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Fortin operator and discrete compactness for edge elements

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Summary. The basic properties of the edge elements are proven in the original papers by Nédélec [22,23]. In the two-dimensional case the edge elements are isomorphic to the face elements (the well-known Raviart–Thomas elements [24]), so that all known results concerning face elements can be easily formulated for edge elements. In three-dimensional domains this is not the case. The aim of the present paper is to show how to construct a Fortin operator which converges uniformly to the identity in the spirit of [5,4]. The construction is given for any order tetrahedral edge elements in general geometries. We relate this result to the well-known *commuting diagram* property and apply it to improve the error estimate for a mixed problem which involves edge elements. Finally we show that this result can be applied to the analysis of the approximation of the time-harmonic Maxwell's system.

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1. Introduction

The so-called edge elements are widely used in the approximation of problems which arise from electromagnetics. They are also referred to as Whitney or Nédélec elements. Actually, the original idea is due to Whitney [25] in the framework of differential forms, while Nédélec [22] introduced them as a new family of finite element spaces. They are well-suited for problems involving the curl operator, as it can be seen in a general way in the framework of differential forms [7].

In the two-dimensional case, the edge elements are nothing else than the classical Raviart–Thomas (face) elements [24] rotated by the angle $\pi/2$. So that all known results on face element can be easily extended to edge elements by changing the role of the divergence with the curl. For an exhaustive description of Raviart–Thomas elements, see for instance [24, 9] and the references therein.

In the three-dimensional case, edge and face elements still have some analogies, but are no longer isomorphic. For instance, the numbers of degrees of freedom are different (e.g., six and four respectively for the lowest-order elements on tetrahedra). Some properties of edge elements are contained in the original papers by Nédélec [22, 23]. In particular basic approximation properties are proven, as well as an inf-sup condition (see [8]) in the case of a convex polyhedral domain; this implies that edge elements are well-suited for source problems associated with Maxwell's equations. See also [16] for a nice abstract setting concerning differential forms, div and curl operators, face and edge elements.

The main goal of this paper is the construction of a Fortin operator (see [14]) for tetrahedral edge elements of any order in the case of a polyhedral domain. We shall discuss later the minimal hypotheses on the domain, which, in particular, may be non-convex. Moreover, we shall prove that the Fortin operator converges uniformly to the identity, in a sense which will be made precise. This fills a gap between face and edge elements in three dimensional domains, in particular for what the commuting diagram property is concerned (see Sects. 2 and 3 for a parallel between the two families of spaces).

A Fortin operator for edge elements has been already used by other authors (see, for instance [12, 20, 21, 10, 17] and the references therein), however they usually need the mesh to be quasiuniform and more regularity hypotheses in order to prove the convergence of such an operator to the identity. In particular the estimate of Theorem 1 is new and, as we show in the last section, it is of fundamental importance for proving the absence of spurious solutions in the approximation of Maxwell's eigenproblem.

Girault and Raviart in their book [15] (see (5.50) p. 273) conjectured a similar property for the Fortin operator; our result is a partial answer to that question in a more general setting.

In this way, we prove in particular that the edge elements fit the hypotheses of [5, 4], showing that they are well-suited for eigenvalue problems arising from Maxwell's system (see [6]). Results of this nature are usually referred to as discrete compactness properties (see [18]).

An outline of the paper is as follows. In Sect. 2 we describe face and edge finite element spaces in 3D. We try to emphasize analogies and differences. In Sect. 3 we introduce the differential problem we are dealing with. Once again we shall do a parallel between face and edge elements. In particular we show why the construction of a Fortin operator for edge elements cannot be obtained as one usually does for face elements and we outline some properties related to the existence of a Fortin operator. In Sect. 4, we state and prove our main results concerning the Fortin operator. Finally, in Sect. 5, we apply the abstract setting to the approximation of Maxwell's eigenproblem.

2. Face and edge elements in three dimensions

In this section we recall the definitions and the basic properties of face and edge elements on tetrahedra. For convenience of the reader we state all results we are using in the next sections. As basic references concerning these results let us quote, for instance, [15,9,24,22].

Let Ω be a Lipschitz polyhedral (possibly non-convex) domain in $\mathbb{R}^3$, $\partial\Omega$ its boundary and $\underline{n}$ the outward unit normal vector.

Let $k \geq 0$ be a fixed integer number and given an open set $K \subset \mathbb{R}^3$ let us denote by $\mathcal{P}_k(K)$ the space of polynomials of degree at most k.

Let us introduce a tetrahedral triangulation $\mathcal{T}_h$ of Ω which we suppose to be *regular*. In particular we do not assume the mesh to be quasiuniform. With an abuse of notation, as usual the parameter h is related to the maximum diameter of the elements of $\mathcal{T}_h$.

Let us recall the definition of the Raviart-Thomas face discretization of $H(\mathrm{div};\Omega)$ which we denote by $\tilde{F}_h$ (the letter F stands for "face"). The restriction of an element of $\tilde{F}_h$ to a given tetrahedron K is of the form $\underline{p}(\underline{x}) + r(\underline{x})\underline{x}$ with $\underline{p} \in \mathcal{P}_k(K)^3$ and $r \in \mathcal{P}_k(K)$. The degrees of freedom of $\underline{p} \in \tilde{F}_h$ are the moments of the normal component of $\underline{p}$ of degree at most k on each face of K and the moments of $\underline{p}$ of degree at most $k-1$ ($k > 0$) on the tetrahedron K.

$\tilde{E}_h$ will denote the Nédélec edge discretization of $H(\mathrm{curl};\Omega)$ of index k. The restriction of an element of $\tilde{E}_h$ to a given tetrahedron K is of the form $\underline{p}(\underline{x}) + \underline{r}(\underline{x})$ with $\underline{p} \in \mathcal{P}_k(K)^3$ and $\underline{r} \in \mathcal{P}_{k+1}(K)^3$ such that $\underline{r} \cdot \underline{x} \equiv 0$. The degrees of freedom of $\underline{\sigma} \in \tilde{E}_h$ are the moments of the tangential component of $\underline{\sigma}$ of degree at most k on each edge of K, the moments of the tangential component of $\underline{\sigma}$ of degree at most $k-1$ ($k > 0$) on each face of K and the moments of $\underline{\sigma}$ of degree at most $k-2$ ($k > 1$) on the tetrahedron K.

Let us denote with π_h^F and π_h^E the interpolation operators which act to $\tilde{F}_h$ and $\tilde{E}_h$, respectively, using the appropriate degrees of freedom.

The following approximation properties hold true (see [1])

$$
(1)\qquad
\begin{array}{ll}
\|\underline{p}-\pi_h^F\underline{p}\|_0 \le Ch^s|\underline{p}|_s & 1/2 < s \le k+1\\
\|\underline{\sigma}-\pi_h^E\underline{\sigma}\|_0 \le Ch^s|\underline{\sigma}|_s & 1/2 < s \le k+1\\
\|\operatorname{div}\underline{p}-\operatorname{div}\pi_h^F\underline{p}\|_0 \le Ch^t|\operatorname{div}\underline{p}|_t & 0 < t \le k+1\\
\|\operatorname{curl}\underline{\sigma}-\operatorname{curl}\pi_h^E\underline{\sigma}\|_0 \le Ch^t|\operatorname{curl}\underline{\sigma}|_t & 0 < t \le k+1.
\end{array}
$$

The following relation between face and edge elements is of great importance

$$
(2)\qquad \operatorname{curl}\tilde{E}_h \subset \tilde{F}_h.
$$

The following relation expresses the so-called *commuting diagram* property for edge elements

$$
(3)\qquad \operatorname{curl}\pi_h^E\underline{\sigma} = \pi_h^F\operatorname{curl}\underline{\sigma}.
$$

Let us denote with P_{k+1}^c the space of *continuous* piecewise polynomials of degree at most $k+1$ and with P_k the space of arbitrary piecewise polynomials of degree at most k. The standard interpolation operator for P_k will be denoted with π_h^P.

The following relation characterizes the divergences of face elements

$$
(4)\qquad \operatorname{div}\tilde{F}_h = P_k
$$

and the following expression gives the *commuting diagram* property for face elements

$$
(5)\qquad \operatorname{div}\pi_h^F\underline{p} = \pi_h^P\operatorname{div}\underline{p}.
$$

According to the boundary condition of the problem we shall deal with, we introduce the notation E_h denoting the subspace of $\tilde{E}_h$ consisting of vector fields $\underline{\sigma}$ such that $\underline{\sigma}\times\underline{n}=0$ on $\partial\Omega$. In a similar way, F_h will denote the subspace of $\tilde{F}_h$ containing vector fields $\underline{p}$ with $\underline{p}\cdot\underline{n}=0$ on the boundary.

We shall also make use of the space $\dot{P}_{k+1}^c$ of continuous piecewise polynomials of degree at most $k+1$ which vanish on the boundary $\partial\Omega$ and of the space Q_h defined by $Q_h=\operatorname{div}F_h\subset P_k$.

Another useful space is $W_h=\operatorname{curl}E_h\subset F_h$.

The following inclusion is well-known

$$
(6)\qquad \operatorname{grad}\dot{P}_{k+1}^c \subset E_h
$$

and the following discrete Helmholtz decomposition has been used by many authors (see [2] for a nice and simple approach)

$$
(7)\qquad
\begin{array}{l}
A=\{\underline{\tau}\in E_h : (\underline{\tau},\operatorname{grad}p)=0\ \forall p\in\dot{P}_{k+1}^c\},\\
E_h = A\oplus\operatorname{grad}\dot{P}_{k+1}^c.
\end{array}
$$

3. Setting of the problem

In order to fix our notation, let us recall the definitions of the spaces we are using.

$$
(8)\qquad
\begin{array}{ll}
L^2(\Omega) & = \{v : \Omega \to \mathbb{R} \mid \int_\Omega v^2 < +\infty\} \\
L^2_0(\Omega) & = \{v \in L^2(\Omega) \mid \int_\Omega v = 0\} \\
H^1(\Omega) & = \{v \in L^2(\Omega) \mid \mathrm{grad}\, v \in L^2(\Omega)^3\} \\
H^1_0(\Omega) & = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\} \\
H(\mathrm{curl};\Omega) & = \{\underline{v} \in L^2(\Omega)^3 \mid \mathrm{curl}\, \underline{v} \in L^2(\Omega)^3\} \\
H_0(\mathrm{curl};\Omega) & = \{\underline{v} \in H(\mathrm{curl};\Omega) \mid \underline{v} \times \underline{n} = 0 \text{ on } \partial\Omega\} \\
H(\mathrm{div};\Omega) & = \{\underline{v} \in L^2(\Omega)^3 \mid \mathrm{div}\, \underline{v} \in L^2(\Omega)\} \\
H_0(\mathrm{div};\Omega) & = \{\underline{v} \in H(\mathrm{div};\Omega) \mid \underline{v} \cdot \underline{n} = 0 \text{ on } \partial\Omega\} \\
H(\mathrm{div}^0;\Omega) & = \{\underline{v} \in H(\mathrm{div};\Omega) \mid \mathrm{div}\, \underline{v} = 0 \text{ in } \Omega\} \\
H_0(\mathrm{div}^0;\Omega) & = H_0(\mathrm{div};\Omega) \cap H(\mathrm{div}^0;\Omega)
\end{array}
$$

We observe that it is standard to show, using suitable Green's formulae, that the traces involved in the definition of $H^1_0(\Omega)$, $H_0(\mathrm{curl};\Omega)$ and $H_0(\mathrm{div};\Omega)$ exist in the sense of [19]. Moreover, we remark that the spaces recalled in (8) are endowed with their usual norms, which we denote, in a natural way, by $||\cdot||_0$ (for both $L^2(\Omega)$ and $L^2(\Omega)^3$), $||\cdot||_1$, $||\cdot||_{\mathrm{curl}}$ and $||\cdot||_{\mathrm{div}}$; we shall denote the scalar product of $L^2(\Omega)^n$ ($n = 1, 3$), as usual, by $(\cdot,\cdot)$. In addition to the spaces introduced in (8), we shall make use of the fractional Sobolev spaces $H^s(\Omega)$ ($s > 0$), whose norm will be denoted by $||\cdot||_s$. For a definition of those spaces, see for instance [19].

Throughout the paper we implicitly assume that a curl-free vector field is a gradient. This is true, for instance, if Ω is simply connected. In presence of cavities within Ω our analysis can be extended following the guidelines of [1].

Let us introduce the following notation

$$
(9)\qquad
\begin{array}{l}
F = H_0(\mathrm{div};\Omega),\ \ Q = L^2_0(\Omega), \\
E = H_0(\mathrm{curl};\Omega),\ W = H_0(\mathrm{div}^0;\Omega)
\end{array}
$$

and consider the following two problems:

$$
(10)\qquad
\begin{array}{l}
\text{given } f \in Q, \text{ find } (\underline{p}, u) \in F \times Q \text{ such that} \\
\begin{cases}
(\underline{p}, \underline{q}) + (\mathrm{div}\, \underline{q}, u) = 0\ \forall \underline{q} \in F \\
(\mathrm{div}\, \underline{p}, v) = -(f, v) \quad \forall v \in Q.
\end{cases}
\end{array}
$$

and

$$
(11)\qquad
\begin{array}{l}
\text{given } \underline{g} \in W, \text{ find } (\underline{\sigma}, \underline{u}) \in E \times W \text{ such that} \\
\begin{cases}
(\underline{\sigma}, \underline{\tau}) + (\mathrm{curl}\, \underline{\tau}, \underline{u}) = 0\ \forall \underline{\tau} \in E \\
(\mathrm{curl}\, \underline{\sigma}, \underline{v}) = -(\underline{g}, \underline{v}) \quad \forall \underline{v} \in W.
\end{cases}
\end{array}
$$

It is not difficult to see that if the solutions of (10) and (11) are regular enough then they satisfy the following equations in strong form with the corresponding Neumann-type boundary conditions (where in order to obtain the second one we used the identity $-\underline{\Delta}\cdot = \operatorname{curl}\operatorname{curl}\cdot - \operatorname{grad}\operatorname{div}\cdot$ and the fact that $\operatorname{div}\underline{u} = 0$)

$$(12)\qquad \begin{array}{ll} -\Delta u = f \text{ in } \Omega & \\ \int_\Omega u = 0 & \\ \frac{\partial u}{\partial \underline{n}} = 0 & \text{on } \partial\Omega \\ \underline{p} = \operatorname{grad} u \text{ in } \Omega & \end{array} \qquad \begin{array}{ll} -\underline{\Delta}\,\underline{u} = \underline{g} & \text{in } \Omega \\ \operatorname{div}\underline{u} = 0 & \text{in } \Omega \\ \underline{u}\cdot\underline{n} = 0 & \text{on } \partial\Omega \\ \operatorname{curl}\underline{u}\times\underline{n} = 0 & \text{on } \partial\Omega \\ \underline{\sigma} = -\operatorname{curl}\underline{u} & \text{in } \Omega \end{array}$$

The analogies between problems (10) and (11) are evident; the same is true for their associated equations (12). In particular in two dimensions they are basically the same problem; actually in two variables the curl and the div operator are isomorphic.

The main goal of this section is to show that this is not the case when Ω is a three-dimensional domain; besides the analogies some major differences arise which make the analysis of the corresponding approximating schemes not equivalent.

In particular the face elements and their divergences are well-suited for the approximation of problem (10) and the involved numerical analysis consists of results which can be considered classical in this framework (see for instance [9] for a review of them).

On the other hand the edge elements and their curls could be used for the approximation of (11); the analysis of the corresponding numerical scheme has not been yet completed and in some case, as we shall see, it cannot follow the lines used for analyzing face elements.

Using the notation of the previous section, let us introduce the discrete problems which correspond to (10) and (11), respectively.

$$(13)\qquad \begin{array}{l} \text{given } f \in L^2_0(\Omega)\text{, find } (\underline{p}_h, u_h) \in F_h \times Q_h \text{ such that} \\ \begin{cases} (\underline{p}_h, \underline{q}) + (\operatorname{div}\underline{q}, u_h) = 0 \ \forall \underline{q} \in F_h \\ (\operatorname{div}\underline{p}_h, v) = -(f, v) \qquad \forall v \in Q_h. \end{cases} \end{array}$$

and

$$(14)\qquad \begin{array}{l} \text{given } \underline{g} \in H_0(\operatorname{div}^0;\Omega)\text{, find } (\underline{\sigma}_h, \underline{u}_h) \in E_h \times W_h \text{ such that} \\ \begin{cases} (\underline{\sigma}_h, \underline{\tau}) + (\operatorname{curl}\underline{\tau}, \underline{u}_h) = 0 \ \forall \underline{\tau} \in E_h \\ (\operatorname{curl}\underline{\sigma}_h, \underline{v}) = -(\underline{g}, \underline{v}) \qquad \forall \underline{v} \in W_h. \end{cases} \end{array}$$

Let us describe now some of the known properties concerning problems (13) and (14). For the face element we refer to [9] and to the references therein, for the edge elements we shall give the appropriate reference for each property.

3.1. The ellipticity in the kernel

The ellipticity in the kernel for problems (13) and (14) are immediate to obtain. Let us define the two kernels

$$K_1 = \{\underline{p} \in F_h : (\operatorname{div} \underline{p}, u) = 0 \;\forall u \in Q_h\}$$
$$K_2 = \{\underline{\sigma} \in E_h : (\operatorname{curl} \underline{\sigma}, \underline{u}) = 0 \;\forall \underline{u} \in W_h\}. \tag{15}$$

It is clear by the definitions of the discrete spaces that vectors in K_1 are divergence free while vectors in K_2 have vanishing curl, so that

$$(\underline{p}, \underline{p}) = ||\underline{p}||^2_{\text{div}} \quad \forall \underline{p} \in K_1,$$
$$(\underline{\sigma}, \underline{\sigma}) = ||\underline{\sigma}||^2_{\text{curl}} \;\forall \underline{\sigma} \in K_2. \tag{16}$$

3.2. The inf-sup condition

The following inf-sup condition is standard for face elements

$$\inf_{u \in Q_h} \sup_{\underline{p} \in F_h} \frac{(\operatorname{div} \underline{p}, u)}{||\underline{p}||_{\text{div}} ||u||_0} \geq C. \tag{17}$$

The corresponding inf-sup condition for edge elements

$$\inf_{\underline{u} \in W_h} \sup_{\underline{\sigma} \in E_h} \frac{(\operatorname{curl} \underline{\sigma}, \underline{u})}{||\underline{\sigma}||_{\text{curl}} ||\underline{u}||_0} \geq C \tag{18}$$

has been proven by Nédélec [22] in the case when Ω is convex and follows from Proposition 4.6 of [1] in the general case (see also [2]).

3.3. The Fortin operator

Given $f \in L^2_0(\Omega)$ and $\underline{g} \in H_0(\operatorname{div}^0; \Omega)$ let us consider the spaces F^0 and E^0 containing all $\underline{p}$ and $\underline{\sigma}$ which solve problems (10) and (11), respectively. We endow F^0 and E^0 with their natural norms.

A Fortin operator Π_1 for problem (13) has to satisfy

$$\Pi_1 : F^0 \to F_h$$
$$(\operatorname{div}(\underline{p} - \Pi_1 \underline{p}), u) = 0 \;\forall u \in Q_h, \;\forall \underline{p} \in F^0 \tag{19}$$
$$||\Pi_1 \underline{p}||_{\text{div}} \leq C ||\underline{p}||_{F^0} \quad \forall \underline{p} \in F^0.$$

It is well-known that the interpolation operator π^F_h fulfills conditions (19), so that can be chosen as a Fortin operator for problem (13). Moreover, if F^0 is contained in $H^s(\Omega)^3$ for $s > 1/2$ then we can easily obtain the estimate

$$||\underline{p} - \Pi_1 \underline{p}||_0 \le Ch^t ||\underline{p}||_s \tag{20}$$

where $t = \min(s, k+1)$.

Let us now consider the corresponding Fortin operator Π_2 associated with problem (14) which has to satisfy

$$\begin{aligned} &\Pi_2 : E^0 \to E_h \\ &(\operatorname{curl}(\underline{\sigma} - \Pi_2\underline{\sigma}), \underline{u}) = 0 \ \forall \underline{u} \in W_h, \ \forall \underline{\sigma} \in E^0 \\ &||\Pi_2\underline{\sigma}||_{\operatorname{curl}} \le C||\underline{\sigma}||_{E^0} \quad \forall \underline{\sigma} \in E^0. \end{aligned} \tag{21}$$

It turns out that the interpolation operator π_h^E does not meet the conditions described in (21), in particular one has for a general $\underline{\sigma} \in E^0$ and $\underline{u} \in W_h$

$$(\operatorname{curl}(\underline{\sigma} - \pi_h^E \underline{\sigma}), \underline{u}) \ne 0 \tag{22}$$

as can be verified by integration by parts. On the other hand the discrete problem (14) can be used in order to define a Fortin operator Π_2. Given $\underline{\sigma} \in E^0$ let $\underline{g} = -\operatorname{curl}\underline{\sigma}$ and *define* $\Pi_2\underline{\sigma} = \underline{\sigma}_h$, where $\underline{\sigma}_h$ solves (14). We state this construction in the following

Definition 1. *Given $\underline{\sigma} \in E^0$, let $\Pi_2\underline{\sigma} \in E_h$ be the first component $\underline{\sigma}_h$ of the solution of problem (14) with $\underline{g} = -\operatorname{curl}\underline{\sigma}$, that is for some $\underline{u}_h \in W_h$ it holds true*

$$\begin{cases} (\Pi_2\underline{\sigma}, \underline{\tau}) + (\operatorname{curl}\underline{\tau}, \underline{u}_h) = 0 \ \forall \underline{\tau} \in E_h \\ (\operatorname{curl}\Pi_2\underline{\sigma}, \underline{v}) = (\operatorname{curl}\underline{\sigma}, \underline{v}) \quad \forall \underline{v} \in W_h. \end{cases} \tag{23}$$

Let us check that Π_2 satisfies (21). Indeed from the second equation of (23) it follows the first property in (21). The second bound of (21) is a consequence of the stability of (23). Moreover it can be shown that $\Pi_2\underline{\sigma}$ is the unique solution $\underline{\sigma}_h$ of the following problem

$$\begin{aligned} &\text{find } \underline{\sigma}_h \in A \text{ such that} \\ &(\operatorname{curl}\underline{\sigma}_h, \operatorname{curl}\underline{\tau}) = (\operatorname{curl}\underline{\sigma}, \operatorname{curl}\underline{\tau}) \quad \forall \underline{\tau} \in A. \end{aligned} \tag{24}$$

In a similar way (making explicit the definition of A) it can be shown that $\Pi_2\underline{\sigma}$ is the first component $\underline{\sigma}_h$ of the solution of the following *stable* mixed problem

$$\begin{aligned} &\text{find } (\underline{\sigma}_h, p_h) \in E_h \times \dot{P}^c_{k+1} \text{ such that} \\ &\begin{cases} (\operatorname{curl}\underline{\sigma}_h, \operatorname{curl}\underline{\tau}) + (\underline{\tau}, \operatorname{grad} p_h) = (\operatorname{curl}\underline{\sigma}, \operatorname{curl}\underline{\tau}) \ \forall \underline{\tau} \in E_h \\ (\underline{\sigma}_h, \operatorname{grad} q) = 0 \qquad\qquad \forall q \in \dot{P}^c_{k+1}. \end{cases} \end{aligned} \tag{25}$$

Remark 1. The definition of the Fortin operator Π_2 relies on the well-posedness of problem (23). This is quite unusual; actually, it is more common to construct a Fortin operator in order to prove a mixed problem to be well-posed. On the other hand our analysis shows the necessity for a Fortin operator to exist whenever a mixed problem is stable. And the existence of a Fortin operator is needed, for instance, in order to analyze the eigenvalue problem (51) (see the final section).

Let us now try to prove the analogous of (20) for the operator Π_2 using the standard error estimates for problems (14) and (25).

Let us consider problems (11) and (14) with $\underline{g} = -\operatorname{curl}\underline{\sigma}$. It is then clear that $\underline{\sigma}$ solves problem (11) with $\underline{u}$ such that $\operatorname{curl}\underline{u} = -\underline{\sigma}$. Since $\Pi_2\underline{\sigma}$ by definition solves problem (14) from the standard a priori error estimate of mixed methods (see [8]) we have

$$
\begin{aligned}
||\underline{\sigma} - \Pi_2\underline{\sigma}||_0 &\leq ||\underline{\sigma} - \Pi_2\underline{\sigma}||_{\mathrm{curl}} \\
&\leq \inf_{\underline{\tau}\in E_h, \underline{v}\in W_h} (||\underline{\sigma} - \underline{\tau}||_{\mathrm{curl}} + ||\underline{u} - \underline{v}||_0).
\end{aligned} \tag{26}
$$

It turns out that the second term in the right hand side of (26) can be estimated in an uniform way by using the regularity of $\underline{u}$. On the other hand in order to deal with the first term we cannot use any extra regularity of $\underline{\sigma}$. In particular if $\underline{\sigma}$ belongs to E^0 then its curl by definition is in $L^2(\Omega)^3$ and we cannot use estimate (1) to obtain an uniform bound for $||\underline{\sigma} - \Pi_2\underline{\sigma}||_{\mathrm{curl}}$.

Analogous troubles arise of course when we try to use the error estimate of problem (25).

Actually, we cannot hope to do better if we consider the $H(\mathrm{curl})$–norm of the difference $\underline{\sigma} - \Pi_2\underline{\sigma}$. The same problem arises when estimating the $H(\mathrm{div})$–norm of the difference $\underline{p} - \Pi_1\underline{p}$. However, if we confine ourselves to the L^2–norm then we would like to get an estimate of the type

$$
||\underline{\sigma} - \Pi_2\underline{\sigma}||_0 \leq C||\underline{\sigma} - \pi_h^E\underline{\sigma}||_0 \tag{27}
$$

This is usually obtained by using the *commuting diagram property* that we are going to analyze in the next subsection.

3.4. The commuting diagram

The commuting diagram property for face elements is summarized in the equation

$$
\operatorname{div}\pi_h^F\underline{p} = \pi_h^P\operatorname{div}\underline{p} \qquad \forall\underline{p} \text{ “smooth enough”} \tag{28}
$$

From this property one can obtain the fundamental error estimate for problem (13)

$$
||\underline{p} - \underline{p}_h||_0 \leq C||\underline{p} - \pi_h^F\underline{p}||_0. \tag{29}
$$

Moreover it is clear that property (28) is strictly related to the fact that the Fortin operator Π_1 can be chosen to be equal to the interpolation operator π_h^F.

Let us see what happens when edge elements are considered for the approximation of problem (11). We would like to obtain the analogous of (29) which reads

$$||\underline{\sigma} - \underline{\sigma}_h||_0 \leq C||\underline{\sigma} - \pi_h^E \underline{\sigma}||_0. \tag{30}$$

Unfortunately (30) cannot be obtained in general as it has been pointed out in [2]. In order to get (30) we would need the following commuting diagram property which is false:

$$\operatorname{curl} \pi_h^E \underline{\sigma} = \pi_h^{W_h} \operatorname{curl} \underline{\sigma}, \tag{31}$$

where $\pi_h^{W_h}$ denotes the projection onto the space W_h, that is

$$(\pi_h^{W_h} \underline{u}, \underline{v}) = (\underline{u}, \underline{v}) \qquad \forall \underline{v} \in W_h \ . \tag{32}$$

Actually the following commuting diagram property holds true for edge elements

$$\operatorname{curl} \pi_h^E \underline{\sigma} = \pi_h^F \operatorname{curl} \underline{\sigma} \ . \tag{33}$$

In [2] however it has been proven the following estimate for problem (14) (actually, estimate (34) has been obtained in [2] under the hypothesis on Ω to be convex, but the same proof also works in our case)

$$||\underline{\sigma} - \underline{\sigma}_h||_0 \leq ||\underline{\sigma} - \pi_h^E \underline{\sigma}||_0, \qquad \text{when } g \in W_h. \tag{34}$$

We shall make use of the previous estimate in the next section in order to prove that the Fortin operator Π_2 converges uniformly to the identity in the L^2–norm.

4. Fortin operator for edge elements

This section contains the main result of the present paper concerning the Fortin operator.

From the previous section (see subsection 3.3) we know that there exists an operator $\Pi_2 : E^0 \to E_h$ such that

$$\begin{aligned} &(\operatorname{curl}(\underline{\sigma} - \Pi_2 \underline{\sigma}), \underline{v}_h) = 0 \ \forall \underline{\sigma} \in E^0, \ \forall \underline{v}_h \in W_h, \\ &||\Pi_2 \underline{\sigma}||_{\mathrm{curl}} \leq C||\underline{\sigma}||_{E^0} \qquad \forall \underline{\sigma} \in E^0. \end{aligned} \tag{35}$$

We know moreover that Π_2 can be defined from E^0 to A as the solution $\underline{\sigma}_h$ of the following problem

$$\begin{aligned} &\text{given } \underline{\sigma} \in E^0 \text{ find } \underline{\sigma}_h \in A \text{ such that} \\ &(\operatorname{curl} \underline{\sigma}_h, \operatorname{curl} \underline{\tau}) = (\operatorname{curl} \underline{\sigma}, \operatorname{curl} \underline{\tau}) \quad \forall \underline{\tau} \in A \ . \end{aligned} \tag{36}$$

We recall (see (26)) that we have the following *pointwise* approximation property:

(37) $$||\underline{\sigma} - \Pi_2\underline{\sigma}||_{\mathrm{curl}} \to 0 \quad \forall \underline{\sigma} \in E^0.$$

The aim of the main theorem of this section is to prove that if we replace the $H(\mathrm{curl};\Omega)$-norm with the $L^2(\Omega)$-one then the convergence becomes uniform.

Remark 2. As it has been pointed out in the previous section, if we know some specific regularity on $\underline{\sigma}$ and $\mathrm{curl}\,\underline{\sigma}$, then the following result is basically known in the case of quasiuniform meshes (see e.g. [12,20,10]) and easy to obtain without the quasiuniformity assumption using for instance the standard error estimate for mixed problems applied to problem (14). However our result holds true under the very weak hypothesis that $\underline{\sigma}$ belongs to $H^s(\Omega)^3$ for some $s > 1/2$ and $\mathrm{curl}\,\underline{\sigma}$ is in $L^2(\Omega)^3$. As we shall see in Sect. 5, this is a crucial point when we are interested in the approximation of eigenvalue problems. In that case if we do not show the uniform convergence of Π_2 to the identity without any additional regularity on $\mathrm{curl}\,\underline{\sigma}$ then we cannot prove the absence of spectrum pollution.

Before stating the theorem concerning the Fortin operator, we make some remark and some hypotheses concerning the solution of problem (11) and its approximation (14).

We recall that E^0 is the space containing all the vector fields $\underline{\sigma}$ which solve (11) with $g \in H_0(\mathrm{div}^0;\Omega)$, that is

(38) $$E^0 = \{\underline{\sigma} \in H_0(\mathrm{curl};\Omega) : \mathrm{div}\,\underline{\sigma} = 0\}.$$

In a similar way we denote by W^0 the space of $\underline{u}$ solving (11) with $g \in H_0(\mathrm{div}^0;\Omega)$, that is

(39) $$W^0 = \{\underline{u} \in H_0(\mathrm{div}^0;\Omega) : -\underline{\Delta}\,\underline{u} \in L^2(\Omega)^3, \\ \mathrm{curl}\,\underline{u} \times \underline{n} = 0 \text{ on } \partial\Omega\}.$$

We shall make the following assumption

(40) $$[\mathrm{REG}] \qquad \begin{array}{l} E^0 \subset H^s(\Omega)^3, \\ W^0 \subset H^s(\Omega)^3, \end{array}$$

for $s > 1/2$ and with continuous embeddings.

Remark 3. In [1] it has been proven that [REG] holds true if Ω is a polyhedral domain.

Theorem 1. *Let Ω be a polyhedral domain or more generally let us assume that [REG] is fulfilled with $1/2 < s \le 1$. Then there exists C independent of h such that*

(41) $$||\underline{\sigma} - \Pi_2\underline{\sigma}||_0 \le Ch^s||\underline{\sigma}||_s.$$

Proof. Since $\Pi_2\underline{\sigma} \in A$, equation (36) implies that there exists $\underline{u}_h \in W_h$ such that $(\Pi_2\underline{\sigma}, \underline{u}_h)$ solves (14) with right-hand side $\underline{g} = -\operatorname{curl}\underline{\sigma}$.

Let us consider the solution $(\tilde{\underline{\sigma}}, \tilde{\underline{u}})$ of (11) with $\underline{g} = \operatorname{curl}\Pi_2\underline{\sigma}$. We have

$$||\tilde{\underline{\sigma}}||_s \leq C||\operatorname{curl}\Pi_2\underline{\sigma}||_0 \leq C||\operatorname{curl}\underline{\sigma}||_0 \tag{42}$$

and from (1), (34) and (42) it follows (since of course $\Pi_2\underline{\sigma}$ solves (14) with RHS $\underline{g} = -\operatorname{curl}\Pi_2\underline{\sigma}$):

$$||\tilde{\underline{\sigma}} - \Pi_2\underline{\sigma}||_0 \leq C||\tilde{\underline{\sigma}} - \pi_h^E\tilde{\underline{\sigma}}||_0 \leq Ch^s||\tilde{\underline{\sigma}}||_s \leq Ch^s||\operatorname{curl}\underline{\sigma}||_0. \tag{43}$$

Remark 4. Actually, estimate (43) is not formally correct. Indeed, according to [1], the interpolant $\pi_h^E\tilde{\underline{\sigma}}$ is defined if $\tilde{\underline{\sigma}} \in H^s(\Omega)^3$, $s > 1/2$ and $\operatorname{curl}\tilde{\underline{\sigma}} \in L^p(\Omega)^3$, $p > 2$. In our case $\operatorname{curl}\tilde{\underline{\sigma}}$ is a priori bounded only in $L^2(\Omega)^3$. However, taking advantage of the relation $\operatorname{curl}\tilde{\underline{\sigma}} = \Pi_2\underline{\sigma}$, it is possible to adapt in a straightforward way the technique used in page 856 of [1] to get the required estimate.

So we have to estimate the difference $||\underline{\sigma} - \tilde{\underline{\sigma}}||_0$ in order to conclude the proof by triangular inequality.

Let us introduce the following functions:

$$\phi = \underline{\sigma} - \tilde{\underline{\sigma}}, \quad \psi = \underline{u} - \tilde{\underline{u}}, \tag{44}$$

where we considered $\underline{u}$ so that $(\underline{\sigma}, \underline{u})$ solves (11) with $\underline{g} = \operatorname{curl}\underline{\sigma}$. It follows that:

$$\begin{aligned} &\operatorname{curl}\psi = \phi \\ &||\psi||_s \leq C||\phi||_0 \\ &\operatorname{curl}\phi = \operatorname{curl}(\underline{\sigma} - \tilde{\underline{\sigma}}) = \operatorname{curl}(\underline{\sigma} - \Pi_2\underline{\sigma}). \end{aligned} \tag{45}$$

We are now in position to conclude with the final estimate

$$\begin{aligned} ||\phi||_0^2 &= (\phi, \operatorname{curl}\psi) = (\operatorname{curl}\phi, \psi) \\ &= (\operatorname{curl}(\underline{\sigma} - \Pi_2\underline{\sigma}), \psi) = (\operatorname{curl}(\underline{\sigma} - \Pi_2\underline{\sigma}), \psi - \pi_h^F\psi) \\ &\leq Ch^s||\operatorname{curl}(\underline{\sigma} - \Pi_2\underline{\sigma})||_0||\psi||_s \leq Ch^s||\operatorname{curl}\underline{\sigma}||_0||\phi||_0, \end{aligned} \tag{46}$$

where we used the properties of the Fortin operator Π_2 together with the fact that $\operatorname{div}\psi = 0$, so that $\pi_h^F\psi$ belongs to W_h due to the commuting diagram property.

Remark 5. The result obtained in Theorem 1 is not optimal. In the case of face elements the interpolation operator is a Fortin operator (see (19)) and we would like to obatin also for edge element the following approximation property:

$$||\underline{\sigma} - \Pi_2\underline{\sigma}||_0 \leq ||\underline{\sigma} - \pi_h^E\underline{\sigma}||_0 \tag{47}$$

Our proof however relies on the solution $\tilde{\underline{\sigma}}$ of problem (11) with $\underline{g} = \operatorname{curl}\Pi_2\underline{\sigma}$. The regularity of $\underline{g}$ is just $L^2(\Omega)^3$, so that we cannot hope in general $\tilde{\underline{\sigma}}$ to be more regular than $H^1(\Omega)^3$ even in the case of a convex domain. This means that the constraint on s to be not bigger than 1 cannot be avoided.

5. Application to eigenvalue problems

In this section we apply the Fortin operator analyzed in Theorem 1 to the study of the approximation of the following eigenvalue problem which arises from problem (11). This result fits the framework of [5] and [4] and is an improvement of the one which has been announced in [6].

We are dealing with the following eigenvalue problem

$$\text{(48)}\qquad \begin{array}{l} \text{find } (\lambda, \underline{u}) \in \mathbb{C} \times H_0(\mathrm{div}^0; \Omega), \\ \text{with } \underline{u} \neq 0, \text{ such that } \exists \underline{\sigma} \in H_0(\mathrm{curl}; \Omega): \\ \begin{cases} (\underline{\sigma}, \underline{\tau}) + (\mathrm{curl}\, \underline{\tau}, \underline{u}) = 0 \;\forall \underline{\tau} \in H_0(\mathrm{curl}; \Omega) \\ (\mathrm{curl}\, \underline{\sigma}, \underline{v}) = -\lambda(\underline{u}, \underline{v}) \quad \forall \underline{v} \in H_0(\mathrm{div}^0; \Omega). \end{cases} \end{array}$$

It is clear that the operator which maps $\underline{g}$ to $\underline{u}$ (see (11)) is self-adjoint and moreover, due to the compact embedding of $H_0(\mathrm{curl}; \Omega) \cap H(\mathrm{div}^0; \Omega)$ into $L^2(\Omega)^3$ (see, for instance, for general compactness results [13]), it is also compact. Hence the eigenvalues of (48) are real and can be ordered in an increasing sequence, which tends to the infinity. We denote them by λ_i ($i \in \mathbb{N}$), in such a way that

$$\text{(49)}\qquad 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \cdots$$

and all the eigenvalues have multiplicity equal to one (notice that, with our notation, to a multiple eigenvalue correspond several distinct λ_i with the same value). Moreover we denote by $\mathcal{E}_i$ the eigenspace associated with λ_i.

Following [5], the following map $m : \mathbb{N} \to \mathbb{N}$ will be useful to analyze the eigenvalues convergence.

$$\text{(50)}\qquad \begin{array}{l} m(1) = \dim\{\oplus_i \mathcal{E}_i : \lambda_i = \lambda_1\}, \\ m(N+1) = m(N) + \dim\{\oplus_i \mathcal{E}_i : \lambda_i = \lambda_{m(N)+1}\}. \end{array}$$

Hence, $\lambda_{m(1)}, \ldots, \lambda_{m(N)}$ will be the first N distinct eigenvalues of (48).

Given two sequences of finite dimensional subspaces Σ_h and W_h of $H_0(\mathrm{curl}; \Omega)$ and $H_0(\mathrm{div}^0; \Omega)$, respectively, the finite element discretization of (48) reads as follows:

$$\text{(51)}\qquad \begin{array}{l} \text{find } \lambda_h \in \mathbb{R} \text{ and } \underline{u}_h \in W_h, \text{ with } \underline{u}_h \neq 0, \text{ such that } \exists \underline{\sigma}_h \in \Sigma_h: \\ \begin{cases} (\underline{\sigma}_h, \underline{\tau}_h) + (\mathrm{curl}\, \underline{\tau}_h, \underline{u}_h) = 0 \;\forall \underline{\tau}_h \in \Sigma_h \\ (\mathrm{curl}\, \underline{\sigma}_h, \underline{v}_h) = -\lambda_h(\underline{u}_h, \underline{v}_h) \;\forall \underline{v}_h \in W_h. \end{cases} \end{array}$$

We denote the eigenvalues of (51) by $\lambda_{h,i}$ ($i \in \mathbb{N}$) so that they are ordered in an increasing sequence

$$\text{(52)}\qquad 0 < \lambda_{h,1} \leq \lambda_{h,2} \leq \ldots \leq \lambda_{h,i} \leq \ldots,$$

the corresponding discrete eigenspace will be denoted by $\mathcal{E}_{h,i}$.

Before recalling the hypotheses introduced in [4] for the convergence of eigenvalues/eigenvectors of (51) to those of (48), let us set some notation.

We recall the definition of E^0 and W^0 as the spaces of the solutions σ and $\underline{u}$ of problem (11), with $\underline{g}$ varying in $H_0(\mathrm{div}^0;\Omega)$. If Ω is a general (possibly non-convex) polyhedral domain then is has been proven in [1] that E^0 and W^0 are contained in $H^s(\Omega)^3$. We shall denote by $||\cdot||_{E^0}$ and $||\cdot||_{W^0}$ their natural norms.

The discrete kernel IK is defined by

$$IK = \{\underline{\tau}_h \in \Sigma_h : (\mathrm{curl}\,\underline{\tau}_h, \underline{v}_h) = 0\ \forall \underline{v}_h \in W_h\}. \tag{53}$$

The following hypotheses have been introduced in [4] in an abstract framework (see definitions 3, 4 and 5).

H1: the *weak approximability* of W^0 is satisfied if there exists $\omega_1(h)$ tending to zero as h goes to zero such that for every $\underline{u} \in W_0$ and for every $\underline{\tau}_h \in IK$

$$(\mathrm{curl}\,\underline{\tau}_h, \underline{u}) \le \omega_1(h)||\underline{\tau}_h||_0||\underline{u}||_{W^0}. \tag{54}$$

H2: the *strong approximability* of W^0 is satisfied if there exists $\omega_2(h)$ tending to zero as h goes to zero such that for every $\underline{u} \in W^0$ there exists $\underline{u}^I \in W_h$ such that

$$||\underline{u} - \underline{u}^I||_{\mathrm{div}} \le \omega_2(h)||\underline{u}||_{W^0}. \tag{55}$$

H3: $\Pi_h : E^0 \to \Sigma_h$ is called *Fortin operator* (see also the previous section) if it satisfies:

$$\begin{aligned}&(\mathrm{curl}(\underline{\sigma} - \Pi_h\underline{\sigma}), \underline{v}_h) = 0\ \forall \underline{\sigma} \in E^0,\ \forall \underline{v}_h \in W_h,\\ &||\Pi_h\underline{\sigma}||_{\mathrm{curl}} \le C||\underline{\sigma}||_{E^0} \quad \forall \underline{\sigma} \in E^0.\end{aligned} \tag{56}$$

H4: A Fortin operator Π_h *converges uniformly to the identity* if there exists $\omega_3(h)$ tending to zero as h goes to zero such that

$$||\underline{\sigma} - \Pi_h\underline{\sigma}||_0 \le \omega_3(h)||\underline{\sigma}||_{E^0} \quad \forall \underline{\sigma} \in E^0. \tag{57}$$

In [5, 4] it has been proven that H1–H4 imply that the discrete eigenvalues converge to the continuous ones, that is

$$\begin{aligned}&\forall \varepsilon > 0,\ \forall N \in \mathbb{N}\ \exists h_0 > 0 \text{ such that } \forall h \le h_0\\ &\max_{i=1,\dots,m(N)} |\lambda_i - \lambda_{h,i}| \le \varepsilon,\\ &\hat{\delta}(\oplus_{i=1}^{m(N)} \mathcal{E}_i, \oplus_{i=1}^{m(N)} \mathcal{E}_{h,i}) \le \varepsilon,\end{aligned} \tag{58}$$

where $\hat{\delta}(A, B)$, for A and B linear subspaces of $L^2(\Omega)$, denotes as usual the gap between A and B.

Using the notation of Sect. 3 let us now define $\Sigma_h = E_h$. In the previous section it has been proved the validity of H3 and H4 under weak regularity hypotheses.

In order to apply the abstract theory of [4] it remains to check hypotheses H1 and H2.

It is immediate to see that hypothesis H1 holds true; indeed the left-hand side in (54) is always equal to zero. Actually, an element $\underline{\tau}_h$ in IK satisfies $\operatorname{curl} \underline{\tau}_h = 0$. For the proof of hypothesis H2, we use the following standard way. Let us define $\underline{u}^I$ to be the face interpolant of $\underline{u}$, that us $\underline{u}^I = \pi_h^F \underline{u}$. Then, thanks to the commuting diagram property for edge elements (3), $\operatorname{div} \underline{u}^I = 0$ and $\underline{u}^I$ belongs to W_h. Thus estimate (55) follows from the classical approximation properties of face elements (see (1)), together with the regularity results on $\underline{u}$ with $\omega_2(h) = Ch^s$.

It follows that the following convergence theorem holds true (see [3], Thm.'s 11.1 and 11.2 and [5]).

Theorem 2. *Let λ_i and $\lambda_{h,i}$ ($i \in \mathbb{N}$) be the eigenvalues of (48) and (51), respectively. Assume that the space E^0 is contained in $H^s(\Omega)^3$ for $s > 1/2$. For the sake of simplicity let us suppose that λ_i corresponds to a simple eigenvalue; this means that $\lambda_i \neq \lambda_j$ for $i \neq j$. Let us denote by $\underline{\sigma}_i$ and $\underline{u}_i$ the corresponding eigenfunctions ($\underline{\sigma}_{h,i}$ and $\underline{u}_{h,i}$ the discrete ones), normalized in such a way that $||\underline{u}_i||_0 = ||\underline{u}_{h,i}||_0 = 1$. Then (58) is fulfilled. Moreover the following estimates hold true*

$$
\begin{aligned}
&|\lambda_i - \lambda_{h,i}| \leq C \inf_{\underline{\tau}\in E_h, \underline{v}\in W_h} \\
&\times \left(||\underline{\sigma}_i - \underline{\tau}||_0^2 + ||\underline{\sigma}_i - \underline{\tau}||_{\text{curl}} ||\underline{u}_i - \underline{v}||_0 + ||\underline{u}_i - \underline{v}||_0^2 \right) \\
&||\underline{u}_i - \underline{u}_{h,i}||_0 \leq \inf_{\underline{\tau}\in E_h, \underline{v}\in W_h} (||\underline{\sigma} - \underline{\tau}||_{\text{curl}} + ||\underline{u}_i - \underline{v}||_0).
\end{aligned}
\tag{59}
$$

Remark 6. Theorem 2 deals with simple eigenvalues. This has been done in order to make the presentation simpler, however it is not difficult to extend the result to the general case according to [3].

Remark 7. As far as the regularity hypothesis $E^0 \subset H^s(\Omega)^3$ is concerned, it has been proven in [1] that it holds true if the domain is a polyhedron. Actually it seems to be satisfied in most practical application, except when different materials may be involved (see [11]).

We point out however that this is the minimum regularity for which the interpolation operator π_h^E can be defined.

In [6] it has been shown that the eigensolution of problem (48) can be related to the solution to the Maxwell's cavity problem.

In the original Maxwell's eigenproblem, $\underline{\sigma}_i$ is the eigenfunction component we are interested in, so that it would be useful to estimate the error $||\underline{\sigma}_i - \underline{\sigma}_{h,i}||_{\text{curl}}$ and to give a bound for $|\lambda_i - \lambda_{h,i}|$ only in terms of $\underline{\sigma}_i$. This is the aim of the following theorem

Theorem 3. *Under the same hypotheses and notation as in Theorem 2, let us assume that* $\inf_{\underline{v}\in W_h} ||\underline{u}_i - \underline{v}||_0$ *is negligible with respect to* $\inf_{\underline{\tau}\in E_h} ||\underline{\sigma}_i - \underline{\tau}||_{\mathrm{curl}}$*, that is*

$$\inf_{\underline{v}\in W_h} ||\underline{u}_i - \underline{v}||_0 \le C \inf_{\underline{\tau}\in E_h} ||\underline{\sigma}_i - \underline{\tau}||_{\mathrm{curl}}. \tag{60}$$

Then one has

$$\begin{aligned} &|\lambda_i - \lambda_{h,i}| \le C \inf\nolimits_{\underline{\tau}\in E_h} ||\underline{\sigma}_i - \underline{\tau}||^2_{\mathrm{curl}} \\ &||\underline{\sigma}_i - \underline{\sigma}_{h,i}||_{\mathrm{curl}} \le C \inf\nolimits_{\underline{\tau}\in E_h} ||\underline{\sigma}_i - \underline{\tau}||_{\mathrm{curl}}. \end{aligned} \tag{61}$$

Proof. The first estimate is a consequence of (59) and (60).

Let us denote by ϵ_h the quantity $\inf_{\underline{\tau}\in E_h} ||\underline{\sigma}_i - \underline{\tau}||_{\mathrm{curl}}$. In order to bound $||\underline{\sigma}_i - \underline{\sigma}_{h,i}||_{\mathrm{curl}}$, by triangular inequality we introduce the term $\Pi_2\underline{\sigma}_i$ (Π_2 is the Fortin operator analyzed in the previous section). The difference $||\underline{\sigma}_i - \Pi_2\underline{\sigma}_i||_{\mathrm{curl}}$ can be estimated using the stability of the problem (23) which defines $\Pi_2\underline{\sigma}$. Thanks to (60) this gives

$$||\underline{\sigma}_i - \Pi_2\underline{\sigma}_i||_{\mathrm{curl}} \le C\epsilon_h. \tag{62}$$

Then we observe that

$$\begin{aligned} \mathrm{curl}\,\underline{\sigma}_i &= -\lambda_i \underline{u}_i \\ \mathrm{curl}\,\underline{\sigma}_{h,i} &= -\lambda_{h,i}\underline{u}_{h,i}. \end{aligned} \tag{63}$$

Whence we have in particular $||\,\mathrm{curl}\,\underline{\sigma}_i - \mathrm{curl}\,\underline{\sigma}_{h,i}||_0 \le C\epsilon_h$.

The remaining term is estimated using the bound of Proposition 4.6 in [1] in the following way

$$\begin{aligned} &||\Pi_2\underline{\sigma}_i - \underline{\sigma}_{h,i}||_{\mathrm{curl}} \le C||\,\mathrm{curl}(\Pi_2\underline{\sigma}_i - \underline{\sigma}_{h,i})||_0 \\ &\quad \le ||\,\mathrm{curl}\,\Pi_2\underline{\sigma}_i - \mathrm{curl}\,\underline{\sigma}_i||_0 + ||\,\mathrm{curl}\,\underline{\sigma}_i - \mathrm{curl}\,\underline{\sigma}_{h,i}||_0 \le C\epsilon_h, \end{aligned} \tag{64}$$

where we made use of (63), (60) and of the fact that $\mathrm{curl}\,\Pi_2\underline{\sigma}_i$ is a good approximation of $\mathrm{curl}\,\underline{\sigma}_i$.

Remark 8. As far as hypothesis (60) is concerned, we point out that it is usually fulfilled. Actually $\underline{\sigma}_i = \mathrm{curl}\,\underline{u}_i$ and $\underline{u}_i$ is not less regular than $\underline{\sigma}_i$, so that the $L^2(\Omega)$ error for $\underline{u}_i$ is usually not larger than the $H(\mathrm{curl})$ one for $\underline{\sigma}_i$.

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