}(\Gamma)} = O(\Delta t^p) + O(\Delta x^{m+\frac{1}{2}}) \ ,$$

uniformly over bounded time intervals.

Proof. We split

$$\varphi_{\Delta x}^{\Delta t} - \varphi = \left(\varphi_{\Delta x}^{\Delta t} - \varphi_{\Delta x}\right) + \left(\varphi_{\Delta x} - \varphi\right) \ .$$

The first term equals

$$\varphi_{\Delta x}^{\Delta t} - \varphi_{\Delta x} = V_{\Delta x}^{-1}(\partial_t^{\Delta t}) P_{\Delta x}\, g - V_{\Delta x}^{-1}(\partial_t) P_{\Delta x}\, g$$

with $V_{\Delta x}(s)$ of (5.15). Using the bound (5.17) in Theorem 3.1 or 3.2, this is seen to be $O(\Delta t^p)$ as required. The second term is the spatial discretization error, and the result follows from Lemma 5.3. □

It remains to discuss the actual implementation of (5.18). Let χ_i ($i = 1, \ldots, I$) be the basis functions chosen for the boundary element space $X_{\Delta x}$. For Re $s > 0$, we denote the Galerkin matrix by

(5.19) $$A(s) = \left(a_{ij}(s)\right) \in \mathbb{C}^{I \times I} \ , \qquad a_{ij}(s) = \langle V(s)\chi_j, \chi_i \rangle \ , \qquad i, j = 1, \ldots, I \ .$$

Data vectors are written for $n = 0, 1, \ldots, N$ as

$$g^n = \left(g_i^n\right) \in \mathbb{R}^I \ , \qquad g_i^n = \langle g(\cdot, n\Delta t), \chi_i \rangle \ , \qquad i = 1, \ldots, I \ .$$

The unknown coefficients of the approximate solution

$$\varphi^n = \sum_{i=1}^{I} \phi_i^n \chi_i$$

are collected in a vector $\phi^n \in \mathbb{R}^I$. Finally, let $A^n = \omega_n(\Delta t, A) \in \mathbb{R}^{I \times I}$ be defined by (3.2) with $A(s)$ in the role of $K(s)$. Then, the method (5.18) is equivalent to the recursion

(5.20) $$\sum_{j=0}^{n} A^{n-j}\, \phi^j = g^n \ , \qquad n = 0, 1, \ldots, N \ .$$

In every time step we thus have to solve a linear system with the same symmetric positive definite matrix $A^0 = A(\delta(0)/\Delta t)$. Moreover, the exponential decay of the fundamental solution for real $s > 0$ implies that the entries of A^0 behave like $a_{ij}^0 = O(\mathrm{e}^{-d_{ij} \cdot \delta(0)/\Delta t})$ where d_{ij} denotes the distance of the supports of the basis functions χ_i and χ_j. The other recursion matrices A^n ($n = 1, \ldots, N$) are computed to arbitrary precision by (3.10). With $L = 2N$ in (3.10), this requires the computation of Galerkin matrices $A(s)$ for $N + 1$ different complex values of s (a pleasure on massively parallel computers!), and $O(N \cdot \log N)$ arithmetical operations for the computation of the entries a_{ij}^n ($n = 1, \ldots, N$) using FFT. The recursion (5.20) itself can be solved using $O(N \cdot (\log N)^2)$ matrix-vector multiplications by the technique of [HaLS]. The computational complexity is thus almost linear in the number of time steps.

In practice, the Galerkin matrices $A(s)$ have to be approximated by perturbed matrices $\tilde{A}(s)$. These variational crimes should be committed systematically, such that $\tilde{A}(s)$ still depends analytically on s (except for very small perturbations). The recursion matrix $\tilde{A}^n$ computed by (3.10) is then an accurate approximation of $\omega_n(\Delta t, \tilde{A})$. The analysis of the effect of perturbations such as the approximation of the boundary can be largely extended from elliptic boundary integral equations to the present parameter-dependent case, see Sect. 6 of [BaH1] for a result of that type. Typically, the following situation arises: There is a small parameter ϵ characterizing the perturbation such that the perturbed matrix $A_\epsilon(s)$ has an inverse bounded by $C \cdot |s|^2$ (as in (2.20) and (5.17)) only for $|\epsilon s| \leq 1$. Here the following conditional stability lemma is useful, applied with $K_\epsilon(s) = A_\epsilon^{-1}(s) - A^{-1}(s) = -A_\epsilon^{-1}(s) \cdot (A_\epsilon(s) - A(s)) \cdot A^{-1}(s)$. We remark that this lemma allows us to derive more favorable error bounds than those of [BaH1, Theorem 8], as there is no loss of powers of Δt.

Lemma 5.5. *Let $K_\epsilon(s)$, $\epsilon > 0$, be a family of functions analytic on the half-disks $\{\text{Re } s > \sigma_0\} \cap \{|\epsilon s| \leq 1\}$, and there bounded by*

$$|K_\epsilon(s)| \leq M_\epsilon \cdot |s|^\mu \ .$$

Let the time discretization method satisfy the conditions of Theorem 3.1, and assume $B := \sup_{|\zeta|<1} |\delta(\zeta)| < \infty$. Under the restriction

$$h \geq B \cdot \epsilon$$

we have for data $g \in H_0^r(0,T)$ with $r \geq \mu$ and $r > \frac{1}{2}$ the stability bound

$$\Big(h \sum_{n=0}^{N} |K_\epsilon(\partial_t^h)\, g(nh)|^2 \Big)^{1/2} \leq C \cdot M_\epsilon \cdot \|g\|_{H^r} \ ,$$

uniformly for bounded $T = Nh$ and $h \leq h_0$. The constant C is independent of ϵ, h, and g.

We omit the proof which is based on the Poisson summation formula and Parseval's formula similarly to Theorem 3.3.

References

[An] Antes, H. (1988): Anwendungen der Methode der Randelemente in der Elastodynamik und der Fluiddynamik. Teubner, Stuttgart

[BaH1] Bamberger, A., Ha Duong, T. (1986): Formulation variationnelle espace-temps pour le calcul par potentiel retardé de la diffraction d'une onde acoustique. Math. Meth. Appl. Sci. **8**, 405–435

[BaH2] Bamberger, A., Ha Duong, T. (1986): Formulation variationnelle pour le calcul de la diffraction d'une onde acoustique par une surface rigide. Math. Meth. Appl. Sci. **8**, 598–608

[Be] Becache, E. (1991): Résolution par une méthode d'équations intégrales d'un problème de diffraction d'ondes élastiques transitoires par une fissure. Thèse de doctorat, Université de Paris 6

[Br] Brebbia C.A., ed. (1985): Topics in boundary element research, Vol. 2. Time-dependent and vibration problems. Springer, Berlin Heidelberg New York

[Da] Dahlquist, G. (1963): A special stability problem for linear multistep methods. BIT **3**, 27–43

[Eg] Eggermont, P.P.B. (1992): On the quadrature error in operational quadrature methods for convolutions. Numer. Math. **62**, 35–48

[EgL] Eggermont, P.P.B., Lubich, Ch. (1993): Operational quadrature methods for Wiener-Hopf integral equations. Math. Comp. **60**, 699–718

[HaLS] Hairer, E., Lubich, Ch., Schlichte, M. (1985): Fast numerical solution of nonlinear Volterra convolution equations. SIAM J. Sci. Stat. Comp. **6**, 532–541

[Ka] Katznelson, Y. (1976): An introduction to harmonic analysis. 2nd edition, Dover, New York

[KrL] Kreiss, H.O., Lorenz, J. (1989): Initial-boundary value problems and the Navier-Stokes equations. Academic Press, New York

[KrS] Kreiss, H.O., Scherer, G. (1992): Method of lines for hyperbolic differential equations. SIAM J. Numer. Anal. **29**, 640–646

[LiM] Lions, J.L., Magenes, E. (1972): Non-homogeneous boundary value problems and applications I. Springer, Berlin Heidelberg New York

[Lu1] Lubich, Ch. (1988): Convolution quadrature and discretized operational calculus. I and II. Numer. Math. **52**, 129–145 and 413–425

[Lu2] Lubich, Ch. (1992): On convolution quadrature and Hille-Phillips operational calculus. Appl. Numer. Math. **9**, 187–199

[LuO] Lubich, Ch., Ostermann, A. (1987): Multigrid dynamic iteration for parabolic equations. BIT **27**, 216–234

[LuS] Lubich, Ch., Schneider, R. (1992): Time discretization of parabolic boundary integral equations. Numer. Math. **63**, 455–481

[Ne] Nédélec, J.C. (1990): Approximation of integral equations by finite elements. Ch. XIII. In: R. Dautray, J.L. Lions, eds., Mathematical analysis and numerical methods for science and technology. Springer, Berlin Heidelberg New York

[ReT] Reddy, S.C., Trefethen, L.N. (1990): Lax-stability of fully discrete spectral methods via stability regions and pseudo-eigenvalues. Comp. Meth. Appl. Mech. Eng. **80**, 147–164