Numerische Mathematik (2022) 151:601–625
https://doi.org/10.1007/s00211-022-01297-w

Numerische Mathematik

# A Godunov type scheme and error estimates for scalar conservation laws with Panov-type discontinuous flux

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Received: 10 March 2021 / Revised: 22 September 2021 / Accepted: 10 May 2022 /
Published online: 10 June 2022


**Abstract**

This article concerns a scalar conservation law where the flux is of Panov type and may contain spatial discontinuities. We define a notion of entropy solution and discuss the existence via Godunov type finite volume approximation. We further show that our numerical scheme converges the entropy solution at an optimal rate of $\mathcal{O}(\sqrt{\Delta t})$. To the best of our knowledge, the error estimates of the numerical scheme are the first of its kind for conservation laws with discontinuous flux where spatial discontinuities can accumulate. We present numerical examples that illustrate the theory.

**Mathematics Subject Classification** 35L65 · 35B44 · 35A01 · 65M06 · 65M08

## 1 Introduction

In this article we study the initial value problem for the following scalar conservation law,

$$u_t + A(x,u)_x = 0 \qquad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}, \tag{1}$$
$$u(0,x) = u_0(x) \qquad \text{for } x \in \mathbb{R}, \tag{2}$$

where the flux $A : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of Panov type, as in [38], i.e., $A(x,u) = g(\beta(x,u))$, where $g$ can be a locally Lipschitz continuous real-valued function and $\beta(x,\cdot)$ is a

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monotone function for each $x \in \mathbb{R}$. Thus in this article we do not impose any restriction on the shape of $u \mapsto A(x,u)$ and thereby extending the one dimensional convergence analysis discussed in [24, 26, 41]. Furthermore, the flux function $A$ can have infinitely many spatial discontinuities with accumulation points. Optimal rate of convergence $1/2$ is achieved under the assumption that $\beta(x,u) = u + r(x)$.

Mathematical analysis of these type of equations is complicated due to the presence of discontinuities in the spatial variable of the flux function $A(\cdot,\cdot)$. It is well known that when $x \mapsto A(x,u)$ is not sufficiently smooth, the classical Kruzkov inequality,

$$\begin{aligned} &\partial_t|u-k| + \partial_x\left[\operatorname{sgn}(u-k)(A(x,u)-A(x,k))\right] \\ &\quad + \operatorname{sgn}(u-k)\partial_x A(x,k) \le 0, \quad \forall k \in \mathbb{R} \text{ in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}), \end{aligned}$$

does not make sense due to the term $\operatorname{sgn}(u-k)\partial_x A(x,k)$. When the spatial discontinuities are discrete, the uniqueness of weak solutions is obtained by imposing certain additional conditions (known as interface entropy conditions) along the spatial discontinuities of the flux, which require the existence of traces. Various types of entropy conditions can be chosen depending on the underlying physics of the problem, details of which can be found in [1, 3–5, 7, 8, 14–17, 40] and the references therein. However, when the spatial discontinuities accumulate, the traces do not exist in general. To overcome this obstacle, the notion of *adapted entropy solutions* has been proposed, first in [13] for a monotone flux, and then in [10] for monotone or unimodal flux. The adapted entropy approach to uniqueness can be seen as a generalization of the classical Kruzkov theory. Adapted entropy conditions use a certain class of spatially dependent steady state solutions $k = k(x)$ chosen so that the term $\operatorname{sgn}(u-k(x))\partial_x A(x,k(x))$ vanishes. This work was later generalized in [37] to $A(x,u)$ of the form $g(\beta(x,u))$. In addition, uniqueness results for solutions of (1)–(2) have been further generalized to fluxes possessing degeneracy, see [25]. The convergence analysis of the numerical schemes for these kind of fluxes was open for a quite a long time and recently this has been answered in [24, 26, 41].

One of the important objective of this article is to study the error analysis of our numerical method. From a practical point of view, along with the convergence, it is also important to understand how fast the scheme converges, i.e. how fast the error of approximation of the exact solution $u$ by the numerical approximation $||u^{\Delta}(T,\cdot) - u(T,\cdot)||_{L^1}$ goes to zero as mesh size $\Delta$ goes to zero. This can be measured in terms of the $\alpha$ which satisfies the following

$$||u^{\Delta}(T,\cdot) - u(T,\cdot)||_{L^1} \le C\Delta t^{\alpha}. \tag{3}$$

In addition, convergence rates can also be used for a posteriori error based mesh adaptation [42] and optimal design of multilevel Monte Carlo methods [11]. In the case of a spatially independent flux with $d = 1$, using the doubling of the variable argument, Kuznetsov [35] proved that monotone schemes converge to the weak solution satisfying the Kruzkov entropy condition with $\alpha = 1/2$. Reference [33] shows that these results are indeed true in several spatial dimensions (for flux function independent of space variable). Sabac constructed explicit examples in [39] which imply that this estimate is optimal. Of late, [23] proves the convergence rates of monotone schemes for conservation laws for Holder continuous initial data with Holder exponent greater than 1/2, where bounded variation of the initial data is not required. For unilateral constrained problem [18] provides error estimate for the Godunov approximation of the problem to be $\mathcal{O}(\Delta t^{\frac{1}{3}})$. However, the rates can be shown to be the optimal rate of $\mathcal{O}(\sqrt{\Delta t})$ provided bounds on the temporal total variation of the finite volume approximation exists in the cells adjacent to the point where the constraint is imposed. The techniques introduced in this paper can be adapted to scalar conservation laws with discontinuous flux (with finitely many discontinuities) and the rate of convergence depends on the temporal total variation bounds of the finite volume approximation in the cells adjacent to the spatial of discontinuities of the flux (see section 7.3, [18]). Such bounds on temporal variation can be easily obtained for Riemann data, however, such bounds were not known for general data. Very recently, the bounds on the temporal total variation of the finite volume approximation are proved for the case of strictly monotone fluxes [12] and thus the rates are shown to be 1/2 for monotone fluxes with finitely many spatial discontinuities. These estimates are obtained based on the idea that, for the case of monotone fluxes, problem of discontinuous flux can be treated as boundary value problem with a $BV$ boundary data, where Kuznetsov's type arguments can be invoked and combining the boundary value problems, error estimates can be obtained for the IVP (1)–(2), which allows to estimate the boundary terms in space at the discontinuities that appear when applying the classical Kuznetsov theory to problem.

To the best of our knowledge proofs for the optimal rate $1/2$ are not known for general $BV$ data for non monotone flux even in the case of single discontinuity. Also, no results on error estimates are available when spatial discontinuities of the flux are allowed to be infinite, which in turn may accumulate. In this article, for a certain class of fluxes we prove that Godunov type schemes converge to the adapted entropy solution with the optimal rate 1/2, thus dispensing with the assumption of strict monotonicity and finitely many points of discontinuity of [12] to obtain the optimal rate 1/2. Since the methods of [12] are not applicable when the set of spatial discontinuities contains accumulation points, we prove a Kuznetsov type lemma based on adapted entropy formulation to obtain the error estimates. To the best of our knowledge, this is the first error estimate for conservation laws with discontinuous flux where the set of spatial discontinuities of $A(x, u)$ is infinite and may also contain accumulation points.

One-dimensional conservation laws with discontinuous flux have been the subject of a large literature over the past several decades. The multidimensional case has received less attention, see e.g., [6, 9, 21, 22, 29, 31, 34, 37, 38]. The notion of interface entropy condition was then generalized to several dimensions in [9] and existence of such solutions was established via the vanishing viscosity method. However, the convergence of finite volume approximations remains open for the multidimensional problem even for the case of single discontinuity. For the case of homogeneous flux (no spatial dependence), convergence of numerical approximations is established by the so-called dimension splitting techniques see for example, [20, 30]. The classical dimensional splitting arguments cannot be used when the fluxes are discontinuous because the solutions do not satisfy the TVD property in general [2, 25, 27, 28]. In the case of several dimensions, if we further assume that $\beta(x, u) = u + r(x)$ and

$x \in \mathbb{R}^d$, we can use the dimension splitting techniques to prove the $\beta-$TVD property for general $g$ and use it to establish the convergence of the dimension splitting method for $\beta(x,u) = u + r(x)$. This technique also implies the existence of a BV bound on the solution for the class fluxes which are under consideration, which is of independent interest.

In Sect. 2 we define the relevant notion of the entropy solution and discuss the uniqueness and convergence of finite volume approximation which establishes the well-posedness. Section 3 presents rate convergence estimate, obtained by a Kuznetsov-type analysis. Section 4 presents various numerical simulations that illustrate the theory.

## 2 Adapted entropy condition and well-posedness

We denote by $Q = \mathbb{R}^+ \times \mathbb{R}$. Consider the flux function of the form $A(x,u) = g(\beta(x,u))$, where $g$ and $\beta$ satisfy the following assumptions.

A-1 For $u, v \in [-M, M]$, $M > 0$

$$|\beta(x,v) - \beta(x,u)| \leq \mathcal{K}_1(M)|u - v|, \tag{4}$$

for some continuous $\mathcal{K}_1 : \mathbb{R} \to [0,\infty)$. Also,

$$|\beta(x,u) - \beta(y,u)| \leq \mathcal{K}_2(u)|r(x) - r(y)|, \tag{5}$$

where $\mathcal{K}_2 : \mathbb{R} \to [0,\infty)$ is continuous and $r \in BV(\mathbb{R})$.

A-2 For some $\mathcal{K}_3 > 0$, independent of $x$,

$$|\beta(x,u) - \beta(x,v)| \geq \mathcal{K}_3|u - v|. \tag{6}$$

In other words, let $\tilde{\beta}(x,\cdot)$ denote the inverse of the map $\xi \mapsto \beta(x,\xi)$, then

$$|\tilde{\beta}(x,u) - \tilde{\beta}(x,v)| \leq \tilde{\mathcal{K}_3}|u - v|.$$

A-3 The function $g(z)$ is (locally) Lipschitz-continuous, i.e.,

$$|g(z_1) - g(z_2)| \leq \mathcal{K}_4(M)|z_1 - z_2| \text{ for } z_1, z_2 \in [-M, M],\ M > 0, \tag{7}$$

where $\mathcal{K}_4 : \mathbb{R} \to [0,\infty)$ is continuous.

A-4 For strictly increasing functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ such that $\lim_{|u|\to\infty} |h_1(u)| = \infty$, for any fixed $u$, $h_1(u) \leq \beta(x,u) \leq h_2(u)$, for all $x \in \mathbb{R}$.

**Definition 1** (*Adapted Entropy Condition*) A function $u \in C([0,T]; L^1_{loc}(\mathbb{R})) \cap L^\infty(Q)$ is said to be an adapted entropy solution of the IVP (1)–(2) if the following holds:

$$\partial_t |u(t,x) - k_\alpha(x)| + \partial_x \left[\operatorname{sgn}(u - k_\alpha(x))(A(u,x) - g(\alpha))\right] \leq 0, \text{ in } \mathcal{D}'(Q) \tag{8}$$

for $\alpha \in \mathbb{R}$. Or equivalently, for all $0 \le \phi \in C_c^\infty(Q)$,

$$\begin{aligned} \int_Q & |u(t,x) - k_\alpha(x)|\phi_t(t,x) \\ & + \operatorname{sgn}(u(t,x) - k_\alpha(x))(A(x,u(t,x)) - g(\alpha))\phi_x(t,x)\,dxdt \\ & + \int_{\mathbb{R}} |u_0(x) - k_\alpha(x)|\phi(0,x)\,dx \ge 0, \end{aligned} \tag{9}$$

where $k_\alpha : \mathbb{R} \to \mathbb{R}$ is a stationary state defined by $\beta(x, k_\alpha(x)) := \alpha$.

***Remark 1*** If the flux function $A(x,u)$ is unimodal, then the above definition of *adapted entropy solutions* can be viewed as the generalization of the definition given in [10], in the following sense:

Let $\Psi_A(x,u)$ denote the singular map corresponding to $A(x,u)$. Then the flux can be written in the Panov form $A(x,u) = g(\beta(x,u))$, with $g(u) = |u|$ and $\beta(x,u) = \Psi_A(x,u)$. Now, for $\alpha \in \mathbb{R}$, we have,

$$k_\alpha(x) = \begin{cases} k_\alpha^+(x), & \alpha \ge 0, \\ k_{|\alpha|}^-(x), & \alpha \le 0. \end{cases}$$

Here, $k_\alpha^\pm(x) := (A^\pm)^{-1}(x,\alpha)$ for $\alpha > 0$.

**Theorem 1** *Let $u, v \in C([0,T]; L^1_{loc}(\mathbb{R})) \cap L^\infty(Q)$ be entropic solutions to the IVP (1)–(2) with initial data $u_0, v_0 \in L^\infty(\mathbb{R})$. Assume the flux satisfies the hypothesis (A-1)–(A-4). Then for $t \in [0,T]$ the following holds,*

$$\int_a^b |u(t,x) - v(t,x)|dx \le \int_{a-\overline{M}t}^{b+\overline{M}t} |u_0(x) - v_0(x)|dx, \tag{10}$$

*where $-\infty \le a < b \le \infty$ and $\overline{M} := \sup\{|A_u(x,u(t,x))|;\ x \in \mathbb{R}, 0 \le t \le T\}$.*

***Proof*** Note that an adapted entropy solution $u$ satisfies the following distributional inequality:

$$|\tilde\beta(x, v(t,x)) - \tilde\beta(x,k)|_t + [\operatorname{sgn}(v-k)(g(v) - g(k))]_x \le 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}),$$

where $v(t,x) := \beta(x, u(t,x))$ and $\tilde\beta(x,\cdot)$ is the inverse of the map $\xi \mapsto \beta(x,\xi)$. Suppose $u_1$ and $u_2$ are the two adapted entropy solutions, by repeating the doubling of the variable arguments, we get:

$$\begin{aligned} & |\tilde\beta(x, v_1(t,x)) - \tilde\beta(x, v_2(t,x))|_t + [\operatorname{sgn}(v_1(t,x) - v_1(t,x))(g(v_1(t,x)) - g(v_2(t,x)))]_x \\ & \quad \le 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}). \end{aligned}$$

By appropriately choosing a sequence of test functions and passing to the limit (see [37] for details) we get the following contraction estimate:

$$\int_a^b |\tilde{\beta}(x, v_1(t, x)) - \tilde{\beta}(x, v_2(t, x))| dx \leq \int_{a-\overline{M}t}^{b+\overline{M}t} |\tilde{\beta}(x, v_1(0, x)) - \tilde{\beta}(x, v_2(0, x))| dx,$$

which is same as (10). □

### 2.1 Godunov type scheme and its convergence

We briefly present the convergence analysis for a general $g$. Most of the proofs are in the spirit of [26].

For $\Delta x$, $\Delta t > 0$, consider equidistant spatial grid points $x_i := i\Delta x$ for $i \in \mathbb{Z}$ and temporal grid points $t^n := n\Delta t$ for integers $0 \leq n \leq N$, such that $T \in [t^N, t^{N+1})$. Let $\lambda := \Delta t/\Delta x$. Let $\chi(x)$ denote the indicator function of $C_i := [x_i - \Delta x/2, x_i + \Delta x/2)$, and let $\chi^n(t)$ denote the indicator function of $C^n := [t^n, t^{n+1})$. We approximate the initial data according to:

$$u_0^\Delta(x) := \sum_{i \in \mathbb{Z}} \chi(x) u_i^0, \quad \text{where } u_i^0 = u_0(y_i) \text{ for } i \in \mathbb{Z}. \tag{11}$$

The approximations generated by the scheme are denoted by $u_i^n$, where $u_i^n \approx u(x_i, t^n)$. The grid function $\{u_i^n\}$ is extended to a function defined on $\Pi_T = \mathbb{R} \times [0, T]$ via

$$u^\Delta(x, t) = \sum_{n=0}^{N} \sum_{i \in \mathbb{Z}} \chi(x) \chi^n(t) u_i^n.$$

Similarly, we define another grid function $\beta_i^n = \beta(x_i, u_i^n) \approx \beta(x_i, u(x_i, t^n))$, and is extended to a function defined on $\Pi_T$ via

$$\beta^\Delta(x, t) = \sum_{n=0}^{N} \sum_{i \in \mathbb{Z}} \chi(x) \chi^n(t) \beta_i^n.$$

We use the symbols $\Delta_\pm$ to denote spatial difference operators:

$$\Delta_+ z_i = z_{i+1} - z_i, \quad \Delta_- z_i = z_i - z_{i-1}. \tag{12}$$

We use the Godunov type scheme given by:

$$u_i^{n+1} = u_i^n - \lambda \Delta_- \bar{A}(u_i^n, u_{i+1}^n, x_i, x_{i+1}), \quad i \in \mathbb{Z}, n = 0, 1, 2, \ldots, \tag{13}$$

where the numerical flux $\bar{A}$ is the generalized Godunov flux of [26]:

$$\bar{A}(u, v, x_i, x_{i+1}) := \bar{g}\left(\beta(x_i, u), \beta(x_{i+1}, v)\right) \tag{14}$$

and

$$\bar{g}(p, q) = \begin{cases} \min_{w\in[p,q]} g(w), & p \le q, \\ \max_{w\in[q,p]} g(w), & p \ge q. \end{cases} \tag{15}$$

Let $\mathcal{S} = \sup_{|u|\le\mathcal{M}, x\in\mathbb{R}} |\beta(x, u)|$, and define $L_\beta = \mathcal{K}_1(\mathcal{M})$, $L_g = \mathcal{K}_4(\mathcal{S})$. Hereafter the ratio $\lambda = \frac{\Delta t}{\Delta x}$ is fixed and satisfies the CFL condition:

$$\lambda L_g L_\beta \le 1/2. \tag{16}$$

**Lemma 1** *Under the* CFL *condition (16), the following properties hold:*

*i. The scheme is monotone and the Godunov approximations are bounded:*

$$|u_i^n| \le \mathcal{M}, \quad i \in \mathbb{Z}, n = 0, 1, 2, \ldots. \tag{17}$$

*ii. Discrete time continuity estimates:*

$$\sum_{i\in\mathbb{Z}} |u_i^{n+1} - u_i^n| \le \mathcal{K}_5 \,\mathrm{TV}(\beta^0), n = 0, 1, 2, \ldots \tag{18}$$

*where $\mathcal{K}_5 > 0$ is independent of the mesh size $\Delta$.*

*iii.* TVD *property with respect to $\{\beta_i^n\}$ :*

$$\sum_{i\in\mathbb{Z}} |\beta_{i+1}^{n+1} - \beta_i^{n+1}| \le \sum_{i\in\mathbb{Z}} |\beta_{i+1}^n - \beta_i^n|. \tag{19}$$

*iv. Discrete entropy inequality:*

$$|u_i^{n+1} - k_i^\alpha| \le |u_i^n - k_i^\alpha| - \lambda(\mathcal{P}_{i+1/2}^n - \mathcal{P}_{i-1/2}^n), \text{ for all } i \in \mathbb{Z}, n = 0, 1, 2, \ldots, \tag{20}$$

*where*

$$\mathcal{P}_{i+1/2}^n = \bar{A}(u_i^n \vee k_i^\alpha, u_{i+1}^n \vee k_{i+1}^\alpha, x_i, x_{i+1}) - \bar{A}(u_i^n \wedge k_i^\alpha, u_{i+1}^n \wedge k_{i+1}^\alpha, x_{\Delta x}, x_{i+1}).$$

***Proof*** The key step in obtaining compactness is the $\beta-$TVD property and can be proved as below. Using the marching formula of $u$ (13), we can express $\beta$ as follows:

$$\beta_i^{n+1} = \beta_i^n - \lambda\theta_i^{n+1/2} \Delta_- \bar{g}_{j+1/2}^n, \tag{21}$$

where

$$\theta_i^{n+1/2} = \begin{cases} \frac{\beta_i^{n+1}-\beta_i^n}{u_i^{n+1}-u_i^n}, & u_i^{n+1} - u_i^n \neq 0, \\ 0, & u_i^{n+1} - u_i^n = 0. \end{cases} \tag{22}$$

Note that, $0 \leq \theta_i^{n+1/2} \leq L_\beta$. Hence we can write (21) in incremental form:

$$\beta_i^{n+1} = \beta_i^n + \mathcal{C}_{j+1/2}^n \Delta_+ \beta_i^n - \mathcal{D}_{j-1/2}^n \Delta_- \beta_i^n, \tag{23}$$

where

$$\begin{aligned} \mathcal{C}_{j+1/2}^n &= \begin{cases} -\lambda\theta_i^{n+1/2} \left( \frac{\bar{g}(\beta_j^n, \beta_{j+1}^n) - \bar{g}(\beta_j^n, \beta_i^n)}{\beta_{j+1}^n - \beta_i^n} \right), & \beta_{j+1}^n - \beta_i^n \neq 0, \\ 0, & \beta_{j+1}^n - \beta_i^n = 0, \end{cases} \\ \mathcal{D}_{j-1/2}^n &= \begin{cases} \lambda\theta_i^{n+1/2} \left( \frac{\bar{g}(\beta_j^n, \beta_i^n) - \bar{g}(\beta_{j-1}^n, \beta_j^n)}{\beta_j^n - \beta_{j-1}^n} \right), & \beta_j^n - \beta_{j-1}^n \neq 0, \\ 0, & \beta_j^n - \beta_{j-1}^n = 0. \end{cases} \end{aligned} \tag{24}$$

Recalling (2.1) and that $\bar{g}(\cdot,\cdot)$ is a monotone numerical flux [19, 36], we have:

$$0 \leq \mathcal{C}_{j+1/2}^n, \mathcal{D}_{j-1/2}^n \leq \lambda L_g L_\beta \leq 1/2. \tag{25}$$

Thus $\mathcal{C}_{j+1/2}^n + \mathcal{D}_{j+1/2}^n \leq 1$, and we can apply Harten's lemma [36][Theorem 6.1] to get the $\beta-$TVD property. □

**Theorem 2** *Assume that the flux function $A(x,u)$ satisfies the assumptions (A-1) through (A-4), and that $u_0 \in BV(\mathbb{R})$. Then as the mesh size $\Delta \to 0$, the approximations $u^\Delta$ generated by the Godunov scheme described above converge in $L^1_{\mathrm{loc}}(Q)$ and pointwise a.e. in $Q$ to the unique adapted entropy solution $u \in L^\infty(Q) \cap C([0,T] : L^1_{\mathrm{loc}}(\mathbb{R}))$ corresponding to the Cauchy problem (1), (2) with initial data $u_0$. In addition, the total variation $u(\cdot,t)$ is uniformly bounded for $t \geq 0$.*

***Proof*** Proof is same as the one presented in [26]. □

## 3 Error estimates

In this section, we estimate the rate of convergence of the numerical methods introduced in the previous section. The idea is to prove the Kuznetsov type lemma based on the adapted entropy formulation. We begin by listing some of the technical tools required to prove the Kuznetsov lemma. We assume that $u_0, r \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$ and the fluxes satisfy the assumptions detailed in the previous section.

**Definition 2** Let $\Pi_T = \mathbb{R} \times [0,T]$. We define $\Phi^{\eta,\epsilon} : {\Pi_T}^2 \to \mathbb{R}$ by,

$$\Phi^{\eta,\epsilon}(t,x,s,y) = \omega_\epsilon(t-s)\omega_\eta(x-y),$$

where for $\mathbf{z} \in \mathbb{R}$, $\omega_\eta(z) := \frac{1}{\eta}\omega\left(\frac{z}{\eta}\right)$ is a mollifier such that $\omega \in C^\infty(\mathbb{R};\mathbb{R})$ is an even function and satisfies the following:

$$\text{supp}(w) \subset [0,1], \quad 0 \le \omega(z) \le 1 \text{ and } \int\limits_{\mathbb{R}} w(z)dz = 1. \tag{26}$$

For further calculations, we note the following properties of $\Phi^{\eta,\epsilon}$:

1.

$$\Phi_x^{\eta,\epsilon} = \frac{\partial}{\partial x}\Phi^{\eta,\epsilon}(t,x,s,y) = -\frac{\partial}{\partial y}\Phi^{\eta,\epsilon}(t,x,s,y) = -\Phi_y^{\eta,\epsilon}. \tag{27}$$

2.

$$\begin{aligned}\Phi_t^{\eta,\epsilon} &= \frac{\partial}{\partial t}\Phi^{\eta,\epsilon}(t,x,s,y) = \omega_\epsilon^{'}(t-s)\omega_\eta(x-y)\\ &= -\frac{\partial}{\partial s}\Phi^{\eta,\epsilon}(t,x,s,y) = -\Phi_s^{\eta,\epsilon}.\end{aligned} \tag{28}$$

3.

$$\Phi^{\eta,\epsilon}(t,x,s,y) = \Phi^{\eta,\epsilon}(t,y,s,x) = \Phi^{\eta,\epsilon}(s,x,t,y) = \Phi^{\eta,\epsilon}(s,y,t,x). \tag{29}$$

4.

$$\int\limits_{\mathbb{R}} w_\eta(x-y)dy = 1 \text{ and } \int\limits_0^T w_\epsilon(t-s)ds \le 1, \quad \text{for all } x \in \mathbb{R}, t \ge 0, \tag{30}$$

5. There exists $C$ independent of $\eta$ and $\epsilon$ such that,

$$\begin{aligned}&\int\limits_{\mathbb{R}} |\partial_x w_\eta(x-y)|dy \le \frac{C}{\eta} \text{ and } \int\limits_0^T |w_\epsilon^{\prime}|(t-s)ds \le \frac{C}{\epsilon},\\ &\text{for all } x \in \mathbb{R}, t \ge 0.\end{aligned} \tag{31}$$

**Definition 3** For $\sigma > 0$, define the following

i. $\kappa := \{u : \Pi_T \to \mathbb{R} : ||u(\cdot,t)||_{L^\infty} \le k, |u(\cdot,t)|_{BV} \le k\}$.
ii. $\nu_t(u,\sigma) := \sup_{|\tau|\le\sigma} ||u(t+\tau) - u(t)||_1$.
iii. $\nu(u,\sigma) := \sup_{0<t<T} \nu_t(u,\sigma) = \sup_{t\in(0,T)} \sup_{|\tau|\le\sigma} ||u(t+\tau) - u(t)||_1$.

***Remark*** If $u_0 \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$ then there exists $L$ such that adapted entropy solution satisfies $\nu(u,\sigma) \le L\sigma$.

**Definition 4**

$$
\begin{aligned}
\Lambda_T(u,\phi,k_\alpha) := \int_{\Pi_T} &\Big(|u(t,x)-k_\alpha(x)|\phi_t + \operatorname{sgn}(u(t,x)-k_\alpha(x)) \\
&\times\Big(A(x,u(t,x))-\alpha\Big)\phi_x\Big)dxdt \\
&-\int_{\mathbb{R}} |u(T,x)-k_\alpha(x)|\phi(T,x)dx + \int_{\mathbb{R}} |u_0(x)-k_\alpha(x)|\phi(0,x)dx.
\end{aligned} \tag{32}
$$

$$
\Lambda_{\eta,\epsilon}(u,v) := \int_{\Pi_T} \Lambda_T(u(\cdot,\cdot),\phi^{\eta,\epsilon}(\cdot,\cdot,s,y),\tilde{v}(s,y,x))dyds. \tag{33}
$$

$$
\Lambda_{\eta,\epsilon}(v,u) := \int_{\Pi_T} \Lambda_T(v(\cdot,\cdot),\Phi^{\eta,\epsilon}(t,x,\cdot,\cdot),\tilde{u}(t,x,y))dxdt. \tag{34}
$$

**Lemma 2** *Let $v$ be the solution of IVP (1)–(2) and $u \in \kappa$. For $0<\epsilon<T$ and $\eta>0$, then*

$$
\begin{aligned}
\|u(\cdot,T)-v(\cdot,T)\|_{L^1(\mathbb{R})} \le \|u_0-v_0\|_{L^1(\mathbb{R})} + C\Big[&L\epsilon + \mathrm{TV}(r)|\eta| + \mathrm{TV}(v)|\eta| \\
&+\nu(u,\epsilon)\Big] - \Lambda_{\eta,\epsilon}(u,v).
\end{aligned} \tag{35}
$$

*where $C$ is independent of the mesh size $\Delta$.*

***Proof*** Adding $\Lambda_{\eta,\epsilon}(v,u)$ and $\Lambda_{\eta,\epsilon}(u,v)$, we get the following

$$
\begin{aligned}
&\Lambda_{\eta,\epsilon}(v,u)+\Lambda_{\eta,\epsilon}(u,v) \\
&\int_{\Pi_T}\Big(|u(t,x)-\tilde{v}(y,s,x)|\Phi_t^{\eta,\epsilon}dxdtdyds \\
&\quad+\int_{\Pi_T}\big[(\operatorname{sgn}(u(t,x)-\tilde{v}(y,s,x))(A_i(x,u(x,t))-A_i(y,v(y,s))))\big]\Phi_{x_i}^{\eta,\epsilon}dxdtdyds \\
&\quad-\int_{\Pi_T}\int_{\mathbb{R}}|u(T,x)-\tilde{v}(s,y,x)|\Phi^{\eta,\epsilon}(x,T,y,s)dxdyds \\
&\quad+\int_{\Pi_T}\int_{\mathbb{R}}|u_0(x)-\tilde{v}(s,y,x)|\Phi^{\eta,\epsilon}(x,0,y,s)dxdyds \\
&\quad+\int_{\Pi_T}\Big(|v(s,y)-\tilde{u}(t,x,y)|\Phi_s^{\eta,\epsilon}dydsdxdt \\
&\quad+\int_{\Pi_T}\big[\operatorname{sgn}(v(s,y)-\tilde{u}(t,x,y))(A_i(y,v(s,y))-A(x,u(t,x)))\big]\Phi_y^{\eta,\epsilon}dydsdxdt \\
&\quad-\int_{\Pi_T}\int_{\mathbb{R}}|v(T,y)-\tilde{u}(t,x,y)|\Phi^{\eta,\epsilon}(x,t,y,T)dydxdt \\
&\quad+\int_{\Pi_T}\int_{\mathbb{R}}|v_0(y)-\tilde{u}(t,x,y)|\Phi^{\eta,\epsilon}(x,t,y,0)dydxdt.
\end{aligned}
$$

From (27), terms involving $\Phi_x^{\eta,\epsilon}$ and $\Phi_y^{\eta,\epsilon}$ cancel each other. Now invoking symmetry of $\Phi^{\eta,\epsilon}$ given by (27)–(29), we have the following

$$\Lambda_{\eta,\epsilon}(u,v) = -\Lambda_{\eta,\epsilon}(v,u) - \mathcal{A} + \mathcal{B} + \mathcal{C},$$

where

$$\begin{aligned}
\mathcal{A} &= \int_{\Pi_T}\int_{\mathbb{R}} \Big(|u(T,x) - \tilde{v}(s,y,x)| + |v(T,y) - \tilde{u}(t,x,y)|\Big)\Phi^{\eta,\epsilon}(x,s,y,T)dydxds \\
&= \int_0^T w_\epsilon(T-s)\int_{\mathbb{R}^2} \Big(|u(t,x) - \tilde{v}(s,y,x)| + |v(T,y) - \tilde{u}(t,x,y)|\Big)w_\eta(x-y)dydxds. \\
\mathcal{B} &= \int_{\Pi_T}\int_{\mathbb{R}} \Big(|u_0(x) - \tilde{v}(s,y,x)| + |v_0(y) - \tilde{u}(t,x,y)|\Big)\Phi^{\eta,\epsilon}(x,s,y,0)dxdyds \\
&= \int_0^T w_\epsilon(T-s)\int_{\mathbb{R}^2} \Big(|u_0(x) - \tilde{v}(s,y,x)| + |v_0(y) - \tilde{u}(t,x,y)|\Big)w_\eta(x-y)dxdyds. \\
\mathcal{C} &= \int\limits_{{\Pi_T}^2} \Big(|u(t,x) - \tilde{v}(s,y,x)| - |v(s,y) - \tilde{u}(t,x,y)|\Big)w'_\epsilon(t-s)w_\eta(x-y)dxdydsdt.
\end{aligned}$$

since $v$ is the solution, $\Lambda_{\eta,\epsilon}(v,u) \geq 0$, implying that

$$\mathcal{A} \leq \mathcal{B} + \mathcal{C} - \Lambda_{\eta,\epsilon}(u,v). \tag{36}$$

Claim 1 We have the following lower bound on $\mathcal{A}$ :

$$\mathcal{A} \geq \|u(\cdot,T) - v(\cdot,T)\|_{L^1(\mathbb{R})} - C\Big(L\epsilon + \mathrm{TV}(r)|\eta| + \mathrm{TV}(v)|\eta| + \nu(u,\epsilon)\Big). \tag{37}$$

To prove the claim we make the following estimates.

(a) Estimation of $|u(T,x) - \tilde{v}(s,y,x)|$: Consider,

$$\begin{aligned}
|u(T,x) - v(T,x)| &= |u(T,x) - \tilde{v}(s,y,x) + \tilde{v}(s,y,x) \\
&\quad - \tilde{v}(T,y,x) + \tilde{v}(T,y,x) - v(T,x)| \\
&\leq |u(T,x) - \tilde{v}(s,y,x)| + |\tilde{v}(s,y,x) \\
&\quad - \tilde{v}(T,y,x)| + |\tilde{v}(T,y,x) - v(T,x)|.
\end{aligned}$$

Thus we have,

$$\begin{aligned}
|u(T,x) - \tilde{v}(s,y,x)| \geq\ & |u(T,x) - v(T,x)| - |\tilde{v}(s,y,x) \\
& - \tilde{v}(T,y,x)| - |\tilde{v}(T,y,x) - v(T,x)|.
\end{aligned}$$

Using the definition of $\tilde{v}$ we get,

$$|\tilde{v}(T,y,x) - \tilde{v}(s,y,x)| = |\tilde{\beta}(x,\beta(y,v(T,y))) - \tilde{\beta}(x,\beta(y,v(s,y)))|$$

$$
\begin{aligned}
&\leq C|\beta(y, v(T, y)) - \beta(y, v(s, y))| \\
&\leq C|v(T, y) - v(s, y)|.
\end{aligned}
$$

Invoking the properties of $\beta$, we get the following

$$
\begin{aligned}
|\tilde{v}(T, y, x) - \tilde{v}(T, x, x)| &= |\tilde{\beta}(x, \beta(y, v(T, y))) - \tilde{\beta}(x, \beta(x, v(T, x)))| \\
&\leq C|\beta(y, v(T, y)) - \beta(x, v(T, x))| \\
&= C|\beta(y, v(T, y)) - \beta(x, v(T, y)) \\
&\quad + \beta(x, v(T, y)) - \beta(x, v(T, x))|. \\
&\leq C\left[|r(x) - r(y)| + |v(T, y) - v(T, x)|\right].
\end{aligned}
$$

Combining all these estimates we get,

$$
\begin{aligned}
|u(T, x) - \tilde{v}(s, y, x)| \geq{}& |u(T, x) - v(T, x)| - |v(T, y) - v(s, y)| \\
&- C\left[|r(x) - r(y)| + |v(T, y) - v(T, x)|\right].
\end{aligned} \tag{38}
$$

(b) Estimation of $|v(T, y) - \tilde{u}(t, x, y)|$: Consider $|u(T, x) - v(T, x)|$, add and subtract $\tilde{u}(s, x, y)$ and $v(T, y) = \tilde{v}(T, y, y)$ to get,

$$
\begin{aligned}
|u(T, x) - v(T, x)| &= |u(T, x) - \tilde{u}(s, x, y) \\
&\quad + \tilde{u}(s, x, y) - v(T, y) + v(T, y) - v(T, x)| \\
&\leq |u(T, x) - \tilde{u}(s, x, y)| \\
&\quad + |\tilde{u}(s, x, y) - v(T, y)| + |v(T, y) - v(T, x)|.
\end{aligned}
$$

Thus we have,

$$
\begin{aligned}
|\tilde{u}(s, x, y) - v(T, y)| &\geq |u(T, x) - v(T, x)| \\
&\quad - |u(T, x) - \tilde{u}(s, x, y)| - |v(T, y) - v(T, x)|. \\
|u(T, x) - \tilde{u}(s, x, y)| &= |\tilde{\beta}(x, \beta(x, u(T, x))) \\
&\quad - \tilde{\beta}(y, \beta(x, u(s, x)))| \\
&\leq C|r(x) - r(y)| + |u(T, x) - u(s, x)|.
\end{aligned}
$$

Combining all these estimates we get,

$$
\begin{aligned}
|v(T, y) - \tilde{u}(t, x, y)| \geq{}& |u(T, x) - v(T, x)| - |v(T, y) - v(T, x)| \\
&- C\left[|r(x) - r(y)| + |u(T, x) - u(s, x)|\right].
\end{aligned} \tag{39}
$$

Adding (38) and (39), for some $C > 0$ we get the following estimate:

$$
\begin{aligned}
&|u(T, x) - \tilde{v}(s, y, x)| + |\tilde{u}(s, x, y) - v(T, y)| \\
&\quad \geq 2|u(T, x) - v(T, x)|C|v(T, y) - v(s, y)|
\end{aligned}
$$

$$
\begin{aligned}
&-C\left[|r(x)-r(y)|+|v(T,y)-v(T,x)|\right]\\
&-C\left[|r(x)-r(y)|+|u(T,x)-u(s,x)|\right]-|v(T,y)-v(T,x)|\\
\geq\ & 2|u(T,x)-v(T,x)|\\
&-C\Big[|v(T,y)-v(s,y)|+|r(x)-r(y)|+|v(T,y)-v(T,x)|+|u(T,x)-u(s,x)|\Big].
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathcal{A} &= \int_0^T w_\epsilon(T-s)\int_{\mathbb{R}^2}\Big(|u(t,x)-\tilde{v}(s,y,x)|+|v(T,y)-\tilde{v}(t,x,y)|\Big)w_\eta(x-y)dydxds\\
&\geq \int_0^T w_\epsilon(T-s)\int_{\mathbb{R}^2} 2|u(T,x)-v(T,x)|-C\Big(|v(T,y)-v(s,y)|+|r(x)-r(y)|\\
&\quad +|v(T,y)-v(T,x)|+|u(T,x)-u(s,x)|\Big)w_\eta(x-y)dydxds. \qquad (40)
\end{aligned}
$$

To obtain the desired lower bound on $\mathcal{A}$, we estimate terms appearing on the right side of (40) as follows:

i. Consider the integral $\int_0^T \Big[w_\epsilon(T-s)\int_{\mathbb{R}^2}\Big(|u(T,x)-v(T,x)|\Big)w_\eta(x-y)\, dydx\Big]ds$. By symmetry of $w$ we have

$$
\int_0^T \omega_\epsilon(T-s)ds=\int_0^T \omega_\epsilon(s)ds=\frac{1}{2},
$$

Now applying Fubini-Tonellis's theorem we get,

$$
\begin{aligned}
&\int_0^T \Big[w_\epsilon(T-s)\int_{\mathbb{R}^2}\Big(|u(T,x)-v(T,x)|\Big)w_\eta(x-y)dydx\Big]ds\\
&\quad=\frac{1}{2}\|u(T,\cdot)-v(T,\cdot)\|_{L^1(\mathbb{R})}.
\end{aligned}
$$

ii. Consider the integral $\int_0^T w_\epsilon(T-s)\int_{\mathbb{R}^2}\Big(|v(T,y)-v(s,y)|\Big)w_\eta(x-y)dydxds$. Since the support of $w_\epsilon\subset[-\epsilon,\epsilon]$, using the time continuity of $v$ we get,

$$
\int_0^T w_\epsilon(T-s)\int_{\mathbb{R}^2}\Big(|v(T,y)-v(s,y)|\Big)w_\eta(x-y)dydxds\leq\frac{1}{2}L\epsilon.
$$

iii. Consider the integral $\int_0^T w_\epsilon(T-s)\int_{\mathbb{R}^2}\Big(|r(x)-r(y)|\Big)w_\eta(x-y)dydxds$. Note that,

$$
\int_{\mathbb{R}^2}\omega_\eta(x-y)|r(x)-r(y)|dxdy\leq|\eta|\,\mathrm{TV}(r),
$$

and thus we have,

$$\int_0^T w_\epsilon(T-s)\int_{\mathbb{R}^2}\Big(|r(x)-r(y)|\Big)w_\eta(x-y)dydxds \le \frac{1}{2}\,\mathrm{TV}(r)|\eta|.$$

iv. Consider the integral $\int_0^T w_\epsilon(T-s)\int_{\mathbb{R}^2}\Big(|v(T,x)-v(T,y)|\Big)w_\eta(x-y)dydxds$. Since $v(T,\cdot)$ has bounded variation, repeating the arguments as in the previous step, we get,

$$\int_0^T w_\epsilon(T-s)\int_{\mathbb{R}^2}\Big(|v(T,y)-v(T,x)|\Big)w_\eta(x-y)dydxds \le \frac{1}{2}\,\mathrm{TV}(v)||\eta|.$$

v. Consider the integral $\int_0^T w_\epsilon(T-s)\int_{\mathbb{R}^2}\Big(|u(T,x)-u(s,x)|\Big)w_\eta(x-y)dydxds$. Note that $w_\epsilon(T-s)$ is zero for $T-s>\epsilon$. Thus invoking the definition of $\nu(u,\epsilon)$ we get,

$$\int_0^T w_\epsilon(T-s)\int_{\mathbb{R}^2}\Big(|u(T,x)-u(s,x)|\Big)w_\eta(x-y)dydxds \le \frac{1}{2}\nu(u,\epsilon).$$

Combining all these estimates, we get the desired lower bound on $\mathcal{A}$.

Claim 2 We have the following upper bound on $\mathcal{B}$.

$$\mathcal{B} \le \|u(\cdot,0)-v(\cdot,0)\|_{L^1(\mathbb{R})} + C\Big(L\epsilon + \mathrm{TV}(r)|\eta| + \mathrm{TV}(v)|\eta| + \nu(u,\epsilon)\Big). \tag{41}$$

Claim follows by repeating the arguments done in the estimation of $\mathcal{A}$, for $|u_0(x)-\tilde{v}(s,y,x)|+|v_0(y)-\tilde{v}(t,x,y)|$.

Claim 3

$$\mathcal{C}=0. \tag{42}$$

Suppose $\beta(x,u)=au+r(x)$ for $a\neq 0$, using the definition of $\tilde{u}$ and $\tilde{v}$ we have,

$$\beta(x,u(t,x)) = au(t,x)+r(x) = a\tilde{u}(t,x,y)+r(y) = \beta(y,\tilde{u}(t,x,y)),$$
$$\beta(x,\tilde{v}(s,y,x)) = a\tilde{v}(s,y,x)+r(x) = av(s,y)+r(y) = \beta(y,v(s,y)).$$

Which implies

$$u(t,x)-\tilde{v}(s,y,x)=\tilde{u}(t,x,y)-v(s,y),$$

and hence

$$|u(t,x)-\tilde{v}(s,y,x)|=|\tilde{u}(t,x,y)-v(s,y)|.$$

Thus we have $\mathcal{C} = 0$ and claim is proved.

Substituting the values of (37)–(42) in (36), we have

$$\|u(T,\cdot) - v(T,\cdot)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})} \\ +C\Big[L\epsilon + \mathrm{TV}(r)|\eta| + \mathrm{TV}(v)|\eta| + \nu(u,\epsilon)\Big] - \Lambda_{\eta,\epsilon}(u,v).$$

which completes the proof of the lemma. □

***Remark 2*** The terms involving $\mathrm{TV}(r)$ are absent in the original Kuznetsov lemma where the flux is homogeneous.

Before moving on to the proof of the error estimate, we introduce the following notations:

$$\eta_i^n := |u_i^n - k_\alpha^i|,$$
$$p_i^n := \mathrm{sgn}(u_i^n - k_\alpha^i)\big(A(x_i,u_i^n) - A(x_i,k_\alpha^i)\big) = A(x_i,u_i^n \vee k_\alpha^i) - A(x_i,u_i^n \wedge k_\alpha^i).$$

Now we state and prove the convergence rate theorem.

**Theorem 3** *(*Convergence rate for conservation laws with discontinuous flux*) Let $u$ be the entropy solution of (1)–(2) and $u^\Delta$ the numerical solution given by (13)–(14). Then we have the following convergence rate:*

$$\left\|u^\Delta(T,\cdot) - v(T,\cdot)\right\|_{L^1(\mathbb{R})} = \mathcal{O}(\sqrt{\Delta t}),$$

*for some constant $C$ independent of $\Delta t$.*

***Proof*** Let $\eta = \epsilon = \sqrt{\Delta t}$. In view of the previous lemma, it is enough to show the following:

$$\nu(u^\Delta, \sqrt{\Delta t}) = \mathcal{O}(\sqrt{\Delta t}), \tag{43}$$
$$-\Lambda_{\sqrt{\Delta t},\sqrt{\Delta t}}(u,v) = \mathcal{O}(\sqrt{\Delta t}). \tag{44}$$

Note that (43) follows from the time estimate (18). Now it remains to prove (44). Let $x \in \mathbb{R}$ and $u^\Delta$ be a piecewise constant function obtained by the numerical scheme. Consider,

$$-\Lambda_T^\Delta(u^\Delta,\phi,k_\alpha^\Delta) = -\sum_{n=0}^{N-1}\Bigg(\sum_i\Bigg[\int_{C_i}\int_{C^n}\left(\eta_i^n\phi_t(x,s) + p_i^n\phi_x(x,s)\right)ds dx \\ -\int_{C_i}\eta_i^0\phi(x,0)dydx + \sum_i\int_{C_i}\eta_i^N\phi(x,t)dydx\Bigg).$$

Fundamental theorem of calculus followed by summation by parts imply,

$$-\Lambda_T^{\Delta}(u^{\Delta},\phi,k_{\alpha}^{\Delta}) = \sum_i \sum_{n=0}^{N-1} \Bigg[ \left(\eta_i^{n+1}-\eta_i^n\right) \int\limits_{C_i} \phi(x,t^{n+1})dxdy + \left(p_i^n - p_{i-1}^n\right) \int\limits_{C^n} \phi(x_{i-\frac{1}{2}},s)dyds \Bigg].$$

Using the discrete entropy inequality (20) in the above equation, we get

$$-\Lambda_T^{\Delta}(u^{\Delta},\phi,k_{\alpha}^{\Delta}) \leq \sum_i \sum_{n=0}^{N-1} \Bigg[ -\Lambda \left(\mathcal{P}^n_{i+\frac{1}{2}} - \mathcal{P}^n_{i-\frac{1}{2}}\right) \int\limits_{C_i} \phi(x,t^{n+1})dx + \left(p_i^n - p_{i-1}^n\right) \int\limits_{C^n} \phi(x_{i-\frac{1}{2}},s)ds \Bigg],$$

which on rearrangement implies that

$$-\Lambda_T^{\Delta}(u^{\Delta},\phi,k_{\alpha}^{\Delta}) \leq \sum_i \sum_{n=0}^{N-1} \Bigg[ \lambda |\mathcal{P}^n_{i+\frac{1}{2},j} - p_i^n| \int\limits_{C_i} |\phi(x+\Delta x,t^{n+1}) - \phi(x,t^{n+1})|dydx + |p_i^n - p_{i-1}^n| \Big| \int\limits_{C^n} \phi(x_{i-\frac{1}{2}},y,s)dyds - \lambda \int\limits_{C_i} \phi(x,t^{n+1})dxdy \Big| \Bigg].$$

Adding and subtracting

$$\lambda \int_{C_i} \phi(x_{i-\frac{1}{2}},y,t^{n+1})dx = \int_{C^n} \phi(x_{i-\frac{1}{2}},y,t^{n+1})dt$$

in the term

$$\left| \int_{C^n} \phi(x_{i-\frac{1}{2}},y,s)ds - \lambda \int_{C_i} \phi(x,t^{n+1})dx \right|$$

respectively, we get

$$\begin{aligned} -\Lambda_T^{\Delta}(u^{\Delta},\phi,k_{\alpha}^{\Delta}) &\leq \sum_i \sum_{n=0}^{N-1} \Bigg[ \lambda \mathcal{G}_1^{\phi} |\mathcal{P}^n_{i+\frac{1}{2},j} - p_i^n| + \left|p_i^n - p_{i-1}^n\right| \left(\mathcal{G}_2^{\phi} + \lambda_x \mathcal{G}_3^{\phi}\right) \Bigg] \\ &:= \sum_i \sum_{n=0}^{N-1} \Bigg[ \mathcal{G}_1^{\phi} K_1^{i,n} + \mathcal{G}_2^{\phi} K_2^{i,n} + \mathcal{G}_3^{\phi} K_3^{i,n} \Bigg], \end{aligned}$$

where

$$\mathcal{G}_1^{\phi} = \int\limits_{C_i} \phi(x+\Delta x, t^{n+1}) - \phi(x, t^{n+1}) dx,$$
$$\mathcal{G}_2^{\phi} = \int\limits_{C^n} |\phi(x_{i-1/2}, t) - \phi(x_{i-1/2}, t^{n+1})| ds dx,$$
$$\mathcal{G}_3^{\phi} = \int\limits_{C_i} |\phi(x, t^{n+1}) - \phi(x_{i-1/2}, t^{n+1})| dy dx,$$

For each $(\overline{x}, s) \in \Pi_T$, consider the test function $\phi(x,t) := \Phi^{\sqrt{\Delta t},\sqrt{\Delta t}}(x, t, \overline{x}, s)$ and $\alpha = \beta(\overline{x}, v(, s))$.

Using the properties of $\Phi^{\sqrt{\Delta t},\sqrt{\Delta t}}$, the following estimate can be obtained (see [32] for the details):

$$\int\limits_{\Pi_T} \mathcal{G}_l^{\Phi^{\sqrt{\Delta t},\sqrt{\Delta t}}(\cdot,\cdot,\cdot,\overline{x},s)} d\overline{x} ds \leq \mathcal{O}(\Delta t^{5/2}). \tag{45}$$

Our assumptions on the flux function $A$ imply the following:

$$|p_i^n - p_{i-1}^n| \leq C\Big[|u_i^n - u_{i-1,j}^n| + |r_i - r_{i-1,j}|\Big],$$
$$|p_i^n - \mathcal{P}^n_{i,j+\frac{1}{2}}| \leq C \sum_{k=-1}^{1} |u^n_{i+k,j} - u_i^n|.$$

Since the numerical approximations are uniformly total variation bounded, the above inequalities imply that, $\Delta t \sum_i K_l^{i,n}$ is uniformly bounded for $l \in \{1, 2, \ldots, 4\}$, $n = 0, 1, 2, \ldots, N-1, \alpha \in \mathbb{R}$ and $\Delta > 0$.

Now (45) implies the following

$$\begin{aligned}
\Lambda^{\Delta}_{\sqrt{\Delta t},\sqrt{\Delta t}}(u^{\Delta}, v) &= \int_{\Pi_T} \Lambda_T^{\Delta}(u^{\Delta}, \Phi^{\sqrt{\Delta t},\sqrt{\Delta t}}(\cdot,\cdot,\cdot,\overline{x}, s), k^{\Delta}_{\beta(y,v(y,s))}) \\
&= \left(\sum_i \sum_{n=0}^{N-1} K_l^{i,n}\right) \int_{\Pi_T} G_l^{\Phi^{\sqrt{\Delta t},\sqrt{\Delta t}}(\cdot,\cdot,\cdot,\overline{x},s)} d\overline{x} ds \\
&= \mathcal{O}(\Delta t^{-2})\mathcal{O}(\Delta t^{5/2}) = \mathcal{O}(\sqrt{\Delta t}).
\end{aligned} \tag{46}$$

Note that,

$$\begin{aligned}
\Lambda_T^{\Delta}(u^{\Delta}, \phi, k_{\alpha}^{\Delta}) = \int_{\Pi_T} \Big(&|u^{\Delta}(t,x) - k_{\alpha}^{\Delta}(x)|\phi_t \\
&+ \operatorname{sgn}(u^{\Delta}(t,x) - k_{\alpha}^{\Delta}(x))\Big(g(\beta^{\Delta}(x, u^{\Delta}(t,x))) - \alpha\Big)\phi_x\Big) dx dt
\end{aligned}$$

$$
-\int_{\mathbb{R}^2} |u^{\Delta}(T,x) - k_{\alpha}^{\Delta}(x)|\phi(T,x)dx + \int_{\mathbb{R}^2} |u_0(x) \\
-k_{\alpha}^{\Delta}(x)|\phi(0,x)dx.
$$

Thus, we have,

$$
\begin{aligned}
|\Lambda_{\sqrt{\Delta t},\sqrt{\Delta t}}(u^{\Delta}, v) - \Lambda^{\Delta}_{\sqrt{\Delta t},\sqrt{\Delta t}}(u^{\Delta}, v)| \leq C\Bigg( &\int\limits_{\Pi_T{}^2} |k_\alpha - k_\alpha^{\Delta}| \left|\Phi_t^{\sqrt{\Delta t},\sqrt{\Delta t}} + \Phi_x^{\sqrt{\Delta t},\sqrt{\Delta t}}\right| \\
&+ \int\limits_{\mathbb{R}^4} |k_\alpha - k_\alpha^{\Delta}| \\
&\left|\Phi^{\sqrt{\Delta t},\sqrt{\Delta t}}(\cdot, 0) + \Phi^{\sqrt{\Delta t},\sqrt{\Delta t}}(\cdot, T)\right|\Bigg).
\end{aligned}
$$

Since, $\left\|k - k^{\Delta}\right\|_{L^1(\mathbb{R}^2)} = \mathcal{O}(\Delta t)$, using (31) and (46) in the above inequality, we get

$$
\Lambda_{\sqrt{\Delta t},\sqrt{\Delta t}}(u^{\Delta}, v) = \Lambda^{\Delta}_{\sqrt{\Delta t},\sqrt{\Delta t}}(u^{\Delta}, v) + \mathcal{O}(\sqrt{\Delta t}) = \mathcal{O}(\sqrt{\Delta t}).
$$

This completes the proof of the theorem. □

***Remark 3*** In Theorem 3 we proved that the rate of convergence is not less than 1/2. This result has to be considered as the worst case estimate in the sense that rate cannot be less than 1/2. An example due to Sabac [39] shows that in general this result cannot be improved as the rate $1/2$ is achieved for the example. However, the method in many cases exhibits rates much higher than $1/2$.

***Remark 4*** For general $\beta$, where the flux satisfies conditions (A-1)–(A-4), (42) is replaced by

$$
\mathcal{C} \leq C_\beta \frac{\epsilon}{\eta}.
$$

As a result, we get the following Kuznetsov type estimate:

$$
\begin{aligned}
\|u(\cdot,T) - v(\cdot,T)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})} + C\Big[&L\epsilon + \mathrm{TV}(r)|\eta| + \mathrm{TV}(v)|\eta| \\
&+ \nu(u,\epsilon)\Big] - \Lambda_{\eta,\epsilon}(u,v) + C_\beta \frac{\epsilon}{\eta}.
\end{aligned} \tag{47}
$$

Repeating the arguments of the theorem 3 for any $\epsilon$ and $\eta$ we get

$$
-\Lambda_{\eta,\epsilon}(u,v) \leq CT\Delta x\left(\frac{1}{\epsilon} + \frac{1}{\eta}\right). \tag{48}
$$

Hence, we get the convergence rate $1/3$ by choosing $\epsilon = (\Delta t)^{2/3}$ and $\eta = (\Delta t)^{1/3}$ in (47)–(48).

***Remark 5*** Consider the conservation law in several dimensions given by,

$$u_t + \operatorname{div} A(x,u) = 0 \qquad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^d, \tag{49}$$
$$u(0,x) = u_0(x) \qquad \text{for } x \in \mathbb{R}^d, \tag{50}$$

where the flux $A : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is of Panov type, as in [38], i.e., $A(x,u) = g(\beta(x,u))$, where $g$ can be a locally Lipschitz continuous real-valued function and $\beta(x,\cdot)$ is a monotone function for each $x \in \mathbb{R}^d$ along with the other properties mentioned in Sect. 2. For $d > 1$, we further assume that $\beta(x,u) = u + r(x)$. The notion of adapted entropy condition (Definition 1)can be easily extended to several space dimensions and the uniqueness can be proved as in Theorem 1. Note that when the flux function contains discontinuities in the space variable, numerical approximations $(u^{\Delta})$ do not satisfy TVD property, as a result the usual dimension splitting arguments can not be applied directly to conservation laws with discontinuous flux. However, in the previous section, we observed that our numerical scheme satisfies $\beta-$TVD property. For $\beta(x,u) = au + r(x)$, $L^1$ contractivity of $u$ implies the $L^1$ contractivity of $\beta(\cdot, u(t,\cdot))$. Thus, by repeating the TVD arguments of the homogeneous case on $\beta$, $\beta-$TVD property can be proved in several dimensions as well. Now, the limit of the numerical approximation can be shown to be the adapted entropy solution by a Crandall-Majda type argument [20] which establishes the existence of the adapted entropy solution. Lemma 2 and Theorem 3 remains valid in several space dimensions and hence we get the desired error estimates.

## 4 Numerical simulations

This section present numerical simulations of the Godunov type scheme for various types of flux functions and initial data.

***Example 1*** We consider the IVP (49)–(50) with fluxes as defined below:

$$A_i(x,y,u) := g_i(u + r(x)), \quad \text{for } i = 1, 2.$$

$$g_1(u) = u^2/2, \quad g_2(u) = sin(u) \quad \text{and} \quad r(x) = \begin{cases} p, & x < 1, \\ pq^{n-1}, & x \in C_n, n \in \mathbb{N}, \\ 0, & x > a_\infty, \end{cases} \tag{51}$$

where $p = 4$, $q = 0.8$ and for each $n \in \mathbb{N}$, $C_n = [a_n, a_{n+1}]$, with

$$a_1 = 1 \text{ and } a_n = 1 + \sum_{i=1}^{n-1} \tilde{a}_i \text{ for } n \geq 2$$

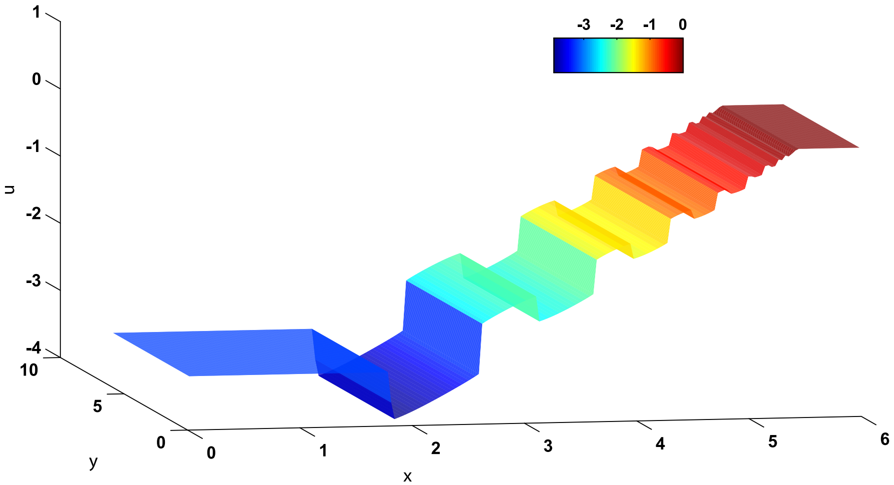


**Fig. 1** Example 1. The solution at $t = 6$ with mesh size $\Delta x = \Delta y = 6/200$. Solution contains infinitely many shocks along the spatial discontinuities of the flux, which accumulates along the plane $x = 5$.

with

$$\tilde{a}_n = \begin{cases} pq^{n-1} - pq^n, & \text{if } n \text{ is odd,} \\ pq^{n-2} - pq^{n-1}, & \text{if } n \text{ is even.} \end{cases}$$

Define and consider a piecewise constant initial data

$$u_0(x, y) = \begin{cases} -pq, & x < a_2, \\ -pq^n, & x \in C_n \text{ and } n \text{ odd,} \\ -pq^{n-2}, & x \in C_n \text{ and } n \text{ even,} \\ 0, & x > a_\infty. \end{cases} \tag{52}$$

At $t = 1$, the solution is given by,

$$u(1, x, y) = \begin{cases} -pq, & x < a_2, \\ x - a_n - pq^{n-1}, & x \in C_n \text{ and } n \text{ odd,} \\ x - a_{n+1} - pq^{n-1}, & x \in C_n \text{ and } n \text{ even,} \\ 0, & x > a_\infty. \end{cases} \tag{53}$$

Numerical experiments are performed on the spatial domain $[0, 6] \times [0, 6]$ with $M = 50, 100, 200$ and $400$ uniformly spaced spatial grid points along the $x$ and $y$ directions. Figure 1 plots the numerical solutions at the final time $t = 1$ for the mesh size $\Delta x = \Delta y = 6/200$. It can be seen that the scheme captures both stationary shocks and rarefactions efficiently.

**Table 1** Approximate $L^1$ error and total variation at $t = 1$ for Example 1

| M | $e_\Delta$ | $\mathrm{TV}(u^\Delta(\cdot, 1))$ | $\mathrm{TV}(\beta(\cdot, u^\Delta(1, \cdot))$ |
|---|---|---|---|
| 50 | 1.3464 | 32.9298 | 33.9876 |
| 100 | 0.9618 | 34.4796 | 35.9166 |
| 200 | 0.6282 | 37.7934 | 40.0374 |
| 400 | 0.4038 | 39.4704 | 41.8296 |

Clearly, the solutions are the extensions of the solutions obtained in the one dimensional case (see Example 4.1, [26]), more precisely $u(1, x, y) = u(1, x)$, for $(x, y) \in [0, 6] \times [0, 6]$. Thus, the values listed in the above table are approximately six times of those obtained in the corresponding 1D simulations (see Table 1, [26]).

***Example 2*** We consider the IVP (49)–(50) with $u_0(x, y) = 2$ and fluxes as defined below:

$$A_i(x, y, u) := g_i(u + r(x)) \quad \text{for } i = 1, 2,$$

where

$$g_1(u) = \begin{cases} -u - 1, & u < -1, \\ 0, & u \in (-1, 0), \\ u, & u > 1, \end{cases} \quad g_2(u) = \sin(u) \quad \text{and}$$

$$r(x) = \begin{cases} 2, & x < 1, \\ r_n \chi_{[a_n, a_{n+1}]}(x), & x \in (1, 5), \\ 1, & x > 5, \end{cases}$$

with

$$a_n = 5(1 - 0.8^n), r_n = 1 - (-0.8)^n.$$

The flux considered here admits infinitely many spatial discontinuities which accumulates along the plane $x = 5$. Solution at $t = 6$ is given by,

$$u(6, x, y) = r(x) \quad \text{for } (x, y) \in [0, 6] \times [0, 6].$$

Numerical experiments are performed on the spatial domain $[0, 6] \times [0, 6]$ with $M = 50, 100, 200$ and $400$ uniformly spaced spatial grid points along the $x$ and $y$ directions. Figure 2 plots the numerical solutions at the final time $t = 6$ for the mesh size $\Delta x = \Delta y = 6/200$. It can be seen that the scheme captures both stationary shocks efficiently.

As in the previous example, the values listed in the above table are approximately six times of those obtained in the corresponding 1D simulation (see Table 2, [26]).

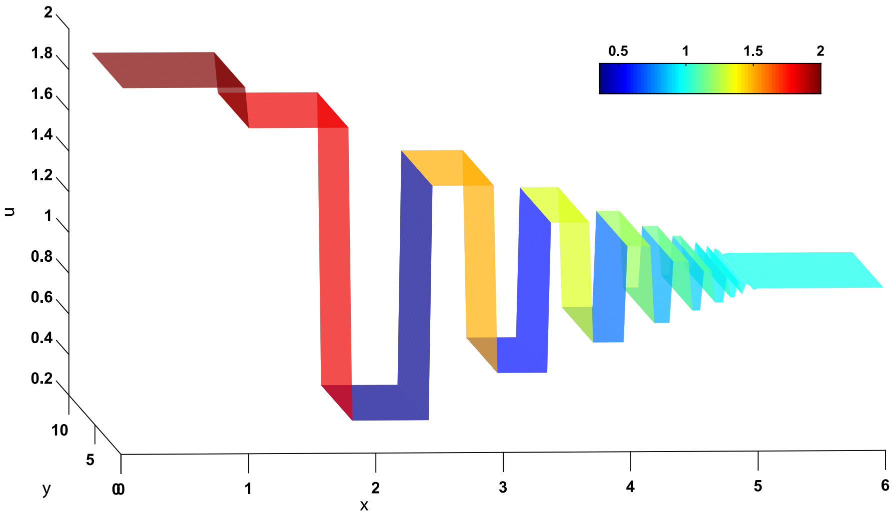


**Fig. 2** Example 2. The solution at $t = 6$ with mesh size $\Delta x = \Delta y = 6/200$. Solution contains infinitely many shocks along the spatial discontinuities of the flux, which accumulate along the plane $x = 5$

**Table 2** Approximate $L^1$ error and total variation at $t = 6$ for Example 2

| M | $e_\Delta$ | $\mathrm{TV}(u^\Delta(6,\cdot))$ | $\mathrm{TV}(\beta(\cdot, u^\Delta(6,\cdot))$ |
|---|---|---|---|
| 50 | 2.7933e-02 | 40.701 | 8.7198e-02 |
| 100 | 2.559e-03 | 41.914 | 1.3788e-02 |
| 200 | 1.1147e-04 | 43.4088 | 1.07436e-03 |
| 400 | 3.5146e-07 | 43.6824 | 6.3834e-06 |

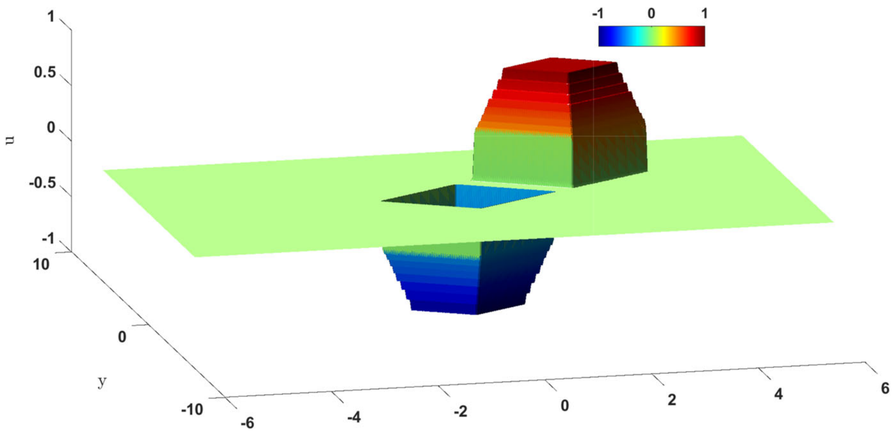


**Fig. 3** Example 3. The solution at $t = 2$ with mesh size $\Delta x = \Delta y = 12/200$

**Table 3** Approximate $L^1$ error and total variation at $t = 2$ for Example 3

| M | $e_\Delta$ | $\mathrm{TV}(u^\Delta(2,\cdot))$ | $\mathrm{TV}(\beta(\cdot,u^\Delta(2,\cdot))$ |
|---|---|---|---|
| 100 | 0.5175 | 22.0051 | 10.8616 |
| 200 | 0.4424 | 22.7040 | 11.2148 |
| 400 | 0.2732 | 22.9291 | 11.3466 |
| 800 | 0.1790 | 23.1841 | 11.4962 |

***Example 3*** In this example, we consider the IVP (49)–(50) with constant initial data $u_0 = 0$ and the flux function $A_1(x, y, u) = A_2(x, y, u) = u + r(|x| + |y|)$, where $r$ is as defined in Example 1 with $p = 1$ and $q = 0.9$. The solution of the IVP at time $t$ is given by $u(t, x, y) = r(|x - t| + |y - t|) - r(|x| + |y|)$.

Numerical experiments are performed on the spatial domain $[-6, 6] \times [-6, 6]$ with $M = 100, 200, 400$ and 800 uniformly spaced spatial grid points along the $x$ and $y$ directions. Figure 3 plots the numerical solution at finial time $t = 2$ for mesh size $\Delta x = \Delta y = 12/200$. Table 3 compares the $L^1$ error and the total variation of $u^\Delta$ and $\beta(u^\Delta)$ for various mesh sizes.

**Acknowledgements** First and last authors, would like to thank Department of Atomic Energy, Government of India, under Project No. 12-R&D-TFR-5.01-0520. First author would also like to acknowledge Inspire faculty-research grant DST/INSPIRE/04/2016/000237. We thank the anonymous referees for their careful reading of the paper and constructive inputs which improved the scope of the study.

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