



Resolution of the implicit Euler scheme for the Navier–Stokes equation through a least-squares method

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Abstract

This work analyzes a least-squares method in order to solve implicit time schemes associated to the 2D and 3D Navier–Stokes system, introduced in 1979 by Bristeau, Glowinski, Periaux, Perrier and Pironneau. Implicit time schemes reduce the numerical resolution of the Navier–Stokes system to multiple resolutions of steady Navier–Stokes equations. We first construct a minimizing sequence (by a gradient type method) for the least-squares functional which converges strongly and quadratically toward a solution of a steady Navier–Stokes equation from any initial guess. The method turns out to be related to the globally convergent damped Newton approach applied to the Navier–Stokes operator. Then, we apply iteratively the analysis on the fully implicit Euler scheme and show the convergence of the method uniformly with respect to the time discretization. Numerical experiments for 2D examples support our analysis.

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1 Introduction–motivation

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$ be a bounded connected open set whose boundary $\partial\Omega$ is Lipschitz and $T > 0$. We endow $H_0^1(\Omega)$ with the scalar product $\langle v, w \rangle_{H_0^1(\Omega)} = \int_{\Omega} \nabla v \cdot \nabla w$ and the associated norm and we endow the dual $H^{-1}(\Omega)$ of $H_0^1(\Omega)$ with the dual norm of $H_0^1(\Omega)$. We denote by $\mathcal{V} = \{v \in \mathcal{D}(\Omega)^d, \nabla \cdot v = 0\}$, \mathbf{H} the

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closure of \mathcal{V} in $L^2(\Omega)^d$ and V the closure of \mathcal{V} in $H_0^1(\Omega)^d$ endowed with the norm of $H_0^1(\Omega)^d$.

The Navier–Stokes system describes a viscous incompressible fluid flow in the bounded domain Ω during the time interval $(0, T)$ submitted to the external force F . It reads as follows:

$$\begin{cases} u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p = F, & \nabla \cdot u = 0 \quad \text{in } \Omega \times (0, T), \\ u = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0 \quad \text{in } \Omega, \end{cases} \tag{1}$$

where u is the velocity of the fluid, p its pressure and ν is the viscosity constant assumed smaller than one. We refer to [19]. This work is concerned with the approximation of (1) through the time marching fully implicit Euler scheme

$$\begin{cases} y^0 = u_0 \quad \text{in } \Omega, \\ \frac{y^{n+1} - y^n}{\delta t} - \nu \Delta y^{n+1} + (y^{n+1} \cdot \nabla)y^{n+1} + \nabla \pi^{n+1} = \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} F(\cdot, s) ds, \quad n \geq 0, \\ \nabla \cdot y^{n+1} = 0 \quad \text{in } \Omega, \quad n \geq 0, \\ y^{n+1} = 0 \quad \text{on } \partial\Omega, \quad n \geq 0, \end{cases} \tag{2}$$

where $\{t_n\}_{n=0\dots N}$, for a given $N \in \mathbb{N}$, is a uniform discretization of the time interval $(0, T)$. $\delta t = T/N$ is the time discretization step. This also-called backward Euler scheme is studied for instance in [19, chapter 3, section 4]. It is proved there that the piecewise linear interpolation (in time) of $\{y^n\}_{n \in [0, N]}$ weakly converges in $L^2(0, T; V)$ toward a solution u of (1) as δt goes to zero. It achieves a first order convergence with respect to δt . We also refer to [20] for a stability analysis of the scheme in long time. We refer to [17] for Crank-Nicolson schemes achieving second order convergence.

The determination of y^{n+1} from y^n requires the resolution of a nonlinear partial differential equation. Precisely y^{n+1} together with the pressure π^{n+1} solve the following problem: find $y \in V$ and $\pi \in L_0^2(\Omega)$, solution of

$$\begin{cases} \alpha y - \nu \Delta y + (y \cdot \nabla)y + \nabla \pi = f + \alpha g, & \nabla \cdot y = 0 \quad \text{in } \Omega, \\ y = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3}$$

with

$$\alpha := \frac{1}{\delta t} > 0, \quad f := \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} F(\cdot, s) ds, \quad g = y^n. \tag{4}$$

Recall that for any $f \in H^{-1}(\Omega)^d$ and $g \in L^2(\Omega)^d$, there exists one solution $(y, \pi) \in V \times L_0^2(\Omega)$ of (3), unique if $\|g\|_{L^2(\Omega)^d}^2 + \alpha^{-1} \nu^{-1} \|f\|_{H^{-1}(\Omega)^d}^2$ is small enough (see

Proposition 1 for a precise statement). $L^2_0(\Omega)$ stands for the space of functions in $L^2(\Omega)^d$ with zero means.

A weak solution $y \in \mathbf{V}$ of (3) solves the formulation $F(y, w) = 0$ for all $w \in \mathbf{V}$ where F is defined by

$$\begin{aligned}
 F(y, z) &:= \int_{\Omega} \alpha y \cdot z + \nu \nabla y \cdot \nabla z + (y \cdot \nabla) y \cdot z \\
 &\quad - \langle f, z \rangle_{H^{-1}(\Omega)^d \times H^1_0(\Omega)^d} - \alpha \int_{\Omega} g \cdot z = 0, \quad \forall z \in \mathbf{V}.
 \end{aligned}
 \tag{5}$$

If $D_y F$ is invertible, one may approximate a weak solution through the iterative Newton method: a sequence $\{y_k\}_{k \in \mathbb{N}} \in \mathbf{V}$ is constructed as follows

$$\begin{cases}
 y_0 \in \mathbf{V}, \\
 D_y F(y_k, w) \cdot (y_{k+1} - y_k) = -F(y_k, w), \quad \forall w \in \mathbf{V}, k \geq 0.
 \end{cases}
 \tag{6}$$

If the initial guess y_0 is close enough to a weak solution of (3), i.e. a solution satisfying $F(y, w) = 0$ for all w , then the sequence $\{y_k\}_{k \in \mathbb{N}}$ converges. We refer to [15, Section 10.3], [4, Chapter 6]) and for some numerical aspects to [10].

Alternatively, we may also employ least-squares methods which consist in minimizing a quadratic functional, which measures how an element y is close to the solution. For instance, we may introduce the extremal problem : $\inf_{y \in \mathbf{V}} E(y)$ with $E : \mathbf{V} \rightarrow \mathbb{R}^+$ defined by

$$E(y) := \frac{1}{2} \int_{\Omega} \alpha |v|^2 + \nu |\nabla v|^2
 \tag{7}$$

where the corrector v , together with the pressure, is the unique solution in $\mathbf{V} \times L^2_0(\Omega)$ of the linear boundary value problem:

$$\begin{cases}
 \alpha v - \nu \Delta v + \nabla \pi + (\alpha y - \nu \Delta y + (y \cdot \nabla) y - f - \alpha g) = 0, \quad \nabla \cdot v = 0 \quad \text{in } \Omega, \\
 v = 0 \quad \text{on } \partial \Omega.
 \end{cases}
 \tag{8}$$

$E(y)$ vanishes if and only if $y \in \mathbf{V}$ is a weak solution of (3), equivalently a zero of $F(y, w) = 0$ for all $w \in \mathbf{V}$. As a matter of fact, the infimum is reached. Least-squares methods to solve nonlinear boundary value problems have been the subject of intensive developments in the last decades, as they present several advantages, notably on computational and stability viewpoints. We refer to [1], [7]. The minimization of the functional E over \mathbf{V} leads to a so-called weak least squares method. Precisely, the equality $\sqrt{2E(y)} = \sup_{w \in \mathbf{V}, w \neq 0} \frac{F(y, w)}{\|w\|_{\mathbf{V}}}$ —where $\|w\|_{\mathbf{V}}$ is defined in (9)—shows that E is equivalent to the \mathbf{V}' norm of the Navier–Stokes equation (see Remark 1). The terminology “ H^{-1} -least-squares method” is employed in [2] where the minimization of E has been introduced and numerically implemented to approximate solutions of

(1) through the scheme (2). We also mention [4, Chapter 4, Section 6] which studied the use of a least-squares strategy to solve a steady Navier–Stokes equation without incompressibility constraint.

The first objective of the present work is to analyze rigorously the method introduced in [2] and show that one may construct minimizing sequences in V for E that converge strongly toward a solution of (3). The second objective is to justify the use of that least-squares method to solve iteratively a weak formulation of the scheme (2), leading to an approximation of the solution of (1). This requires to show some convergence properties of a minimizing sequence for E , uniformly with respect to the parameter n related to the time discretization. As we shall see, this requires smallness assumptions on the data u_0 and F .

The rest of the paper is organized as follows. Section 2 is devoted to the analyze the least-squares method (7)–(8) associated to weak solutions of (3). We show that E is differentiable over V and that any critical point for E in the ball $\mathbb{B} := \{y \in V, \tau_d(y) < 1\}$ (see Definition 1) is also a zero of E . This is done by introducing a descent direction Y_1 for E at any point $y \in V$ for which $E'(y) \cdot Y_1$ is proportional to $E(y)$. Then, assuming that there exists at least one solution of (3) in \mathbb{B} , we show that any minimizing sequence $\{y_k\}_{k \in \mathbb{N}}$ for E in \mathbb{B} strongly converges to a solution of (14). Such limit belongs to \mathbb{B} and is actually the unique solution. Eventually, we construct a minimizing sequence (defined in (30)) based on the element Y_1 and initialized with g assumed in V . If α is large enough, we show that this particular sequence belongs to \mathbb{B} and converges (quadratically after a finite number of iterates related to the values of ν and α) strongly to the solution of (3) (see Theorem 1). This specific sequence coincides with the one obtained from the damped Newton method, a globally convergent generalization of (6). Then, in Sect. 3, as an application, we consider the least-squares approach to solve iteratively the backward Euler scheme (see (51)), weak formulation of (2). For each $n > 0$, in order to approximate y^{n+1} , we define a minimizing sequence $\{y_k^{n+1}\}_{k \geq 0}$ based on Y_1^{n+1} and initialized with y^n . Adapting the global convergence result of Sect. 2, we then show, assuming $\|u_0\|_{L^2(\Omega)^d} + \|F\|_{L^2(0,T;H^{-1}(\Omega)^d)}$ small enough, the strong convergence of the minimizing sequences, uniformly with respect to the time discretization parameter n (see Theorem 4). The analysis is performed for $d = 2$ for both weak and regular solutions and for $d = 3$ for regular solutions. Our analysis justifies the use of Newton type methods to solve implicit time schemes for (1), as mentioned in [15, Section 10.3]. To the best of our knowledge, such analysis of convergence is original. In Sect. 4, we discuss numerical experiments based on finite element approximations in space for two 2D geometries: the celebrated example of the channel with a backward facing step and the semi-circular driven cavity introduced in [5]. We notably exhibit for small values of the viscosity constant the robustness of the damped Newton method (compared to the Newton one).

2 Analysis of a Least-squares method for a steady Navier–Stokes equation

We analyse in this section a least-squares method to solve the steady Navier–Stokes Eq. (3) assuming $\alpha > 0$: we extend [11] where the particular case $\alpha = 0$ is addressed.

2.1 Technical preliminary results

In the sequel $\|\cdot\|_2$ stands for the norm $\|\cdot\|_{L^2(\Omega)^d}$. We shall also use the following notations

$$\|y\|_V^2 := \alpha \|y\|_2^2 + \nu \|\nabla y\|_2^2, \quad \forall y \in V \tag{9}$$

and $\langle y, z \rangle_V := \alpha \int_{\Omega} y \cdot z + \nu \int_{\Omega} \nabla y \cdot \nabla z$ so that $\langle y, z \rangle_V \leq \|y\|_V \|z\|_V$ for any $y, z \in V$.

In the sequel, we repeatedly use the following classical estimates (see [19]).

Lemma 1 *Let any $u, v \in V$. If $d = 2$, then*

$$-\int_{\Omega} u \cdot \nabla u \cdot v = \int_{\Omega} u \cdot \nabla v \cdot u \leq \sqrt{2} \|u\|_2 \|\nabla v\|_2 \|\nabla u\|_2. \tag{10}$$

If $d = 3$, then there exists a constant $c = c(\Omega)$ such that

$$\int_{\Omega} u \cdot \nabla v \cdot u \leq c \|u\|_2^{1/2} \|\nabla v\|_2 \|\nabla u\|_2^{3/2}. \tag{11}$$

Definition 1 For any $y \in V, \alpha > 0$ and $\nu > 0$, we define

$$\tau_d(y) := \begin{cases} \frac{\|y\|_{H_0^1(\Omega)^2}}{\sqrt{2\alpha\nu}}, & \text{if } d = 2, \\ \frac{M \|y\|_{H_0^1(\Omega)^3}}{(\alpha\nu^3)^{1/4}}, & \text{if } d = 3 \end{cases}$$

with $M := \frac{3^{3/4}}{4}c$ and c from (11).

We shall also repeatedly use the following Young type inequalities.

Lemma 2 *For any $u, v \in V$, the following inequalities hold true:*

$$\sqrt{2} \|u\|_2 \|\nabla v\|_2 \|\nabla u\|_2 \leq \tau_2(v) \|u\|_V^2 \tag{12}$$

if $d = 2$ and

$$c \|u\|_2^{1/2} \|\nabla v\|_2 \|\nabla u\|_2^{3/2} \leq \tau_3(v) \|u\|_V^2 \tag{13}$$

if $d = 3$.

Let $f \in H^{-1}(\Omega)^d, g \in L^2(\Omega)^d$ and $\alpha \in \mathbb{R}_+^*$. The weak formulation of (3) reads as follows: find $y \in V$ solution of

$$\int_{\Omega} \alpha y \cdot w + \nu \nabla y \cdot \nabla w + y \cdot \nabla y \cdot w$$

$$= \langle f, w \rangle_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} + \alpha \int_{\Omega} g \cdot w, \quad \forall w \in V. \tag{14}$$

The following result holds true (we refer to [13]).

Proposition 1 *Assume $\Omega \subset \mathbb{R}^d$ is bounded and Lipschitz. There exists at least one solution y of (14) satisfying*

$$\|y\|_V^2 \leq \frac{1}{\nu} \|f\|_{H^{-1}(\Omega)^d}^2 + \alpha \|g\|_2^2. \tag{15}$$

If moreover Ω is C^2 and $f \in L^2(\Omega)^d$, then any solution $y \in V$ of (14) belongs to $H^2(\Omega)^d$.

Lemma 3 *Assume that a solution $y \in V$ of (14) satisfies $\tau_d(y) < 1$. Then, such solution is the unique solution of (14).*

Proof Let $y_1 \in V$ and $y_2 \in V$ be two solutions of (14). Set $Y = y_1 - y_2$. Then,

$$\alpha \int_{\Omega} Y \cdot w + \nu \int_{\Omega} \nabla Y \cdot \nabla w + \int_{\Omega} y_2 \cdot \nabla Y \cdot w + \int_{\Omega} Y \cdot \nabla y_1 \cdot w = 0 \quad \forall w \in V.$$

We now take $w = Y$ and use that $\int_{\Omega} y_2 \cdot \nabla Y \cdot Y = 0$. If $d = 2$, we use (10) and (12) to get

$$\|Y\|_V^2 = - \int_{\Omega} Y \cdot \nabla y_1 \cdot Y \leq \tau_2(y_1) \|Y\|_V^2$$

leading to $(1 - \tau_2(y_1)) \|Y\|_V^2 \leq 0$. Consequently, if $\tau_2(y_1) < 1$ then $Y = 0$ and the solution of (14) is unique. In view of (15), this holds if the data satisfy $\nu \|g\|_2^2 + \frac{1}{\alpha} \|f\|_{H^{-1}(\Omega)^d}^2 < 2\nu^2$.

If $d = 3$, we use (11) and (13) to obtain

$$\|Y\|_V^2 = - \int_{\Omega} Y \cdot \nabla y_1 \cdot Y \leq c \|Y\|_2^{\frac{1}{2}} \|\nabla Y\|_2^{\frac{3}{2}} \|\nabla y_1\|_2 \leq \tau_3(y_1) \|Y\|_V^2$$

leading to $(1 - \tau_3(y_1)) \|Y\|_2^2 \leq 0$ and to the uniqueness if $\tau_3(y_1) < 1$. In view of (15), this holds if the data satisfy $\nu \|g\|_2^2 + \frac{1}{\alpha} \|f\|_{H^{-1}(\Omega)^d}^2 < M^{-2} \nu^{7/2} \alpha^{-1/2}$. □

We now introduce our least-squares functional $E : V \rightarrow \mathbb{R}^+$ as follows

$$E(y) := \frac{1}{2} \int_{\Omega} (\alpha |v|^2 + \nu |\nabla v|^2) = \frac{1}{2} \|v\|_V^2 \tag{16}$$

where the corrector $v \in V$ is the unique solution of the linear formulation

$$\begin{aligned} &\alpha \int_{\Omega} v \cdot w + \nu \int_{\Omega} \nabla v \cdot \nabla w \\ &= -\alpha \int_{\Omega} y \cdot w - \nu \int_{\Omega} \nabla y \cdot \nabla w - \int_{\Omega} y \cdot \nabla y \cdot w \\ &\quad + \langle f, w \rangle_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} + \alpha \int_{\Omega} g \cdot w, \quad \forall w \in V. \end{aligned} \tag{17}$$

In particular, for $d = 2$, the corrector v satisfies the estimate:

$$\|v\|_V \leq \|y\|_V \left(1 + \frac{\|y\|_V}{2\sqrt{\alpha\nu}} \right) + \sqrt{\frac{\|f\|_{H^{-1}(\Omega)^2}^2}{\nu} + \alpha \|g\|_2^2}. \tag{18}$$

Conversely, we also have

$$\|y\|_V \leq \|v\|_V + \sqrt{\frac{\|f\|_{H^{-1}(\Omega)^2}^2}{\nu} + \alpha \|g\|_2^2}. \tag{19}$$

The infimum of E is equal to zero and is reached by a solution of (14). In this sense, the functional E is a so-called error functional which measures, through the corrector variable v , the deviation of the pair y from being a solution of the underlying Eq. (14).

A practical way of taking a functional to its minimum is through some (clever) use of descent directions, i.e. the use of its derivative. In doing so, the presence of local minima is something that may dramatically spoil the whole scheme. The unique structural property that discards this possibility is the strict convexity of the functional. However, for non-linear equations like (14), one cannot expect this property to hold for the functional E in (16). Nevertheless, we insist in that for a descent strategy applied to the extremal problem $\min_{y \in V} E(y)$ numerical procedures cannot converge except to a global minimizer leading E down to zero.

Indeed, we would like to show that the critical points for E correspond to solutions of (14). In such a case, the search for an element y solution of (14) is reduced to the minimization of E .

For any $y \in V$, we now look for an element $Y_1 \in V$ solution of the following formulation

$$\begin{aligned} &\alpha \int_{\Omega} Y_1 \cdot w + \nu \int_{\Omega} \nabla Y_1 \cdot \nabla w + \int_{\Omega} (y \cdot \nabla Y_1 + Y_1 \cdot \nabla y) \cdot w \\ &= -\alpha \int_{\Omega} v \cdot w - \nu \int_{\Omega} \nabla v \cdot \nabla w, \end{aligned} \tag{20}$$

for all $w \in V$ where $v \in V$ is the corrector (associated to y) solution of (17). Y_1 enjoys the following property.

Proposition 2 For all $y \in \mathbf{V}$ satisfying $\tau_d(y) < 1$, there exists a unique solution Y_1 of (20) associated to y . Moreover, this solution satisfies

$$(1 - \tau_d(y)) \|Y_1\|_{\mathbf{V}} \leq \sqrt{2E(y)}. \tag{21}$$

Proof We define the bilinear and continuous form $a : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$ by

$$a(Y, w) = \alpha \int_{\Omega} Y \cdot w + \nu \int_{\Omega} \nabla Y \cdot \nabla w + \int_{\Omega} (y \cdot \nabla Y + Y \cdot \nabla y) \cdot w$$

so that $a(Y, Y) = \|Y\|_{\mathbf{V}}^2 + \int_{\Omega} Y \cdot \nabla y \cdot Y$. If $d = 2$, using (12), we obtain $a(Y, Y) \geq (1 - \tau_2(y)) \|Y\|_{\mathbf{V}}^2$ for all $Y \in \mathbf{V}$. Lax-Milgram lemma leads to the existence and uniqueness of Y_1 assuming that $\tau_2(y) < 1$. Then, putting $w = Y_1$ in (20) implies

$$a(Y_1, Y_1) \leq -\alpha \int_{\Omega} v \cdot Y_1 - \nu \int_{\Omega} \nabla v \cdot \nabla Y_1 \leq \|Y_1\|_{\mathbf{V}} \|v\|_{\mathbf{V}} = \|Y_1\|_{\mathbf{V}} \sqrt{2E(y)}$$

leading to (21). If $d = 3$, using (13), we obtain $a(Y, Y) \geq (1 - \tau_3(y)) \|Y\|_{\mathbf{V}}^2$ for all $Y \in \mathbf{V}$ and we conclude as before. \square

We now check the differentiability of the least-squares functional.

Proposition 3 For all $y \in \mathbf{V}$, the map $Y \mapsto E(y + Y)$ is a differentiable function on the Hilbert space \mathbf{V} and for any $Y \in \mathbf{V}$, we have

$$E'(y) \cdot Y = \int_{\Omega} \alpha v \cdot V + \nu \nabla v \cdot \nabla V \tag{22}$$

where $V \in \mathbf{V}$ is the unique solution of

$$\begin{aligned} \int_{\Omega} \alpha V \cdot w + \nu \nabla V \cdot \nabla w &= -\alpha \int_{\Omega} Y \cdot w - \nu \int_{\Omega} \nabla Y \cdot \nabla w \\ &\quad - \int_{\Omega} (y \cdot \nabla Y + Y \cdot \nabla y) \cdot w, \forall w \in \mathbf{V}. \end{aligned} \tag{23}$$

Proof Let $y \in \mathbf{V}$ and $Y \in \mathbf{V}$. We have $E(y + Y) = \frac{1}{2} \| \overline{V} \|_{\mathbf{V}}^2$ where $\overline{V} \in \mathbf{V}$ is the unique solution of

$$\begin{aligned} \alpha \int_{\Omega} \overline{V} \cdot w + \nu \int_{\Omega} \nabla \overline{V} \cdot \nabla w + \alpha \int_{\Omega} (y + Y) \cdot w + \nu \int_{\Omega} \nabla (y + Y) \cdot \nabla w \\ + \int_{\Omega} (y + Y) \cdot \nabla (y + Y) \cdot w - \langle f, w \rangle_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} \\ - \alpha \int_{\Omega} g \cdot w = 0, \forall w \in \mathbf{V}. \end{aligned}$$

If $v \in V$ is the solution of (17) associated to y , $v' \in V$ is the unique solution of

$$\alpha \int_{\Omega} v' \cdot w + v \int_{\Omega} \nabla v' \cdot \nabla w + \int_{\Omega} Y \cdot \nabla Y \cdot w = 0, \quad \forall w \in V \tag{24}$$

and $V \in V$ is the unique solution of (23), then it is straightforward to check that $\bar{V} - v - v' - V \in V$ is solution of

$$\alpha \int_{\Omega} (\bar{V} - v - v' - V) \cdot w + v \int_{\Omega} \nabla(\bar{V} - v - v' - V) \cdot \nabla w = 0, \quad \forall w \in V$$

and therefore $\bar{V} - v - v' - V = 0$. Thus

$$\begin{aligned} E(y + Y) &= \frac{1}{2} \|\|v + v' + V\|\|_{\mathbf{V}}^2 \\ &= \frac{1}{2} \|v\|_{\mathbf{V}}^2 + \frac{1}{2} \|v'\|_{\mathbf{V}}^2 + \frac{1}{2} \|V\|_{\mathbf{V}}^2 + \langle V, v' \rangle_{\mathbf{V}} + \langle V, v \rangle_{\mathbf{V}} + \langle v, v' \rangle_{\mathbf{V}}. \end{aligned} \tag{25}$$

Assume $d = 2$. Then, writing (23) with $w = V$ and using (10), we obtain

$$\begin{aligned} \|\|V\|\|_{\mathbf{V}}^2 &\leq \|V\|_{\mathbf{V}} \|Y\|_{\mathbf{V}} + \sqrt{2}(\|y\|_2 \|\nabla Y\|_2 + \|Y\|_2 \|\nabla y\|_2) \|\nabla V\|_2 \\ &\leq \|V\|_{\mathbf{V}} \|Y\|_{\mathbf{V}} + \frac{\sqrt{2}}{\sqrt{\alpha v}} \|y\|_{\mathbf{V}} \|Y\|_{\mathbf{V}} \|\nabla V\|_2 \end{aligned}$$

leading to $\|\|V\|\|_{\mathbf{V}} \leq \|Y\|_{\mathbf{V}} (1 + \frac{\sqrt{2}}{\sqrt{\alpha v}} \|y\|_{\mathbf{V}})$. Similarly, using (24), we obtain $\|\|v'\|\|_{\mathbf{V}} \leq \frac{1}{\sqrt{2\alpha v}} \|Y\|_{\mathbf{V}}^2$. It follows that $\frac{1}{2} \|\|v'\|\|_{\mathbf{V}}^2 + \frac{1}{2} \|V\|_{\mathbf{V}}^2 + \langle V, v' \rangle_{\mathbf{V}} + \langle v, v' \rangle_{\mathbf{V}} = o(\|Y\|_{\mathbf{V}})$ and from (25) that

$$E(y + Y) = E(y) + \langle v, V \rangle_{\mathbf{V}} + o(\|Y\|_{\mathbf{V}}).$$

Eventually, the estimate $|\langle v, V \rangle_{\mathbf{V}}| \leq \|v\|_{\mathbf{V}} \|V\|_{\mathbf{V}} \leq \sqrt{2}(1 + \frac{\sqrt{2}}{\sqrt{\alpha v}} \|y\|_{\mathbf{V}}) \sqrt{E(y)} \|Y\|_{\mathbf{V}}$ gives the continuity of the linear map $Y \mapsto \langle v, V \rangle_{\mathbf{V}}$. The case $d = 3$ is similar. \square

We are now in position to prove the following result which indicates that, in the ball \mathbb{B} , any critical point for E is also a zero of E .

Proposition 4 For all $y \in V$ satisfying $\tau_d(y) < 1$,

$$(1 - \tau_d(y))\sqrt{2E(y)} \leq \frac{1}{\sqrt{v}} \|E'(y)\|_{H^{-1}(\Omega)^d}.$$

Proof For any $Y \in V$, $E'(y) \cdot Y = \int_{\Omega} \alpha v \cdot V + v \nabla v \cdot \nabla V$ where $V \in V$ is the unique solution of (23). In particular, taking $Y = Y_1$ defined by (20), we obtain an element $V_1 \in V$ solution of

$$\int_{\Omega} \alpha V_1 \cdot w + v \nabla V_1 \cdot \nabla w = -\alpha \int_{\Omega} Y_1 \cdot w - v \int_{\Omega} \nabla Y_1 \cdot \nabla w$$

$$-\int_{\Omega} (y \cdot \nabla Y_1 + Y_1 \cdot \nabla y) \cdot w, \tag{26}$$

for all $w \in V$. Summing (20) and (26), we obtain that $v - V_1 \in V$ solves

$$\alpha \int_{\Omega} (v - V_1) \cdot w + v \int_{\Omega} (\nabla v - \nabla V_1) \cdot w = 0, \quad \forall w \in V.$$

This implies that v and V_1 coincide and then that

$$E'(y) \cdot Y_1 = \int_{\Omega} \alpha |v|^2 + v |\nabla v|^2 = 2E(y), \quad \forall y \in V. \tag{27}$$

It follows that

$$2E(y) = E'(y) \cdot Y_1 \leq \|E'(y)\|_{H^{-1}(\Omega)^d} \|Y_1\|_{H_0^1(\Omega)^d} \leq \|E'(y)\|_{H^{-1}(\Omega)^d} \frac{\|Y_1\|_V}{\sqrt{v}}.$$

Proposition 2 allows to conclude. □

Eventually, we prove the following coercivity type inequality for the functional E .

Proposition 5 *Assume that a solution $\bar{y} \in V$ of (14) satisfies $\tau_d(\bar{y}) < 1$. Then, for all $y \in V$,*

$$\|y - \bar{y}\|_V \leq (1 - \tau_d(\bar{y}))^{-1} \sqrt{2E(y)}. \tag{28}$$

Proof For any $y \in V$, let v be the corresponding corrector and let $Y = y - \bar{y}$. We have

$$\begin{aligned} &\alpha \int_{\Omega} Y \cdot w + v \int_{\Omega} \nabla Y \cdot \nabla w + \int_{\Omega} y \cdot \nabla Y \cdot w + \int_{\Omega} Y \cdot \nabla \bar{y} \cdot w \\ &= -\alpha \int_{\Omega} v \cdot w - v \int_{\Omega} \nabla v \cdot \nabla w \end{aligned} \tag{29}$$

for all $w \in V$. For $w = Y$, this equality rewrites

$$\|Y\|_V^2 = -\int_{\Omega} Y \cdot \nabla \bar{y} \cdot Y - \alpha \int_{\Omega} v \cdot Y - v \int_{\Omega} \nabla v \cdot \nabla Y.$$

Repeating the arguments of the Proof of Proposition 2, the result follows. □

Assuming the existence of a solution of (14) in the ball $\mathbb{B} = \{y \in V, \tau_d(y) < 1\}$, Propositions 4 and 5 imply that any minimizing sequence $\{y_k\}_{k \in \mathbb{N}}$ for E in \mathbb{B} strongly converges to a solution of (14). Remark that, from Lemma 3, such solution is unique. In the next section, assuming the parameter α large enough, we construct such sequence $\{y_k\}_{k \in \mathbb{N}}$.

Remark 1 In order to simplify notations, we have introduced the corrector variable v leading to the functional E . Instead, we may consider the functional $\tilde{E} : V \rightarrow \mathbb{R}$ defined by

$$\tilde{E}(y) := \frac{1}{2} \|\alpha y + v B_1(y) + B(y, y) - f - \alpha g\|_V^2,$$

with $B_1 : V \rightarrow L^2(\Omega)^d$ and $B : V \times V \rightarrow L^2(\Omega)^d$ defined by $(B_1(y), w) := (\nabla y, \nabla w)_2$ and $(B(y, z), w) := \int_{\Omega} y \cdot \nabla z \cdot w$ respectively. E and \tilde{E} are equivalent. Precisely, from the definition of v (see (17)), we deduce that

$$E(y) = \frac{1}{2} \|v\|_V^2 \leq \frac{1}{2v} \|\alpha y + v B_1(y) + B(y, y) - f - \alpha g\|_V^2 = \frac{1}{v} \tilde{E}(y), \quad \forall y \in V.$$

Conversely,

$$\begin{aligned} \|\alpha y + v B_1(y) + B(y, y) - f - \alpha g\|_V &= \sup_{w \in V, w \neq 0} \frac{\int_{\Omega} (\alpha v \cdot w + v \nabla v \cdot \nabla w)}{\|w\|_{H_0^1(\Omega)^d}} \\ &\leq \|v\|_V \sup_{w \in V, w \neq 0} \frac{\|w\|_V}{\|w\|_{H_0^1(\Omega)^d}} \leq \sqrt{\alpha + v} \|v\|_V \end{aligned}$$

so that $\tilde{E}(y) \leq (\alpha + v)E(y)$ for all $y \in V$.

2.2 A strongly convergent minimizing sequence for E

We define in this section a sequence converging strongly to a solution of (14) for which E vanishes. According to Proposition 4, it suffices to define a minimizing sequence for E included in the ball \mathbb{B} . In this respect, the equality (27) shows that $-Y_1$ (see (20)) is a descent direction for the functional E . Therefore, we can define at least formally, for any $m \geq 1$, the minimizing sequence $\{y_k\}_{(k \geq 0)}$:

$$\begin{cases} y_0 \in V \text{ given,} \\ y_{k+1} = y_k - \lambda_k Y_{1,k}, \quad k \geq 0, \\ \lambda_k = \operatorname{argmin}_{\lambda \in [0, m]} E(y_k - \lambda Y_{1,k}) \end{cases} \tag{30}$$

with $Y_{1,k} \in V$ the solution of the formulation

$$\begin{aligned} \alpha \int_{\Omega} Y_{1,k} \cdot w + v \int_{\Omega} \nabla Y_{1,k} \cdot \nabla w + \int_{\Omega} (y_k \cdot \nabla Y_{1,k} + Y_{1,k} \cdot \nabla y_k) \cdot w \\ = -\alpha \int_{\Omega} v_k \cdot w - v \int_{\Omega} \nabla v_k \cdot \nabla w, \quad \forall w \in V \end{aligned} \tag{31}$$

and $v_k \in V$ the corrector (associated to y_k) solution of (17). The algorithm (30) can be expanded as follows:

$$\left\{ \begin{array}{l} y_0 \in V \text{ given,} \\ \int_{\Omega} \alpha y_{k+1} \cdot w + \nu \nabla y_{k+1} \cdot \nabla w + y_k \cdot \nabla y_{k+1} \cdot w + y_{k+1} \cdot \nabla y_k \cdot w \\ = (1 - \lambda_k) \int_{\Omega} \alpha y_k \cdot w + \nu \nabla y_k \cdot \nabla w + y_k \cdot \nabla y_k \cdot w \\ + \lambda_k \int_{\Omega} \alpha g \cdot w + \langle f, w \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \int_{\Omega} y_k \cdot \nabla y_k \cdot w \quad k \geq 0, \quad \forall w \in V. \end{array} \right. \tag{32}$$

From (19), the sequence $\{y_k\}_{k>0}$ is bounded. However, we insist that, in order to justify the existence of the element $Y_{1,k}$, y_k should satisfy $\tau_d(y_k) < 1$ for all k . We proceed in two steps: assuming that the sequence $\{y_k\}_{(k>0)}$ defined by (30) satisfies $\tau_d(y_k) \leq c_1 < 1$ for all k , we show that $E(y_k) \rightarrow 0$ and that $\{y_k\}_{k \in \mathbb{N}}$ converges strongly in V to a solution of (14). Then, we determine sufficient conditions on the initial guess $y_0 \in V$ so that $\tau_d(y_k) < 1$ for all $k \in \mathbb{N}$.

We start with the following lemma which provides the main property of the sequence $\{E(y_k)\}_{(k \geq 0)}$.

Lemma 4 *Assume that the sequence $\{y_k\}_{(k \geq 0)}$ defined by (30) satisfies $\tau_d(y_k) < 1$ for all $k \geq 0$. Then, for all $\lambda \in \mathbb{R}$,*

$$E(y_k - \lambda Y_{1,k}) \leq E(y_k) \left(|1 - \lambda| + \lambda^2 \frac{(1 - \tau_2(y_k))^{-2}}{\sqrt{\alpha \nu}} \sqrt{E(y_k)} \right)^2 \tag{33}$$

if $d = 2$ and

$$E(y_k - \lambda Y_{1,k}) \leq E(y_k) \left(|1 - \lambda| + \lambda^2 \frac{\sqrt{2}}{\sqrt{\nu}} \frac{M}{(\alpha \nu^3)^{1/4}} (1 - \tau_3(y_k))^{-2} \sqrt{E(y_k)} \right)^2 \tag{34}$$

if $d = 3$.

Proof For any real λ and any $y_k, w_k \in V$ we get the expansion

$$\begin{aligned} E(y_k - \lambda w_k) = & E(y_k) - \lambda \langle v_k, \bar{v}_k \rangle_V + \frac{\lambda^2}{2} (\langle \bar{v}_k, \bar{v}_k \rangle_V + 2 \langle v_k, \bar{\bar{v}}_k \rangle_V) \\ & - \lambda^3 \langle \bar{v}_k, \bar{\bar{v}}_k \rangle_V + \frac{\lambda^4}{2} \langle \bar{\bar{v}}_k, \bar{\bar{v}}_k \rangle_V \end{aligned} \tag{35}$$

where $v_k, \bar{v}_k \in V$ and $\bar{\bar{v}}_k \in V$ solves respectively

$$\begin{aligned} \alpha \int_{\Omega} v_k \cdot w + \nu \int_{\Omega} \nabla v_k \cdot \nabla w + \alpha \int_{\Omega} y_k \cdot w + \nu \int_{\Omega} \nabla y_k \cdot \nabla w + \int_{\Omega} y_k \cdot \nabla y_k \cdot w \\ = \langle f, w \rangle_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} + \alpha \int_{\Omega} g \cdot w, \quad \forall w \in V, \end{aligned} \tag{36}$$

$$\alpha \int_{\Omega} \bar{v}_k \cdot w + \nu \int_{\Omega} \nabla \bar{v}_k \cdot \nabla w + \alpha \int_{\Omega} w_k \cdot w + \nu \int_{\Omega} \nabla w_k \cdot \nabla w + \int_{\Omega} w_k \cdot \nabla y_k \cdot w + y_k \cdot \nabla w_k \cdot w = 0, \quad \forall w \in V, \tag{37}$$

and

$$\alpha \int_{\Omega} \bar{\bar{v}}_k \cdot w + \nu \int_{\Omega} \nabla \bar{\bar{v}}_k \cdot \nabla w + \int_{\Omega} w_k \cdot \nabla w_k \cdot w = 0, \quad \forall w \in V. \tag{38}$$

Since the corrector \bar{v}_k associated to $Y_{1,k}$ coincides with the corrector v_k associated to y_k (see Proof of Proposition 4), expansion (35) reduces to

$$\begin{aligned} E(y_k - \lambda Y_{1,k}) &= (1 - \lambda)^2 E(y_k) + \lambda^2 (1 - \lambda) \langle v_k, \bar{v}_k \rangle_V + \frac{\lambda^4}{2} \langle \bar{v}_k, \bar{v}_k \rangle_V \\ &\leq \left(|1 - \lambda| \sqrt{E(y_k)} + \frac{\lambda^2}{\sqrt{2}} \|\bar{v}_k\|_V \right)^2. \end{aligned} \tag{39}$$

If $d = 2$, then (38) leads to $\|\bar{v}_k\|_V \leq \frac{\|Y_{1,k}\|_V^2}{\sqrt{2\alpha\nu}} \leq \sqrt{2}(1 - \tau_2(y_k))^{-2} \frac{E(y_k)}{\sqrt{\alpha\nu}}$ and then to (33). If $d = 3$, then

$$\|\bar{v}_k\|_V \leq \frac{1}{\sqrt{\nu}} \frac{M}{(\alpha\nu^3)^{1/4}} \|Y_{1,k}\|_V^2 \leq \frac{1}{\sqrt{\nu}} \frac{M}{(\alpha\nu^3)^{1/4}} 2(1 - \tau_3(y_k))^{-2} E(y_k)$$

leading to (34). □

We are now in position to prove the convergence of the sequence $\{E(y_k)\}_{(k \geq 0)}$.

Proposition 6 *Let $\{y_k\}_{k \geq 0}$ be the sequence defined by (30). Assume that there exists a constant $c_1 \in (0, 1)$ such that $\tau_d(y_k) \leq c_1$ for all k . Then $E(y_k) \rightarrow 0$ as $k \rightarrow \infty$. Moreover, there exists $k_0 \in \mathbb{N}$ such that the sequence $\{E(y_k)\}_{(k \geq k_0)}$ decays quadratically.*

Proof Consider the case $d = 2$. The inequality $\tau_2(y_k) \leq c_1$ and (33) imply that

$$E(y_k - \lambda Y_{1,k}) \leq E(y_k) \left(|1 - \lambda| + \lambda^2 c_{\alpha,\nu} \sqrt{E(y_k)} \right)^2, \quad c_{\alpha,\nu} := \frac{(1 - c_1)^{-2}}{\sqrt{\alpha\nu}}.$$

Let us denote the function $p_k(\lambda) = |1 - \lambda| + \lambda^2 c_{\alpha,\nu} \sqrt{E(y_k)}$ for all $\lambda \in [0, m]$. We can write

$$\sqrt{E(y_{k+1})} = \min_{\lambda \in [0, m]} \sqrt{E(y_k - \lambda Y_{1,k})} \leq p_k(\tilde{\lambda}_k) \sqrt{E(y_k)}$$

with $p_k(\tilde{\lambda}_k) := \min_{\lambda \in [0, m]} p_k(\lambda)$.

Suppose first that $c_{\alpha, \nu} \sqrt{E(y_0)} \geq 1$ and prove that the set $I := \{k \in \mathbb{N}, c_{\alpha, \nu} \sqrt{E(y_k)} \geq 1\}$ is a finite subset of \mathbb{N} . For all $k \in I$, we get

$$\min_{\lambda \in [0, m]} p_k(\lambda) = \min_{\lambda \in [0, 1]} p_k(\lambda) = p_k\left(\frac{1}{2c_{\alpha, \nu} \sqrt{E(y_k)}}\right) = 1 - \frac{1}{4c_{\alpha, \nu} \sqrt{E(y_k)}}$$

and thus, for all $k \in I$,

$$c_{\alpha, \nu} \sqrt{E(y_{k+1})} \leq \left(1 - \frac{1}{4c_{\alpha, \nu} \sqrt{E(y_k)}}\right) c_{\alpha, \nu} \sqrt{E(y_k)} = c_{\alpha, \nu} \sqrt{E(y_k)} - \frac{1}{4}.$$

Consequently, the sequence $\{c_{\alpha, \nu} \sqrt{E(y_k)}\}_{k \in I}$ strictly decreases and thus, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $c_{\alpha, \nu} \sqrt{E(y_k)} < 1$. Thus I is a finite subset of \mathbb{N} . Then, for all $k \geq k_0$, we get that

$$p_k(\tilde{\lambda}_k) \leq p_k(1) = c_{\alpha, \nu} \sqrt{E(y_k)}$$

and thus, for all $k \geq k_0$,

$$c_{\alpha, \nu} \sqrt{E(y_{k+1})} \leq (c_{\alpha, \nu} \sqrt{E(y_k)})^2 \tag{40}$$

implying that $c_{\alpha, \nu} \sqrt{E(y_k)} \rightarrow 0$ as $k \rightarrow \infty$ with a quadratic rate.

On the other hand, if $c_{\alpha, \nu} \sqrt{E(y_0)} < 1$ (and thus $c_{\alpha, \nu} \sqrt{E(y_k)} < 1$ for all $k \in \mathbb{N}$, since by construction the sequence $\{E(y_k)\}_k$ decreases), then (40) holds true for all $k \geq 0$.

In both cases, remark that $p_k(\tilde{\lambda}_k)$ decreases with respect to k . The case $d = 3$ is similar with

$$c_{\alpha, \nu} = \frac{\sqrt{2}}{\sqrt{\nu}} \frac{M}{(\alpha \nu^3)^{1/4}} (1 - c_1)^{-2}.$$

□

Lemma 5 Assume that the sequence $\{y_k\}_{(k \geq 0)}$ defined by (30) satisfies $\tau_d(y_k) \leq c_1$ for all k and some $c_1 \in (0, 1)$. Then $\lambda_k \rightarrow 1$ as $k \rightarrow \infty$.

Proof In view of (39), we have, as long as $E(y_k) > 0$,

$$(1 - \lambda_k)^2 = \frac{E(y_{k+1})}{E(y_k)} - \lambda_k^2 (1 - \lambda_k) \frac{\langle v_k, \bar{v}_k \rangle_V}{E(y_k)} - \lambda_k^4 \frac{\|\bar{v}_k\|_V^2}{2E(y_k)}.$$

From the Proof of Lemma 4, $\frac{\langle v_k, \bar{v}_k \rangle_V}{E(y_k)} \leq C(\alpha, \nu)(1 - c_1)^{-2} \sqrt{E(y_k)}$ while $\frac{\|\bar{v}_k\|_V^2}{E(y_k)} \leq C(\alpha, \nu)^2 (1 - c_1)^{-4} E(y_k)$. Consequently, since $\lambda_k \in [0, m]$ and $\frac{E(y_{k+1})}{E(y_k)} \rightarrow 0$, we deduce that $(1 - \lambda_k)^2 \rightarrow 0$ as $k \rightarrow \infty$. □

Proposition 7 *Let $\{y_k\}_{(k \geq 0)}$ be the sequence defined by (30). Assume that there exists a constant $c_1 \in (0, 1)$ such that $\tau_d(y_k) \leq c_1$ for all k . Then, $y_k \rightarrow \bar{y}$ in \mathbf{V} where $\bar{y} \in \mathbf{V}$ is the unique solution of (14).*

Proof Remark that we can not use Proposition 5 since we do not know yet that there exists a solution, say z , of (14) satisfying $\tau_d(z) < 1$. In view of $y_{k+1} = y_0 - \sum_{n=0}^k \lambda_n Y_{1,n}$, we write

$$\sum_{n=0}^k |\lambda_n| \|Y_{1,n}\|_{\mathbf{V}} \leq m \sum_{n=0}^k \|Y_{1,n}\|_{\mathbf{V}} \leq m\sqrt{2} \sum_{n=0}^k \frac{\sqrt{E(y_n)}}{1 - \tau_d(y_n)} \leq \frac{m\sqrt{2}}{1 - c_1} \sum_{n=0}^k \sqrt{E(y_n)}.$$

Using that $p_n(\tilde{\lambda}_n) \leq p_0(\tilde{\lambda}_0)$ for all $n \geq 0$, we obtain for all $n > 0$,

$$\sqrt{E(y_n)} \leq p_{n-1}(\tilde{\lambda}_{n-1})\sqrt{E(y_{n-1})} \leq p_0(\tilde{\lambda}_0)\sqrt{E(y_{n-1})} \leq p_0(\tilde{\lambda}_0)^n \sqrt{E(y_0)}.$$

Recalling that $p_0(\tilde{\lambda}_0) = \min_{\lambda \in [0,1]} p_0(\lambda) < 1$ since $p_0(0) = 1$ and $p'_0(0) = -1$, we finally obtain

$$\sum_{n=0}^k |\lambda_n| \|Y_{1,n}\|_{\mathbf{V}} \leq \frac{m\sqrt{2}}{1 - c_1} \frac{\sqrt{E(y_0)}}{1 - p_0(\tilde{\lambda}_0)}$$

from which we deduce that the series $\sum_{k \geq 0} \lambda_k Y_{1,k}$ converges in \mathbf{V} . Then, y_k converges in \mathbf{V} to $\bar{y} := y_0 + \sum_{k \geq 0} \lambda_k Y_{1,k}$. Eventually, the convergence of $E(y_k)$ to 0 implies the convergence of the corrector v_k to 0 in \mathbf{V} ; taking the limit in the corrector Eq. (36) shows that \bar{y} solves (14). Since $\tau_d(\bar{y}) \leq c_1 < 1$, Lemma 3 shows that this solution is unique. □

As mentioned earlier, the remaining and crucial point is to show that the sequence $\{y_k\}_{(k \geq 0)}$ satisfies the uniform property $\tau_d(y_k) \leq c_1$ for some $c_1 < 1$.

Lemma 6 *Assume that $y_0 = g \in \mathbf{V}$. For all $c_1 \in (0, 1)$ there exists $\alpha_0 > 0$, such that, for any $\alpha \geq \alpha_0$, the unique sequence defined by (30) satisfies $\tau_d(y_k) \leq c_1$ for all $k \geq 0$.*

Proof Let $c_1 \in (0, 1)$ and assume that y_0 belongs to \mathbf{V} . Since $\tau_d(y_0) \rightarrow 0$ as $\alpha \rightarrow \infty$, there exists $\alpha_1 > 0$ such that for all $\alpha \geq \alpha_1$ $\tau_d(y_0) \leq \frac{c_1}{2}$.

Moreover, in view of the above computation and using that $\|v\|_{H^1_0(\Omega)^d} \leq \frac{1}{\sqrt{v}} \|v\|_{\mathbf{V}}$ for all $v \in \mathbf{V}$ and $\alpha > 0$, we obtain, for all $k \in \mathbb{N}$

$$\|y_{k+1}\|_{H^1_0(\Omega)^d} \leq \|y_0\|_{H^1_0(\Omega)^d} + \frac{m\sqrt{2}}{\sqrt{v}(1 - c_1)} \frac{\sqrt{E(y_0)}}{1 - p_0(\tilde{\lambda}_0)}$$

where

$$\frac{\sqrt{E(y_0)}}{1 - p_0(\tilde{\lambda}_0)} \leq \begin{cases} \frac{\sqrt{E(y_0)}}{1 - c_{\alpha,v}\sqrt{E(y_0)}}, & \text{if } c_{\alpha,v}\sqrt{E(y_0)} < 1, \\ 4c_{\alpha,v}E(y_0), & \text{if } c_{\alpha,v}\sqrt{E(y_0)} \geq 1. \end{cases}$$

Assume $d = 2$. From (17), we obtain for all $y \in V$ that

$$\|v\|_V^2 \leq \alpha \|g - y\|_2^2 + \frac{1}{v} \left(v \|\nabla y\|_2 + \sqrt{2} \|y\|_2 \|\nabla y\|_2 + \|f\|_{H^{-1}(\Omega)^2} \right)^2.$$

In particular, taking $y = g$ allows to remove the α term and gives

$$E(g) \leq \frac{1}{2v} \left(\|g\|_{H_0^1(\Omega)^d} (v + \sqrt{2} \|g\|_2) + \|f\|_{H^{-1}(\Omega)^2} \right)^2 := \frac{1}{2v} c_2(f, g). \tag{41}$$

If $c_{\alpha_1,v}\sqrt{E(g)} \geq 1$ then for all $\alpha \geq \alpha_1$ such that $c_{\alpha,v}\sqrt{E(g)} \geq 1$ and for all $k \in \mathbb{N}$:

$$\begin{aligned} \|y_{k+1}\|_{H_0^1(\Omega)^d} &\leq \|g\|_{H_0^1(\Omega)^d} + \frac{m\sqrt{2}}{\sqrt{v}(1 - c_1)} \frac{\sqrt{E(g)}}{1 - p_0(\tilde{\lambda}_0)} \\ &\leq \|g\|_{H_0^1(\Omega)^d} + \frac{2m\sqrt{2}}{v^{5/2}\sqrt{\alpha}(1 - c_1)^3} c_2(f, g). \end{aligned} \tag{42}$$

If $c_{\alpha_1,v}\sqrt{E(g)} < 1$ then there exists $0 < K < 1$ such that for all $\alpha \geq \alpha_1$ we have $c_{\alpha,v}\sqrt{E(g)} \leq K$. We therefore have for all $\alpha \geq \alpha_1$

$$\frac{\sqrt{E(g)}}{1 - p_0(\tilde{\lambda}_0)} \leq \frac{\sqrt{E(g)}}{1 - K}$$

and thus for all $k \in \mathbb{N}$:

$$\begin{aligned} \|y_{k+1}\|_{H_0^1(\Omega)^d} &\leq \|g\|_{H_0^1(\Omega)^d} + \frac{m\sqrt{2}}{\sqrt{v}(1 - c_1)} \frac{\sqrt{E(g)}}{1 - p_0(\tilde{\lambda}_0)} \\ &\leq \|g\|_{H_0^1(\Omega)^d} + \frac{m}{v(1 - c_1)(1 - K)} \sqrt{c_2(f, g)}. \end{aligned} \tag{43}$$

On the other hand, there exists $\alpha_0 \geq \alpha_1$ such that, for all $\alpha \geq \alpha_0$ we have

$$\frac{2m\sqrt{2}}{v^{5/2}\sqrt{\alpha}(1 - c_1)^3} c_2(f, g) \leq \frac{c_1}{2} \sqrt{2\alpha v}$$

and

$$\frac{m}{v(1 - c_1)(1 - K)} \sqrt{c_2(f, g)} \leq \frac{c_1}{2} \sqrt{2\alpha v}.$$

We then deduce from (42) and (43) that for all $\alpha \geq \alpha_0$ and for all $k \in \mathbb{N}$:

$$\|y_{k+1}\|_{H_0^1(\Omega)^d} \leq \frac{c_1}{2}\sqrt{2\alpha v} + \frac{c_1}{2}\sqrt{2\alpha v} = c_1\sqrt{2\alpha v}$$

that is $\tau_2(y_{k+1}) \leq c_1$.

Assume $d = 3$. We argue as in the case $d = 2$ and deduce from (17), since $y_0 = g$, that

$$E(g) \leq \frac{1}{2v} \left(\|g\|_{H_0^1(\Omega)^3} (v + c\|y_0\|_2^{1/2} \|g\|_{H_0^1(\Omega)^3}^{1/2}) + \|f\|_{H^{-1}(\Omega)^3} \right)^2 := \frac{1}{2v} c_3(f, g) \tag{44}$$

and thus, if $c_{\alpha_1, v}\sqrt{E(g)} \geq 1$, then for all $\alpha \geq \alpha_1$ such that $c_{\alpha, v}\sqrt{E(g)} \geq 1$ and for all $k \in \mathbb{N}$:

$$\begin{aligned} \|y_{k+1}\|_{H_0^1(\Omega)^3} &\leq \|g\|_{H_0^1(\Omega)^3} + \frac{m\sqrt{2}}{\sqrt{v}(1-c_1)} \frac{\sqrt{E(g)}}{1-p_0(\tilde{\lambda}_0)} \\ &\leq \|g\|_{H_0^1(\Omega)^3} + \frac{4mM}{v^2(\alpha v^3)^{1/4}(1-c_1)^3} c_3(f, g). \end{aligned} \tag{45}$$

If $c_{\alpha_1, v}\sqrt{E(g)} < 1$ then there exists $0 < K < 1$ such that for all $\alpha \geq \alpha_1$ we have $c_{\alpha, v}\sqrt{E(g)} \leq K$. We therefore have for all $\alpha \geq \alpha_1$

$$\frac{\sqrt{E(g)}}{1-p_0(\tilde{\lambda}_0)} \leq \frac{\sqrt{E(y_0)}}{1-K}$$

and thus for all $k \in \mathbb{N}$:

$$\begin{aligned} \|y_{k+1}\|_{H_0^1(\Omega)^3} &\leq \|g\|_{H_0^1(\Omega)^3} + \frac{m\sqrt{2}}{\sqrt{v}(1-c_1)} \frac{\sqrt{E(g)}}{1-p_0(\tilde{\lambda}_0)} \\ &\leq \|g\|_{H_0^1(\Omega)^3} + \frac{m}{v(1-c_1)(1-K)} \sqrt{c_3(f, g)}. \end{aligned} \tag{46}$$

On the other hand, there exists $\alpha_0 \geq \alpha_1$ such that, for all $\alpha \geq \alpha_0$ we have

$$\frac{4mM}{v^2(\alpha v^3)^{1/4}(1-c_1)^3} c_3(f, g) \leq \frac{c_1}{2} \frac{(\alpha v^3)^{1/4}}{M}$$

and

$$\frac{m}{v(1-c_1)(1-K)} \sqrt{c_3(f, g)} \leq \frac{c_1}{2} \frac{(\alpha v^3)^{1/4}}{M}.$$

We then deduce from (45) to (46) that for all $\alpha \geq \alpha_0$ and for all $k \in \mathbb{N}$:

$$\|y_{k+1}\|_{H_0^1(\Omega)^d} \leq \frac{c_1}{2} \frac{(\alpha v^3)^{1/4}}{M} + \frac{c_1}{2} \frac{(\alpha v^3)^{1/4}}{M} = c_1 \frac{(\alpha v^3)^{1/4}}{M}$$

that is $\tau_3(y_{k+1}) \leq c_1$. □

Gathering the previous lemmas and propositions, we deduce the strong convergence of the sequence $\{y_k\}_{k \geq 0}$ defined by (30), initialized by $y_0 = g$.

Theorem 1 *Let $c_1 \in (0, 1)$. Assume that $y_0 = g \in V$ and α is large enough so that*

$$\begin{aligned}
 c_2(f, g) &\leq \max\left(\frac{1 - c_1}{2}, \frac{c_1(1 - K)^2}{m}\right) \frac{c_1}{4m} v^2 (1 - c_1)^2 2\alpha v, \text{ if } d = 2, \\
 c_3(f, g) &\leq \max\left(\frac{1 - c_1}{2}, \frac{c_1(1 - K)^2}{m}\right) \frac{c_1}{4mM^2} v^2 (1 - c_1)^2 (\alpha v^3)^{1/2}, \text{ if } d = 3,
 \end{aligned}
 \tag{47}$$

where $c_2(f, g)$ and $c_3(f, g)$ are defined in (41) and (44), respectively. The sequence $\{y_k\}_{(k \in \mathbb{N})}$ defined by (30) strongly converges to the unique solution y of (14). Moreover, there exists $k_0 \in \mathbb{N}$ such that the sequence $\{y_k\}_{k \geq k_0}$ converges quadratically to y . Moreover, this solution satisfies $\tau_d(y) < 1$.

2.3 Additional comments

1) Estimate (15) is usually used to obtain a sufficient condition on the data f, g to ensure the uniqueness of the solution of (14) (i.e. $\tau_d(y) < 1$): it leads to

$$\begin{aligned}
 \alpha \|g\|_2^2 + \frac{1}{v} \|f\|_{H^{-1}(\Omega)^2}^2 &\leq 2\alpha v^2, \quad \text{if } d = 2, \\
 \alpha \|g\|_2^2 + \frac{1}{v} \|f\|_{H^{-1}(\Omega)^3}^2 &\leq \frac{v(\alpha v^3)^{1/2}}{M^2}, \quad \text{if } d = 3.
 \end{aligned}$$

We emphasize that such (sufficient) conditions are more restrictive than (47), as they impose smallness properties on g : precisely $\|g\|_2^2 \leq 2v^2$ if $d = 2$ and $\|g\|_2^2 \leq \frac{v^{5/2}}{M^2 \alpha^{1/2}}$ if $d = 3$. This latter yields a restrictive condition for α large contrary to (47).

2) Let $\mathcal{F} : V \rightarrow V'$ the application be defined as $\mathcal{F}(y) = \alpha y + vB_1(y) + B(y, y) - f - \alpha g$. The sequence $\{y_k\}_{(k > 0)}$ associated to the Newton method to find the zero of \mathcal{F} is formally defined as follows:

$$\begin{cases} y_0 \in V, \\ D\mathcal{F}(y_k) \cdot (y_{k+1} - y_k) = -\mathcal{F}(y_k), \quad k \geq 0. \end{cases}
 \tag{48}$$

We check that this sequence coincides with the sequence obtained from (30) if λ_k is fixed equal to one. The algorithm (30) which consists in optimizing the parameter $\lambda_k \in [0, m], m \geq 1$, in order to minimize $E(y_k)$, equivalently $\|\mathcal{F}(y_k)\|_{V'}$, corresponds to the so-called damped Newton method for the application \mathcal{F} (see [3]). As the iterates increase, the optimal parameter λ_k converges to one (according to Lemma 5), this globally convergent method behaves like the standard Newton method (for which λ_k is fixed equal to one): this explains the quadratic rate of convergence after a finite number of iterates. To the best of our knowledge, this is the first analysis of the damped

Newton method for a partial differential equation. Among the few numerical works devoted to the damped Newton method for partial differential equations, we mention [16] for computing viscoplastic fluid flows.

3) Section 6, chapter 6 of the book [4] introduces a least-squares method in order to solve an Oseen type equation (without incompressibility constraint). The convergence of any minimizing sequence toward a solution y is proved under the *a priori* assumption that the operator $D\mathcal{F}(y) : V \rightarrow V'$

$$D\mathcal{F}(y) \cdot w = \alpha w - \nu \Delta w + [(w \cdot \nabla)y + (y \cdot \nabla)w] \tag{49}$$

(for some $\alpha > 0$) is an isomorphism. y is then said to be a nonsingular point. According to Proposition 2, a sufficient condition for y to be a nonsingular point is $\tau_d(y) < 1$. Recall that τ_d depends on α . As far as we know, determining a weaker condition ensuring that $D\mathcal{F}(y)$ is an isomorphism is an open question. Moreover, according to Lemma 3, it turns out that this condition is also a sufficient condition for the uniqueness of (14). Theorem 1 asserts that, if α is large enough, then the sequence $\{y_k\}_{k \in \mathbb{N}}$ defined in (30), initialized with $y_0 = g$, is a convergent sequence of nonsingular points.

4) We may also define a minimizing sequence for E using the derivative E' (see (22)):

$$\begin{cases} y_0 \in V \text{ given,} \\ y_{k+1} = y_k - \lambda_k g_k, \quad k \geq 0, \\ \lambda_k = \operatorname{argmin}_{\lambda \in [0, m]} E(y_k - \lambda g_k) \end{cases} \tag{50}$$

with $g_k \in V$ such that $(g_k, w)_V = \langle E'(y_k), w \rangle_{V' \times V}$ for all $w \in V$. In particular, $\|g_k\|_V = \|E'(y_k)\|_{V'}$. Using the expansion (25) with $w_k = g_k$, we can prove the linear decrease of the sequence $\{E(y_k)\}_{k > 0}$ to zero assuming however that $E(y_0)$ is small enough, of the order of ν^2 , independently of the value of α (we refer to [11, Lemma 4.1] in a similar context).

3 Application to the backward Euler scheme

We now use the results of the previous section to discuss the resolution of the backward Euler scheme (2) through a least-squares method. The weak formulation of this scheme reads as follows: given $y^0 = u_0 \in V$, the sequence $\{y^n\}_{n > 0}$ in V is defined by recurrence as follows:

$$\begin{aligned} \int_{\Omega} \frac{y^{n+1} - y^n}{\delta t} \cdot w + \nu \int_{\Omega} \nabla y^{n+1} \cdot \nabla w \\ + \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot w = \langle f^n, w \rangle_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} \end{aligned} \tag{51}$$

with f^n defined by (4) in term of the external force of the Navier–Stokes model (1). We recall that a piecewise linear interpolation in time of $\{y^n\}_{n \geq 0}$ weakly converges in $L^2(0, T; V)$ toward a solution of (1).

As done in [2], one may use the least-squares method (analyzed in Sect. 2) to solve iteratively (51). Precisely, in order to approximate y^{n+1} from y^n , one may consider the following extremal problem

$$\inf_{y \in V} E_n(y), \quad E_n(y) = \frac{1}{2} \|v\|_V^2 \tag{52}$$

where the corrector $v \in V$ solves

$$\begin{aligned} \alpha \int_{\Omega} v \cdot w + v \int_{\Omega} \nabla v \cdot \nabla w &= -\alpha \int_{\Omega} y \cdot w - v \int_{\Omega} \nabla y \cdot \nabla w - \int_{\Omega} y \cdot \nabla y \cdot w \\ &+ \langle f^n, w \rangle_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} + \alpha \int_{\Omega} y^n \cdot w, \quad \forall w \in V \end{aligned} \tag{53}$$

with α and f^n given by (4). For any $n \geq 0$, a minimizing sequence $\{y_k^n\}_{k \geq 0}$ for E_n is defined as follows:

$$\begin{cases} y_0^{n+1} = y^n, \\ y_{k+1}^{n+1} = y_k^{n+1} - \lambda_k Y_{1,k}^{n+1}, \quad k \geq 0, \\ \lambda_k = \operatorname{argmin}_{\lambda \in [0,m]} E_n(y_k^{n+1} - \lambda Y_{1,k}^{n+1}) \end{cases} \tag{54}$$

where $Y_{1,k}^n \in V$ solves (31) for $y_k = y_k^{n+1}$. For each n , algorithm (54) can be expanded as follows

$$\begin{cases} y_0^{n+1} = y^n, \\ \int_{\Omega} \alpha y_{k+1}^{n+1} \cdot w + v \nabla y_{k+1}^{n+1} \cdot \nabla w + y_k^{n+1} \cdot \nabla y_{k+1}^{n+1} \cdot w + y_{k+1}^{n+1} \cdot \nabla y_k^{n+1} \cdot w \\ = (1 - \lambda_k) \int_{\Omega} \alpha y_k^{n+1} \cdot w + v \nabla y_k^{n+1} \cdot \nabla w + y_k^{n+1} \cdot \nabla y_k^{n+1} \cdot w \\ + \lambda_k \int_{\Omega} \alpha y^n \cdot w + \langle f^n, w \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \int_{\Omega} y_k^{n+1} \cdot \nabla y_k^{n+1} \cdot w \quad k \geq 0, \end{cases} \tag{55}$$

for all $w \in V$. In view of Theorem 1, the first element of the minimizing sequence is chosen equal to y^n , i.e. the minimizer of E_{n-1} .

The main goal of this section is to prove that for all $n \in \mathbb{N}$, the minimizing sequence $\{y_k^{n+1}\}_{k \in \mathbb{N}}$ converges to a solution y^{n+1} of (51). Arguing as in Lemma 6, we have to prove the existence of a constant $c_1 \in (0, 1)$, such that $\tau_d(y_k^n) \leq c_1$ for all n and k in \mathbb{N} . Remark that the initialization y_0^{n+1} is fixed as the minimizer of the functional E^{n-1} , obtained at the previous iterate. Consequently, the uniform property $\tau_d(y_k^n) \leq c_1$ is related to the initial guess y_0^0 equal to the initial position u_0 , to the external force F (see (1)) and to the value of α . u_0 and F are given *a priori*. On the other hand, the parameter α , related to the discretization parameter δt , can be chosen as large as necessary. As we shall see, this uniform property, which is essential to set up the least-squares procedure, requires smallness properties on u_0 and F .

We start with the following result analogue to Proposition 1.

Proposition 8 Let $(f^n)_{n \in \mathbb{N}}$ be a sequence in $H^{-1}(\Omega)^d$, $\alpha > 0$ and $y^0 = u_0 \in \mathbf{H}$. For any $n \in \mathbb{N}$, there exists a solution $y^{n+1} \in \mathbf{V}$ of

$$\alpha \int_{\Omega} (y^{n+1} - y^n) \cdot w + \nu \int_{\Omega} \nabla y^{n+1} \cdot \nabla w + \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot w = \langle f^n, w \rangle_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} \tag{56}$$

for all $w \in \mathbf{V}$. Moreover, for all $n \in \mathbb{N}$, y^{n+1} satisfies

$$\| \| y^{n+1} \| \|_{\mathbf{V}}^2 \leq \frac{1}{\nu} \| f^n \|_{H^{-1}(\Omega)^d}^2 + \alpha \| y^n \|_2^2. \tag{57}$$

Moreover, for all $n \in \mathbb{N}^*$:

$$\| y^n \|_2^2 + \frac{\nu}{\alpha} \sum_{k=1}^n \| \nabla y^k \|_2^2 \leq \frac{1}{\nu} \left(\frac{1}{\alpha} \sum_{k=0}^{n-1} \| f^k \|_{H^{-1}(\Omega)^d}^2 + \nu \| u_0 \|_2^2 \right). \tag{58}$$

Proof The existence of y^{n+1} is given in Proposition 1. (58) is obtained by summing (57). □

Remark 2 Arguing as in Lemma 3, if there exists a solution y^{n+1} in \mathbf{V} of (53) satisfying $\tau_d(y^{n+1}) < 1$, then such solution is unique. In view of Proposition 8, this holds true if the quantity $\mathcal{M}(f, \alpha, \nu)$ defined as follows

$$\mathcal{M}(f, \alpha, \nu) = \begin{cases} \frac{1}{\nu^3} \left(\frac{1}{\alpha} \sum_{k=0}^n \| f^k \|_{H^{-1}(\Omega)^2}^2 + \nu \| u_0 \|_2^2 \right), & \text{if } d = 2, \\ \frac{\alpha^{1/2}}{\nu^{7/2}} \left(\frac{1}{\alpha} \sum_{k=0}^n \| f^k \|_{H^{-1}(\Omega)^3}^2 + \nu \| u_0 \|_2^2 \right), & \text{if } d = 3 \end{cases} \tag{59}$$

is small enough.

We now distinguish the case $d = 2$ from the case $d = 3$ and consider weak and regular solutions.

3.1 Two dimensional case

We have the following convergence for weak solutions of (56).

Theorem 2 Suppose $F \in L^2(0, T; H^{-1}(\Omega)^2)$, $u_0 \in \mathbf{V}$. Let α be large enough and f^n be given by (4) for all $n \in \{0, \dots, N - 1\}$. Let $c(u_0, F)$ be defined as follows:

$$c(u_0, F) := \max \left(\frac{1}{\alpha} \| u_0 \|_{H_0^1(\Omega)^d}^2 (\nu + \sqrt{2} \| u_0 \|_2)^2 + \| F \|_{L^2(0, T; H^{-1}(\Omega)^2)}^2, \right. \\ \left. 2 \| F \|_{L^2(0, T; H^{-1}(\Omega)^2)}^2 + 2 \nu \| u_0 \|_2^2 \right).$$

There exists a constant $c > 0$ such that if

$$c(u_0, F) \leq cv^3 \tag{60}$$

then for all $n \in \{0, \dots, N - 1\}$ the solution $y^{n+1} \in V$ of (56) is unique and the minimizing sequence $\{y_k^{n+1}\}_{k \in \mathbb{N}}$ defined by (54) strongly converges in V to y^{n+1} .

Proof According to Proposition 7, we have to prove the existence of a constant $c_1 \in (0, 1)$ such that, for all $n \in \{0, \dots, N - 1\}$ and all $k \in \mathbb{N}$, $\tau_2(y_k^n) \leq c_1$.

For $n = 0$, as in the previous section, it suffices to take α large enough to ensure the conditions (47) with $g = y_0^0 = u_0$ leading to the property $\tau_2(y_k^0) < c_1$ for all $k \in \mathbb{N}$ and therefore $\tau_2(y^1) \leq c_1$.

For the next minimizing sequences, we recall (see Lemma 6) that for all $n \in \{0, \dots, N - 1\}$ and all $k \in \mathbb{N}$

$$\|y_k^{n+1}\|_{H_0^1(\Omega)^2} \leq \|y^n\|_{H_0^1(\Omega)^2} + \frac{m\sqrt{2}}{\sqrt{v}(1 - c_1)} \frac{\sqrt{E_n(y^n)}}{1 - p_{n,0}(\tilde{\lambda}_{n,0})}$$

where $p_{n,0}(\tilde{\lambda}_{n,0})$ is defined as in the Proof of Proposition 4.

First, since for all $n \in \{0, \dots, N - 1\}$, $\|f^n\|_{H^{-1}(\Omega)^2}^2 \leq \alpha \|F\|_{L^2(0,T;H^{-1}(\Omega)^2)}^2$, we can write

$$\begin{aligned} E_0(y^0) = E_0(u_0) &\leq \frac{1}{2v} \left(\|u_0\|_{H_0^1(\Omega)^d} (v + \sqrt{2}\|u_0\|_2) + \|f^0\|_{H^{-1}(\Omega)^2} \right)^2 \\ &\leq \frac{1}{v} \left(\|u_0\|_{H_0^1(\Omega)^d}^2 (v + \sqrt{2}\|u_0\|_2)^2 + \|f^0\|_{H^{-1}(\Omega)^2}^2 \right) \\ &\leq \frac{\alpha}{v} \left(\frac{1}{\alpha} \|u_0\|_{H_0^1(\Omega)^d}^2 (v + \sqrt{2}\|u_0\|_2)^2 + \|F\|_{L^2(0,T;H^{-1}(\Omega)^2)}^2 \right). \end{aligned}$$

Since y^n is solution of (56), it follows from (53) and (58), that for all n in $\{1, \dots, N - 1\}$:

$$\begin{aligned} E_n(y^n) &\leq \frac{1}{2v} \|f^n - f^{n-1}\|_{H^{-1}(\Omega)^2}^2 + \frac{\alpha}{2} \|y^n - y^{n-1}\|_2^2 \\ &\leq \frac{1}{v} \left(\|f^n\|_{H^{-1}(\Omega)^2}^2 + \|f^{n-1}\|_{H^{-1}(\Omega)^2}^2 \right) + \alpha \left(\|y^n\|_2^2 + \|y^{n-1}\|_2^2 \right) \\ &\leq \frac{1}{v} \left(\|f^n\|_{H^{-1}(\Omega)^2}^2 + \|f^{n-1}\|_{H^{-1}(\Omega)^2}^2 \right) + \frac{1}{v} \left(\sum_{k=0}^{n-1} \|f^k\|_{H^{-1}(\Omega)^2}^2 + \sum_{k=0}^{n-2} \|f^k\|_{H^{-1}(\Omega)^2}^2 \right) \\ &\leq \frac{2}{v} \sum_{k=0}^n \|f^k\|_{H^{-1}(\Omega)^2}^2 + 2\alpha \|u_0\|_2^2 \\ &\leq \frac{\alpha}{v} \left(2\|F\|_{L^2(0,T;H^{-1}(\Omega)^2)}^2 + 2v\|u_0\|_2^2 \right). \end{aligned}$$

Therefore, for all $n \in \{0, \dots, N - 1\}$, $E_n(y^n) \leq \frac{\alpha}{v} c(u_0, F)$.

Let $c_1 \in (0, 1)$ and suppose that $c(u_0, F) < (1 - c_1)^4 v^3$. Then, there exists $K \in (0, 1)$ and $\alpha_0 > 0$ such that, for all $\alpha \geq \alpha_0$, $c_{\alpha, v} \sqrt{E_n(y^n)} \leq K < 1$. We therefore have (see Lemma 6), for all $\alpha \geq \alpha_0$, all $n \in \{0, \dots, N - 1\}$ and all $k \in \mathbb{N}$:

$$\begin{aligned} \|y_k^{n+1}\|_{H_0^1(\Omega)^2} &\leq \|y^n\|_{H_0^1(\Omega)^2} + \frac{m\sqrt{2}}{\sqrt{v}(1 - c_1)} \frac{\sqrt{E_n(y^n)}}{1 - c_{\alpha, v} \sqrt{E_n(y^n)}} \\ &\leq \|y^n\|_{H_0^1(\Omega)^2} + \frac{m\sqrt{2}}{\sqrt{v}(1 - c_1)} \frac{\sqrt{E_n(y^n)}}{1 - K} \\ &\leq \|y^n\|_{H_0^1(\Omega)^2} + \frac{m\sqrt{2\alpha}}{v(1 - c_1)(1 - K)} \sqrt{c(u_0, F)}. \end{aligned} \tag{61}$$

From (58) we then obtain, for all $n \in \{0, \dots, N - 1\}$,

$$\|y^n\|_{H_0^1(\Omega)^2} \leq \frac{\sqrt{\alpha}}{v} \sqrt{\frac{1}{\alpha} \sum_{k=0}^{n-1} \|f^k\|_{H^{-1}(\Omega)^2}^2 + v\|u_0\|_2^2}$$

and since $\frac{1}{\alpha} \sum_{k=0}^{n-1} \|f^k\|_{H^{-1}(\Omega)^2}^2 \leq \|F\|_{L^2(0, T; H^{-1}(\Omega)^2)}^2$, we deduce that if

$$\|F\|_{L^2(0, T; H^{-1}(\Omega)^2)}^2 + v\|u_0\|_2^2 \leq \frac{c_1^2}{2} v^3$$

then $\|y^n\|_{H_0^1(\Omega)^2} \leq \frac{c_1}{2} \sqrt{2\alpha v}$. Moreover, assuming $c(u_0, F) \leq \frac{c_1^2(1 - c_1)^2(1 - K)^2}{4m^2} v^3$, we deduce from (61), for all $n \in \{0, \dots, N - 1\}$ and for all $k \in \mathbb{N}$:

$$\|y_k^n\|_{H_0^1(\Omega)^2} \leq \frac{c_1}{2} \sqrt{2\alpha v} + \frac{c_1}{2} \sqrt{2\alpha v} = c_1 \sqrt{2\alpha v}$$

that is $\tau_2(y_k^n) \leq c_1$. The result follows from Proposition 7. □

We emphasize that, for each $n \in \mathbb{N}$, the limit y^{n+1} of the sequence $\{y_k^{n+1}\}_{k \in \mathbb{N}}$ satisfies $\tau_2(y^{n+1}) < 1$ and is therefore the unique solution of (56). Moreover, for α large enough, the condition (60) reads as the following smallness property on the data: $\|F\|_{L^2(0, T; H^{-1}(\Omega)^2)}^2 + v\|u_0\|_2^2 \leq cv^3$. In contrast with the static case of Sect. 2 where the unique condition (47) on the data g is fulfilled as soon as α is large, the iterated case requires a condition on the data u_0 and F , whatever be the amplitude of α . Again, this smallness property is introduced in order to guarantees the condition $\tau_2(y^n) < 1$ for all n . In view of (58), this condition implies notably that $\|y^n\|_2 \leq cv^{3/2}$ for all $n > 0$.

For regular solutions of (56) which we now consider, we may slightly improve the results, notably based on the control of two consecutive elements of the corresponding sequence $\{y^n\}_{n \in \mathbb{N}}$ for the $L^2(\Omega)$ norm.

Proposition 9 *Assume that Ω is C^2 , that $(f^n)_n$ is a sequence in $L^2(\Omega)^2$ and that $u_0 \in V$. Then, for all $n \in \mathbb{N}$, any solution $y^{n+1} \in V$ of (56) belongs to $H^2(\Omega)^2$.*

If moreover, there exists $C > 0$ such that

$$\frac{1}{\alpha} \sum_{k=0}^n \|f^k\|_{H^{-1}(\Omega)^2}^2 + \nu \|y^0\|_2^2 < C\nu^3, \tag{62}$$

then y^{n+1} satisfies

$$\int_{\Omega} |\nabla y^{n+1}|^2 + \frac{\nu}{2\alpha} \sum_{k=1}^{n+1} \int_{\Omega} |P \Delta y^k|^2 \leq \frac{1}{\nu} \left(\frac{1}{\alpha} \sum_{k=0}^n \|f^k\|_2^2 + \nu \|\nabla u_0\|_2^2 \right) \tag{63}$$

where P is the operator of projection from $L^2(\Omega)^d$ into \mathbf{H} .

Proof From Proposition 1, we know that for all $n \in \mathbb{N}^*$, $y^n \in H^2(\Omega)^2 \cap \mathbf{V}$. Thus, integrating by part (56) we obtain, using density argument:

$$\begin{aligned} & \alpha \int_{\Omega} (y^{n+1} - y^n) \cdot w - \nu \int_{\Omega} \Delta y^{n+1} \cdot w \\ & + \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot w = \langle f^n, w \rangle_{H^{-1}(\Omega)^2 \times H_0^1(\Omega)^2} \end{aligned}$$

for all $w \in \mathbf{H}$. Then, taking $w = P \Delta y^{n+1}$ and integrating by part leads to:

$$\begin{aligned} & \alpha \int_{\Omega} |\nabla y^{n+1}|^2 + \nu \int_{\Omega} |P \Delta y^{n+1}|^2 \\ & = - \int_{\Omega} f^n P \Delta y^{n+1} \\ & + \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot P \Delta y^{n+1} + \alpha \int_{\Omega} \nabla y^n \cdot \nabla y^{n+1}. \end{aligned} \tag{64}$$

Recall that

$$\left| \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot P \Delta y^{n+1} \right| \leq \|y^{n+1}\|_{\infty} \|\nabla y^{n+1}\|_2 \|P \Delta y^{n+1}\|_2.$$

We now use (see [18, chapter 25]) that there exist three constants c_1, c_2 and c_3 such that

$$\|\Delta y^{n+1}\|_2 \leq c_1 \|P \Delta y^{n+1}\|_2, \quad \|y^{n+1}\|_{\infty} \leq c_2 \|y^{n+1}\|_2^{\frac{1}{2}} \|\Delta y^{n+1}\|_2^{\frac{1}{2}}$$

and

$$\|\nabla y^{n+1}\|_2 \leq c_3 \|y^{n+1}\|_2^{\frac{1}{2}} \|\Delta y^{n+1}\|_2^{\frac{1}{2}}.$$

This implies that

$$\left| \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot P \Delta y^{n+1} \right| \leq c \|y^{n+1}\|_2 \|P \Delta y^{n+1}\|_2^2$$

with $c = c_1 c_2 c_3$. Recalling (64), it follows that

$$\frac{\alpha}{2} \int_{\Omega} |\nabla y^{n+1}|^2 + \left(\frac{\nu}{2} - c \|y^{n+1}\|_2 \right) \int_{\Omega} |P \Delta y^{n+1}|^2 \leq \frac{1}{2\nu} \|f^n\|_2^2 + \frac{\alpha}{2} \int_{\Omega} |\nabla y^n|^2.$$

But, from estimate (58), the assumption (62) implies that $\|y^{n+1}\|_2 \leq \frac{\nu}{4c}$ and

$$\int_{\Omega} |\nabla y^{n+1}|^2 + \frac{\nu}{2\alpha} \int_{\Omega} |P \Delta y^{n+1}|^2 \leq \frac{1}{\nu\alpha} \|f^n\|_2^2 + \int_{\Omega} |\nabla y^n|^2.$$

Summing then implies (63) for all $n \in \mathbb{N}$. □

Remark 3 Under the hypothesis of Proposition 9, suppose that

$$B_{\alpha,\nu} := (\alpha\nu^5)^{-1} \left(\alpha^{-1} \sum_{k=0}^n \|f^k\|_{H^{-1}(\Omega)^2}^2 + \nu \|y^0\|_2^2 \right) \left(\alpha^{-1} \sum_{k=0}^{n-1} \|f^k\|_2^2 + \nu \|\nabla y^0\|_2^2 \right)$$

is small (which is satisfied as soon as α is large enough). Then, the solution of (56) is unique.

Indeed, let $n \in \mathbb{N}$ and let $y_1^{n+1}, y_2^{n+1} \in V$ be two solutions of (56). Then $Y := y_1^{n+1} - y_2^{n+1}$ satisfies

$$\alpha \int_{\Omega} Y \cdot w + \nu \int_{\Omega} \nabla Y \cdot \nabla w + \int_{\Omega} y_2^{n+1} \cdot \nabla Y \cdot w + \int_{\Omega} Y \cdot \nabla y_1^{n+1} \cdot w = 0 \quad \forall w \in V$$

and in particular, for $w = Y$ (since $\int_{\Omega} y_2^{n+1} \cdot \nabla Y \cdot Y = 0$)

$$\begin{aligned} \alpha \int_{\Omega} |Y|^2 + \nu \int_{\Omega} |\nabla Y|^2 &= - \int_{\Omega} Y \cdot \nabla y_1^{n+1} \cdot Y = \int_{\Omega} Y \cdot \nabla Y \cdot y_1^{n+1} \\ &\leq c \|y_1^{n+1}\|_{\infty} \|\nabla Y\|_2 \|Y\|_2 \\ &\leq c \|y_1^{n+1}\|_2^{1/2} \|P \Delta y_1^{n+1}\|_2^{1/2} \|\nabla Y\|_2 \|Y\|_2 \\ &\leq \alpha \|Y\|_2^2 + \frac{c}{\alpha} \|y_1^{n+1}\|_2 \|P \Delta y_1^{n+1}\|_2 \|\nabla Y\|_2^2 \end{aligned}$$

leading to

$$\left(\nu - \frac{c}{\alpha} \|y_1^{n+1}\|_2 \|P \Delta y_1^{n+1}\|_2 \right) \|\nabla Y\|_2^2 \leq 0.$$

If

$$\|y_1^{n+1}\|_2 \|P \Delta y_1^{n+1}\|_2 < \frac{\nu \alpha}{c}, \tag{65}$$

then $Y = 0$ and the solution is unique. But, from (58) and (63),

$$\begin{aligned} \|y_1^{n+1}\|_2^2 \|P \Delta y_1^{n+1}\|_2^2 &\leq \frac{4\alpha}{\nu^3} \left(\frac{1}{\alpha} \sum_{k=0}^n \|f^k\|_{H^{-1}(\Omega)^2}^2 + \nu \|y^0\|_2^2 \right) \\ &\left(\frac{1}{\alpha} \sum_{k=0}^n \|f^k\|_2^2 + \nu \|\nabla y^0\|_2^2 \right). \end{aligned}$$

Therefore, there exists a constant C such that if $B_{\alpha,\nu} < C$, then (65) holds true.

Proposition 9 then allows to obtain the following estimate of $\|y^{n+1} - y^n\|_2$ in term of the parameter α .

Theorem 3 *We assume that Ω is C^2 , that $\{f^n\}_{n \in \mathbb{N}}$ is a sequence in $L^2(\Omega)^2$ and that $\alpha^{-1} \sum_{k=0}^{+\infty} \|f^k\|_2^2 < +\infty$, that $u_0 \in V$ and that for all $n \in \mathbb{N}$, $y^{n+1} \in H^2(\Omega)^2 \cap V$ is a solution of (56) satisfying $\|y^{n+1}\|_2 \leq \frac{\nu}{4c}$. There exists $C_1 > 0$ such that*

$$\|y^{n+1} - y^n\|_2^2 \leq \frac{C_1}{\alpha \nu^{3/2}}. \tag{66}$$

Proof For all $n \in \mathbb{N}$, $w = y^{n+1} - y^n$ in (56) gives:

$$\begin{aligned} \alpha \|y^{n+1} - y^n\|_2^2 + \nu \|\nabla y^{n+1}\|_2^2 &\leq \left| \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot (y^{n+1} - y^n) \right| \\ &+ \left| \int_{\Omega} f^n \cdot (y^{n+1} - y^n) \right| + \nu \left| \int_{\Omega} \nabla y^n \cdot \nabla y^{n+1} \right|. \end{aligned}$$

Moreover,

$$\begin{aligned} \left| \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot (y^{n+1} - y^n) \right| &\leq c \|\nabla y^{n+1}\|_2^2 (\|\nabla y^{n+1}\|_2 + \|\nabla y^n\|_2), \\ \left| \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot (y^{n+1} - y^n) \right| &\leq c \|\nabla y^{n+1}\|_2^2 \|\nabla (y^{n+1} - y^n)\|_2 \\ &\leq c \|\nabla y^{n+1}\|_2^2 (\|\nabla y^{n+1}\|_2 + \|\nabla y^n\|_2). \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha \|y^{n+1} - y^n\|_2^2 + \nu \|\nabla y^{n+1}\|_2^2 &\leq c \|\nabla y^{n+1}\|_2^2 (\|\nabla y^{n+1}\|_2 \\ &+ \|\nabla y^n\|_2) + \frac{1}{\alpha} \|f^n\|_2^2 + \nu \|\nabla y^n\|_2^2. \end{aligned}$$

But, (63) implies that for all $n \in \mathbb{N}$

$$\int_{\Omega} |\nabla y^{n+1}|^2 \leq \frac{1}{\nu} \left(\frac{1}{\alpha} \sum_{k=0}^{+\infty} \|f^k\|_2^2 + \nu \|\nabla y^0\|_2^2 \right) := \frac{C}{\nu}$$

and thus, since $\nu < 1$

$$\alpha \|y^{n+1} - y^n\|_2^2 + \nu \|\nabla y^{n+1}\|_2^2 \leq \frac{2cC^{3/2}}{\nu^{3/2}} + 2C \leq \frac{C_1}{\nu^{3/2}}$$

leading to $\|y^{n+1} - y^n\|_2^2 = O(\frac{1}{\alpha\nu^{3/2}})$ as announced. □

This result asserts that two consecutive elements of the sequence $\{y^n\}_{n \geq 0}$ defined by recurrence from the scheme (2) are close to each other as soon as δt , the time step discretization, is small enough. In particular, this justifies the choice of the initial term $y_0^{n+1} = y^n$ of the minimizing sequence in order to approximate y^{n+1} .

We end this section devoted to the case $d = 2$ with the analogue of Theorem 2 for regular data.

Theorem 4 *Suppose $F \in L^2(0, T; L^2(\Omega)^2)$, $u_0 \in \mathbf{V}$, for all $n \in \{0, \dots, N - 1\}$, α and f^n are given by (4) and $y^{n+1} \in \mathbf{V}$ solution of (56). If $C(u_0, F) := \|F\|_{L^2(0,T;L^2(\Omega)^2)}^2 + \nu \|u_0\|_{H_0^1(\Omega)^2}^2 \leq C\nu^2$ for some C and α is large enough, then, for any $n \geq 0$, the minimizing sequence $\{y_k^{n+1}\}_{k \in \mathbb{N}}$ defined by (54) strongly converges to the unique of solution of (56).*

Proof As for Theorem 2, it suffices to prove that there exists $c_1 \in (0, 1)$ such that, for all $n \in \{0, \dots, N - 1\}$ and all $k \in \mathbb{N}$, $\tau_2(y_k^n) \leq c_1$. Let us recall that for all $n \in \{0, \dots, N - 1\}$ and all $k \in \mathbb{N}$

$$\|y_{k+1}^{n+1}\|_{H_0^1(\Omega)^2} \leq \|y^n\|_{H_0^1(\Omega)^2} + \frac{m\sqrt{2}}{\nu(1-c_1)} \frac{\sqrt{E_n(y^n)}}{1-p_{n,0}(\tilde{\lambda}_{n,0})}$$

where $p_{n,0}(\tilde{\lambda}_{n,0})$ is defined as in the proof of Proposition 4. From (53), since for all $n \in \{0, \dots, N - 1\}$, $\|f^n\|_2^2 \leq \alpha \|F\|_{L^2(0,T;L^2(\Omega)^2)}^2$:

$$\begin{aligned} E_0(y^0) = E_0(u_0) &\leq \frac{1}{2\nu} \left(\|u_0\|_{H_0^1(\Omega)^2} (\nu + \|u_0\|_2) + \sqrt{\frac{\nu}{\alpha}} \|f^1\|_2 \right)^2 \\ &\leq \frac{1}{\nu} \|u_0\|_{H_0^1(\Omega)^2}^2 (\nu + \|u_0\|_2)^2 + \|F\|_{L^2(0,T;L^2(\Omega)^2)}^2 \end{aligned}$$

and, since y^n is solution of (56), then for all $n \in \{1, \dots, N - 1\}$:

$$\begin{aligned} E_n(y^n) &\leq \frac{1}{\alpha} \|f^n - f^{n-1}\|_2^2 + \alpha \|y^n - y^{n-1}\|_2^2 \\ &\leq 2\|F\|_{L^2(0,T;L^2(\Omega)^2)}^2 + \alpha \|y^n - y^{n-1}\|_2^2. \end{aligned}$$

From the Proof of Theorem 3, we deduce that for all $n \in \{0, \dots, N - 1\}$:

$$\alpha \|y^{n+1} - y^n\|_2^2 \leq \frac{2cC(u_0, F)^{3/2}}{\nu^{3/2}} + 2C(u_0, F)$$

and thus, for all $n \in \{1, \dots, N - 1\}$

$$E_n(y^n) \leq \frac{2cC(u_0, F)^{3/2}}{\nu^{3/2}} + 4C(u_0, F).$$

Moreover, from (63), for all $n \in \{0, \dots, N - 1\}$:

$$\|y^n\|_{H_0^1(\Omega)^2}^2 \leq \frac{1}{\nu} \left(\frac{1}{\alpha} \sum_{k=0}^n \|f^k\|_2^2 + \nu \|u_0\|_{H_0^1(\Omega)^2}^2 \right) \leq \frac{1}{\nu} \left(\|F\|_{L^2(Q_T)^2}^2 + \nu \|u_0\|_{H_0^1(\Omega)^2}^2 \right).$$

Eventually, let $c_1 \in (0, 1)$. Then there exists $\alpha_0 > 0$ such that, for all $\alpha \geq \alpha_0$ $c_{\alpha, \nu} \sqrt{E_n(y^n)} \leq K < 1$. We therefore have (see Theorem 2), for all $\alpha \geq \alpha_0$, all $n \in \{0, \dots, N - 1\}$ and all $k \in \mathbb{N}$:

$$\|y_{k+1}^{n+1}\|_{H_0^1(\Omega)^2} \leq \|y^n\|_{H_0^1(\Omega)^2} + \frac{m\sqrt{2}}{\nu(1 - c_1)} \frac{\sqrt{E_n(y^n)}}{1 - K}$$

which gives a bound of $\|y_{k+1}^{n+1}\|_{H_0^1(\Omega)^2}$ independent of $\alpha \geq \alpha_0$.

Taking $\alpha_1 \geq \alpha_0$ large enough, we deduce that, for all $\alpha \geq \alpha_1$, all $n \in \{0, \dots, N - 1\}$ and all $k \in \mathbb{N}$, $\|y_k^n\|_{H_0^1(\Omega)^2} \leq c_1 \sqrt{2\alpha\nu}$, that is $\tau_2(y_k^n) \leq c_1$. The announced convergence follows from Proposition 7. □

3.2 Three dimensional case

We now consider regular solutions for the case $d = 3$. The following intermediate regularity result holds true.

Proposition 10 *Assume that Ω is C^2 , that $(f^n)_n$ is a sequence in $L^2(\Omega)^3$ and that $u_0 \in V$. Then any solution $y^{n+1} \in V$ of (56) belongs to $H^2(\Omega)^3$.*

If moreover, there exists $C > 0$ such that

$$\frac{1}{\alpha} \sum_{k=0}^n \|f^k\|_2^2 + \nu \int_{\Omega} |\nabla u_0|^2 \leq C\nu^3, \tag{67}$$

then the inequality (63) holds true.

Proof From Proposition 1, we know that for all $n \in \mathbb{N}^*$, $y^n \in H^2(\Omega)^3 \cap V$. Let now P be the operator of projection from $L^2(\Omega)^3$ into H . Taking $w = P\Delta y^{n+1}$ in (56)

leads to:

$$\begin{aligned} \alpha \int_{\Omega} |\nabla y^{n+1}|^2 + \nu \int_{\Omega} |P \Delta y^{n+1}|^2 &= - \int_{\Omega} f^n \cdot P \Delta y^{n+1} \\ &+ \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot P \Delta y^{n+1} \quad (68) \\ &+ \alpha \int_{\Omega} \nabla y^n \cdot \nabla y^{n+1}. \end{aligned}$$

In view of the inequality

$$\left| \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot P \Delta y^{n+1} \right| \leq \|y^{n+1}\|_3 \|\nabla y^{n+1}\|_6 \|P \Delta y^{n+1}\|_2,$$

we use again that there exist constants $c_1, c_2 > 0$ such that

$$\|\Delta y^{n+1}\|_2 \leq c_1 \|P \Delta y^{n+1}\|_2, \quad \|\nabla y^{n+1}\|_6 \leq c_2 \|\Delta y^{n+1}\|_2 \leq c_1 c_2 \|P \Delta y^{n+1}\|_2$$

so that, for $c = c_1 c_2$, we obtain

$$\left| \int_{\Omega} y^{n+1} \cdot \nabla y^{n+1} \cdot P \Delta y^{n+1} \right| \leq c \|y^{n+1}\|_3 \|P \Delta y^{n+1}\|_2^2.$$

It results from (68) that

$$\frac{\alpha}{2} \int_{\Omega} |\nabla y^{n+1}|^2 + \left(\frac{\nu}{2} - c \|y^{n+1}\|_3 \right) \int_{\Omega} |P \Delta y^{n+1}|^2 \leq \frac{1}{2\nu} \|f^n\|_2^2 + \frac{\alpha}{2} \int_{\Omega} |\nabla y^n|^2. \quad (69)$$

Assume that, for all $n \in \mathbb{N}^*$, we have constructed by recurrence an element y^n solution of (56) such that

$$\frac{\nu}{4} - c \|y^n\|_3 > 0. \quad (70)$$

Then, for all $n \in \mathbb{N}$

$$\int_{\Omega} |\nabla y^{n+1}|^2 + \frac{\nu}{2\alpha} \int_{\Omega} |P \Delta y^{n+1}|^2 \leq \frac{1}{\nu\alpha} \|f^n\|_2^2 + \int_{\Omega} |\nabla y^n|^2 \quad (71)$$

and recursively, for all $n \in \mathbb{N}^*$, we get (63).

It remains to construct a sequence $\{y^n\}_{n \in \mathbb{N}^*}$ solution of (56) and satisfying for all $n \in \mathbb{N}^*$ the property (70). Let us first remark that the hypothesis (67) implies that y^0 satisfies (70). Let then $n \in \mathbb{N}$ fixed. Assume now, that we have constructed, for

$k \in \{0, \dots, n\}$ a solution y^k satisfying (56) if $k \geq 1$ and $\frac{\nu}{4} - c\|y^k\|_3 > 0$ for $c = c_1c_2$ introduced above. Let $y_1 \in V$ and let $y_2 \in H^2(\Omega)^3 \cap V$ be the unique solution of

$$\alpha \int_{\Omega} (y_2 - y^n) \cdot w + \nu \int_{\Omega} \nabla y_2 \cdot \nabla w + \int_{\Omega} y_1 \cdot \nabla y_2 \cdot w = \langle f^n, w \rangle_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d}, \quad \forall w \in V.$$

If y_1 satisfies $\|y_1\|_3 \leq \frac{\nu}{4c}$, then in view of (69),

$$\frac{\alpha}{2} \int_{\Omega} |\nabla y_2|^2 + \left(\frac{\nu}{2} - c\|y_1\|_3\right) \int_{\Omega} |P\Delta y_2|^2 \leq \frac{1}{2\nu} \|f^n\|_2^2 + \frac{\alpha}{2} \int_{\Omega} |\nabla y^n|^2$$

and consequently

$$\frac{\alpha}{2} \int_{\Omega} |\nabla y_2|^2 + \frac{\nu}{4} \int_{\Omega} |P\Delta y_2|^2 \leq \frac{1}{2\nu} \|f^n\|_2^2 + \frac{\alpha}{2} \int_{\Omega} |\nabla y^n|^2.$$

(71) then implies

$$\int_{\Omega} |\nabla y_2|^2 + \frac{\nu}{2\alpha} \int_{\Omega} |P\Delta y_2|^2 \leq \frac{1}{\nu} \left(\frac{1}{\alpha} \sum_{k=0}^n \|f^k\|_2^2 + \nu \int_{\Omega} |\nabla y^0|^2 \right). \tag{72}$$

We now use that there exists a constant $c_3 > 0$ such that, for all $n \in \mathbb{N}$ $\|y_2\|_3 \leq c_3\|\nabla y_2\|_2$ to obtain

$$\|y_2\|_3^2 \leq \frac{c_3^2}{\nu} \left(\frac{1}{\alpha} \sum_{k=0}^n \|f^k\|_2^2 + \nu \int_{\Omega} |\nabla u_0|^2 \right).$$

Invoking assumption (67), we conclude that $\|y_2\|_3 \leq \frac{\nu}{4c}$.

Eventually, we introduce the application $T : \mathcal{C} \rightarrow \mathcal{C}$, $y_1 \mapsto y_2$ where \mathcal{C} is the closed convex set of V defined by $\mathcal{C} := \{y \in V, \frac{\nu}{4c} \geq \|y\|_3\}$. Let us check that T is continuous. Let $y_1 \in \mathcal{C}$ et $z_1 \in \mathcal{C}$, $y_2 = T(y_1)$ et $z_2 = T(z_1)$ so that

$$\alpha \int_{\Omega} (z_2 - y_2) \cdot w + \nu \int_{\Omega} \nabla(z_2 - y_2) \cdot \nabla w + \int_{\Omega} y_1 \cdot \nabla(y_2 - z_2) \cdot w + \int_{\Omega} (y_1 - z_1) \cdot \nabla z_2 \cdot w = 0$$

for all $w \in V$ and then for $w = z_2 - y_2$

$$\begin{aligned} \alpha \int_{\Omega} |z_2 - y_2|^2 + \nu \int_{\Omega} |\nabla(z_2 - y_2)|^2 &\leq \left| \int_{\Omega} (y_1 - z_1) \cdot \nabla z_2 \cdot (z_2 - y_2) \right| \\ &\leq c\|\nabla(y_1 - z_1)\|_2 \|\nabla z_2\|_2 \|z_2 - y_2\|_3 \\ &\leq c\|\nabla(y_1 - z_1)\|_2 \end{aligned}$$

using (72); this implies the continuity of \mathcal{T} . On the other hand, since $\mathcal{T}(\mathcal{C})$ is a bounded set of $H^2(\Omega)^3$, \mathcal{T} is relatively compact. The Schauder Theorem allows to affirm that \mathcal{T} has a fixed point $y \in \mathcal{C}$, that is, a solution $y^{n+1} \in \mathcal{C}$ of (56). \square

Remark 4 Under the hypothesis of Proposition 10, suppose moreover that

$$C_{\alpha, \nu} := \nu^{-5/2} \alpha^{-1/2} \left(\alpha^{-1} \sum_{k=0}^{n-1} \|f^k\|_2^2 + \nu \|\nabla y^0\|_2^2 \right)$$

is small enough, then, the solution of (56) is unique.

Indeed, let $n \in \mathbb{N}$ and let $y_1^{n+1}, y_2^{n+1} \in \mathbf{V}$ be two solutions of (56). Let $Y := y_1^{n+1} - y_2^{n+1}$. Then,

$$\alpha \int_{\Omega} Y \cdot w + \nu \int_{\Omega} \nabla Y \cdot \nabla w + \int_{\Omega} y_2^{n+1} \cdot \nabla Y \cdot w + \int_{\Omega} Y \cdot \nabla y_1^{n+1} \cdot w = 0 \quad \forall w \in \mathbf{V}$$

and in particular, for $w = Y$ (since $\int_{\Omega} y_2^{n+1} \cdot \nabla Y \cdot Y = 0$)

$$\begin{aligned} \alpha \int_{\Omega} |Y|^2 + \nu \int_{\Omega} |\nabla Y|^2 &= - \int_{\Omega} Y \cdot \nabla y_1^{n+1} \cdot Y = \int_{\Omega} Y \cdot \nabla Y \cdot y_1^{n+1} \\ &\leq c \|y_1^{n+1}\|_{\infty} \|\nabla Y\|_2 \|Y\|_2 \\ &\leq c \|\nabla y_1^{n+1}\|_2^{1/2} \|P \Delta y_1^{n+1}\|_2^{1/2} \|\nabla Y\|_2 \|Y\|_2 \\ &\leq \alpha \|Y\|_2^2 + \frac{c}{\alpha} \|\nabla y_1^{n+1}\|_2 \|P \Delta y_1^{n+1}\|_2 \|\nabla Y\|_2^2 \end{aligned}$$

and therefore $(\nu - \frac{c}{\alpha} \|\nabla y_1^{n+1}\|_2 \|P \Delta y_1^{n+1}\|_2) \|\nabla Y\|_2^2 \leq 0$. Moreover, from (63),

$$\|\nabla y_1^{n+1}\|_2 \|P \Delta y_1^{n+1}\|_2 \leq \frac{2\alpha^{1/2}}{\nu^{3/2}} \left(\frac{1}{\alpha} \sum_{k=0}^n \|f^k\|_2^2 + \nu \|\nabla y^0\|_2^2 \right).$$

Therefore, there exists a constant $c > 0$ such that if $C_{\alpha, \nu} < c$, then, arguing as in the 2D case, $\|\nabla Y\|_2^2 \leq 0$ and $Y = 0$.

As in the 2D case, Proposition 10 then allows, following the Proof of Theorem 3, to obtain an estimate of $\|y^{n+1} - y^n\|_2$ in term of the parameter α .

Theorem 5 Assume that Ω is C^2 , that $\{f^n\}_{n \in \mathbb{N}}$ is a sequence in $L^2(\Omega)^3$ satisfying $\alpha^{-1} \sum_{k=0}^{+\infty} \|f^k\|_2^2 < +\infty$. Assume moreover that $y^0 \in \mathbf{V}$ and that for all $n \in \mathbb{N}$, $y^{n+1} \in H^2(\Omega)^3 \cap \mathbf{V}$ is a solution of (56) satisfying (63). Then, the sequence $(y^n)_n$ satisfies (66).

Eventually, adapting the Proof of Theorem 4, we get the following convergence result.

Theorem 6 Suppose $F \in L^2(0, T; L^2(\Omega)^3)$, $y^0 \in \mathbf{V}$, for all $n \in \{0, \dots, N - 1\}$, α and f^n are given by (4) and $y^{n+1} \in \mathbf{V}$ solution of (56). If $C(y^0, F) := \|F\|_{L^2(0, T; L^2(\Omega)^3)}^2 + \nu \|y^0\|_{H_0^1(\Omega)^3}^2 \leq C\nu^3$ for some $C > 0$ and α is large enough, then for any $n \geq 0$, the minimizing sequence $\{y_k^{n+1}\}_{k \in \mathbb{N}}$ defined by (54) strongly converges to the unique of solution of (56).

Remark 5 We have considered regular solutions in the case $d = 3$ in order to be able to prove the uniform property $\tau_3(y_k^n) \leq c_1 < 1$ for some c_1 independent of k and n , i.e. $\|y_k^n\|_{H_0^1(\Omega)^3} \leq c_1 M^{-1}(\alpha\nu^3)^{1/4}$. Actually, for regular solutions, Proposition 10 implies that $\|y_k^n\|_{H_0^1(\Omega)^3} \leq C$ for some C independent of α , which is sufficient for α large enough. By considering weak solutions, we can only prove that $\|y_k^n\|_{H_0^1(\Omega)^3} \leq C\alpha^{1/2}$ for some $C > 0$ (see (58)) which does not imply $\tau_3(y_k^n) \leq c_1 < 1$.

4 Numerical illustrations

We discuss in this section numerical experiments based on finite element approximations in space for two geometries of \mathbb{R}^2 : the celebrated channel with a backward facing step and the semi-circular driven cavity introduced in [5]. In both cases, the velocity of the fluid is imposed on the boundary. We first start with the case $\alpha = 0$ in (14) (discussed in [11]) allowing, first to get the solution of (1) as time becomes large and secondly, to enhance the gain of the optimization of the descent step parameter λ_k in (30). Then, for the semi driven cavity, we consider the cases $\alpha = 0$ and $\alpha > 0$ applied to the resolution of the backward Euler scheme (51). In a final part, we briefly compare the computational cost of this least-squares approach with standard explicit and semi-explicit scheme.

The numerical simulations are performed with the FreeFem++ package (see [8]). Regular triangular meshes are used together with the $\mathbb{P}_2/\mathbb{P}_1$ Taylor-Hood finite element, satisfying the Ladyzenskaia-Babushka-Brezzi condition of stability.

4.1 Steady case: two dimensional channel with a backward facing step

We consider in the steady situation the test problem of a two-dimensional channel with a backward facing step, described for instance in Section 45 of [6] (see also [9]). The geometry is depicted in Fig. 1. Dirichlet conditions of the Poiseuille type are imposed on the inflow and outflow sides Γ_1 and Γ_2 of the channel: we impose $y = (4(H - x_2)(x_2 - h)/(H - h)^2, 0)$ on Γ_1 and $y = (4(H - h)x_2(H - x_2)/H^2, 0)$ on Γ_2 , with $h = 1$, $H = 3$, $l = 3$ and $L = 30$. On the remaining part $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$, the fluid flow is imposed to zero. The external force f is zero.

We consider the extremal problem (16) to solve the steady Navier–Stokes Eq. (3) with here $\alpha = 0$. We compare the descent algorithm (30) based on the descent direction Y_1 with the conjugate gradient (CG) algorithm used in [11]. In both cases, the initial guess is defined as the solution of the corresponding Stokes problem. Moreover, the scalar extremal problem with respect to λ_k in (30) is performed with the Newton–Rasphon method.

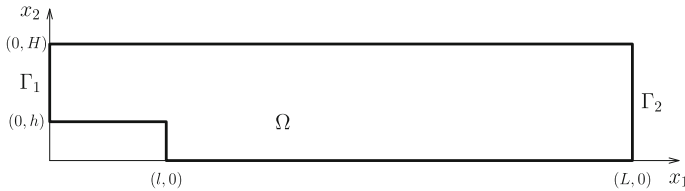


Fig. 1 A two-dimensional channel with a step

We start with a large value of $\nu = 1/150$. Table 1 reports the evolution of the quantity $\|y_{k+1} - y_k\|_{H_0^1(\Omega)^2} / \|y_k\|_{H_0^1(\Omega)^2}$ with respect to the iterate k associated to the algorithms (30), (30) with fixed step $\lambda_k = 1$ and CG respectively. We also consider the so-called by analogy damped quasi newton method

$$\begin{cases} y_0 \in \mathbf{V} \text{ given,} \\ y_{k+1} = y_k - \lambda_k \tilde{Y}_{1,k}, \quad k \geq 0, \\ \lambda_k = \operatorname{argmin}_{\lambda \in [0,m]} E(y_k - \lambda \tilde{Y}_{1,k}) \end{cases} \tag{73}$$

with $\tilde{Y}_{1,k} \in \mathbf{V}$ the solution of the formulation

$$\begin{aligned} & \alpha \int_{\Omega} \tilde{Y}_{1,k} \cdot w + \nu \int_{\Omega} \nabla \tilde{Y}_{1,k} \cdot \nabla w + \int_{\Omega} (y_0 \cdot \nabla \tilde{Y}_{1,k} + \tilde{Y}_{1,k} \cdot \nabla y_0) \cdot w \\ & = -\alpha \int_{\Omega} v_k \cdot w - \nu \int_{\Omega} \nabla v_k \cdot \nabla w, \forall w \in \mathbf{V} \end{aligned}$$

and $v_k \in \mathbf{V}$ the corrector (associated to y_k) solution of (17).

A regular mesh composed of 14 143 triangles and 7 360 vertices is used. Table 2 reports the evolution of the norm of the corrector $\|v_k\|_{H_0^1(\Omega)^2}$, an upper bound of $\|y - y_k\|_{H_0^1(\Omega)^2}$, according to Proposition 5. As expected in view of the results in Sect. 2.2, the descent algorithm (30) based on $Y_{1,k}$ is much faster than the CG algorithm. Moreover, the optimal values for the optimal step λ_k are close to one, so that the Newton method provides a similar speed of convergence. As the norm of $Y_{1,k}$ goes to zero with k , the term factor of λ^2 in (33) gets small, and the optimal λ_k gets close to one. Remark as well that the algorithm (73) offers an excellent speed of convergence. In term of CPU times, algorithms (30) and (73) require about 53 and 108 seconds respectively and leads to the same approximation. We have notably $\|\nabla \cdot y\|_{L^2(\Omega)} / |\Omega| \approx 1.83 \times 10^{-4}$.

For smaller values of ν , the results are qualitatively different. Table 3 reports some norms with respect to k for $\nu = 1/700$. We observe, from the last column, that the Newton method for which λ_k is fixed to one does not converge anymore. Actually, the Newton method, when initialized with the solution of the corresponding Stokes problem, diverges for $\nu \leq 1/250$. On the other hand, the optimization of the step λ_k produces a very fast convergence of the sequence $\{y_k\}_{(k>0)}$. Observe here that the values for the optimal λ_k are not close to one, during the first iterations. We obtain notably $\|\nabla \cdot y\|_{L^2(\Omega)} / |\Omega| \approx 5.78 \times 10^{-2}$. In agreement with Theorem 1, we observe

Table 1 2D channel geometry; $\nu = 1/150$; Evolution of $\|y_{k+1} - y_k\|_{H_0^1(\Omega)^2} / \|y_k\|_{H_0^1(\Omega)^2}$ with respect to k

# iterate k	(30) with $\lambda_k = 1$	(30)	(73)	CG
1	4.44×10^{-1}	3.79×10^{-1} (0.854)	3.79×10^{-1}	5.21×10^{-2}
2	1.95×10^{-1}	1.81×10^{-1} (0.957)	1.59×10^{-1}	4.19×10^{-2}
3	5.60×10^{-2}	4.04×10^{-2} (0.994)	4.37×10^{-2}	3.27×10^{-2}
4	3.98×10^{-3}	2.22×10^{-3} (1.001)	6.05×10^{-3}	2.94×10^{-2}
5	2.08×10^{-5}	5.71×10^{-6} (0.999)	6.80×10^{-3}	2.56×10^{-2}
6	5.91×10^{-10}	4.95×10^{-11} (1)	9.89×10^{-4}	2.29×10^{-2}
7	4.88×10^{-15}	3.29×10^{-15} (1)	9.00×10^{-4}	2.21×10^{-2}
8	—	—	1.48×10^{-4}	2.02×10^{-2}
9	—	—	9.55×10^{-5}	1.95×10^{-2}
10	—	—	2.09×10^{-5}	1.81×10^{-2}
11	—	—	1.39×10^{-5}	1.76×10^{-2}
12	—	—	3.17×10^{-6}	1.72×10^{-2}
13	—	—	1.83×10^{-6}	1.67×10^{-2}
14	—	—	3.80×10^{-7}	1.65×10^{-2}
26	—	—	4.32×10^{-13}	1.12×10^{-2}
200	—	—	—	2.09×10^{-5}

Table 2 2D channel geometry; $\nu = 1/150$; evolution of $\|v_k\|_{H_0^1(\Omega)^2} = \sqrt{2E(y_k)}$ with respect to k

# iterate k	(30) with $\lambda_k = 1$	(30)	(73)	CG
1	5.46×10^{-2}	5.46×10^{-2}	5.47×10^{-2}	5.46×10^{-2}
2	2.39×10^{-2}	2.22×10^{-2}	2.22×10^{-2}	3.70×10^{-2}
3	4.95×10^{-3}	4.60×10^{-3}	5.45×10^{-3}	2.91×10^{-2}
4	3.20×10^{-4}	1.56×10^{-4}	9.32×10^{-4}	2.49×10^{-2}
5	1.53×10^{-6}	5.43×10^{-7}	5.19×10^{-4}	2.20×10^{-2}
6	3.65×10^{-11}	4.22×10^{-12}	1.71×10^{-4}	1.99×10^{-2}
7	6.54×10^{-16}	2.54×10^{-16}	1.71×10^{-4}	1.84×10^{-2}
8	—	—	7.85×10^{-5}	1.70×10^{-2}
9	—	—	2.47×10^{-5}	1.60×10^{-2}
10	—	—	8.95×10^{-6}	1.51×10^{-2}
11	—	—	3.42×10^{-6}	1.43×10^{-2}
12	—	—	1.20×10^{-6}	1.36×10^{-2}
13	—	—	4.25×10^{-7}	1.30×10^{-2}
14	—	—	1.36×10^{-7}	1.24×10^{-2}
26	—	—	1.59×10^{-14}	6.25×10^{-3}
200	—	—	—	1.55×10^{-5}

Table 3 2D channel geometry; $\nu = 1/700$; results for the algorithm (30)

iterate k	$\frac{\ y_{k+1} - y_k\ _{H_0^1(\Omega)^2}}{\ y_k\ _{H_0^1(\Omega)^2}}$	$\sqrt{2E(y_k)}$	λ_k	$\sqrt{2E(y_k)}$ with $\lambda_k = 1$
1	7.15×10^{-1}	5.46×10^{-2}	0.727	5.46×10^{-2}
2	1.42×10^{-4}	2.79×10^{-2}	4.77×10^{-5}	3.45×10^{-2}
3	2.07×10^{-1}	2.79×10^{-2}	2.01×10^{-2}	8.08×10^{-2}
4	3.53×10^{-1}	2.73×10^{-2}	0.958	5.34×10^{-2}
5	9.13×10^{-2}	7.27×10^{-3}	4.81×10^{-6}	2.40
6	6.24×10^{-2}	2.62×10^{-3}	1.73×10^{-3}	6.11×10^{-1}
7	2.02×10^{-2}	1.07×10^{-3}	0.358	3.944
8	3.69×10^{-3}	2.61×10^{-4}	0.521	9.85×10^1
9	7.52×10^{-4}	4.18×10^{-5}	1.098	8.18×10^1
10	9.88×10^{-6}	6.01×10^{-7}	0.963	4.38×10^4
11	3.87×10^{-6}	1.69×10^{-7}	1.032	1.09×10^4
12	6.82×10^{-11}	4.40×10^{-12}	0.9983	3.16×10^4
13	1.28×10^{-10}	2.88×10^{-12}	0.9999	1.57×10^5
14	6.87×10^{-15}	3.26×10^{-16}	1.	4.06×10^4

from Table 3 that the decrease of $\sqrt{E(y_k)}$ to zero is first linear and then becomes quadratic.

The algorithm (73) is a bit more robust than the Newton one as it converges for all ν satisfying $\nu \geq 1/290$ approximately. Finally, as discussed in [11], the CG algorithm converges and produces similar numerical values: the convergence is however much slower since about 350 iterates are needed to achieve $\sqrt{2E(y_k)}$ of the order 10^{-3} .

The algorithm (30) requires however the initial guess to be close enough to the solution. Initialized with the solution of the corresponding Stokes problem, it diverges for $\nu \leq 1/720$. A continuation method with respect to ν is then necessary in that case. Algorithm (30) is also robust with respect to the mesh size: with a twice finer mesh composed of 84 707 triangles and 43 069 vertices, the convergence $\|y^{k+1} - y^k\|_{H_0^1(\Omega)^2} \leq 10^{-12} \|y^k\|_{H_0^1(\Omega)^2}$ is observed after $k = 18$ iterates (instead of 14 for the coarser mesh) leading notably to $\|\nabla \cdot y\|_{L^2(\Omega)}/|\Omega| \approx 3.91 \times 10^{-2}$.

4.2 Steady case: 2D semi-circular cavity

We now consider the test discussed in [5]. The geometry is a semi-disk $\Omega = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 < 1/4, x_2 \leq 0\}$ depicted on Fig. 2. The velocity is imposed to $y = (g, 0)$ on $\Gamma_0 = \{(x_1, 0) \in \mathbb{R}^2, |x_1| < 1/2\}$ with g vanishing at $x_1 = \pm 1/2$ and close to one elsewhere: we take $g(x_1) = (1 - e^{100(x_1-1/2)})(1 - e^{-100(x_1+1/2)})$. On the rest $\Gamma_1 = \{(x_1, x_2) \in \mathbb{R}^2, x_2 < 0, x_1^2 + x_2^2 = 1/4\}$ of the boundary the velocity is fixed to zero.

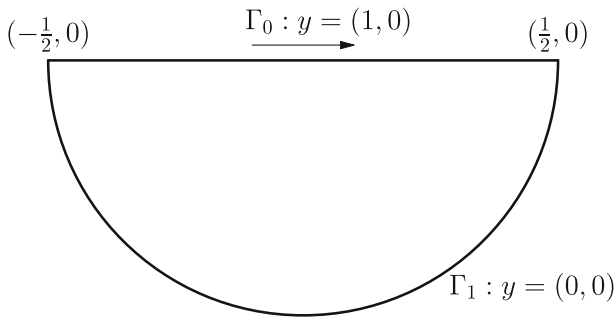


Fig. 2 Semi-disk geometry

For a regular triangular mesh, composed of 79 628 triangles and 40 205 vertices, leading to a mesh size $h \approx 6.23 \times 10^{-3}$, the Newton method ($\lambda_k = 1$) initialized with the corresponding Stokes solution, converges up to $\nu^{-1} \approx 500$. On the other hand, the algorithm (30) still converges up to $\nu^{-1} \approx 910$. Figures 3 depicts the streamlines of the steady state solution corresponding to $\nu^{-1} = 500$ and to $\nu^{-1} = i \times 10^3$ for $i = 1, \dots, 8$. The values used to plot the stream function are given in Table 6. The figures are in very good agreements with those depicted in [5]. The solution corresponding to $\nu^{-1} = 500$ is obtained from the sequence given (30) initialized with the Stokes solution. Eight iterates are necessary to achieve $\sqrt{2E(y)} \approx 3.4 \times 10^{-17}$. The stopping criterion is $\|y_{k+1} - y_k\|_{H_0^1(\Omega)^2} \leq 10^{-12} \|y_k\|_{H_0^1(\Omega)^2}$. Tables 4 and 5 collect some values for $\nu = 1/500$ and $\nu = 1/700$. Then, the other solutions are obtained by a continuation method with respect to ν taking $\delta\nu^{-1} = 500$. For instance, the solution corresponding to $\nu^{-1} = 5000$ is obtained from the algorithm (30) initialized with the steady solution corresponding to $\nu^{-1} = 4500$. Table 9 reports the history of the continuation method and highlights the efficiency of the algorithm (30): up to $\nu^{-1} = 9500$, few iterations achieve the convergence of the minimizing sequence $\{y_k\}_{k \in \mathbb{N}}$. From $\nu^{-1} = 10^4$, with a finer mesh (for which the mesh size is $h \approx 4.37 \times 10^{-3}$), $\delta\nu$ is reduced to $\delta\nu^{-1} = 100$ and leads to convergence beyond $\nu^{-1} = 15, 000$. Table also reports the minimal values of the streamline function ψ which compare very well with those of [5].

The case $\alpha > 0$ leads to similar results, in full agreement with the theoretical Sect. 2. For $\nu = 1/1000$, Table 7 reports results of the algorithm (30) for $\alpha \in \{10^{-1}, 1, 10, 100\}$. As expected, the gain of coercivity of the functional E involves a notable robustness and speed up of the algorithm. Recall that for $\alpha = 0$ and $\nu = 1/1000$, algorithm (30) does not converge. For $\alpha = 10^{-1}$, we observe the convergence after 8 iterates. For a fixed value of ν , this number of iterates decreases as α gets larger. Actually, we observe that when $\alpha\sqrt{\nu} = \mathcal{O}(1)$, algorithm (30) converges after few iterates with λ_k close to 1 for all k . Moreover, when $\alpha\nu = \mathcal{O}(1)$, the convergence is achieved after one iterate only. This behavior suggests that one may recover the solution of the steady Navier–Stokes system (corresponding to $\alpha = 0$) by using a continuation procedure with respect to the parameter α decreasing to zero. We easily check, for any $\alpha \geq 0$, the estimate $\|\nabla(y_\alpha - y_{\alpha=0})\|_2 \leq c_p \sqrt{\frac{\alpha}{\nu}} \|y_{\alpha=0}\|_2$ where y_α solves (5) with $g = 0$, and c_p the Poincaré constant. For $\nu = 1/5000$, Table

Table 4 Semi-disk geometry; $\nu = 1/500$; results for the algorithm (30)

# iterate k	$\frac{\ y_{k+1} - y_k\ _{H_0^1(\Omega)^2}}{\ y_k\ _{H_0^1(\Omega)^2}}$	$\sqrt{2E(y_k)}$	λ_k	$\sqrt{2E(y_k)}$ with $\lambda_k = 1$
1	4.66×10^{-1}	8.51×10^{-3}	0.810	8.51×10^{-3}
2	2.03×10^{-1}	3.57×10^{-3}	0.716	4.09×10^{-3}
3	9.54×10^{-2}	1.36×10^{-3}	0.860	2.26×10^{-3}
4	2.64×10^{-2}	3.38×10^{-4}	0.982	2.51×10^{-4}
5	1.53×10^{-3}	2.43×10^{-5}	1.002	3.47×10^{-6}
6	4.21×10^{-6}	6.36×10^{-8}	1.	1.54×10^{-9}
7	4.20×10^{-11}	6.50×10^{-13}	1.	1.52×10^{-16}
8	3.25×10^{-15}	3.45×10^{-17}	1.	3.39×10^{-17}

Table 5 Semi-disk geometry; $\nu = 1/700$; results for the algorithm (30)

# iterate k	$\frac{\ y_{k+1} - y_k\ _{H_0^1(\Omega)^2}}{\ y_k\ _{H_0^1(\Omega)^2}}$	$\sqrt{2E(y_k)}$	λ_k	$\sqrt{2E(y_k)}$ with $\lambda_k = 1$
1	4.89×10^{-1}	8.51×10^{-3}	0.702	8.51×10^{-3}
2	2.41×10^{-1}	4.43×10^{-3}	0.583	5.75×10^{-3}
3	1.48×10^{-1}	2.15×10^{-3}	0.510	5.27×10^{-3}
4	1.01×10^{-1}	1.15×10^{-3}	0.568	1.23×10^{-2}
5	6.83×10^{-2}	6.50×10^{-4}	0.931	4.40×10^{-3}
6	1.19×10^{-2}	1.76×10^{-4}	1.018	1.52×10^{-2}
7	3.35×10^{-4}	3.42×10^{-6}	0.999	5.06×10^{-3}
8	3.19×10^{-7}	4.22×10^{-9}	1.	1.45×10^{-2}
9	2.97×10^{-13}	2.30×10^{-15}	1.	1.85×10^{-2}
20	—	—	—	5.39

Table 6 Values used to plot the contours of the stream function

$-0.07, -0.0675, -0.065, -.05, -0.04, -0.03, -0.02, -0.01, \pm 10^{-4}, \pm 10^{-5}$
$\pm 10^{-7}, -10^{-10}, 0., 10^{-8}, 10^{-6}, 5 \times 10^{-4}, 10^{-3}, 2 \times 10^{-3}, 3 \times 10^{-3}, 4 \times 10^{-3}$
$5 \times 10^{-3}, 6 \times 10^{-3}, 7 \times 10^{-3}, 8 \times 10^{-3}, 9 \times 10^{-3}, 0.01$

8 reports the history of the continuation approach starting from $\alpha = 1.$ to $\alpha = 0$ with intermediate steps $\alpha \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}.$ Figure 4 depicts the evolution of the sequences $\{\sqrt{2E(y_k)}\}_k$ and $\{\lambda_k\}_k$ obtained from the algorithm (30) with $\alpha = 10^{-3}.$ The algorithm is initialized with the solution corresponding to $\alpha = 10^{-2}.$ The behavior of these sequences fully illustrates Theorem 1 and the robustness of the method: as λ_k increases, the decay of $\sqrt{2E(y_k)},$ initially low, gets larger and becomes

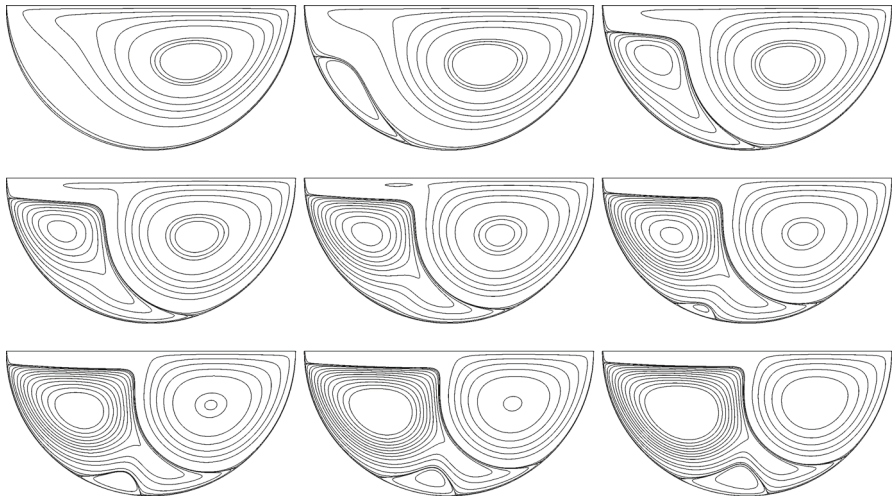


Fig. 3 Streamlines of the steady state solution for $\nu^{-1} = 500, 1000, 2000, 3000, 4000, 5000, 6000, 7000$ and $\nu^{-1} = 8000$

Table 7 Semi-disk geometry; $\nu = 1/1000$; results for the algorithm (30); $\sqrt{2E(y_k)}$ and λ_k with respect to α

k	$\alpha = 0.1$	$\alpha = 1.$	$\alpha = 10.$	$\alpha = 100.$
1	$6.59 \times 10^{-3}(0.65)$	$3.11 \times 10^{-3}(0.87)$	$1.04 \times 10^{-3}(0.99)$	$2.99 \times 10^{-4}(0.99)$
2	$3.54 \times 10^{-3}(0.63)$	$7.59 \times 10^{-4}(0.98)$	$3.68 \times 10^{-5}(0.99)$	$7.81 \times 10^{-7}(1.)$
3	$1.55 \times 10^{-3}(0.64)$	$5.94 \times 10^{-5}(1.00)$	$5.15 \times 10^{-8}(1.)$	$6.92 \times 10^{-12}(1.)$
4	$7.05 \times 10^{-4}(0.74)$	$4.84 \times 10^{-7}(1.)$	1.20×10^{-13}	5.50×10^{-17}
5	$2.76 \times 10^{-4}(1.01)$	$2.63 \times 10^{-11}(1.)$	—	—
6	$2.95 \times 10^{-5}(1.)$	1.03×10^{-17}	—	—
7	$1.55 \times 10^{-7}(1.)$	—	—	—
8	$1.99 \times 10^{-11}(1.)$	—	—	—
9	1.60×10^{-17}	—	—	—

very fast when λ_k is close to one. Figure 5 reports the streamlines of the solution for various values of α .

4.3 Unsteady case: 2D semi-circular cavity

We now use the least-squares method in order to solve iteratively the implicit Euler scheme (2). The parameter $\alpha = 1/\delta t$ is strictly positive. We remind that for ν approximately larger than $1/6600$, the unsteady solution converges as time evolves to the steady solution (corresponding to $\alpha = 0$) obtained in the previous section by a continuation technique. Actually, the iterative process due to the time discretization can also be seen as a continuation approach. We consider the value $\nu = 1/1000$. Fol-

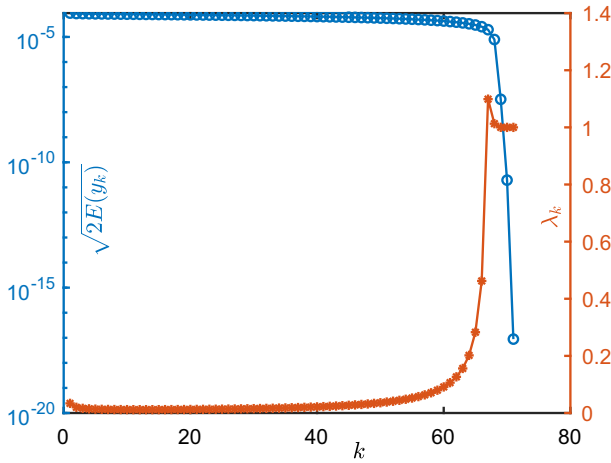


Fig. 4 Semi-disk geometry; $\nu = 1/5000$; evolution of $\sqrt{2E(y_k)}$ and λ_k w.r.t. k for α from 10^{-2} to 10^{-3}

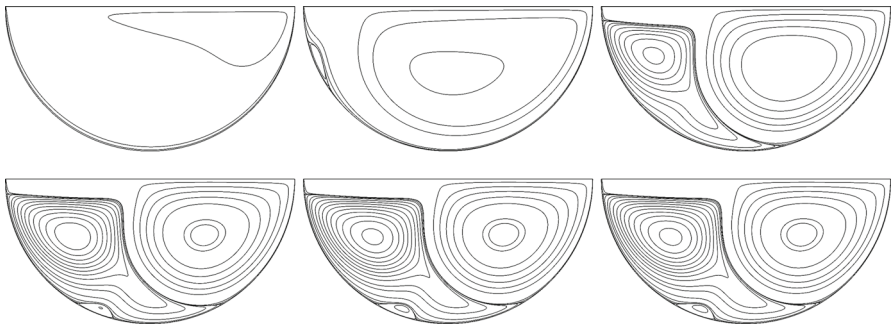


Fig. 5 Streamlines of the α -steady state solution for $\alpha = 1, 10^{-1}, 10^{-2}$ (Top) and $\alpha = 10^{-3}, 10^{-4}$ and 0. (Bottom)

Table 8 Semi-disk geometry; continuation method with respect to α for the solution of the steady Navier–Stokes equation; $\nu = 1/5000$

α	# it.	$\ \pi_\alpha\ _{L^2(\Omega)}$	$\ y_\alpha\ _{H_0^1(\Omega)^2}$	$\ y_\alpha - y_{\alpha=0}\ _{H_0^1(\Omega)^2}$
Stokes $\rightarrow 1$.	6	0.00684	7.659	4.28
$1. \rightarrow 10^{-1}$	37	0.011	7.4896	3.32
$10^{-1} \rightarrow 10^{-2}$	49	0.0297	7.8951	2.67
$10^{-2} \rightarrow 10^{-3}$	70	0.0360	8.0166	4.02×10^{-1}
$10^{-3} \rightarrow 10^{-4}$	12	0.0366	8.0278	4.03×10^{-2}
$10^{-4} \rightarrow 0$	4	0.0367	8.029	4.03×10^{-3}

lowing [5], we take as initial condition u_0 the steady-state solution corresponding to $\nu = 1/500$. For the value $\alpha = 200$ corresponding to the time discretization parameter $\delta t = 5 \times 10^{-3}$, we observe the convergence of the sequence $\{y_k^{n+1}\}_{k>0}$ after at most

Table 9 Semi-disk geometry; continuation method with respect to ν for the solution of the steady Navier–Stokes equation; $\alpha = 0$

ν^{-1}	# it.	$\ \pi\ _{L^2(\Omega)}$	$\ y_1\ _{H_0^1(\Omega)^2}$	$\ y_2\ _{H_0^1(\Omega)^2}$	$\min_{\Omega} \psi$	$\min_{\Omega} \psi$ [5]
Stokes \rightarrow 500	7	4.31×10^{-2}	4.462	2.489	-0.0766	–
500 \rightarrow 1000	7	4.07×10^{-2}	4.919	2.883	-0.0780	-0.0779
1000 \rightarrow 1500	6	3.99×10^{-2}	5.296	3.153	-0.0775	–
1500 \rightarrow 2000	6	3.93×10^{-2}	5.612	3.361	-0.0766	-0.0763
2000 \rightarrow 2500	5	3.88×10^{-2}	5.884	3.531	-0.0756	–
2500 \rightarrow 3000	5	3.83×10^{-2}	6.126	3.675	-0.0744	-0.0742
3000 \rightarrow 3500	5	3.79×10^{-2}	6.345	3.801	-0.0733	–
3500 \rightarrow 4000	5	3.75×10^{-2}	6.545	3.911	-0.0721	–
4000 \rightarrow 4500	6	3.71×10^{-2}	6.731	4.010	-0.0710	–
4500 \rightarrow 5000	6	3.67×10^{-2}	6.903	4.099	-0.0699	-0.0700
5000 \rightarrow 5500	6	3.64×10^{-2}	7.065	4.181	-0.0689	–
5500 \rightarrow 6000	6	3.60×10^{-2}	7.217	4.256	-0.0679	–
6000 \rightarrow 6500	6	3.57×10^{-2}	7.362	4.325	-0.0669	–
6500 \rightarrow 7000	6	3.53×10^{-2}	7.499	4.389	-0.0660	–
7000 \rightarrow 7500	5	3.50×10^{-2}	7.631	4.448	-0.0651	–
7500 \rightarrow 8000	5	3.47×10^{-2}	7.756	4.504	-0.0643	–
8000 \rightarrow 8500	6	3.44×10^{-2}	7.876	4.557	-0.0634	–
8500 \rightarrow 9000	6	3.41×10^{-2}	7.992	4.606	-0.0626	–

three iterations, for each n (except for $n = 0$ requiring 6 iterations). For the value $\alpha = 2000$ corresponding to $\delta t = 5 \times 10^{-4}$ (used in [5]), we observe the convergence of the sequence after one iterate. At time $T = 10$, the unsteady state solution is close to the solution of the steady Navier–Stokes equation: we compute that the sequence $\{y^n\}_{n=0,\dots,2000}$ satisfies $\|y^{2000} - y^{1999}\|_{L^2(\Omega)} / \|y^{2000}\|_{L^2(\Omega)} \approx 1.19 \times 10^{-5}$. $n = 2000$ corresponds to $T = 2000 \times \delta t = 10$. Figures 6 display the streamlines of the unsteady state solution corresponding to $\nu = 1/1000$ at time 0, 1, 2, 3, 4, 5, 6 and 7s to be compared with the streamlines of the steady solution depicted in Fig. 5. These figures are in full agreement with [5].

Before comparing with standard time marching schemes, let us make a comment on the algorithm (55). For α large, the optimal step λ_k in (55) equals one and the convergence with respect to k is achieved after one iterate. The convergent approximation $y^{n+1} := y_1^{n+1}$ then simply solves, for each n the following semi-implicit scheme (mentioned in [15, section 13.4])

$$\begin{aligned}
 & \int_{\Omega} \alpha y^{n+1} \cdot w + \nu \nabla y^{n+1} \cdot \nabla w + y^n \cdot \nabla y^{n+1} \cdot w + y^{n+1} \cdot \nabla y^n \cdot w \\
 & = \int_{\Omega} \alpha y^n \cdot w + \langle f^n, w \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \int_{\Omega} y^n \cdot \nabla y^n \cdot w, \quad \forall w \in V.
 \end{aligned}
 \tag{74}$$

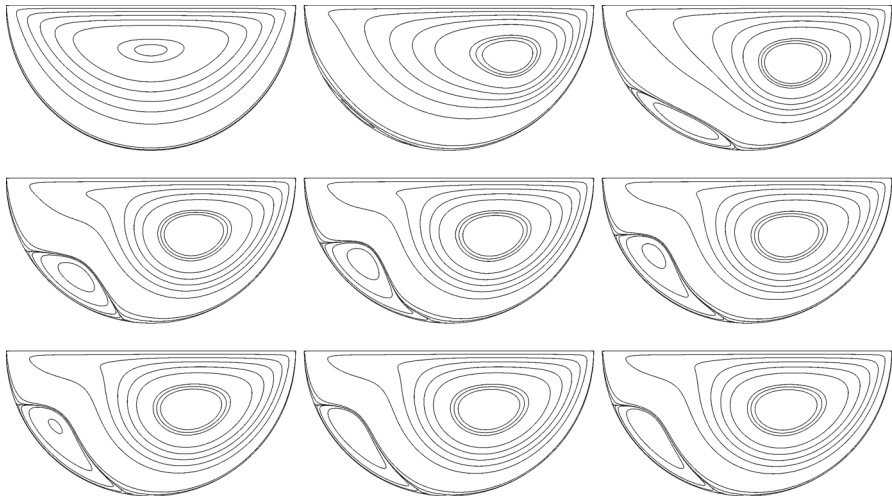


Fig. 6 Streamlines of the unsteady state solution for $\nu^{-1} = 1000$ at time $t = i, i = 0, \dots, 8s$

For δt of the order 10^{-3} , this first-order in time scheme displays, for $\nu = 1/1000$ and the semi-disk geometry of Fig. 2, very similar results than the conditionally stable partially explicit scheme (we refer to [19, Section 5.1])

$$\begin{aligned} & \int_{\Omega} \alpha y^{n+1} \cdot w + \nu \nabla y^{n+1} \cdot \nabla w + y^n \cdot \nabla y^n \cdot w \\ &= \int_{\Omega} \alpha y^n \cdot w + \langle f^n, w \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}, \end{aligned} \tag{75}$$

for all $w \in V$ and than the unconditionally stable scheme

$$\begin{aligned} & \int_{\Omega} \alpha y^{n+1} \cdot w + \nu \nabla y^{n+1} \cdot \nabla w + y^n \cdot \nabla y^{n+1} \cdot w \\ &= \int_{\Omega} \alpha y^n \cdot w + \langle f^n, w \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \end{aligned} \tag{76}$$

for all $w \in V$. In term of computational cost, the scheme (75) is as expected faster than the scheme (74): the ratio of the computational time to perform 10000 iterates (leading to $T = 10$) between (74) and (75) is approximatively equal to 1.65. The regular triangulation used corresponds to a mesh size h of the order of 6.23×10^{-3} making (75) stable. On the other hand, the computational times of (74) and (76) are equivalent: we observe a ratio equal to 1.05. For $\delta t = 10^{-2}$, scheme (75) is unstable. The convergence with respect to k of (55) (not anymore equivalent to (74)) is observed after two iterates. The ratio of computational time between (55) and (76) raises to 1.89. We observe that the approximation for (55) is much less sensitive to the variation of δt : we observe similar results than with $\delta t = 10^{-3}$ and than [5] where $\delta t = 5 \times 10^{-4}$ is used.

5 Conclusions and perspectives

We have rigorously analyzed a weak least-squares method introduced forty years ago in [2] allowing to solve a steady nonlinear Navier–Stokes equation, in the incompressible regime. This equation with a zero order term appears after any fully implicit time discretization of the unsteady Navier–Stokes equation. We have constructed a sequence converging strongly to the solution of the steady equation. Using a particular descent direction very appropriate for the analysis, this convergent sequence turns out to coincide with the sequence obtained using the damped Newton method to solve the underlying variational weak formulation. This globally convergent approach enjoys a quadratic rate of convergence after a finite number of iterates and is in particular much faster than the conjugate gradient method used in [2], [11]. Then, we have shown the convergence of the method, uniformly with respect to the time discretization, to solve the fully implicitly Euler scheme associated to the unsteady Navier–Stokes equation. When the time discretization is fine enough, each step of the damped Newton method simply reduced to the Newton one. In such a case, we obtained a proof of convergence of the Newton scheme to solve the unsteady Navier–Stokes. As far as we know, this proof is original. Numerical experiments have highlighted the robustness of the method, including for values of the viscosity coefficient of order 10^{-4} . We also emphasize that the least-squares approaches, employed here to treat the Navier–Stokes nonlinearity, can be used to solve other nonlinear equations, as formally done in [14] for a sublinear heat equation. Eventually, we may solve the unsteady Navier–Stokes system by a fully $L^2(0, T; H^{-1}(\Omega))$ least-squares approach. The underlying corrector solves an unsteady Stokes type equation; we refer to [12].

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