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Interpolation error estimates of a modified 8-node serendipity finite element

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Summary. Interpolation error estimates for a modified 8-node serendipity finite element are derived in both regular and degenerate cases, the latter of which includes the case when the element is of triangular shape. For $u \in W^{3,p}(K)$ defined over a quadrilateral K, the error for the interpolant $\Pi_K u$ is estimated as $|u - \Pi_K u|_{W^{\alpha,p}(K)} \leq Ch_K^{3-\alpha}|u|_{W^{3,p}(K)}$ $(\alpha = 0, 1)$, where $1 \leq p \leq +\infty$ in the regular case and $1 \leq p < 3$ in the degenerate case, respectively. Thus, the obtained error estimate in the degenerate case is of the same quality as in the regular case at least for $1 \leq p < 3$. Results for some related elements are also given.

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1. Introduction

The finite element method is now a standard tool for numerical analysis of partial differential equations, and the so-called isoparametric family of finite elements is in common use. Among such elements, the 8-node quadrilateral element, often called 8-node serendipity, is one of the most popular ones in 2D finite element libraries [2,3,8].

Recently, various modifications of the 8-node serendipity element have been proposed to overcome defects of the original serendipity. Such modified elements are devised so that they can represent any Cartesian quadratic polynomials even when the elements are of general bilinear isoparametric shape, and have been experimentally shown to have better approximation properties

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than the original serendipity with the degree of freedom unchanged [5–7]. However, mathematical error analysis has not been performed for them yet.

On the other hand, mathematical error analysis of classical isoparametric elements have been done by various investigators. Among them, Ciarlet [3] gave interpolation error estimates of various isoparametric elements in the regular case, i.e., when elements are not too flat and additionally, in the case of quadrilateral elements, their shapes are not close to triangular ones. See also related results by Brenner and Scott [2], which may be convenient for our purposes since the so-called "chunkiness parameter" is effectively used to evaluate various error constants. Moreover, Jamet [4] and Ženišek and Vanmaele [10] derived estimates for the straight 4-node quadrilateral element when it may degenerate to triangles or may become very flat, and their approaches are also very useful as basis of our study.

The aim of this paper is to derive error estimates for a modified 8-node serendipity element of Kikuchi-Okabe-Fujio [6] in both regular and degenerate cases. This element exhibits nice numerical behaviors at least experimentally, and we believe that it is worthy of such theoretical analysis. In the regular case, the estimates can be obtained by the approaches of Ciarlet [3] and Brenner-Scott [2]. However, analysis becomes more difficult in degenerate and nearly degenerate cases where quadrilaterals may be almost degenerated to triangles, and so we will follow the main idea of Jamet [4] and Ženišek-Vanmaele [10] to obtain the desired results. In particular, we can show that, even in degenerate or nearly degenerate cases, the same order of accuracy in some Sobolev (semi-)norms is achieved as that in the regular case.

Furthermore, we can apply our approach to some related elements such as the 4-node quadrilateral element and the 9-node Lagrange one, giving some results generalizing those of [4] and [10] for the former element.

2. Mathematical preliminaries

For a bounded domain Ω in $\boldsymbol{R}^2$ with Lipschitz boundary, we denote by $W^{m,p}(\Omega)$ the Sobolev space of order m, where $m \geq 0$ is an integer and p is any number satisfying $1 \leq p \leq +\infty$ ($p = +\infty$ is included here). The norm of $W^{m,p}(\Omega)$ is given by

$$(1) \quad \|v\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha v(x)|^p dx \right)^{\frac{1}{p}} ; \quad 1 \leq p < +\infty ,$$

$$(2) \quad \|v\|_{W^{m,\infty}(\Omega)} = \max_{|\alpha| \leq m} \left\{ \operatorname*{ess\,sup}_{x \in \Omega} |D^\alpha v(x)| \right\} ; \quad p = +\infty ,$$

where $\alpha = (\alpha_1, \alpha_2)$ is the multi-index whose components α_1 and α_2 are non-negative integers, and $|\alpha| = \alpha_1 + \alpha_2$. For $m = 0$, we also use the standard notation $L_p(\Omega)$ for $W^{0,p}(\Omega)$. Moreover, we will use the semi-norm of $W^{m,p}(\Omega)$ defined by

$$(3) \quad |v|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha v(x)|^p dx \right)^{\frac{1}{p}} \; ; \quad 1 \le p < +\infty \, ,$$

$$(4) \quad |v|_{W^{m,\infty}(\Omega)} = \max_{|\alpha|=m} \left\{ \operatorname*{ess\,sup}_{x\in\Omega} |D^\alpha v(x)| \right\} \; ; \quad p = +\infty \, .$$

In this paper, we will use the so-called interpolation operators. For such operators to be definable, we need some results of the Sobolev imbedding theorem: here we will mainly use the inclusion relations

$$(5) \qquad W^{2,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \quad (1 \le \forall p \le +\infty) \, ,$$

$$(6) \qquad W^{3,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \quad (1 \le \forall p \le +\infty) \, ,$$

where $C(\overline{\Omega})$ is the space of continuous functions defined over $\overline{\Omega}$(=the closure of Ω) with the same norm as that of $L_\infty(\Omega)$, and $\hookrightarrow$ denotes the notation of continuous imbedding. Relation (5) for $p = 1$ is somewhat delicate to hold for general domains but is valid for the present type of special domains [1].

3. Modified 8-node serendipity element

The 8-node serendipity element was modified by MacNeal-Harder [7] and Kikuchi [5] to overcome the defect that it cannot represent arbitrary Cartesian quadratic polynomials in general convex "bilinear" quadrilateral shape. Then it was further modified by Kikuchi-Okabe-Fujio [6] so that it has better behavior when it is degenerated to triangles. We will briefly explain the essence of the last element in what follows, see [6] for details.

Let K be a convex open quadrilateral element with straight edges in the usual Cartesian xy-coordinates. We employ 4 vertices and 4 midpoints of edges as nodes. We denote the vertices of K by $\boldsymbol{z}_i = (x_i, y_i)$ $(1 \le i \le 4)$, while each midpoint node $\boldsymbol{z}_{i+4}$ $(1 \le i \le 4)$ is specified to be on the edge $\boldsymbol{z}_i\boldsymbol{z}_j$ with $j = 1 + \mathrm{mod}(i, 4)$. Moreover, let $\hat{K}$ be a unit square $0 < \xi, \eta < 1$ in the $\xi\eta$-coordinates, whose vertices $\hat{\boldsymbol{z}}_i$ for $1 \le i \le 4$ are in turn $(\xi, \eta) = (0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$. See Fig. 1 for configurations of K and $\hat{K}$, where $\hat{\boldsymbol{z}}_i$ for $5 \le i \le 8$ are midpoint nodes of $\hat{K}$.

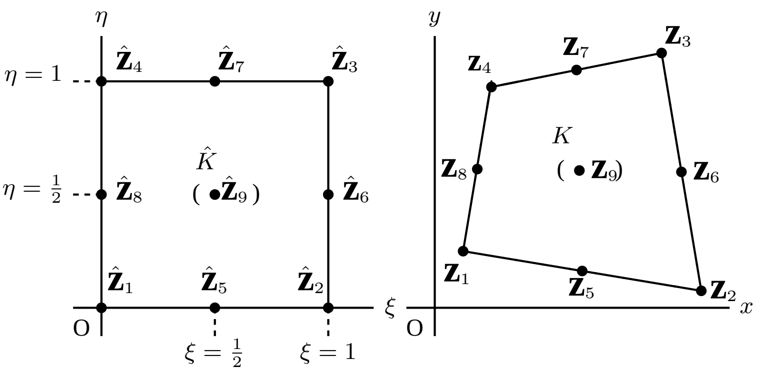

Fig. 1. Configuration of $\hat{K}$ and K

The bilinear isoparametric transformation between $\hat{\boldsymbol{z}} = (\xi, \eta) \in \hat{K}$ and $\boldsymbol{z} = (x, y) \in K$ is given by

$$\begin{aligned} \boldsymbol{z} &= (1-\xi)(1-\eta)\boldsymbol{z}_1 + \xi(1-\eta)\boldsymbol{z}_2 + \xi\eta\boldsymbol{z}_3 + (1-\xi)\eta\boldsymbol{z}_4 \\ &= \sum_{i=1}^{4} L_i(\xi, \eta)\boldsymbol{z}_i \ , \end{aligned} \tag{7}$$

where L_i's are the so-called bilinear shape functions defined by

$$\begin{aligned} &L_1(\xi, \eta) = (1-\xi)(1-\eta), \ L_2(\xi, \eta) = \xi(1-\eta), \\ &L_3(\xi, \eta) = \xi\eta, \ L_4(\xi, \eta) = (1-\xi)\eta \ . \end{aligned} \tag{8}$$

Then the present transformation from $\hat{K}$ to K is one-to-one and onto (i. e. bijective) so long as K is a convex quadrilateral [3]. Moreover, it is also bijective between $\overline{\hat{K}}$ and $\overline{K}$(=the closures of $\hat{K}$ and K) unless K is degenerated to a triangle. With this transformation, we can identify a function in K with that in $\hat{K}$ and vice versa. Hereafter, we will frequently use this convention with the notations of coordinate transformations and composite functions omitted. In particular, we may use both $f(x, y)$ and $f(\xi, \eta)$ for the same f.

For $\hat{K}$ and the associated K considered above, let us define some real function spaces for a non-negative integer k :

$$\begin{aligned} &Q_k(\xi, \eta; \hat{K}) \\ &\quad = \text{linear space spanned by } \{\xi^m \eta^n\}_{0 \le m,n \le k} \text{ over } \hat{K} \ , \end{aligned} \tag{9}$$

$$P_k(\xi, \eta; \hat{K})$$

(10) $= \text{linear space of } \xi\eta\text{-polynomials of order} \leq k \text{ over } \hat{K}$,

$P_k(x, y; K)$

(11) $= \text{linear space of } xy\text{-polynomials of order} \leq k \text{ over } K$.

Under the bilinear transformation and the above-mentioned convention, it holds for each k that [9]

$$P_k(x, y; K) \subset Q_k(\xi, \eta; \hat{K}) \ . \tag{12}$$

The shape functions for the modified 8-node serendipity element proposed in [6] are given by, for $1 \leq i \leq 4$,

$$M_i := N_i - \left\{ \frac{1}{4} + \frac{D_k - D_i}{8(D_i + D_k)} \right\} N_9 \ ,$$
$$M_{i+4} := N_{i+4} + \left\{ \frac{1}{2} + \frac{D_m - D_i}{4(D_i + D_k)} \right\} N_9 \ , \tag{13}$$

where $N_1, \cdots, N_9$ are the 9-node Lagrange shape functions and are nothing but basis functions of $Q_2(\xi, \eta; \hat{K})$, D_i is the Jacobian $\partial(x, y)/\partial(\xi, \eta)$ at i-th vertex ($1 \leq i \leq 4$), and (i, j, k, m) is each of the cyclic permutations of $(1, 2, 3, 4)$. More specifically, each N_i is associated with $\boldsymbol{z}_i$ for $1 \leq i \leq 8$, while N_9 is associated with the ninth node $\boldsymbol{z}_9$ which is the image of $\hat{\boldsymbol{z}}_9 = (1/2, 1/2) \in \hat{K}$ for transformation (7). The explicit expressions of N_i's are well known and may be found e. g. in [8]. It is also to be noted that D_i's are either all positive or all negative. If K is degenerated to a triangle, some of D_i's may become zero, but never change their signs. It is also to be noted that expressions in (13) are valid even in such cases since $D_i + D_k \neq 0$ unless K is degenerated to segments. Of course, these shape functions satisfy the interpolation property

$$M_i(\boldsymbol{z}_j) = \delta_{ij} \quad (1 \leq i, j \leq 8) \ , \tag{14}$$

where $\boldsymbol{z}_j$'s are nodes of K already explained. In particular, the linear space U spanned by the above $M_1, \cdots, M_8$ are constructed so that

$$P_2(x, y; K) + P_2(\xi, \eta; \hat{K}) \subset U \subset Q_2(\xi, \eta; \hat{K}) \ ;$$
$$U := \text{ linear hull of } \{M_i\}_{i=1}^{8} \ , \tag{15}$$

where "$+$" denotes the summation notation for two linear spaces, and the convention in (12) is again used. To check the above inclusion, it is sufficient to show that all monomials in $P_2(x, y; K)$ and $P_2(\xi, \eta; \hat{K})$ can be correctly interpolated, and the following identities are useful for such purposes $(1 \leq i \leq 4)$:

$$D_i + D_k = D_j + D_m \ , x_i D_k + x_k D_i = x_j D_m + x_m D_j \ ,$$
$$y_i D_k + y_k D_i = y_j D_m + y_m D_j \ , \tag{16}$$

where notations are the same as were already defined. Moreover, we can obtain the usual 6-node quadratic triangular element by using the so-called node degeneration technique to the present element.

The interpolant $\Pi_K u \in U$ for $u \in C(\overline{K})$ is defined as follows by using nodal values of u :

$$\Pi_K u = \sum_{i=1}^{8} u(\boldsymbol{z}_i) M_i \ . \tag{17}$$

Since $v \in U$ may be considered to belong to $C(\overline{K})$, we have the following fundamental property for Π_K :

$$\Pi_K v = v \ \text{ for } \ v \in U \ . \tag{18}$$

4. Interpolation error analysis

This section is devoted to deriving the estimate of the form, for $\alpha = 0, 1$ and appropriate p,

$$|u - \Pi_K u|_{W^{\alpha,p}(K)} \leq C h_K^{3-\alpha} |u|_{W^{3,p}(K)}; \quad \forall u \in W^{3,p}(K) \ , \tag{19}$$

where h_K is the diameter of K and C is a positive constant independent of u and h_K. To this end, we will make some preparations in the first subsection and then derive our main results in the last two subsections.

In this paper, notations C, C_1, C_2, etc. will be used as generic positive constants which may take different values at different places. If necessary, we will use notations such as $C(\alpha, p, \cdots)$ to specify the dependence on various parameters.

4.1. Geometric properties of K

We will summarize some geometric properties of convex quadrilaterals obtained by Jamet [4].

For a convex quadrilateral K, let us define :

$$\begin{aligned} h_K &= \text{the diameter of } K \\ &= \text{maximum among the largest edge length} \\ &\qquad \text{and the largest diagonal length of } K \ , \end{aligned} \tag{20}$$

$$h'_K = \text{the smallest edge length of } K \ , \tag{21}$$

$$\begin{aligned} \rho_K &= \text{the supremum of the diameters} \\ &\qquad \text{of discs contained in } K \ , \end{aligned} \tag{22}$$

$$\rho_i = \text{the diameter of the inscribed circle}$$

(23) $\quad \text{for } \triangle \boldsymbol{z}_i \boldsymbol{z}_j \boldsymbol{z}_m \ (1 \le i \le 4)$,

(24) $\quad \beta_i = \text{the interior angle for the vertex } \boldsymbol{z}_i \text{ of } K \ (1 \le i \le 4)$,

(25) $\quad \mu_K = \max_{1 \le i \le 4} |\cos \beta_i|$,

where $\{i, j, k, m\}$ is each of the cyclic permutations of $\{1, 2, 3, 4\}$. It is to be noted here that D_i's introduced in Sect. 3 satisfy

$$\frac{1}{2}|D_i| = \text{the area of } \triangle \boldsymbol{z}_i \boldsymbol{z}_j \boldsymbol{z}_m \quad (1 \le i \le 4) . \tag{26}$$

Then we can give the definition of a regular family of convex quadrilaterals. That is, a family of convex quadrilaterals $\{K\}$ is said to be *regular* provided that there exist positive constants σ and μ such that

$$\frac{h'_K}{h_K} \ge \sigma \ (> 0) , \quad (0 \le) \ \mu_K \le \mu < 1 ; \quad \forall K , \tag{27}$$

and, in addition, there is a sequence in $\{K\}$ such that $h_K \to 0$. As was noted by Jamet [4], it follows from the regularity conditions that there exists a positive constant σ^* such that

$$\frac{\rho_K}{h_K} \ge \sigma^* \ (> 0) ; \ \forall K , \tag{28}$$

where σ^* can be expressed in terms of σ and μ. However, its converse does not necessarily hold, that is, (27) does not follow from (28). More specifically, the regularity conditions exclude the cases where K becomes almost to be degenerated to a triangle since neither any edge length of K can approach zero faster than σh_K nor any interior angle of K can tend to π. On the other hand, (28) permits such excluded cases but still excludes the possibility that K becomes too flat.

In what follows, we will present some results required to analyze the above-mentioned degenerate or nearly degenerate cases, which were essentially shown by Jamet [4]. For such purposes, it is sufficient to consider the case where $\rho_2 \ge \rho_4$ and either β_2 or β_4 is the maximum of β_i's $(1 \le i \le 4)$.

For the triangle $T := \triangle \boldsymbol{z}_1 \boldsymbol{z}_2 \boldsymbol{z}_3$ (Fig. 2), whose diameter of the inscribed circle is $\rho_2 (\ge \rho_4)$, we find that

$$\rho_2 > \rho_K / 2 . \tag{29}$$

For $\sigma^* > 0$ in (28), define θ_0 by

$$\theta_0 := 2 \arctan \frac{\sigma^*}{4} . \tag{30}$$

Since σ^* must be less than unity for (28) to be meaningful, we find that

$$0 < \theta_0 < 2 \arctan \frac{1}{4} < \frac{1}{2} . \tag{31}$$

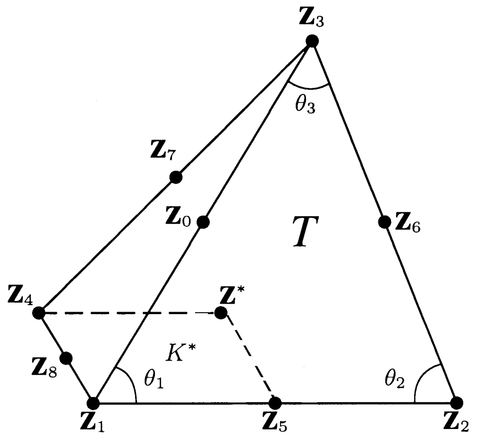

Fig. 2. Triangle $T = \triangle \boldsymbol{z}_1\boldsymbol{z}_2\boldsymbol{z}_3$ and parallelogram $K^* = \boldsymbol{z}_4\boldsymbol{z}_1\boldsymbol{z}_5\boldsymbol{z}^*$

Let θ_i $(i = 1,2,3)$ be the interior angle of T associated with vertex $\boldsymbol{z}_i$, where $\theta_2 = \beta_2$. Then it holds that

(32) $$\theta_0 < \theta_i < \pi - 2\theta_0 \quad (i = 1,2,3)\ .$$

Furthermore, by noting that either β_2 or β_4 is the maximum of β_i's, we can show that the interior angles β_1 and β_3 of K satisfy

(33) $$\theta_0 < \min\{\beta_1, \beta_3\} \le \max\{\beta_1, \beta_3\} < \pi - \theta_0\ .$$

For the edge lengths of T, we have from (28) and (29) that

(34) $$\min\{|\boldsymbol{z}_1\boldsymbol{z}_2|, |\boldsymbol{z}_1\boldsymbol{z}_3|, |\boldsymbol{z}_2\boldsymbol{z}_3|\} > \rho_2 > \frac{\rho_K}{2} \ge \frac{\sigma^* h_K}{2}\ .$$

4.2. Estimates in regular case

Under the regularity conditions for a family of convex quadrilaterals $\{K\}$, we can obtain the following interpolation error estimates for the present modified 8-node serendipity by means of the standard techniques of interpolation error analysis [2],[3].

Theorem 1. *Assume that the family of quadrilaterals $\{K\}$ satisfies the regularity conditions (27). Then the interpolant $\Pi_K u$ defined by (17) for $u \in W^{3,p}(K) \hookrightarrow C(\overline{K})$ $(1 \le p \le +\infty)$ satisfies*

(35) $$|u - \Pi_K u|_{W^{\alpha,p}(K)} \le C h_K^{3-\alpha} |u|_{W^{3,p}(K)}\ ; \quad \alpha = 0,1,2\ ,$$

where $C = C(\alpha, p, \sigma, \mu)$ is a positive constant independent of h_K and u.

Remark 1. This type of estimation is obtainable for some other type of modified 8-node elements such as those of MacNeal-Harder [7] and Kikuchi [5] so long as the regularity conditions hold, since the proof is essentially the same as the present one. However, it is not so for the original serendipity element, which cannot necessarily represent arbitrary Cartesian quadratic polynomials, and we cannot obtain (a) and (b) in the proof below, unless K is a parallelogram. For the original serendipity, the error estimates become one order less than (35) with respect to h_K, where it is required for u to belong to $W^{2,p}(K)$ $(1 \le p \le +\infty)$ only.

Remark 2. We can generalize (35) as, for $1 \le p, q \le +\infty$,

$$(36)\quad |u - \mathit{\Pi}_K u|_{W^{\alpha,p}(K)} \le C h_K^{3-\alpha+\frac{2}{p}-\frac{2}{q}} |u|_{W^{3,q}(K)}\ ; \quad \alpha = 0, 1, 2\ ,$$

where $2/p$ for example is interpreted as 0 for $p = +\infty$. We will not repeat such comments on the results to be given later, since the case $p = q$ appears to be essential for usual purposes.

Proof. Theorem 1 can be proved by using the standard methods such as those of Ciarlet [3] and Brenner-Scott [2], and we will essentially follow the approach of Brenner-Scott here. We will not repeat the details of such process, but just present the sketch to make clear the difference of the proof in the regular case from that in the degenerate or nearly degenerate cases.

1. By using the regularity of $\{K\}$, we can first show the existence of $\hat{u} \in P_2(x, y; K)$ for each $u \in W^{3,p}(K)$ such that

$$(a)\qquad |u - \hat{u}|_{W^{\alpha,p}(K)} \le C_1 h_K^{3-\alpha} |u|_{W^{3,p}(K)}\ ; \quad \alpha = 0, 1, 2\ ,$$

$$(b)\qquad \|u - \hat{u}\|_{L_\infty(K)} \le C_1 h_K^{3-\frac{2}{p}} |u|_{W^{3,p}(K)}\ ,$$

where $C_1 = C_1(\alpha, p, \sigma, \mu) > 0$ is independent of h_K and u. More specifically, the dependence of C_1 on two parameters σ and μ can be arranged to that on a single parameter σ^* in (28), which is a function of σ and μ under (27). It is to be noted here that ρ_K/h_K in (28) is twice the inverse of the so-called "chunkiness" parameter in [2].

2. By (15) and (18), we find for the above $\hat{u} \in P_2(x, y; K) \subset U$ that $\mathit{\Pi}_K \hat{u} = \hat{u}$. Then, we have by the triangle inequality that

$$\begin{aligned} |u - \mathit{\Pi}_K u|_{W^{\alpha,p}(K)} &\le |u - \hat{u}|_{W^{\alpha,p}(K)} + |\hat{u} - \mathit{\Pi}_K u|_{W^{\alpha,p}(K)} \\ &= |u - \hat{u}|_{W^{\alpha,p}(K)} + |\mathit{\Pi}_K \hat{u} - \mathit{\Pi}_K u|_{W^{\alpha,p}(K)} \\ (c)\qquad &= |u - \hat{u}|_{W^{\alpha,p}(K)} + |\mathit{\Pi}_K (u - \hat{u})|_{W^{\alpha,p}(K)}\ . \end{aligned}$$

3. Let us consider the term $|\Pi_K(u-\hat{u})|_{W^{\alpha,p}(K)}$. By (17), we have

$$
\begin{aligned}
|\Pi_K(u-\hat{u})|_{W^{\alpha,p}(K)} &\le \sum_{i=1}^{8} |u(\boldsymbol{z}_i)-\hat{u}(\boldsymbol{z}_i)| \cdot |M_i|_{W^{\alpha,p}(K)} \\
(d) \qquad &\le 8||u-\hat{u}||_{L_\infty(K)} \max_{1\le i\le 8} |M_i|_{W^{\alpha,p}(K)} \, .
\end{aligned}
$$

Thus, $|\Pi_K(u-\hat{u})|_{W^{\alpha,p}(K)}$ may be estimated by evaluating $|M_i|_{W^{\alpha,p}(K)}$.

4. In the estimation of $|M_i|_{W^{\alpha,p}(K)}$, we use the chain rule for the derivatives of M_i under the bilinear transformation (7). In this process, it is essential to evaluate the Jacobian $J(\xi,\eta)=\partial(x,y)/\partial(\xi,\eta)$ associated with the transformation (7), which is expressed in $\hat{K}$ by

$$
\begin{aligned}
J(\xi,\eta) &= \sum_{i=1}^{4} D_i L_i(\xi,\eta) \\
&= D_1(1-\xi)(1-\eta) + D_2\xi(1-\eta) + D_3\xi\eta + D_4(1-\xi)\eta \, .
\end{aligned}
$$

Since $0 \le \xi,\eta \le 1$ in $\hat{K}$, we have $\min_{1\le i\le 4} |D_i| \le |J(\xi,\eta)| \le \max_{1\le i\le 4} |D_i|$. Thus, by directly estimating D_i's with (26), (27) and (28) used, we obtain

$$
C_2 h_K^2 \le |J(\xi,\eta)| \le h_K^2 \, ,
$$

where $C_2 = C_2(\sigma,\mu) > 0$ is independent of h_K. Estimating $M_i(\xi,\eta)$ and the Jacobian matrix associated with (7) as in [4], we have

$$
(e) \quad |M_i|_{W^{\alpha,p}(K)} \le C_3 h_K^{\frac{2}{p}-\alpha} \;\; (\alpha=0,1,2;\ 1\le p\le +\infty; 1\le i\le 8) \, ,
$$

where $C_3 = C_3(\alpha,p,\sigma,\mu) > 0$ is independent of h_K.

5. Substituting (b) and (e) into (d), we have

$$
(f) \qquad |\Pi_K(u-\hat{u})|_{W^{\alpha,p}(K)} \le 8C_1C_3 h_K^{3-\alpha} |u|_{W^{3,p}(K)} \, .
$$

It is now straightforward to obtain (35) by substituting (a) and (f) into (c), and the proof is complete. □

4.3. Estimates including degenerate and nearly degenerate cases

As was already noted, the regularity conditions exclude the cases where K is degenerated or nearly degenerated to triangles, since in such cases h'_K is much smaller than h_K or $\max_{1\le i\le 4}\beta_i$ is almost π. So we will analyze such cases in this subsection. In this respect, several researchers have derived error estimates of the 4-node quadrilateral element in such degenerate or nearly degenerate cases [4], [10], and we will perform such analysis for the present modified 8-node serendipity. As was already mentioned, we can obtain the usual 6-node triangular element when K is fully degenerated to a triangle by overlapping a pair of neighboring vertices.

We can state the main results as follows.

Theorem 2. *Assume that the family of quadrilaterals $\{K\}$ satisfies the condition of the form (28) :*

$$\frac{\rho_K}{h_K} \geq \sigma^* > 0\ ; \quad \forall K\ , \tag{37}$$

where σ^* is a positive constant dependent only on the family. Then the interpolant $\Pi_K u$ for $u \in W^{3,p}(K) \hookrightarrow C(\overline{K})$ satisfies

$$\|u - \Pi_K u\|_{L_p(K)} \leq C h_K^3 |u|_{W^{3,p}(K)} \quad \textit{if} \quad 1 \leq p \leq +\infty\ , \tag{38}$$

$$|u - \Pi_K u|_{W^{1,p}(K)} \leq C h_K^2 |u|_{W^{3,p}(K)} \quad \textit{if} \quad 1 \leq p < 3\ , \tag{39}$$

where $C = C(p, \sigma^*)$ is a positive constant which is independent of h_K and u.

Remark 3. It appears to be difficult to obtain the above type of estimations for other type of modified serendipity elements such as given in [5] and [7]. The difficulty essentially lies in deriving various bounds of the associated shape functions, which are valid for the present element as will be stated in Lemma 4. □

We will essentially follow the approaches of [4] and [10] to prove Theorem 2, which may be difficult to prove by means of the standard approaches of [2] and [3]. First we will explain the outline of proof below.

As was discussed in 4.1, we can assure under (37) that there exists a triangle which is not too flat and made of certain three vertices of K, and such a triangle may be specified as $T = \triangle \boldsymbol{z}_1 \boldsymbol{z}_2 \boldsymbol{z}_3$ without loss of generality. Then K itself may be degenerated to T, while such degeneration cannot occur under the regularity conditions. For T, we can consider the usual 6-node quadratic element with its three vertices and three midpoints of edges as nodes. Let us denote the midpoint of the segment $\boldsymbol{z}_1 \boldsymbol{z}_3$ by $\boldsymbol{z}_0$, see Fig. 2.

To prove Theorem 2, we use another interpolant $I_K u$ for $u \in C(\overline{K})$, which is a Cartesian quadratic polynomial such that

$$(I_K u)(\boldsymbol{z}_i) = u(\boldsymbol{z}_i) \qquad \text{for} \qquad i = 0, 1, 2, 3, 5, 6\ . \tag{40}$$

Such a polynomial exists uniquely for each u, and is nothing but the interpolant of u for the 6-node quadratic element associated with T, cf. [3]. Thus $I_K u \in P_2(x, y; K)$, and we have by (15) and (18) that

$$\Pi_K I_K u = I_K u \quad \text{for } \ u \in C(\overline{K})\ . \tag{41}$$

Now we find by the triangle inequalities as well as (40) and (41) that

$$
\begin{aligned}
|u - \Pi_K u|_{W^{\alpha,p}(K)} &\le |u - I_K u|_{W^{\alpha,p}(K)} + |\Pi_K u - I_K u|_{W^{\alpha,p}(K)} \\
&= |u - I_K u|_{W^{\alpha,p}(K)} + |\Pi_K (u - I_K u)|_{W^{\alpha,p}(K)} \\
&\le |u - I_K u|_{W^{\alpha,p}(K)} \\
&\quad + \sum_{i=4,7,8} |(u - I_K u)(\boldsymbol{z}_i)| \cdot |M_i|_{W^{\alpha,p}(K)}.
\end{aligned}
\tag{42}
$$

The above inequality implies that Theorem 2 may be proved if the quantities in its right-hand side are appropriately evaluated. To this end, we will present several lemmas below.

First, we estimate the first term $|u - I_K u|_{W^{\alpha,p}(K)}$ of (42) by means of the techniques in [2].

Lemma 1. *Under assumption (37) of Theorem 2, the interpolant $I_K u$ defined by (40) for $u \in W^{3,p}(K) \hookrightarrow C(\overline{K})$ $(1 \le p \le \infty)$ satisfies*

$$
|u - I_K u|_{W^{\alpha,p}(K)} \le C h_K^{3-\alpha} |u|_{W^{3,p}(K)} \;;\; \alpha = 0, 1, 2 \,, \tag{43}
$$

$$
\|u - I_K u\|_{L^\infty(K)} \le C h_K^{3-\frac{2}{p}} |u|_{W^{3,p}(K)} \,, \tag{44}
$$

where $C = C(p, \sigma^*)$ is a positive constant independent of h_K and u.

Proof. The proving process is essentially the same as that of Theorem 1, and we give only some comments below. The main difference is that Π_K should be replaced with I_K and the shape functions for the 6-node triangular element should be used instead of M_i's. The new shape functions may be expressed in terms of the so-called barycentric coordinates associated with T, and then the Jacobian of this coordinate transformation becomes a constant function, cf. [3]. Moreover, we need to estimate the shape functions outside T (i. e. over $K \setminus T$) as well, which process is not serious under the condition (37), cf. Lemma 2.1 of Jamet [4]. □

To evaluate terms $|(u - I_K u)(\boldsymbol{z}_i)|$ $(i = 4, 7, 8)$, we will use not only Lemma 1 but also some inequalities on traces along edges of K as we will see in the proof of the following lemma.

Lemma 2. *Under (37), it holds for any $u \in W^{3,p}(K) \hookrightarrow C(\overline{K})$ $(1 \le p \le +\infty)$ that*

$$
\max_{i=4,7,8} |(u - I_K u)(\boldsymbol{z}_i)| \le C \varepsilon_K^{1-\frac{1}{p}} h_K^{2-\frac{1}{p}} |u|_{W^{3,p}(K)} \,, \tag{45}
$$

where $\varepsilon_K = \min\{|\boldsymbol{z}_1\boldsymbol{z}_4|, |\boldsymbol{z}_3\boldsymbol{z}_4|\}$, and $C = C(p, \sigma^*)$ is a positive constant depending only on p and σ^*.

Remark 4. The arguments employed by Jamet [4] to derive his results corresponding to the present lemma appear to be insufficient for our purposes. So we essentially rely on the approach of Ženišek-Vanmaele [10], but their results are slightly generalized to include the cases other than $p = 2$.

Proof. Without loss of generality, we will consider the case where $|\boldsymbol{z}_1\boldsymbol{z}_4| \le |\boldsymbol{z}_3\boldsymbol{z}_4|$, that is, $\varepsilon_K = |\boldsymbol{z}_1\boldsymbol{z}_4|$. For simplicity, we will prove the lemma only for $i = 4$: the other cases can be dealt with similarly.

We will consider two separate cases where $\varepsilon_K \ge l^* := \frac{1}{4}\sigma^* h_K \sin\theta_0$ and $\varepsilon_K < l^*$, respectively, in which θ_0 is defined by (30).

i) $\varepsilon_K \ge l^*$: We have from (44) that

$$|(u - I_K u)(\boldsymbol{z}_4)| \le C h_K^{3-\frac{2}{p}} |u|_{W^{3,p}(K)} = C h_K^{1-\frac{1}{p}} h_K^{2-\frac{1}{p}} |u|_{W^{3,p}(K)} \,.$$

In this case, $\varepsilon_K \ge \dfrac{\sigma^* \sin\theta_0}{4} h_K$, i.e. $h_K \le \dfrac{4}{\sigma^* \sin\theta_0}\varepsilon_K$, and hence we have

$$|(u - I_K u)(\boldsymbol{z}_4)| \le C \left(\frac{4}{\sigma^* \sin\theta_0}\right)^{1-\frac{1}{p}} \varepsilon_K^{1-\frac{1}{p}} h_K^{2-\frac{1}{p}} |u|_{W^{3,p}(K)}$$

by noting that $1 - \frac{1}{p} \ge 0$ since $p \ge 1$. This inequality is of the form (45) if C is modified appropriately.

ii) $\varepsilon_K < l^*$: Let us consider a parallelogram $K^* = \boldsymbol{z}_4\boldsymbol{z}_1\boldsymbol{z}_5\boldsymbol{z}^*$ in Fig. 2, two edges of which are $\boldsymbol{z}_1\boldsymbol{z}_5$ and $\boldsymbol{z}_1\boldsymbol{z}_4$. Then the fourth vertex $\boldsymbol{z}^*$ is shown to lie in $T \subset K$ so long as $\varepsilon_K < l^*$. That is, we have for the length of edge $\boldsymbol{z}_5\boldsymbol{z}^*$ that $|\boldsymbol{z}_5\boldsymbol{z}^*| = |\boldsymbol{z}_1\boldsymbol{z}_4| = \varepsilon_K$, while the lengths of the perpendiculars from $\boldsymbol{z}_5$ to $\boldsymbol{z}_1\boldsymbol{z}_3$ and $\boldsymbol{z}_0\boldsymbol{z}_6$ are evaluated as

$$\min\{|\boldsymbol{z}_1\boldsymbol{z}_5|\sin\theta_1, |\boldsymbol{z}_0\boldsymbol{z}_1|\sin\theta_1\} > \frac{1}{2}\cdot\frac{1}{2}\sigma^* h_K \sin\theta_0 = \frac{1}{4}\sigma^* h_K \sin\theta_0 = l^*$$

by (32) and (34). Thus the parallelogram K^* is contained in K in the present case.

We will now derive some estimates of traces associated with the segment $\boldsymbol{z}_1\boldsymbol{z}_4$, and we will denote the norm of L_p-space on $\boldsymbol{z}_1\boldsymbol{z}_4$ by $\|\cdot\|_{p,\boldsymbol{z}_1\boldsymbol{z}_4}$ $(1 \le p \le +\infty)$. Moreover, we use the notation $h^* := |\boldsymbol{z}_1\boldsymbol{z}_5|$. As (62) of Ženišek-Vanmaele [10], we first obtain for $\forall v \in W^{1,p}(K)$ $(1 \le p < +\infty)$ that

$$\begin{aligned}\|v\|_{p,\boldsymbol{z}_1\boldsymbol{z}_4}^p &\le \frac{2^{p-1}}{\sin\beta_1}\left\{(h^*)^{-1}\|v\|_{L_p(K^*)}^p + (h^*)^{p-1}|v|_{W^{1,p}(K^*)}^p\right\} \\ &\le \frac{2^{p-1}}{\sin\beta_1}\left\{(h^*)^{-1}\|v\|_{L_p(K)}^p + (h^*)^{p-1}|v|_{W^{1,p}(K)}^p\right\} \,,\end{aligned}$$

where v in the left-hand side is actually the trace of v on $\boldsymbol{z}_1\boldsymbol{z}_4$, which is well defined as an element of $L_p(\boldsymbol{z}_1\boldsymbol{z}_4)$ for $v \in W^{1,p}(K)$. Thus we have

$$\|v\|_{p,\boldsymbol{z}_1\boldsymbol{z}_4} \leq 2\left(\frac{1}{2\sin\beta_1}\right)^{\frac{1}{p}}\left\{(h^*)^{-\frac{1}{p}}\|v\|_{L_p(K)} + (h^*)^{1-\frac{1}{p}}|v|_{W^{1,p}(K)}\right\} ,$$

which is valid for $p = +\infty$ as well. From (33) and (34) in 4.1, we find that

$$\sin\beta_1 > \sin\theta_0\ (>0)\ , \quad \frac{\sigma^* h_K}{4} < h^* = \frac{1}{2}|\boldsymbol{z}_1\boldsymbol{z}_2| \leq \frac{h_K}{2} < h_K\ .$$

By these estimates, the original trace estimation for $v \in W^{1,p}(K)$ $(1 \leq p \leq +\infty)$ becomes

$$\|v\|_{p,\boldsymbol{z}_1\boldsymbol{z}_4} \leq C_1\left\{h_K^{-\frac{1}{p}}\|v\|_{L_p(K)} + h_K^{1-\frac{1}{p}}|v|_{W^{1,p}(K)}\right\}\ ;$$

$$(a)\qquad C_1 = C_1(p,\sigma^*) > 0\ .$$

For $\forall w \in W^{3,p}(K) \hookrightarrow C(\overline{K})$ $(1 \leq p \leq +\infty)$, we can show that

$$(b)\qquad |w(\boldsymbol{z}_4) - w(\boldsymbol{z}_1)| = \left|\int_{\boldsymbol{z}_1\boldsymbol{z}_4}\frac{\partial w}{\partial s}(s)ds\right| \leq \varepsilon_K^{1-\frac{1}{p}}\left\|\frac{\partial w}{\partial s}\right\|_{p,\boldsymbol{z}_1\boldsymbol{z}_4} ,$$

where s is the linear coordinate on $\boldsymbol{z}_1\boldsymbol{z}_4$ and $\partial/\partial s$ is the differentiation in the direction of $\boldsymbol{z}_1\boldsymbol{z}_4$. Clearly, we have for the above $\partial w/\partial s$ that

$$\left\|\frac{\partial w}{\partial s}\right\|_{p,\boldsymbol{z}_1\boldsymbol{z}_4} \leq \left\|\frac{\partial w}{\partial x}\right\|_{p,\boldsymbol{z}_1\boldsymbol{z}_4} + \left\|\frac{\partial w}{\partial y}\right\|_{p,\boldsymbol{z}_1\boldsymbol{z}_4} ,$$

and (b) becomes, using (a) with $v = \dfrac{\partial w}{\partial x}$ or $v = \dfrac{\partial w}{\partial y}$,

$$(c)\ \ |w(\boldsymbol{z}_4) - w(\boldsymbol{z}_1)| \leq 2C_1\varepsilon_K^{1-\frac{1}{p}}\left\{h_K^{-\frac{1}{p}}|w|_{W^{1,p}(K)} + h_K^{1-\frac{1}{p}}|w|_{W^{2,p}(K)}\right\} .$$

By applying (40) to $u \in W^{3,p}(K)$ $(1 \leq p \leq +\infty)$, we have an identity

$$(u - I_K u)(\boldsymbol{z}_4) = (u - I_K u)(\boldsymbol{z}_4) - (u - I_K u)(\boldsymbol{z}_1)\ .$$

Thus, using (c) with $w = u - I_K u \in W^{3,p}(K)$, we find that

$$\begin{aligned}&|(u - I_K u)(\boldsymbol{z}_4)|\\ &\qquad \leq 2C_1\varepsilon_K^{1-\frac{1}{p}}\left\{h_K^{-\frac{1}{p}}|u - I_K u|_{W^{1,p}(K)} + h_K^{1-\frac{1}{p}}|u - I_K u|_{W^{2,p}(K)}\right\}\ .\end{aligned}$$

Substituting (43) with $\alpha = 1, 2$ into the right-hand side of this inequality, we obtain

$$|(u - I_K u)(\boldsymbol{z}_4)| \leq 4C_1 C\varepsilon_K^{1-\frac{1}{p}}h_K^{2-\frac{1}{p}}|u|_{W^{3,p}(K)}\ ,$$

which is of the form (45) since $4C_1C$ is dependent on p and σ^* only. □

To estimate $|M_i|_{W^{\alpha,p}(K)}$ for $i = 4, 7, 8$, we first evaluate integrals of the Jacobian $J(\xi,\eta) = \partial(x,y)\ /\partial(\xi,\eta)$ associated with the bilinear transformation (7). To this end, we can generalize Lemma 2.5 of Jamet [4], probably simplifying his proof for $p = 2$, and obtain the following lemma.

Lemma 3. *Under (37), the Jacobian $J(\xi,\eta) = \partial(x,y)/\partial(\xi,\eta)$ of the transformation (7) satisfies, for $1 \le p < 3$,*

$$\iint_{\hat{K}} |J(\xi,\eta)|^{1-p} d\xi d\eta \le C\varepsilon_K^{\frac{1-p}{2}} h_K^{\frac{3(1-p)}{2}} , \tag{46}$$

where $\varepsilon_K = \min\{|\boldsymbol{z}_1\boldsymbol{z}_4|, |\boldsymbol{z}_3\boldsymbol{z}_4|\}$ and $C = C(p,\sigma^*)$ is a positive constant independent of h_K and ε_K.

Remark 5. In the degenerate cases, the above integral may be singular for $p > 1$ and divergent for $p \ge 3$.

Proof. Since K is convex, D_i's $(1 \le i \le 4)$ are either all non-negative or all non-positive as noted in Sect. 3, and hence we will only consider the former case without loss of generality. In such a case, $J(\xi,\eta) = \sum_{i=1}^{4} D_i L_i(\xi,\eta)$ is non-negative all over $\hat{K}$.

Let us divide the required integral into two parts :

$$(a) \quad I := \iint_{\hat{K}} |J(\xi,\eta)|^{1-p} d\xi d\eta = \iint_{\hat{K}} \{J(\xi,\eta)\}^{1-p} d\xi d\eta = I_1 + I_2 ,$$

where

$$I_1 := \int_0^1 \left(\int_0^{1-\xi} \{J(\xi,\eta)\}^{1-p} d\eta \right) d\xi ,$$
$$I_2 := \int_0^1 \left(\int_{1-\xi}^{1} \{J(\xi,\eta)\}^{1-p} d\eta \right) d\xi .$$

We will first estimate I_1, while I_2 can be estimated similarly.

Using the identity $D_1 + D_3 = D_2 + D_4$ in $J(\xi,\eta) = \sum_{i=1}^{4} D_i L_i(\xi,\eta)$, we have

$$(b) \qquad J(\xi,\eta) = D_1(1-\xi-\eta) + D_2\xi + D_4\eta .$$

Thus, for ξ and η such that $0 \le \xi \le 1$, $0 \le \eta \le 1-\xi$ as is required for I_1, it holds that

$$J(\xi,\eta) \ge D_1(1-\xi-\eta) + D_2\xi \ge 0 \ \text{ with } \ 1-\xi-\eta \ge 0 \,, \ \ \xi \ge 0 \,,$$

and, by the inequality for the arithmetic and geometric means, we have for such ξ and η that

$$J(\xi,\eta) \ge 2D_1^{\frac{1}{2}} D_2^{\frac{1}{2}} (1-\xi-\eta)^{\frac{1}{2}} \xi^{\frac{1}{2}} .$$

Since $p \geq 1$, we have the estimate

$$I_1 \leq 2^{1-p} D_1^{\frac{1-p}{2}} D_2^{\frac{1-p}{2}} \int_0^1 \xi^{\frac{1-p}{2}} \left\{ \int_0^{1-\xi} (1-\xi-\eta)^{\frac{1-p}{2}} d\eta \right\} d\xi \ ,$$

provided that the integral in the right-hand side is convergent. This integral is in fact convergent for $1 \leq p < 3$, and is estimated as follows by the use of the beta function $B(\cdot,\cdot)$:

$$(c) \qquad I_1 \leq 2^{1-p} D_1^{\frac{1-p}{2}} D_2^{\frac{1-p}{2}} B\left(\tfrac{3-p}{2}, \tfrac{5-p}{2}\right) B\left(1, \tfrac{3-p}{2}\right) \ .$$

For estimating I_2, we should use the identity for $J(\xi,\eta)$:

$$J(\xi,\eta) = D_3(\xi+\eta-1) + D_2(1-\eta) + D_4(1-\xi) \ ,$$

which may be derived as (b). Then we can obtain similarly to (c) that

$$(d) \qquad I_2 \leq 2^{1-p} D_3^{\frac{1-p}{2}} D_2^{\frac{1-p}{2}} B\left(\tfrac{3-p}{2}, \tfrac{3-p}{2}\right) B\left(1, 3-p\right) \ .$$

From geometric relations (32), (33) and (34) for K and T, we have

$$(e) \qquad D_1 \geq C_1 \varepsilon_K h_K \ , \ \ D_3 \geq C_1 \varepsilon_K h_K \ , \ \ D_2 \geq C_2 h_K^2 \ ,$$

where C_1 and C_2 are positive constants depending on σ^* only. Applying (c), (d) and (e) to (a), we can obtain (46), and the proof is complete. □

It remains to evaluate bounds of shape functions $\{M_i(\xi,\eta)\}_{1\leq i\leq 8}$ and their derivatives.

Lemma 4. *Under (37), there exists a positive constant* $C = C(p,\sigma^*)$ *such that*

$$(47) \qquad \|M_j\|_{L_p(K)} \leq C h_K^{\frac{2}{p}} \quad (1 \leq p \leq +\infty, 1 \leq j \leq 8) \ ,$$

$$(48) \qquad |M_j|_{W^{1,p}(K)} \leq C \varepsilon_K^{\frac{1}{2p}-\frac{1}{2}} h_K^{\frac{3}{2p}-\frac{1}{2}} \quad (1 \leq p < 3, \ j = 4,7,8) \ ,$$

where ε_K *is defined in the preceding lemma.*

Proof. 1. We first prove (47). It is clear that $N_i(\xi,\eta)$ for $1 \leq i \leq 9$ are uniformly bounded for $0 \leq \xi,\eta \leq 1$. Moreover, by geometric consideration, we have

$$0 \leq \frac{D_k}{D_i+D_k} \leq 1 \ , \ \ 0 \leq \frac{D_i}{D_i+D_k} \leq 1 \ ,$$
$$0 \leq \frac{D_m}{D_i+D_k} = \frac{D_m}{D_j+D_m} \leq 1 \ ,$$

where (i, j, k, m) is each of cyclic permutation of $(1, 2, 3, 4)$ as before. Thus, from their definitions (13), $M_j(\xi, \eta)$ $(1 \leq j \leq 8)$ are uniformly bounded for $0 \leq \xi, \eta \leq 1$. By (26), it is also noted that

$$|J(\xi, \eta)| \leq \max_{1 \leq i \leq 4} |D_i| \leq h_K^2 \quad \text{for} \quad 0 \leq \xi, \eta \leq 1 \; .$$

Applying the above considerations to the identity

$$\|M_j\|_{L_p(K)}^p = \iint_K |M_j(x, y)|^p dxdy = \iint_{\hat{K}} |M_j(\xi, \eta)|^p |J(\xi, \eta)| d\xi d\eta \; ,$$

we have (47).

2. For simplicity, we will use notations for derivatives such as $M_{j,x}$, $M_{j,\xi}$ etc., that is,

$$M_{j,x} = \frac{\partial M_j}{\partial x} \; , \;\; M_{j,\xi} = \frac{\partial M_j}{\partial \xi} \quad \text{etc.}$$

As in 1., it is easy to show that $M_{j,\xi}$ and $M_{j,\eta}$ for $1 \leq j \leq 8$ are uniformly bounded for $0 \leq \xi, \eta \leq 1$. Furthermore, we have for $x_{,\xi} = \partial x / \partial \xi$ etc. that

$$0 \leq |x_{,\xi}| \; , \; |x_{,\eta}| \; , \; |y_{,\xi}| \; , \; |y_{,\eta}| \leq h_K \;\text{ for }\; 0 \leq \xi, \eta \leq 1 \; .$$

Noting the identities

$$M_{j,x} = \frac{1}{J}(M_{j,\xi} y_{,\eta} - M_{j,\eta} y_{,\xi}) \; , \;\; M_{j,y} = \frac{1}{J}(-M_{j,\xi} x_{,\eta} + M_{j,\eta} x_{,\xi})$$

and the relation

$$\begin{aligned} |M_j|_{W^{1,p}(K)}^p &= \iint_K (|M_{j,x}|^p + |M_{j,y}|^p) \, dxdy \\ &= \iint_{\hat{K}} (|M_{j,x}|^p + |M_{j,y}|^p) \, |J(\xi, \eta)| d\xi d\eta \; , \end{aligned}$$

we can obtain (48) by the use of Lemma 3 in essentially the same fashion as 1. □

Proof of Theorem 2. Once the above lemmas are proved, it is now straightforward to show (38) and (39) of Theorem 2 by means of (42).

First, we have

$$\|u - \Pi_K u\|_{L_p(K)} \leq C \left(h_K^3 + \varepsilon_K^{1-\frac{1}{p}} h_K^{2+\frac{1}{p}} \right) |u|_{W^{3,p}(K)} \; , \tag{49}$$

from which (38) follows since $\varepsilon_K \leq h_K$.

Similarly, we can obtain (39) from the estimate

$$|u - \Pi_K u|_{W^{1,p}(K)} \leq C \left(h_K^2 + \varepsilon_K^{\frac{1}{2}-\frac{1}{2p}} h_K^{\frac{3}{2}+\frac{1}{2p}} \right) |u|_{W^{3,p}(K)} \, . \quad \square \tag{50}$$

Remark 6. In the degenerate cases, either $\boldsymbol{z}_4$ lies on segment $\boldsymbol{z}_1\mathbf{z}_3$ or ε_K vanishes. The present analysis remains to be valid in the former case. In the latter case, the results of Theorem 2 still hold true if $\Pi_K u$ is replaced with $I_K u$, as may be seen to be natural from the degeneration process of [5] and [6].

5. Results for some related elements

5.1. 4*-node quadrilateral element*

In [4] and [10], some results corresponding to Theorem 2 were proved for the 4-node quadrilateral element only in the case of $p = 2$. For such an element, the interpolation operator $\Pi_K^{(4)} : C(\overline{K}) \to Q_1(\xi, \eta; \hat{K})$ is defined by

$$\Pi_K^{(4)} u = \sum_{i=1}^{4} u(\boldsymbol{z}_i) L_i \quad (u \in C(\overline{K})) \,, \tag{51}$$

where L_i's are given by (8). Here we can extend the results to the cases of $p \neq 2$ under condition (37) by means of the techniques employed in the proof of the Theorem 2 :

$$\|u - \Pi_K^{(4)} u\|_{L_p(K)} \leq C h_K^2 |u|_{W^{2,p}(K)} \; ; \quad (1 \leq p \leq +\infty) \,, \tag{52}$$

$$\left| u - \Pi_K^{(4)} u \right|_{W^{1,p}(K)} \leq C h_K |u|_{W^{2,p}(K)} \; ; \quad (1 \leq p < 3) \,, \tag{53}$$

where $u \in W^{2,p}(K) \hookrightarrow C(\overline{K})$ (cf. (5)), and $C = C(p, \sigma^*)$ is a positive constant independent of h_K and u. Of course, in the regular case, (53) holds for any p with $1 \leq p \leq +\infty$ as an analog of Theorem 1.

5.2. 9*-node quadrilateral element*

Moreover, for the 9-node Lagrange element based on $Q_2(\xi, \eta; \hat{K})$, we can obtain essentially the same results as those of Theorems 1 and 2. In this case, the interpolation operator $\Pi_K^{(9)} : C(\overline{K}) \to Q_2(\xi, \eta; \hat{K})$ is defined by

$$\Pi_K^{(9)} u = \sum_{i=1}^{9} u(\boldsymbol{z}_i) N_i \quad (u \in C(\overline{K})) \,, \tag{54}$$

where each N_i $(1 \leq i \leq 9)$ is the shape function associated with node $\boldsymbol{z}_i$ of the 9-node element. Since the interpolation operator Π_K for the 8-node serendipity satisfies that $\Pi_K u \in Q_2(\xi,\eta;\hat{K})$, it is trivial that the present element has at least the same approximation capability as the modified 8-node element. However, it is not necessarily easy to show the analog of Theorem 2 for the interpolation operator $\Pi_K^{(9)}$ associated with the 9-node element, although the analog of Theorem 1 can be obtained with ease. As far as the authors are aware, analysis of the 9-node element has not been performed in the degenerate or nearly degenerate cases.

In the proof, we additionally need slightly complicated estimations for the "bubble function" N_9 associated with the ninth node $\boldsymbol{z}_9$:

(55) $$N_9(\xi,\eta) = 16\xi(1-\xi)\eta(1-\eta)\ .$$

We can easily obtain the analog of (42), where M_i should be replaced with N_i and a new term $|(u - I_K u)(\boldsymbol{z}_9)| \cdot |N_9|_{W^{\alpha,p}(K)}$ appears in the summation. By (44), we have

(56) $$|(u - I_K u)(\boldsymbol{z}_9)| \le C h_K^{3-\frac{2}{p}} |u|_{W^{3,p}(K)}\ ;\ \ C = C(p,\sigma^*) > 0\ ,$$

while estimation of $|N_9|_{W^{\alpha,p}(K)}$ must be made carefully for $\alpha = 1$ since the above estimate does not contain a desired factor such as $\varepsilon_K^{1-\frac{1}{p}}$. We will present the essence of such process of evaluating $|N_9|_{W^{1,p}(K)}$ below under the same conditions for K and T as those in 4.3 and in the case where $\varepsilon_K = |\boldsymbol{z}_1\boldsymbol{z}_4|$.

Estimation of $|N_9|_{W^{1,p}(K)}$ **:** First we have, for the derivatives of N_9,

$$N_{9,x} = N_{9,\xi}\xi_{,x} + N_{9,\eta}\eta_{,x} = \frac{1}{J}(y_{,\eta}N_{9,\xi} - y_{,\xi}N_{9,\eta})\ ,$$
$$N_{9,y} = N_{9,\xi}\xi_{,y} + N_{9,\eta}\eta_{,y} = \frac{1}{J}(-x_{,\eta}N_{9,\xi} + x_{,\xi}N_{9,\eta})\ ,$$

where

$$N_{9,\xi}(\xi,\eta) = 16(1-2\xi)(\eta-\eta^2)\ ,\ \ N_{9,\eta}(\xi,\eta) = 16(\xi-\xi^2)(1-2\eta)\ ,$$

$$x_{,\xi} = (x_2 - x_1)(1-\eta) + (x_3 - x_4)\eta\ ,$$
$$x_{,\eta} = (x_4 - x_1)(1-\xi) + (x_3 - x_2)\xi\ ,$$
$$y_{,\xi} = (y_2 - y_1)(1-\eta) + (y_3 - y_4)\eta\ ,$$
$$y_{,\eta} = (y_4 - y_1)(1-\xi) + (y_3 - y_2)\xi\ .$$

Thus it is sufficient to evaluate the L_p-norms of the following quantities :

$$a(\xi,\eta) := \frac{1}{|J|}(|x_{,\eta}| + |y_{,\eta}|)|N_{9,\xi}|\ ,\ \ b(\xi,\eta) := \frac{1}{|J|}(|x_{,\xi}| + |y_{,\xi}|)|N_{9,\eta}|\ .$$

To evaluate $b(\xi,\eta)$, notice that

$$|x_{,\xi}|+|y_{,\xi}|\le 2h_K\ ,\ \ |N_{9,\eta}|\le 16\xi\ ,\ \ |J|\ge |D_2|\xi\ge C_1h_K^2\xi$$

over $\hat{K}$, where $C_1=C_1(\sigma^*)>0$ is depending on σ^* only. Thus we have

$$\begin{aligned}&\int_{\hat{K}}\{b(\xi,\eta)\}^p|J(\xi,\eta)|d\xi d\eta\\ &\qquad\le 32^pC_1^{1-p}h_K^{p+2(1-p)}\iint_{\hat{K}}\xi^{p+(1-p)}d\xi d\eta\le C_2h_K^{2-p}\ ,\end{aligned}\tag{a}$$

where $C_2=C_2(p,\sigma^*)>0$ is depending on p and σ^* only.

To evaluate $a(\xi,\eta)$, notice first that

$$\begin{aligned}&|x_{,\eta}|+|y_{,\eta}|\le 2(\varepsilon_K+h_K\xi)\ ,\ \ |N_{9,\xi}|\le 4\ ,\\ &|J|\ge 2|D_1|^{\frac12}|D_2|^{\frac12}\xi^{\frac12}(1-\xi-\eta)^{\frac12}\ge C_3h_K^{\frac32}\varepsilon_K^{\frac12}\xi^{\frac12}(1-\xi-\eta)^{\frac12}\\ &\qquad\qquad\text{for }\ 0\le\xi\le 1\ ,\ 0\le\eta\le 1-\xi\ ,\\ &|J|\ge 2|D_2|^{\frac12}|D_3|^{\frac12}(1-\eta)^{\frac12}(\xi+\eta-1)^{\frac12}\\ &\quad\ge C_3h_K^{\frac32}\varepsilon_K^{\frac12}(1-\eta)^{\frac12}(\xi+\eta-1)^{\frac12}\\ &\qquad\qquad\text{for }\ 0\le\xi\le 1\ ,\ 1-\xi\le\eta\le 1\end{aligned}$$

over $\hat{K}$, where $C_3=C_3(\sigma^*)>0$ is depending on σ^* only. Then we have

$$\begin{aligned}\iint_{\hat{K}}\{a(\xi,\eta)\}^p|J(\xi,\eta)|d\xi d\eta&\le C_4\iint_{\hat{K}}(\varepsilon_K+h_K\xi)^p|J(\xi,\eta)|^{1-p}d\xi d\eta\\ &\le C_5\iint_{\hat{K}}(\varepsilon_K^p+h_K^p\xi^p)|J(\xi,\eta)|^{1-p}d\xi d\eta\ ,\end{aligned}$$

where C_4 and C_5 are positive constants depending on C_3 and p only. Then we should estimate the terms in the right-hand side of the above inequality.

First, we obtain

$$\begin{aligned}\iint_{\hat{K}}\xi^p|J(\xi,\eta)|^{1-p}d\xi d\eta&\le C_1^{1-p}h_K^{2(1-p)}\iint_{\hat{K}}\xi^{p+(1-p)}d\xi d\eta\\ &=\frac12C_1^{1-p}h_K^{2-2p}\ .\end{aligned}\tag{b}$$

Secondly, as in the proof of Lemma 3, we have for $1\le p<3$ that

$$\begin{aligned}&\iint_{\hat{K}}|J(\xi,\eta)|^{1-p}d\xi d\eta\\ &\qquad=\int_0^1\left\{\int_0^{1-\xi}|J(\xi,\eta)|^{1-p}\right\}d\xi d\eta\end{aligned}$$

$$+\int_0^1 \left\{ \int_{1-\xi}^1 |J(\xi,\eta)|^{1-p} \right\} d\xi d\eta$$

$$\le C_3^{1-p} h_K^{\frac{3-3p}{2}} \varepsilon_K^{\frac{1-p}{2}} \left[\int_0^1 \left\{ \int_0^{1-\xi} \xi^{\frac{1-p}{2}} (1-\xi-\eta)^{\frac{1-p}{2}} d\eta \right\} d\xi \right.$$

$$\left. + \int_0^1 \left\{ \int_{1-\xi}^1 (1-\eta)^{\frac{1-p}{2}} (\xi+\eta-1)^{\frac{1-p}{2}} d\eta \right\} d\xi \right]$$

$$= C_3^{1-p} h_K^{\frac{3-3p}{2}} \varepsilon_K^{\frac{1-p}{2}} \left\{ B\left(\tfrac{3-p}{2}, \tfrac{5-p}{2}\right) B\left(1, \tfrac{3-p}{2}\right) \right.$$

$$(c) \qquad \left. + B(3-p,1) B\left(\tfrac{3-p}{2}, \tfrac{3-p}{2}\right) \right\} .$$

From (b) and (c), we have, by noting that $\varepsilon_K \le h_K$,

$$(d) \qquad \iint_{\hat{K}} \{\alpha(\xi,\eta)\}^p |J(\xi,\eta)| d\xi d\eta \le C_8 h_K^{2-p} \; ; \;\; C_8 = C_8(p,\sigma^*) > 0 \, .$$

Combining (d) with (a), we have the desired estimation for $1 \le p < 3$:

$$(57) \qquad |N_9|_{W^{1,p}(K)} \le C h_K^{\frac{2}{p}-1} \; ; \;\; C = C(p,\sigma^*) > 0 \, ,$$

which together with (56) is sufficient for obtaining the analog of Theorem 2.
□

6. Concluding remarks

In this paper, we have given some error estimates for the modified 8-node serendipity finite element of Kikuchi-Okabe-Fujio [6] in both regular and degenerate cases. In particular, even in degenerate and nearly degenerate cases, we can show the same order of accuracy in some Sobolev (semi-) norms as that in the regular case. Thus we have given some theoretical background to the use of such a modified serendipity element. We have also obtained error estimates for some related elements such as the 4-node quadrilateral and 9-node Lagrange elements.

It also appears to be important to perform numerical experiments to check the present theoretical results, which we are planning to publish in due course. Moreover, we will try to perform error analysis of various other finite elements in both regular and degenerate cases as well as the present element used as a fully isoparametric one.

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