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Stability of Runge-Kutta methods for linear delay differential equations

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Summary. This paper investigates the stability of Runge-Kutta methods when they are applied to the complex linear scalar delay differential equation $y'(t) = ay(t) + by(t-1)$. This kind of stability is called $\tau-$stability. We give a characterization of $\tau-$ stable Runge-Kutta methods and then we prove that implicit Euler method is $\tau-$stable.

Mathematics Subject Classification (1991): 65L05

1 Introduction

Let us consider the Delay Differential Equation (DDE)

$$\begin{cases} y'(t) = ay(t) + by(t-1) & t \geq 0 \\ y(t) = \varphi(t) & -1 \leq t \leq 0, \end{cases} \tag{1}$$

where $a, b \in \mathbb{C}$ and $\varphi \in C([-1,0], \mathbb{C})$. It is well known that $y(t) \to 0$, as $t \to \infty$, for all initial functions φ iff

$$\lambda - a - be^{-\lambda} = 0 \Rightarrow \mathrm{Re}(\lambda) < 0. \tag{2}$$

Now consider a Runge-Kutta (RK) method with abscissae $c_1, ..., c_s$ as applied to (1). A constant step size $1/m$, $m \geq 1$ integer, is used and, at the $(n+1)-$th step ($n = 0, 1, 2, ...$), the delayed term $y(t_n + c_i/m - 1)$ ($i = 1, ..., s$) is approximated by the stage value $Y_i^{(n+1-m)}$ computed in the past at the $(n+1-m)-$th step (by $\varphi(t_n + c_i/m - 1)$ if $n+1 \leq m$).

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Then, it is known that the numerical solution asymptotically converges to zero for all initial function φ iff

$$\xi = R\left(\frac{1}{m}\left(a + \frac{b}{\xi^m}\right)\right) \Rightarrow |\xi| < 1 \tag{3}$$

where R is the stability function of the RK method (see [5] and [7]). It is not difficult to see that (3) is equivalent to

$$\lambda - a - \frac{b}{R\left(\frac{1}{m}\lambda\right)^m} = 0 \Rightarrow \left|R\left(\frac{1}{m}\lambda\right)^m\right| < 1. \tag{4}$$

The condition

$$\mathrm{Re}\,(a) < -|b| \tag{5}$$

implies (2) and leads to the definition of $P-$stability (see [1]).

Definition 1.1 *An RK method is called $P-$stable if for all $a, b \in \mathbb{C}$ and $m \geq 1$, m integer, (5) implies (3).*

In [7] it is proved that an RK method is $P-$stable iff it is $A-$ stable.

More interesting is to know whether the numerical solution of (1) asymptotically vanishes whenever (2) holds. This is much more complicated than analyzing $P-$stability. The subject was neglected for long time except for some numerical experiments accomplished in [2] and confined to the real coefficient case.

Recently the notion has been reconsidered in [3], [4] and [5] where the following definition is given.

Definition 1.2 *An RK method is called $\tau-$stable if for all $a, b \in \mathbb{C}$ and $m \geq 1$, m integer, (2) implies (3).*

In [3], [4] and [5] the stability is analyzed for the simpler case of real coefficients a and b ($\tau\,(0)-$stability). In [3], $\theta-$methods are considered and the $\tau\,(0)-$stability is proved for all $A-$stable $\theta-$methods, i.e. for $\theta \geq \frac{1}{2}$. In [4], a necessary condition for the $\tau\,(0)-$stability is given and Lobatto IIIC methods are proved to be not $\tau\,(0)-$stable. Finally, in [5] symmetric methods (e.g. Gaussian collocation methods) and two and three stage Radau IIA methods are proved to be $\tau\,(0)-$stable.

The general complex case is considered in [3] where the $\tau-$ stability of the implicit Euler is conjectured and the trapezoidal rule is proved to be not $\tau-$stable. Moreover in the paper [6] all symmetric methods are proved to be not $\tau-$stable.

In this paper we investigate the $\tau-$stability of $A-$stable RK methods. To this aim,we introduce the following definition.

Definition 1.3 *An $A-$stable ($A-$acceptable) function R is called τ_1- stable if for all $a, b \in \mathbb{C}$, (2) implies (4) with $m = 1$.*

Therefore an $A-$stable RK method with stability function R is $\tau-$ stable iff the functions $R\left(\frac{1}{m}\lambda\right)^m$ are τ_1-stable for all $m \geq 1$ integer.

The paper is organized as follows. In section two we describe, for fixed $b \in \mathbb{C}$, the set of $a \in \mathbb{C}$ such that (2) holds. This description, in addition to be interesting by itself, is needed in the proof of Theorem 3.2. In section three we give a characterization of τ_1-stable functions. In section four we prove that if R is τ_1-stable , then $R\left(\frac{1}{m}\lambda\right)^m$ is τ_1-stable for all $m \geq 1$, m integer. Hence, an RK method is $\tau-$stable iff its stability function is τ_1-stable. Finally, in section five we prove that the implicit Euler method is $\tau-$stable.

2 Description of the stability set for the DDE

Let $b \in \mathbb{C}$ and let

(6) $$S_b := \left\{\lambda - be^{-\lambda} \mid \lambda \in \mathbb{C},\ \mathrm{Re}\,(\lambda) \geq 0\right\}.$$

It is clear that condition (2) holds iff $a \notin S_b$.

If $b = 0$, then $S_b = \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,(\lambda) \geq 0\}$. So let $b \neq 0$ and let $b = Re^{i\theta}$, $R = |b|$. In order to describe S_b, we remark that

(7) $$S_b = S_R + i\theta,$$

and

(8) $$S_R = \bigcup_{k \in Z} \left(S_R^0 + i2\pi k\right)$$

where $S_R^0 := \left\{\lambda - Re^{-\lambda} \mid \lambda \in \mathbb{C}, \mathrm{Re}\,(\lambda) \geq 0, \mathrm{Im}\,(\lambda) \in [-\pi, \pi]\right\}$.

The set S_R^0 is shown in Fig. 1 ($R \leq 1$) and in Fig. 2 ($R > 1$). The border curve B_R is given by

(9) $$\begin{aligned} B_R &:= \left\{iy - Re^{-iy} \mid y \in [-\pi, \pi]\right\} \\ &= \{x \pm f\,(R, x) \mid x \in [-R, R]\} \end{aligned}$$

where

(10) $$f\,(R, x) := \pi - \arccos\left(\frac{x}{R}\right) + \sqrt{R^2 - x^2}, \quad x \in [-R, R]\,.$$

In Fig. 2, the curve B_R intersects the lines $\{\lambda \mid \mathrm{Im}\,(\lambda) = \pm\pi\}$ at points of real abscissa $x_1 = x_1\,(R) = R\cos\varphi_1$, where $\varphi_1 = \varphi_1\,(R)$ is the solution of

(11) $$\varphi = R\sin\varphi$$

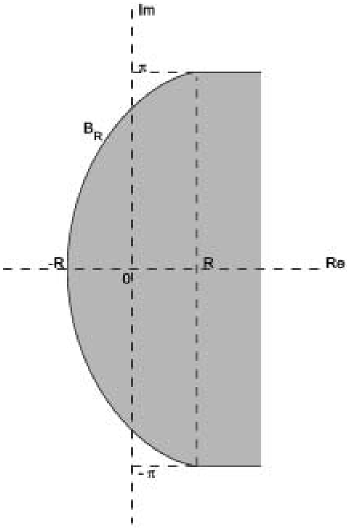

Fig. 1. The set S_R^0 (shaded) in the case $R \leq 1$

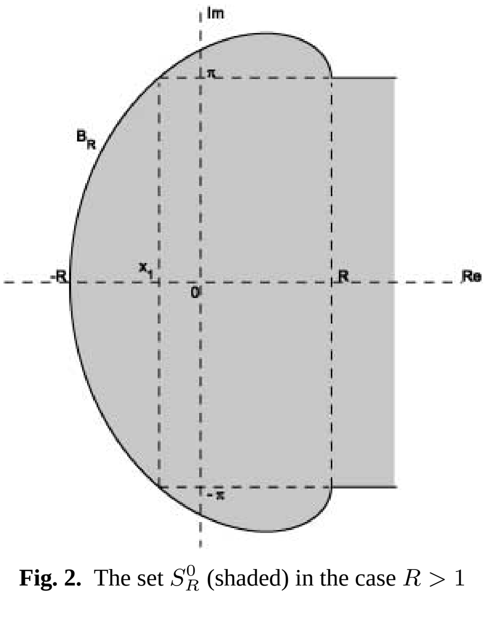

Fig. 2. The set S_R^0 (shaded) in the case $R > 1$

in $(0, \pi)$.

By (8), we obtain that S_R is the set shown in Fig. 3 where the real abscissa $D = D(R)$ is given by

$$D = \begin{cases} R & if\ R \leq 1, \\ x_1 & if\ R > 1. \end{cases}$$

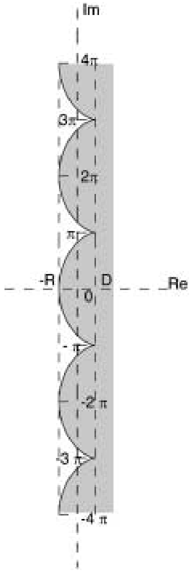

Fig. 3. The set S_R (shaded)

Eventually, by (7), the set S_b is obtained by shifting S_R along the imaginary axis. The set of complex numbers a such that the condition (2) holds is the complementary set of S_b and it is the union of the half-plane described by the condition (5) and of a sequence of 'crests' with period 2π.

Since we have $S_{-R} = S_R + i\pi$, the graph of the function $D = D(R)$, $R > 0$, is the border of the well-known asymptotic stability region of (1) in the real coefficient case with $b < 0$ (see Fig. 4).

3 A characterization of τ_1- stable functions

Let R be an $A-$stable function and let $\mathcal{D} := \{\lambda \in \mathbb{C} \mid |R(\lambda)| \leq 1\}$ be the relevant $A-$stability region.

Next theorem gives a characterization of the τ_1-stability.

Theorem 3.1 *For every $c \geq 0$, let us define the set*

$$H_c := \{h \in \mathbb{C} \mid \forall v \in \mathbb{C},\ \exists z \in \mathbb{C}\ such\ that$$

$$\mathrm{Re}(z) \geq -c\ and\ z + vh - ve^{-z} = 0\Big\}.$$

The $A-$stable function R is τ_1-stable iff

$$\left\{\frac{e^{\lambda}}{R(\lambda)} \mid \lambda \in \mathcal{D}',\ \mathrm{Re}(\lambda) = c\right\} \subseteq H_c \quad for\ all\ c \geq 0,$$

where $\mathcal{D}'$ is the complementary set of $\mathcal{D}$.

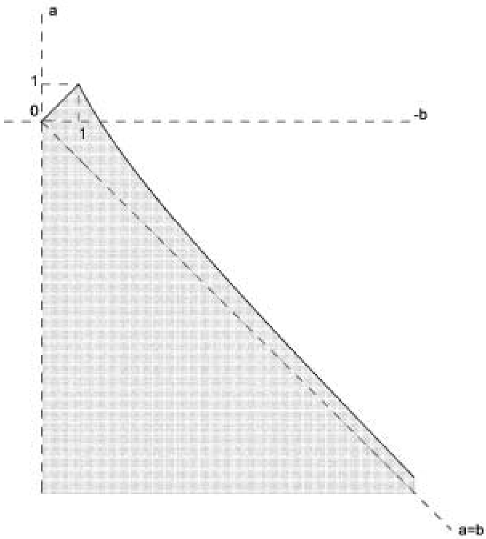

Fig. 4. Asymptotic stability region (shaded) in the $(-b, a)$ −plane for the DDE (1) in the real coefficient case and $b < 0$

Proof. For $b \in \mathbb{C}$ define the set

$$\widetilde{S}_b := \left\{ \lambda - \frac{b}{R(\lambda)} \mid \lambda \in \mathcal{D}' \right\}.$$

The function R is τ_1−stable iff $\widetilde{S}_b \subseteq S_b$ for all $b \in \mathbb{C}$ or, equivalently, iff

$$\forall \lambda \in \mathcal{D}',\ \forall b \in \mathbb{C},\ \exists x \in \mathbb{C}\ such\ that$$

$$\operatorname{Re}(x) \geq 0\ and\ \lambda - \frac{b}{R(\lambda)} = x - be^{-x}$$

By introducing $z := x - \lambda$ and $v := be^{-\lambda}$, we have

$$\lambda - \frac{b}{R(\lambda)} = x - be^{-x} \Leftrightarrow z + v\frac{e^{\lambda}}{R(\lambda)} - ve^{-z} = 0$$

and the theorem follows. □

Next theorem gives an explicit description of the sets H_c in polar representation.

Theorem 3.2 *If* $c \geq 2$*, then*

$$H_c = \left\{ re^{i\varphi} \mid 0 \leq r \leq e^c \right\}.$$

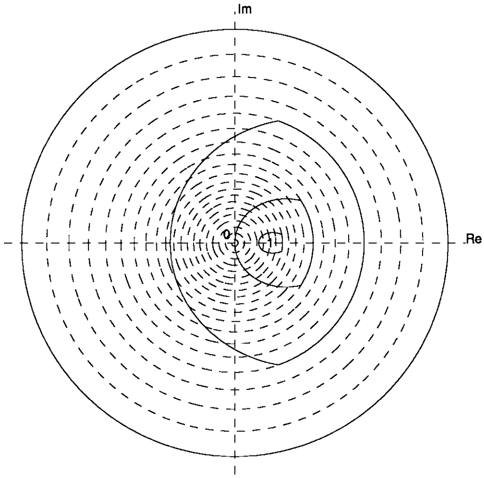

Fig. 5. The sets H_c for $c = 1/2, 1, 3/2, 2$

If $1 \le c < 2$, *then*

$$H_c = \Big\{ re^{i\varphi} \mid 0 \le r \le e^c \; and$$

$$r > e^c (c-1) \Rightarrow |\varphi| \le \arccos\left(\frac{1-c}{\frac{r}{e^c}}\right) - \sqrt{\left(\frac{r}{e^c}\right)^2 - (1-c)^2} \Bigg\}.$$

If $0 \le c < 1$, *then*

$$H_c = \Big\{ re^{i\varphi} \mid 0 \le r \le e^c \; and$$

$$r \ge e^c (1-c) \; and \; |\varphi| \le \arccos\left(\frac{1-c}{\frac{r}{e^c}}\right) - \sqrt{\left(\frac{r}{e^c}\right)^2 - (1-c)^2} \Bigg\}.$$

The proof is very technical and therefore is left in appendix one.

Note that $H_0 = \{1\}$. Moreover, by the definition, it is clear that $H_{c_1} \subseteq H_{c_2}$ for $c_1 \le c_2$. In Fig. 5 the sets H_c are shown for some values of c.

Since $H_0 = \{1\}$ we have the following negative result (see also [6]).

Theorem 3.3 *Let R be the stability function of a symmetric RK method. For all $m \ge 1$, m integer, the function $R\left(\frac{1}{m}\lambda\right)^m$ is not τ_1−stable.*

Proof. Suppose that $R\left(\frac{1}{m}\lambda\right)^m$ is τ_1−stable for some m. Since the set $\left\{\lambda \in \mathbb{C} \mid \left|R\left(\frac{1}{m}\lambda\right)\right|^m \ge 1, \mathrm{Re}\,(\lambda) = 0\right\}$ coincides with the imaginary axis and $H_0 = \{1\}$, Theorem 3.1 yields $e^{iy} = R\left(\frac{1}{m}iy\right)^m$ for all $y \in \mathbb{R}$, which is not true. □

By combining Theorems 3.1 and 3.2, we obtain the following theorem.

Theorem 3.4 *Let R be the stability function of an $A-$stable RK method. For every $0 \leq c < 2$ let us define the functions*

$$r_c(y) = \left| \frac{1}{R(c+iy)} \right|, \ \varphi_c(y) = \arg\left(\frac{1}{R(c+iy)} \right)$$

with domain $(\mathcal{D}')^c := \{y \geq 0 \mid c+iy \in \mathcal{D}'\}$. The function R is τ_1- stable iff for all $1 \leq c < 2$ we have

$$r_c(y) > c-1 \ \Rightarrow \ \exists k \in \mathbb{Z} \ such \ that$$

$$|y+\varphi_c(y) - 2\pi k| \leq \arccos\left(\frac{1-c}{r_c(y)}\right) - \sqrt{r_c(y)^2 - (1-c)^2}$$

and for all $0 \leq c < 1$ we have

$$r_c(y) \geq 1-c \ and \ \exists k \in \mathbb{Z} \ such \ that$$

$$|y+\varphi_c(y) - 2\pi k| \leq \arccos\left(\frac{1-c}{r_c(y)}\right) - \sqrt{r_c(y)^2 - (1-c)^2}.$$

Proof. First, note that for $\lambda = c+iy \in \mathcal{D}'$, $y \leq 0$, we have $\overline{\lambda} \in \mathcal{D}'$ and

$$\frac{e^{\lambda}}{R(\lambda)} \in H_c \Leftrightarrow \frac{e^{\overline{\lambda}}}{R(\overline{\lambda})} \in H_c$$

(provided that H_c is symmetric with respect to the real axis). Now for $\lambda = c+iy \in \mathcal{D}'$, $y \geq 0$, we have

$$\frac{e^{\lambda}}{R(\lambda)} = e^c r_c(y) e^{i(y+\varphi_c(y))}$$

and the theorem follows by Theorems 3.1 and 3.2. □

4 $\tau-$stability means τ_1-stability

The fact that an RK method is $\tau-$stable iff its $A-$stability function is τ_1-stable is a consequence of the following property of the sets H_c.

Theorem 4.1 *For all $c \geq 0$ and $m \geq 1$, m integer, we have*

$$h \in H_c \Rightarrow h^m \in H_{mc}.$$

Proof. Let $m \geq 2$ and let $h = re^{i\varphi} \in H_c$ (where $r \geq 0$ and $\varphi \in [-\pi, \pi]$). We will prove that $h^m \in H_{mc}$ by using the description of the set H_{mc} given by Theorem 3.2.

First, let us observe that $r \leq e^c$ implies $r^m \leq e^{mc}$, and therefore $h^m \in H_{mc}$ for $mc \geq 2$. For $mc < 2$, we have $c < 1$ and then, by Theorem 3.2,

$$r \geq e^c (1-c), \ |\varphi| \leq \arccos\left(\frac{1-c}{\frac{r}{e^c}}\right) - \sqrt{\left(\frac{r}{e^c}\right)^2 - (1-c)^2}.$$

In particular, if $1 \leq mc < 2$ and $r^m > e^{mc}(mc-1)$, then Lemma 7.1 in appendix two yields

$$|m\varphi| \leq \arccos\left(\frac{1-mc}{\frac{r^m}{e^{mc}}}\right) - \sqrt{\left(\frac{r^m}{e^{mc}}\right)^2 - (1-mc)^2} \tag{12}$$

and then $h^m \in H_{mc}$.

On the contrary, if $0 \leq mc < 1$, then $r^m \geq e^{mc}(1-c)^m \geq e^{mc}(1-mc)$ and Lemma 7.1 yields again (12). Thus $h^m \in H_{mc}$. □

Now we can establish the equivalence between $\tau-$stability and τ_1-stability.

Theorem 4.2 *Let R be an $A-$stable function. If R is τ_1-stable, then $R\left(\frac{1}{m}\lambda\right)^m$ is τ_1-stable for all $m \geq 1$, m integer.*

Proof. Let $c \geq 0$ and let $\lambda \in \mathbb{C}$ such that $\left|R\left(\frac{1}{m}\lambda\right)^m\right| \geq 1$ and $\mathrm{Re}(\lambda) = c$. Then $\frac{1}{m}\lambda \in \mathcal{D}'$ and $\mathrm{Re}\left(\frac{1}{m}\lambda\right) = \frac{c}{m}$. By Theorem 3.1 we obtain $\frac{e^{\frac{1}{m}\lambda}}{R\left(\frac{1}{m}\lambda\right)} \in H_{\frac{c}{m}}$ and, by Theorem 4.1, $\frac{e^{\lambda}}{R\left(\frac{1}{m}\lambda\right)^m} \in H_c$. □

5 $\tau-$stability for the implicit Euler method and conclusions

Now we are in position to prove the conjecture stated in [3] for the implicit Euler method.

Theorem 5.1 *The implicit Euler method is $\tau-$stable.*

Proof. We prove that the stability function $R(\lambda) = (1-\lambda)^{-1}$ for implicit Euler method is τ_1-stable. For $0 \leq c < 2$ we have (see the notations in Theorem 3.4)

$$r_c(y) = \sqrt{(1-c)^2 + y^2}, \ \varphi_c(y) = -\arccos\left(\frac{1-c}{r_c(y)}\right)$$

for $y \in (\mathcal{D}')^c = \left[0, \sqrt{1 - |1-c|^2}\right]$. Therefore $r_c(y) \geq |1-c|$ and

$$y + \varphi_c(y) = -\left[\arccos\left(\frac{1-c}{r_c(y)}\right) - \sqrt{r_c(y)^2 - (1-c)^2}\right]$$

and conditions for τ_1-stability in Theorem 3.4 are satisfied. □

In this paper the equivalence between $\tau-$stability and τ_1- stability is proved. This implies that, for a given RK method, one and only one of the following three situations occurs:

a) The RK method is $\tau-$stable, i.e. (2) implies (3), for all $m \geq 1$. Only one method is known to belong to this class, i.e. implicit Euler method.

b) The RK method is not $\tau-$stable but there exists $\overline{m} > 1$ integer (which depends only on the method) such that for all m, m multiple of $\overline{m}$, (2) implies (3). So far, no methods are known to belong to this class.

c) There does not exist m such that (2) implies (3). Symmetric methods belong to this class.

By looking for methods reproducing correct asymptotic behavior, one could be satisfied with methods falling into the class b). Now the problem is to see whether there exist methods such that the function $R\left(\frac{1}{m}\lambda\right)^m$ is τ_1-stable for some $m > 1$ but not for $m = 1$.

6 Appendix one

This appendix contains the proof of Theorem 3.2.

We start by observing that for a fixed $c \geq 0$ we have $H_c = e^c \overline{H}_c$ where

$$\overline{H}_c = \{h \in \mathbb{C} \mid \forall v \in \mathbb{C} \;\; c - vh \in S_v\}$$

and S_v is defined in (6). Our aim is to describe the sets

$$\overline{H}_{c,r} := \left\{h \in \overline{H}_c \mid |h| = r\right\}$$

for all $r \geq 0$.

As for $r = 0$ we have

Proposition 6.1 $0 \in \overline{H}_c$ *iff* $c \geq 1$.

Proof. We have $0 \in \overline{H}_c$ iff $c \in S_v$ for all $v \in \mathbb{C}$. Now, by the desciption of S_v given in section two, observe that $[1, +\infty) \subseteq S_v \cap \mathbb{R}$ for all $v \in \mathbb{C}$ and $[1, +\infty) = S_v \cap \mathbb{R}$ for $v = -1$. □

Suppose $r > 0$. We have

$$\overline{H}_{c,r} = \left\{ re^{i\varphi} \mid \forall v \in \mathbb{C} \;\; c - ve^{i\varphi} \in S_{\frac{1}{r}v} \right\}.$$

Define, for $R \geq 0$,

$$\overline{H}_{c,r,R} = \left\{ re^{i\varphi} \mid \forall v \in \mathbb{C} \textit{ such that } |v| = R, \; c - ve^{i\varphi} \in S_{\frac{1}{r}v} \right\}.$$

For $R = 0$ we have $\overline{H}_{c,r,R} = \{h \in \mathbb{C} \mid |h| = r\}$. For $R > 0$ we have

$$\overline{H}_{c,r,R} = \left\{ re^{i\varphi} \mid B_R + i\varphi \subseteq -c + S_{\frac{1}{r}R} \right\}$$

where B_R is the curve defined in (9). The curve B_R is shown in Figures 1 and 2. The set $-c + S_{\frac{1}{r}R}$ is shown in Fig. 6 where the border C_R, restricted to the strip $\{z \mid -\pi \leq \mathrm{Im}\,(z) \leq \pi\}$, is given by

$$C_R = \left\{ x \pm f\left(\frac{1}{r}R, x + c\right) \mid x \in \left[-c - \frac{1}{r}R, -c + D\left(\frac{1}{r}R\right) \right] \right\}$$

with f the function defined in (10).

Remark that, since B_R and $-c + S_{\frac{1}{r}R}$ are symmetric with respect to the real axis, we have, for $\varphi \leq 0$, $re^{i\varphi} \in \overline{H}_{c,r,R}$ iff $re^{i(-\varphi)} \in \overline{H}_{c,r,R}$. Therefore, in order to check $re^{i\varphi} \in \overline{H}_{c,r,R}$ it is sufficient to consider the case $\varphi \geq 0$.

Proposition 6.2 *If* $r > 1$*, then* $H_{c,r} = \emptyset$.

Proof. If $r > 1$, then $-R < -c - \frac{1}{r}R$ for sufficiently large R. For such an R we have

$$-R + i\varphi \in (B_R + i\varphi) \setminus \left(-c + S_{\frac{1}{r}R} \right)$$

for all $\varphi \in [0, \pi]$. Hence $\overline{H}_{c,r,R} = \emptyset$ and then $\overline{H}_{c,r} = \emptyset$. □

The case $r \leq 1$ needs some preliminary lemmas.

Lemma 6.1 *Let* $c \geq 0$ *and* $0 < r \leq 1$. *We have*

(i) $-c + D\left(\frac{1}{r}R\right) \leq D(R)$ for all $R > 0$ iff $r \geq 1 - c$.
(ii) Let $r \geq 1 - c$. We have $-c + D\left(\frac{1}{r}R\right) \leq -R$ for all $R > 0$ iff $r \leq c - 1$. Moreover if $r > c - 1$, then the set $\left\{ R > 0 \mid -R < -c + D\left(\frac{1}{r}R\right) \right\}$ is an interval $I := (\gamma, \delta)$ which contains r with $\gamma = \frac{c}{1 + \frac{1}{r}}$. As for δ, we have $\delta = +\infty$ iff $c = 0$.

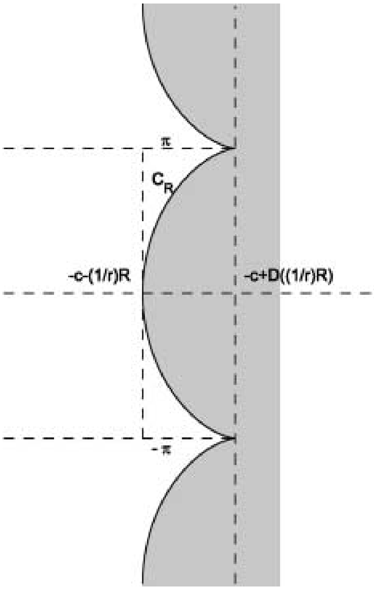

Fig. 6. The set $-c + S_{\frac{1}{r}R}$ (shaded)

(iii) Let $r \geq 1 - c$ and $r > c - 1$. Define

$$\Omega = \Omega(c, r) := \left\{ (R, x) \mid R \in I,\ x \in \left[-R, -c + D\left(\frac{1}{r}R\right) \right] \right\}$$

(see Fig. 7) and

$$d(R, x) := f\left(\frac{1}{r}R, x + c\right) - f(R, x),\ (R, x) \in \Omega.$$

where the function f is given by (10). We have

$$\min_{(R,x)\in\Omega} d(R, x) = d(r, 1 - c) = \pi - f(r, 1 - c) \geq 0 \ .$$

Proof. (i) The function $G(R) := D(R) - D\left(\frac{1}{r}R\right)$, $R > 0$, takes at $R = r$ the minimum value $G(r) = r - 1$ (use the fact that D is decreasing in $[1, +\infty)$). Therefore $-c \leq G(R)$ for all $R > 0$ iff $-c \leq G(r) = r - 1$.

(ii) The function $G(R) := -R - D\left(\frac{1}{r}R\right)$, $R > 0$, is decreasing in $(0, r]$ and increasing in $[r, +\infty)$ (use the fact that $D'(R) < -1$ for $R > 1$). Therefore $-c \leq G(R)$ for all $R > 0$ iff $-c \leq G(r) = -r - 1$. Moreover if $G(r) < -c$, then $\{R > 0 \mid G(R) < -c\}$ is an interval (γ, δ) which contains r where γ is given by $-c = -\gamma - \frac{1}{r}\gamma$. Furthermore, since $G(R) \approx \left(\frac{1}{r} - 1\right) R$ for $r < 1$ and $G(R) \to 0$ for $r = 1$, as $R \to +\infty$, we have $\delta =+\infty$ iff $c = 0$.

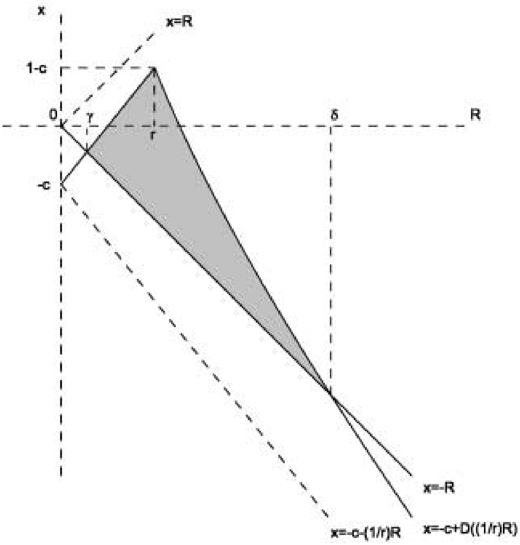

Fig. 7. The set $\Omega = \Omega\,(c, r)$ (shaded) (Lemma 6.1)

(iii) If $c = 0$ the result is trivial. So let $c > 0$. The minimum of d can be attained only on the border. In fact, at every point $(R, x) \in \overset{\circ}{\Omega}$ the function d is differentiable and it is not difficult to see that

$$\frac{\partial d}{\partial R}\,(R, x) = 0 \; and \; \frac{\partial d}{\partial x}\,(R, x) = 0$$

is impossible.

On the lower border of Ω, i.e. the set $\{(R, x) \mid R \in I,\ x = -R\}$, we have

$$\lim_{x \to -R} \frac{\partial d}{\partial x}\,(R, x) = -\infty$$

and then the minimum of d can be attained only on the upper border of Ω, i.e. the set $\left\{(R, x) \mid R \in I,\ x = -c + D\left(\frac{1}{r}R\right)\right\}$. On the upper border we have

$$d\,(R, x) = \pi - f\,(R, x) = \beta\,(R)$$

where

$$\beta\,(R) := \arccos\left(\frac{-c + D\left(\frac{1}{r}R\right)}{R}\right) - \sqrt{R^2 - \left(-c + D\left(\frac{1}{r}R\right)\right)^2}, \quad R \in I.$$

The function β is decreasing in $(0, r)$ and increasing in $(r, +\infty)$ (compute the derivative of β and use the formula

$$D'(R) = \frac{D(R) - R^2}{R(1 - D(R))}, \quad R > 1$$

which can be obtained by implicit differentiation of (11)). So β takes the minimum value at $R = r$. $\square$

Lemma 6.2 *If* $-c + D\left(\frac{1}{r}R\right) \leq -R$, *then* $\overline{H}_{c,r,R} = \{h \in \mathbb{C} \mid |h| = r\}$.

Proof. For every $\varphi \in [0, \pi]$ and $x \in [-R, R]$ we have

$$\mathrm{Re}\,(x \pm if(x) + i\varphi) = x \geq -R \geq -c + D\left(\frac{1}{r}R\right).$$

Therefore $B_R + i\varphi \subseteq -c + S_{\frac{1}{r}R}$ for all $\varphi \in [0, \pi]$. $\square$

By the previous lemmas we get the following description of the sets $\overline{H}_{c,r}$.

Proposition 6.3 *Let* $r \leq 1$. *We have*

(i) If $r < 1 - c$, then $\overline{H}_{c,r} = \emptyset$.

(ii) Let $r \geq 1 - c$. If $r \leq c - 1$, then $\overline{H}_{c,r} = \{h \in \mathbb{C} \mid |h| = r\}$.

(iii) Let $r \geq 1 - c$. If $r > c - 1$, then

$$\overline{H}_{c,r} = \left\{ re^{i\varphi} \mid |\varphi| \leq \arccos\left(\frac{1-c}{r}\right) - \sqrt{r^2 - (1-c)^2} \right\}.$$

Proof. (i) If $r < 1 - c$, then, by Lemma 6.1 (i), we have $D(R) < -c + D\left(\frac{1}{r}R\right)$ for some $R > 0$. Now for every $\varphi \in [0, \pi]$ there exists $x \in [-R, D(R)]$ such that $f(R, x) + \varphi = \pi$ and then

$$x + if(R, x) + i\varphi = x + i\pi \in (B_R + i\varphi) \setminus \left(-c + S_{\frac{1}{r}R}\right).$$

Therefore $\overline{H}_{c,r,R} = \emptyset$ and $\overline{H}_{c,r} = \emptyset$.

(ii) If $r \leq c - 1$, then, by Lemma 6.1 (ii), we have $-c + D\left(\frac{1}{r}R\right) \leq -R$ for all $R > 0$. Now, the assertion follows by Lemma 6.2.

(iii) Consider I, Ω and d as defined in Lemma 6.1. Define

$$\overline{d}(R) := \min_{x \in \left[-R, -c + D\left(\frac{1}{r}R\right)\right]} d(R, x), \ R \in I.$$

and note that

$$\pi \geq f\left(\frac{1}{r}R, -R + c\right) = d(R, -R) \geq \overline{d}(R)$$

and, by Lemma 6.1 (iii),

$$\overline{d}\,(R) \geq \min_{(R,x)\in\Omega} d\,(R,x) \geq 0.$$

Let $R \in I$. If $\varphi \in \left[0, \overline{d}\,(R)\right]$, then

$$-f\left(\frac{1}{r}R, x+c\right) \leq -f\,(R,x) \leq f\,(R,x) + \varphi \leq f\left(\frac{1}{r}R, x+c\right)$$

for all $x \in \left[-R, -c + D\left(\frac{1}{r}R\right)\right]$. Therefore $B_R + i\varphi \in -c + S_{\frac{1}{r}R}$.

If $\varphi \in \left(\overline{d}\,(R)\,, \pi\right]$, let $\overline{x} \in \left[-R, -c + D\left(\frac{1}{r}R\right)\right)$ such that $d\,(R, \overline{x}) < \varphi$. If $f\,(R, \overline{x}) + \varphi \leq \pi$, then

$$\overline{x} + if\,(R,\overline{x}) + i\varphi \in (B_R + i\varphi) \setminus \left(-c + S_{\frac{1}{r}R}\right). \qquad (13)$$

Finally, if $f\,(R, \overline{x}) + \varphi > \pi$, there exists $\overline{x}_1 \in [-R, \overline{x})$ such that $f\,(R, \overline{x}_1) + \varphi = \pi$ and we have (13) with $\overline{x}$ replaced by $\overline{x}_1$.

Therefore we have proved that $\overline{H}_{c,r,R} = \left\{re^{i\varphi} \mid |\varphi| \leq \overline{d}\,(R)\right\}$ for $R \in I$. Since, by Lemma 6.2, $\overline{H}_{c,r,R} = \{h \in \mathbb{C} \mid |h| = r\}$ for $R \notin I$, we obtain

$$\overline{H}_{c,r} = \left\{re^{i\varphi} \mid |\varphi| \leq \min_{R>0} \overline{d}\,(R) = \min_{(R,x)\in\Omega} d\,(R,x)\right\}$$

where, by Lemma 6.1 (iii),

$$\begin{aligned}\min_{(R,x)\in\Omega} d\,(R,x) &= \pi - f\,(r, 1-c) \\ &= \arccos\left(\frac{1-c}{r}\right) - \sqrt{r^2 - (1-c)^2}.\square\end{aligned}$$

By the foregoing description of the sets $\overline{H}_c$ we immediately obtain the description of the sets H_c in Theorem 3.2.

7 Appendix two

This appendix contains the lemma used in the proof of Theorem 4.1.

Lemma 7.1 *Let $m \geq 2$, m integer, and let*

$$\Omega = \Omega\,(m) := \left\{(k,c) \in \mathbb{R}^2 \mid 0 < k \leq 1,\ 0 \leq c \leq \frac{2}{m},\ 1-c \leq k \textit{ and } \right.$$

$$\left. c \geq \frac{1}{m} \Rightarrow k \geq (mc-1)^{\frac{1}{m}}\right\}.$$

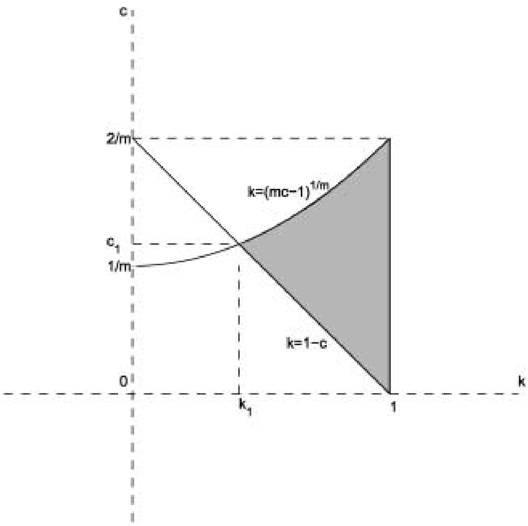

Fig. 8. The set $\Omega = \Omega\,(m)$ (shaded) (Lemma 7.1)

(see Fig. 8). For every $(k,c) \in \Omega$*, we have*

$$\arccos\left(\frac{1-c}{k}\right) - \sqrt{k^2 - (1-c)^2}$$
$$\leq \frac{1}{m}\left[\arccos\left(\frac{1-mc}{k^m}\right) - \sqrt{k^{2m} - (1-mc)^2}\right].$$

Proof. First, note that

$$\Omega = \left\{(k,c) \mid k \in [k_1, 1]\,,\; c \in \left[1-k, \frac{k^m+1}{m}\right]\right\}$$

where k_1 is the unique solution of

$$1 - k = \frac{k^m + 1}{m}.$$

Moreover, observe that in Fig. 8 we have $c_1 := 1 - k_1 = \frac{k_1^m+1}{m}$.

Consider the function

$$d\,(k,c) := \frac{1}{m}\left[\arccos\left(\frac{1-mc}{k^m}\right) - \sqrt{k^{2m} - (1-mc)^2}\right] +$$

$$-\left[\arccos\left(\frac{1-c}{k}\right) - \sqrt{k^2 - (1-c)^2}\right], \quad (k,c) \in \Omega.$$

Now, it remains to prove that

$$\min_{(k,c)\in\Omega} d(k,c) = d(1,0) = 0.$$

The minimum of d can be attained only on the border of Ω. In fact, at a point $(k,c) \in \overset{\circ}{\Omega}$ the function d is differentiable and it is not difficult to see that

$$\frac{\partial d}{\partial k}(k,c) = 0 \; and \; \frac{\partial d}{\partial c}(k,c) = 0$$

is impossible.

For a fixed $k \in [k_1, 1]$, the function

$$c \mapsto d(k,c), \quad c \in \left[1-k, \frac{k^m+1}{m}\right]$$

takes the minimum value at the point

$$\overline{c}(k) := \frac{k^{2m} - m^2 k^2 + m^2 - 1}{2m(m-1)}.$$

We have $\overline{c}(k_1) = c_1$, $\overline{c}(1) = 0$ and $(k, \overline{c}(k)) \in \overset{\circ}{\Omega}$ for $k \in (k_1, 1)$. Since the minimum of d can be obtained only on the border of Ω we have

$$\begin{aligned}\min_{(r,x)\in\Omega} d(k,c) &= \min_{k\in[k_1,1]} d(k, \overline{c}(k)) \\ &= \min\{d(k_1,c_1), d(1,0)\} = d(1,0) = 0. \square\end{aligned}$$

References

1. V. K. Barwell, Special stability problems for functional differential equations. BIT **15**, 130–135 (1975)
2. A. N. Al-Mutib, Stability properties of numerical methods for solving delay differential equations. Jo of Comput. App. Math. **10**, 71–79 (1984)
3. N. Guglielmi, Delay dependent stability regions of $\theta-$methods for delay differential equations. IMA J. Numer. Anal **18**, 399–418 (1998)
4. N. Guglielmi, On the asymptotic stability properties for Runge-Kutta methods for delay differential equations. Numer Math. **77**, 467–485 (1997)
5. N. Guglielmi, E. Hairer, Order stars and stability for delay differential equations. Numer. Math. **83**, 371–383 (1999)
6. N. Guglielmi, E. Hairer, Geometric proofs of numerical stability for delay equations. (preprint)
7. M. Zennaro, P-stability properties of Runge-Kutta methods for delay differential equations Numer. Math. **49**, 305–318 (1986)