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Pointwise error estimates of linear finite element method for Neumann boundary value problems in a smooth domain

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Abstract

Pointwise error analysis of the linear finite element approximation for $-\Delta u + u = f$ in Ω, $\partial_n u = \tau$ on $\partial\Omega$, where Ω is a bounded smooth domain in $\mathbb{R}^N$, is presented. We establish $O(h^2|\log h|)$ and $O(h)$ error bounds in the L^∞- and $W^{1,\infty}$-norms respectively, by adopting the technique of regularized Green's functions combined with local H^1- and L^2-estimates in dyadic annuli. Since the computational domain Ω_h is only polyhedral, one has to take into account non-conformity of the approximation caused by the discrepancy $\Omega_h \neq \Omega$. In particular, the so-called Galerkin orthogonality relation, utilized three times in the proof, does not exactly hold and involves domain perturbation terms (or boundary-skin terms), which need to be addressed carefully. A numerical example is provided to confirm the theoretical result.

Mathematics Subject Classification Primary 65N30; Secondary 65N15

1 Introduction

We consider the following Poisson equation with a non-homogeneous Neumann boundary condition:

$$-\Delta u + u = f \quad \text{in} \quad \Omega, \quad \partial_n u = \tau \quad \text{on} \quad \Gamma := \partial\Omega, \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary Γ of C^∞-class, f is an external force, τ is a prescribed Neumann data, and ∂_n means the directional derivative with respect to the unit outward normal vector n to Γ. The linear (or P_1) finite element approximation to (1.1) is quite standard. Given an approximate polyhedral domain Ω_h whose vertices lie on Γ, one can construct a triangulation $\mathcal{T}_h$ of Ω_h, build a finite dimensional space V_h consisting of piecewise linear functions, and seek for $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h)_{\Omega_h} + (u_h, v_h)_{\Omega_h} = (\tilde{f}, v_h)_{\Omega_h} + (\tilde{\tau}, v_h)_{\Gamma_h} \quad \forall v_h \in V_h, \tag{1.2}$$

where $\Gamma_h := \partial\Omega_h$, and $\tilde{f}$ and $\tilde{\tau}$ denote extensions of f and τ, respectively. Then, the main result of this paper is the following pointwise error estimates in the L^∞- and $W^{1,\infty}$-norms:

$$\begin{aligned} \|\tilde{u} - u_h\|_{L^\infty(\Omega_h)} &\le Ch^2 |\log h| \, \|u\|_{W^{2,\infty}(\Omega)}, \\ \|\tilde{u} - u_h\|_{W^{1,\infty}(\Omega_h)} &\le Ch \, \|u\|_{W^{2,\infty}(\Omega)}, \end{aligned} \tag{1.3}$$

where h denotes the mesh size of $\mathcal{T}_h$, and $\tilde{u}$ is an arbitrary extension of u (of course, the way of extension must enjoy some stability, cf. Sect. 2.3 below).

Regarding pointwise error estimates of the finite element method, there have been many contributions since 1970s (for example, see the references in [14]), and, consequently, standard methods to derive them are now available. The strategy of those methods is briefly explained as follows. By duality, analysis of L^∞- or $W^{1,\infty}$-error of $u - u_h$ may be reduced to that of $W^{1,1}$-error between a regularized Green's function g, with singularity near $x_0 \in \Omega$, and its finite element approximation g_h. To deal with $\|\nabla(g - g_h)\|_{L^1(\Omega)}$ in terms of energy norms, it is estimated either by $\sum_{j=0}^J d_j^{N/2} \|\nabla(g - g_h)\|_{L^2(\Omega\cap A_j)}$ or by $\|\sigma^{N/2}\nabla(g - g_h)\|_{L^2(\Omega)}$, where $\{d_j\}_{j=0}^J$ are radii of dyadic annuli A_j shrinking to x_0 with the minimum $d_J = Kh$, whereas $\sigma(x) := (|x - x_0|^2 + \kappa h^2)^{1/2}$. The two strategies may be regarded as using discrete and continuous weights, respectively, and basically lead to the same results. In this paper, we employ the first approach, in which scaling heuristics seem to work easier (in the second approach one actually needs to introduce an artificial parameter $\lambda \in (0, 1)$ to avoid singular integration, which makes the weighted norm slightly complicated, cf. Remark 8.4.4 of [3]).

The main difficulty of our problem lies in the non-conformity $V_h \not\subset H^1(\Omega)$ arising from the discrepancy $\Omega_h \neq \Omega$ and $\Gamma_h \neq \Gamma$, which we refer to as *domain perturbation*. In fact, the so-called Galerkin orthogonality relation (or consistency) does not exactly hold, and hence the standard methodology of error estimate cannot be directly applied. This issue was already considered in classical literature (see [18, Section 4.4] or [5, Section 4.4]) as long as energy-norm (i.e. H^1) error estimates for a Dirichlet problem are concerned. However, there are much fewer studies of error analysis in other norms or for other boundary value problems, which take into account domain perturbation. For example, Barrett and Elliott [2], Čermák [4] gave optimal L^2-error estimates for a Robin boundary value problem.

As for pointwise error estimates, the issue of domain perturbation was mainly treated only for a homogeneous Dirichlet problem in a convex domain. In this case, one has a conforming approximation $V_h \cap H_0^1(\Omega_h) \subset H_0^1(\Omega)$ with the aid of the zero extension, which makes error analysis simpler. This situation was studied for elliptic problems in [1,17] and for parabolic ones in [8,19]. Although an idea to treat $\Omega_h \not\subset \Omega$ in the case of L^∞-analysis is found in [17, p. 2], it does not seem to be directly applicable to $W^{1,\infty}$-analysis or to Neumann problems. In [8,14,16], they considered Neumann problems in a smooth domain assuming that triangulations exactly fit a curved boundary, where one need not take into account domain perturbation. This assumption, however, excludes the use of usual Lagrange finite elements. The P_2-isoparametric finite element analysis for a Dirichlet problem ($N = 2$) was shown in [20], where the rate of convergence $O(h^{3-\epsilon})$ in the L^∞-norm was obtained.

The aim of this paper is to present pointwise error analysis of the finite element method taking into account full non-conformity caused by domain perturbation. We emphasize that a rigorous proof of such results for Neumann problems remained open even in the simplest setting, i.e., the linear finite element approximation. Therefore, in the present paper, we focus on showing how the issues of domain perturbation can be managed and confine ourselves to the linear approximation. Our main result (1.3) implies that domain perturbation does not affect the rate of convergence in the L^∞- and $W^{1,\infty}$-norms known for the case $\Omega_h = \Omega$ when P_1-elements are used to approximate both a curved domain and a solution. We would like to extend this to higher order cases (e.g. isoparametric finite elements) in future work, by adopting the strategy developed in this paper to manage domain perturbation.

Finally, let us make a comment concerning the opinion that the issue of $\Omega_h \neq \Omega$ is similar to that of numerical integration (see [16, p. 1356]). As mentioned in the same paragraph there, if a computational domain is extended (or transformed) to include Ω and a restriction (or transformation) operator to Ω is applied, then one can disregard the effect of domain perturbation (higher-order schemes based on such a strategy are proposed e.g. in [6]). On the other hand, since implementing such a restriction operator precisely for general domains is non-trivial in practical computation, some approximation of geometric information for Ω should be incorporated in the end. Thereby one needs to more or less deal with domain perturbation in error analysis, and, in our opinion, its rigorous treatment would be quite different from that of numerical integration.

The organization of this paper is as follows. Basic notations are introduced in Sect. 2, together with boundary-skin estimates and a concept of dyadic decomposition. In Sect. 3, we present the main result (Theorem 3.1) and reduce its proof to $W^{1,1}$-error estimate of $g - g_h$. The weighted H^1- and L^2-error estimates of $g - g_h$ are shown in Sects. 4 and 5, respectively, which are then combined to complete the proof of Theorem 3.1 in Sect. 6. A numerical example is given to confirm the theoretical result in Sect. 7. Throughout this paper, $C > 0$ will denote generic constants which may be different at each occurrence; its dependency (or independency) on other quantities will often be mentioned as well. However, when it appears with sub- or super-scripts (e.g., C_{0E}, C'), we do not treat it as generic.

2 Preliminaries

2.1 Basic notation

Recall that $\Omega \subset \mathbb{R}^N$ is a bounded C^∞-domain. We employ the standard notation of the Lebesgue spaces $L^p(\Omega)$, Sobolev spaces $W^{s,p}(\Omega)$ (in particular, $H^s(\Omega) := W^{s,2}(\Omega)$), and Hölder spaces $C^{m,\alpha}(\overline{\Omega})$. Throughout this paper we assume the regularity $u \in W^{2,\infty}(\Omega)$ for (1.1), which is indeed true if $f \in C^\alpha(\overline{\Omega})$ and $\tau \in C^\alpha(\overline{\Gamma})$ for some $\alpha \in (0, 1)$.

Given a bounded domain $D \subset \mathbb{R}^N$, both of the N-dimensional Lebesgue measure of D and the $(N-1)$-dimensional surface measure of ∂D are simply denoted by $|D|$ and $|\partial D|$, as far as there is no fear of confusion. Furthermore, we let $(\cdot, \cdot)_D$ and $(\cdot, \cdot)_{\partial D}$ be the $L^2(D)$- and $L^2(\partial D)$-inner products, respectively, and define the bilinear form

$$a_D(u, v) := (\nabla u, \nabla v)_D + (u, v)_D, \quad u, v \in H^1(D),$$

which is simply written as $a(u, v)$ when $D = \Omega$, and as $a_h(u, v)$ when $D = \Omega_h$ (to be defined below).

Letting Ω_h be a polyhedral domain, we consider a family of triangulations $\{\mathcal{T}_h\}_{h\downarrow 0}$ of Ω_h which consist of closed and mutually disjoint simplices. We assume that $\{\mathcal{T}_h\}_{h\downarrow 0}$ is quasi-uniform, that is, every $T \in \mathcal{T}_h$ contains (resp. is contained in) a ball with the radius ch (resp. h), where $h := \max_{T\in\mathcal{T}_h} h_T$ with $h_T := \operatorname{diam} T$. The boundary mesh on $\Gamma_h := \partial\Omega_h$ inherited from $\mathcal{T}_h$ is denoted by $\mathcal{S}_h$, namely, $\mathcal{S}_h = \{S \subset \Gamma_h \mid S$ is an $(N-1)$-dimensional face of some $T \in \mathcal{T}_h\}$. We then assume that the vertices of every $S \in \mathcal{S}_h$ belong to Γ, that is, Γ_h is essentially a linear interpolation of Γ.

The linear (or P_1) finite element space V_h is given in a standard manner, i.e.,

$$V_h = \left\{v_h \in C(\overline{\Omega}_h) : v_h|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h\right\},$$

where $P_k(T)$ stands for the polynomial functions defined in T with degree $\le k$.

Let us recall a well-known result of an interpolation operator (also known as a local regularization operator) $\mathcal{I}_h : H^1(\Omega_h) \to V_h$ satisfying the following property (see [3, Section 4.8]):

$$\|\nabla^k (v - \mathcal{I}_h v)\|_{L^p(T)} \le C_{\mathcal{I}} h_T^{m-k} \|\nabla^m v\|_{L^p(M_T)} \quad k = 0, 1, \; m = 1, 2, \; v \in W^{m,p}(\Omega_h),$$

where $M_T := \bigcup\{T' \in \mathcal{T}_h : T' \cap T \neq \emptyset\}$ is a macro-element of $T \in \mathcal{T}_h$. The constant $C_{\mathcal{I}}$ depends on c, k, m, p and on a reference element; especially it is independent of v and h_T. We also use the trace estimate

$$\|v\|_{L^2(\Gamma_h)} \le C \|v\|_{L^2(\Omega_h)}^{1/2} \|v\|_{H^1(\Omega_h)}^{1/2},$$

where C depends on the $C^{0,1}$-regularity of Ω_h and thus it is uniformly bounded by that of Ω for $h \le 1$.

2.2 Boundary-skin estimates

To examine the effects due to the domain discrepancy $\Omega_h \neq \Omega$, we introduce a notion of tubular neighborhoods $\Gamma(\delta) := \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Gamma) \le \delta\}$. It is known that (see [9, Section 14.6]) there exists $\delta_0 > 0$, which depends on the $C^{1,1}$-regularity of Ω, such that each $x \in \Gamma(\delta_0)$ admits a unique representation

$$x = \bar{x} + tn(\bar{x}), \quad \bar{x} \in \Gamma,\ t \in [-\delta_0, \delta_0].$$

We denote the maps $\Gamma(\delta_0) \to \Gamma;\ x \mapsto \bar{x}$ and $\Gamma(\delta_0) \to \mathbb{R};\ x \mapsto t$ by $\pi(x)$ and $d(x)$, respectively (actually, π is an orthogonal projection to Γ and d agrees with the signed-distance function). The regularity of Ω is inherited to that of π, d, and n (cf. [7, Section 7.8]).

In [12, Section 8] we proved that $\pi|_{\Gamma_h}$ gives a homeomorphism (and piecewisely a diffeomorphism) between Γ and Γ_h provided h is sufficiently small, taking advantage of the fact that Γ_h can be regarded as a linear interpolation of Γ (recall the assumption on $\mathcal{S}_h$ mentioned above). If we write its inverse map $\pi^* : \Gamma \to \Gamma_h$ as $\pi^*(x) = \bar{x} + t^*(\bar{x})n(\bar{x})$, then t^* satisfies the estimates $\|\nabla_\Gamma^k t^*\|_{L^\infty(\Gamma)} \le C_{kE} h^{2-k}$ for $k = 0, 1, 2$, where ∇_Γ means the surface gradient along Γ and where the constant depends on the $C^{1,1}$-regularity of Ω. This in particular implies that $\Omega_h \triangle \Omega := (\Omega_h \setminus \Omega) \cup (\Omega \setminus \Omega_h)$ and $\Gamma_h \cup \Gamma$ are contained in $\Gamma(\delta)$ with $\delta := C_{0E} h^2 < \delta_0$. We refer to $\Omega_h \triangle \Omega$, $\Gamma(\delta)$ and their subsets as *boundary-skin layers* or more simply as *boundary skins*.

Furthermore, we know from [12, Section 8] the following boundary-skin estimates:

$$\begin{aligned}
\left| \int_\Gamma f\, d\gamma - \int_{\Gamma_h} f \circ \pi\, d\gamma_h \right| &\le C\delta \|f\|_{L^1(\Gamma)}, \\
\|f\|_{L^p(\Gamma(\delta))} &\le C(\delta^{1/p} \|f\|_{L^p(\Gamma)} + \delta \|\nabla f\|_{L^p(\Gamma(\delta))}), \\
\|f - f \circ \pi\|_{L^p(\Gamma_h)} &\le C\delta^{1-1/p} \|\nabla f\|_{L^p(\Gamma(\delta))},
\end{aligned} \tag{2.1}$$

where one can replace $\|f\|_{L^1(\Gamma)}$ in $(2.1)_1$ by $\|f\|_{L^1(\Gamma_h)}$. As a version of $(2.1)_2$, we also need

$$\|f\|_{L^p(\Omega_h \setminus \Omega)} \le C(\delta^{1/p} \|f\|_{L^p(\Gamma_h)} + \delta \|\nabla f\|_{L^p(\Omega_h \setminus \Omega)}), \tag{2.2}$$

whose proof will be given in Lemma A.1. Finally, denoting by n_h the outward unit normal to Γ_h, we notice that its error compared with n is estimated as $\|n \circ \pi - n_h\|_{L^\infty(\Gamma_h)} \le Ch$ (see [12, Section 9]).

2.3 Extension operators

We let $\tilde{\Omega} := \Omega \cup \Gamma(\delta) = \Omega_h \cup \Gamma(\delta)$ with $\delta = C_{0E} h^2$ given above. For $u \in W^{2,\infty}(\Omega)$, $f \in L^\infty(\Omega)$, and $\tau \in L^\infty(\Gamma)$, we assume that there exist extensions $\tilde{u} \in W^{2,\infty}(\tilde{\Omega})$, $\tilde{f} \in L^\infty(\tilde{\Omega})$, and $\tilde{\tau} \in L^\infty(\tilde{\Omega})$, respectively, which are stable in the sense that the norms of the extended quantities can be controlled by those of the original ones, e.g., $\|\tilde{u}\|_{W^{2,\infty}(\tilde{\Omega})} \le C\|u\|_{W^{2,\infty}(\Omega)}$. We emphasize that (1.1) would not hold any longer in the extended region $\tilde{\Omega} \setminus \overline{\Omega}$.

We also need extensions whose behavior in $\Gamma(\delta)\backslash\Omega$ can be completely described by that in $\Gamma(c\delta)\cap\Omega$ for some constant $c>0$. To this end we introduce an extension operator $P\colon W^{k,p}(\Omega)\to W^{k,p}(\tilde{\Omega})$ ($k=0,1,2$, $p\in[1,\infty]$) as follows. For $x\in\Omega\backslash\Gamma(\delta)$ we let $Pf(x)=f(x)$; for $x=\bar{x}+tn(\bar{x})\in\Gamma(\delta)$ we define

$$Pf(\bar{x}+tn(\bar{x}))=\begin{cases} f(\bar{x}+tn(\bar{x})) & (-\delta_0\le t<0),\\ 3f(\bar{x}-tn(\bar{x}))-2f(\bar{x}-2tn(\bar{x})) & (0\le t\le\delta_0),\end{cases}\quad \bar{x}\in\Gamma.$$

Proposition 2.1 *The extension operator P satisfies the following stability condition:*

$$\|Pf\|_{W^{k,p}(\Gamma(\delta))}\le C\|f\|_{W^{k,p}(\Omega\cap\Gamma(2\delta))}\quad (k=0,1,2),\quad p\in[1,\infty],$$

where C is independent of δ and f.

The proof of this proposition will be given in Theorem A.1.

2.4 Dyadic decomposition

We introduce a dyadic decomposition of a domain according to [14]. Let $B(x_0;r)=\{x\in\mathbb{R}^N\colon |x-x_0|\le r\}$ and $A(x_0;r,R)=\{x\in\mathbb{R}^N\colon r\le|x-x_0|\le R\}$ denote a closed ball and annulus in $\mathbb{R}^N$ respectively.

Definition 2.1 For $x_0\in\mathbb{R}^N$, $d_0>0$, $J\in\mathbb{N}_{\ge 0}$, the family of sets $\mathcal{A}(x_0,d_0,J)=\{A_j\}_{j=0}^J$ defined by

$$A_0=B(x_0;d_0),\quad A_j=A(x_0;d_{j-1},d_j),\quad d_j=2^jd_0\quad (j=1,\dots,J)$$

is called the *dyadic J annuli with the center x_0 and the initial stride d_0*.

With a center and an initial stride specified, one can assign to a given domain a unique decomposition by dyadic annuli as follows.

Lemma 2.1 *For a bounded domain $D\subset\mathbb{R}^N$, let $x_0\in D$, $0<d_0<\operatorname{diam} D$, and J be the smallest integer that is greater than $J':=\frac{\log(\operatorname{diam} D/d_0)}{\log 2}$. Then we have $\overline{D}\subset\bigcup\mathcal{A}(x_0,d_0,J)$.*

Proof Since $2^{J'}d_0=\operatorname{diam} D$ and $J'<J\le J'+1$, one has $\operatorname{diam} D<d_J\le 2\operatorname{diam} D$. For arbitrary $x\in D$ we see that $|x-x_0|\le\operatorname{diam} D<d_J$, which implies $\overline{D}\subset B(x_0;d_J)=\bigcup\mathcal{A}(x_0,d_0,J)$. □

Definition 2.2 We define the *decomposition of D into dyadic annuli with the center x_0 and the initial stride d_0 by $\mathcal{A}_D(x_0,d_0)=\{D\cap A_j\}_{j=0}^J$*, where $\{A_j\}_{j=0}^J=\mathcal{A}(x_0,d_0,J)$ are the dyadic annuli given in Lemma 2.1. We also use the terminology *dyadic decomposition* for abbreviation.

For $\mathcal{A}(x_0,d_0,J)=\{A_j\}_{j=0}^J$ and $s\in[0,1]$, we consider expanded annuli $\mathcal{A}^{(s)}(x_0,d_0,J)=\{A_j^{(s)}\}_{j=0}^J$, where

$$A_0^{(s)} = B(x_0; (1+s)d_0),$$
$$A_j^{(s)} := A(x_0; (1-\frac{s}{2})d_{j-1}, (1+s)d_j) \quad (j = 1, \ldots, J).$$

In particular, for $s = 1$ one has $A_j^{(1)} = A_{j-1} \cup A_j \cup A_{j+1}$ where we set $A_{-1} := \emptyset$ and $A_{J+1} := A(x_0; d_J, d_{J+1})$ with $d_{J+1} := 2d_J$.

We collect some basic properties of weighted L^p-norms defined on a dyadic decomposition.

Lemma 2.2 *For a dyadic decomposition* $\mathcal{A}_D(x_0, d_0) = \{D \cap A_j\}_{j=0}^J$ *of* D *and* $p \in [1, \infty]$*, the following estimates hold:*

$$\|f\|_{L^1(D)} \le \alpha_N^{1/p'} \sum_{j=0}^J d_j^{N/p'} \|f\|_{L^p(D\cap A_j)}, \tag{2.3}$$

$$\sum_{j=0}^J d_j^{N/p'} \|f\|_{L^p(D\cap A_j^{(1)})} \le (2^{N/p'} + 1 + 2^{-N/p'}) \sum_{j=0}^J d_j^{N/p'} \|f\|_{L^p(D\cap A_j)}. \tag{2.4}$$

Here, $\alpha_N = \frac{2\pi^{N/2}}{N\Gamma(N/2)}$ *means the volume of the* N*-dimensional unit ball and* $p' = p/(p-1)$.

Proof It follows from the Hölder inequality that

$$\|f\|_{L^1(D)} = \sum_{j=0}^J \|f\|_{L^1(D\cap A_j)} \le \sum_{j=0}^J |A_j|^{1/p'} \|f\|_{L^p(D\cap A_j)},$$

which combined with $|A_j| = (1-2^{-N})d_j^N \alpha_N$ yields (2.3). The estimate (2.4) follows from the fact that

$$\|f\|_{L^p(D\cap A_j^{(1)})} \le \|f\|_{L^p(D\cap A_{j-1})} + \|f\|_{L^p(D\cap A_j)} + \|f\|_{L^p(D\cap A_{j+1})},$$

together with $D \cap A_{-1} = D \cap A_{J+1} = \emptyset$. □

Setting now D to be Ω_h introduced in Sect. 2.1, we consider its dyadic decomposition $\mathcal{A}_{\Omega_h}(x_0, d_0) = \{\Omega_h \cap A_j\}_{j=0}^J$ and its triangulation $\mathcal{T}_h$. At this stage, each triangle in $\mathcal{T}_h$ can simultaneously intersect with some annuls A and its complement A^c; however, we have the following lemma:

Lemma 2.3 *Let* $\mathcal{A}_{\Omega_h}(x_0, d_0) = \{\Omega_h \cap A_j\}_{j=0}^J$ *be a dyadic decomposition of* Ω_h *with* $x_0 \in \Omega_h$ *and* $d_0 \in [16h, 1]$*, and let* $s \in [0, 3/4]$.

- (i) *If* $T \in \mathcal{T}_h$ *satisfies* $T \cap A_j^{(s)} \ne \emptyset$ *then* $M_T \subset A_j^{(s+1/4)}$*, where* M_T *is the macro element of* T.
- (ii) *If* $T \in \mathcal{T}_h$ *satisfies* $T \setminus A_j^{(s+1/4)} \ne \emptyset$ *then* $M_T \subset (A_j^{(s)})^c$.

Proof We only prove (i) since item (ii) can be shown similarly. Let $x \in M_T$ be arbitrary. By assumption there exists $x' \in T \cap A_j^{(s)}$; in particular, $(1-s/2)d_{j-1} \le |x'-x_0| \le (1+s)d_j$. Also, by definition of M_T, $|x-x'| \le 2h$. Then we have $(7/8-s/2)d_{j-1} \le |x-x_0| \le (5/4+s)d_j$ as a result of triangle inequalities, which implies $x \in A_j^{(s+1/4)}$. □

Corollary 2.1 *Under the assumption of* Lemma 2.3, *let* $v \in H^1(\overline{\Omega}_h)$ *satisfy* $\operatorname{supp} v \subset A_j^{(s)}$. *Then we have* $\operatorname{supp} \mathcal{I}_h v \subset A_j^{(s+1/4)}$.

Proof It suffices to show $\mathcal{I}_h v(x) = 0$ for all $x \in \Omega_h \backslash A_j^{(s+1/4)}$. In fact, since there exists $T \in \mathcal{T}_h$ such that $x \in T$, one has $M_T \cap A_j^{(s)} = \emptyset$ as a result of Lemma 2.3(ii). Hence $v|_{M_T} = 0$, so that $\mathcal{I}_h v|_T = 0$. □

Finally, notice that for any dyadic decomposition $\mathcal{A}_{\Omega_h}(x_0, d_0)$ we have

$$\sum_{j=0}^{J} d_j^\beta \le \begin{cases} Cd_J^\beta & (\beta > 0), \\ C(1+|\log d_0|) & (\beta = 0), \\ Cd_0^\beta & (\beta < 0), \end{cases} \tag{2.5}$$

where $C = C(N, \Omega, \beta)$ is independent of x_0, d_0, and J (for the case $\beta = 0$, recall Lemma 2.1 to estimate J). Moreover, since $d_j \le d_J \le 2 \operatorname{diam} \Omega_h$, one has

$$d_j^\alpha + d_j^\beta \le C d_j^{\min\{\alpha,\beta\}}, \quad 0 \le j \le J, \ \alpha, \beta \in \mathbb{R}, \ C = C(N, \Omega, \alpha, \beta),$$

which will not be emphasized in the subsequent arguments.

3 Main theorem and its reduction to $W^{1,1}$-analysis

Let us state the main result of this paper.

Theorem 3.1 *Let* $u \in W^{2,\infty}(\Omega)$ *and* $u_h \in V_h$ *be the solutions of* (1.1) *and* (1.2) *respectively. Then there exists* $h_0 > 0$ *such that for all* $h \in (0, h_0]$ *and* $v_h \in V_h$ *we have*

$$\|\tilde{u} - u_h\|_{L^\infty(\Omega_h)} \le Ch|\log h| \, \|\tilde{u} - v_h\|_{W^{1,\infty}(\Omega_h)} + Ch^2|\log h| \, \|u\|_{W^{2,\infty}(\Omega)},$$
$$\|\tilde{u} - u_h\|_{W^{1,\infty}(\Omega_h)} \le C\|\tilde{u} - v_h\|_{W^{1,\infty}(\Omega_h)} + Ch \, \|u\|_{W^{2,\infty}(\Omega)},$$

where C *is independent of* h, u, *and* v_h.

Remark 3.1 (i) By taking $v_h = \mathcal{I}_h \tilde{u}$, we immediately obtain (1.3).
(ii) The factor $h\|\tilde{u} - v_h\|_{W^{1,\infty}(\Omega_h)}$ in the L^∞-estimate could be replaced by $\|\tilde{u} - v_h\|_{L^\infty(\Omega_h)}$ (cf. [14, p. 889]), which will be discussed elsewhere.

(iii) The above error estimates cannot be improved even if one employs a higher order finite element, as far as the boundary Γ is only linearly approximated. In fact, the domain perturbation term I_4 (see Lemmas 3.2 and 3.5 below) gives rise to $O(h^2|\log h|)$- and $O(h)$-contributions for L^∞- and $W^{1,\infty}$-error estimates respectively, regardless of the choice of V_h. Both of the approximation of functions and that of the boundary must be made higher order in a proper manner to achieve better accuracy (a typical way to do this is the use of isoparametric elements).

Let us reduce pointwise error estimates to $W^{1,1}$-error analysis for regularized Green's functions, which is now a standard approach in this field. For arbitrary $T \in \mathcal{T}_h$ and $x_0 \in T$ we let $\eta = \eta_{x_0} \in C_0^\infty(T)$, $\eta \geq 0$ be a regularized delta function such that

$$\int_T \eta(x) v_h(x)\, dx = v_h(x_0) \quad \forall v_h \in P_1(T), \quad \|\nabla^k \eta\|_{L^\infty(T)} \leq Ch^{-k}\ (k = 0, 1, 2),$$
$$\mathrm{dist}(\mathrm{supp}\, \eta, \partial T) \geq Ch, \tag{3.1}$$

where C is independent of T, h, and x_0 (see [15] for construction of η).

Remark 3.2 (i) The quasi-uniformity of meshes are needed to ensure the last two properties of (3.1).
(ii) We have $\mathrm{supp}\, \eta \cap \Gamma(2\delta) = \emptyset$ with $\delta = C_{0E} h^2$, provided that h is sufficiently small.

We consider two kinds of regularized Green's functions $g_0, g_1 \in C^\infty(\overline{\Omega})$ satisfying the following PDEs:

$$-\Delta g_0 + g_0 = \eta \quad \text{in} \quad \Omega, \quad \partial_n g_0 = 0 \quad \text{on} \quad \Gamma,$$

and

$$-\Delta g_1 + g_1 = \partial\eta \quad \text{in} \quad \Omega, \quad \partial_n g_1 = 0 \quad \text{on} \quad \Gamma,$$

where ∂ stands for an arbitrary directional derivative. Accordingly, we let $g_{0h}, g_{1h} \in V_h$ be the solutions for finite element approximate problems as follows:

$$a_h(v_h, g_{0h}) = (v_h, \eta)_{\Omega_h} \quad \forall v_h \in V_h, \quad \text{and} \quad a_h(v_h, g_{1h}) = (v_h, \partial\eta)_{\Omega_h} \quad \forall v_h \in V_h.$$

The goal of this section is then to reduce Theorem 3.1 to the estimate

$$\|\tilde{g}_m - g_{mh}\|_{W^{1,1}(\Omega_h)} \leq C(h|\log h|)^{1-m}, \quad m = 0, 1, \tag{3.2}$$

where C is independent of h, x_0, and ∂, and $\tilde{g}_m := P g_m$ means the extension defined in Sect. 2.3. To observe this fact, we represent pointwise errors at x_0, with the help of η, as

$$\tilde{u}(x_0) - u_h(x_0) = (\tilde{u} - v_h)(x_0) + (v_h - \tilde{u}, \eta)_{\Omega_h} + (\tilde{u} - u_h, \eta)_{\Omega_h},$$
$$\partial(\tilde{u} - u_h)(x_0) = \partial(\tilde{u} - v_h)(x_0) + (\partial(v_h - \tilde{u}), \eta)_{\Omega_h} - (\tilde{u} - u_h, \partial\eta)_{\Omega_h},$$

for all $v_h \in V_h$. Since the first two terms on the right-hand sides are bounded by $2\|\tilde{u} - v_h\|_{L^\infty(\Omega_h)}$ and $2\|\nabla(\tilde{u} - v_h)\|_{L^\infty(\Omega_h)}$, in order to prove Theorem 3.1 it suffices to show that

$$|(\tilde{u} - u_h, \eta)_{\Omega_h}| \le Ch|\log h|\|\tilde{u} - v_h\|_{W^{1,\infty}(\Omega_h)} + Ch^2|\log h|\|u\|_{W^{2,\infty}(\Omega)},$$
$$|(\tilde{u} - u_h, \partial\eta)_{\Omega_h}| \le C\|\tilde{u} - v_h\|_{W^{1,\infty}(\Omega_h)} + Ch\|u\|_{W^{2,\infty}(\Omega)}.$$

With this aim we prove:

Proposition 3.1 *For $m = 0, 1$ and arbitrary $v_h \in V_h$, one obtains*

$$|(\tilde{u} - u_h, \partial^m\eta)_{\Omega_h}| \le C(\|\tilde{u} - v_h\|_{W^{1,\infty}(\Omega_h)} + Ch\|u\|_{W^{2,\infty}(\Omega)})\|\tilde{g}_m - g_{mh}\|_{W^{1,1}(\Omega_h)} + Ch(h|\log h|)^{1-m}\|u\|_{W^{2,\infty}(\Omega)}.$$

It is immediate to conclude Theorem 3.1 from Proposition 3.1 combined with (3.2). The rest of this section is thus devoted to the proof of Proposition 3.1, whereas (3.2) will be established in Sects. 4–6 below. From now on, we suppress the subscript m of g_m and g_{mh} for simplicity, as far as there is no fear of confusion.

Let us proceed to the proof of Proposition 3.1. Define functionals for $v \in H^1(\Omega_h)$, which will represent "residuals" of Galerkin orthogonality relation, by

$$\mathrm{Res}_u(v) = (-\Delta\tilde{u} + \tilde{u} - \tilde{f}, v)_{\Omega_h\setminus\Omega} + (\partial_{n_h}\tilde{u} - \tilde{\tau}, v)_{\Gamma_h},$$
$$\mathrm{Res}_g(v) = (v, -\Delta\tilde{g} + \tilde{g})_{\Omega_h\setminus\Omega} + (v, \partial_{n_h}\tilde{g})_{\Gamma_h}.$$

If in addition $v \in H^1(\tilde{\Omega})$ in the expanded domain $\tilde{\Omega} = \Omega \cup \Gamma(\delta)$, then $\mathrm{Res}_u(v)$ admits another expression. To observe this, we introduce "signed" integration defined as follows:

$$(\phi, \psi)'_{\Omega_h\triangle\Omega} := (\phi, \psi)_{\Omega_h\setminus\Omega} - (\phi, \psi)_{\Omega\setminus\Omega_h},$$
$$(\phi, \psi)'_{\Gamma_h\cup\Gamma} := (\phi, \psi)_{\Gamma_h} - (\phi, \psi)_{\Gamma},$$
$$a'_{\Omega_h\triangle\Omega}(\phi, \psi) := (\nabla\phi, \nabla\psi)'_{\Omega_h\triangle\Omega} + (\phi, \psi)'_{\Omega_h\triangle\Omega}.$$

Lemma 3.1 *For $v \in H^1(\tilde{\Omega})$ we have*

$$\mathrm{Res}_u(v) = -(\tilde{f}, v)'_{\Omega_h\triangle\Omega} - (\tilde{\tau}, v)'_{\Gamma_h\cup\Gamma} + a'_{\Omega_h\triangle\Omega}(\tilde{u}, v).$$

Proof Notice that the following integration by parts formula holds:

$$(-\Delta\tilde{u}, v)'_{\Omega_h\triangle\Omega} = (\nabla\tilde{u}, \nabla v)'_{\Omega_h\triangle\Omega} - (\partial_{n_h}\tilde{u}, v)_{\Gamma_h} + (\partial_n u, v)_{\Gamma}.$$

From this formula and (1.1) it follows that

$$(-\Delta\tilde{u}, v)_{\Omega_h\setminus\Omega} + (\partial_{n_h}\tilde{u}, v)_{\Gamma_h} = (-\Delta u, v)_{\Omega\setminus\Omega_h} + (\nabla\tilde{u}, \nabla v)'_{\Omega_h\triangle\Omega} + (\partial_n u, v)_{\Gamma}$$
$$= -(u - f, v)_{\Omega\setminus\Omega_h} + (\nabla\tilde{u}, \nabla v)'_{\Omega_h\triangle\Omega} + (\tau, v)_{\Gamma}.$$

Substituting this into the definition of $\mathrm{Res}_u(v)$ leads to the desired equality. □

Now we show that $\mathrm{Res}_u(\cdot)$ and $\mathrm{Res}_g(\cdot)$ represent residuals of Galerkin orthogonality relation for $\tilde{u} - u_h$ and $\tilde{g} - g_h$, respectively.

Lemma 3.2 *For all $v_h \in V_h$ we have*

$$a_h(\tilde{u} - u_h, v_h) = \mathrm{Res}_u(v_h), \quad a_h(v_h, \tilde{g} - g_h) = \mathrm{Res}_g(v_h),$$

and

$$\begin{aligned}(\tilde{u} - u_h, \partial^m \eta)_{\Omega_h} &= a_h(\tilde{u} - v_h, \tilde{g} - g_h) - \mathrm{Res}_g(\tilde{u} - v_h) - \mathrm{Res}_u(\tilde{g} - g_h) + \mathrm{Res}_u(\tilde{g}) \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Proof From integration by parts and from the definitions of u and u_h we have

$$\begin{aligned}a_h(\tilde{u} - u_h, v_h) &= (-\Delta\tilde{u} + \tilde{u}, v_h)_{\Omega_h} + (\partial_{n_h}\tilde{u}, v_h)_{\Gamma_h} - (\tilde{f}, v_h)_{\Omega_h} - (\tilde{\tau}, v_h)_{\Gamma_h} \\ &= \mathrm{Res}_u(v_h).\end{aligned}$$

The second equality is obtained in the same way. To show the third equality, we observe that

$$\begin{aligned}(v_h - u_h, \partial^m \eta)_{\Omega_h} &= a_h(v_h - u_h, g_h) = a_h(v_h - \tilde{u}, g_h) + a_h(\tilde{u} - u_h, g_h) \\ &= a_h(\tilde{u} - v_h, \tilde{g} - g_h) - a_h(\tilde{u} - v_h, \tilde{g}) + \mathrm{Res}_u(g_h).\end{aligned}$$

It follows from integration by parts, $-\Delta g + g = \partial^m \eta$ in Ω, and $\operatorname{supp} \eta \subset \Omega_h \cap \Omega$, that

$$\begin{aligned}a_h(\tilde{u} - v_h, \tilde{g}) &= (\tilde{u} - v_h, -\Delta\tilde{g} + \tilde{g})_{\Omega_h} + (\tilde{u} - v_h, \partial_{n_h}\tilde{g})_{\Gamma_h} \\ &= (u - v_h, \partial^m \eta)_{\Omega_h \cap \Omega} + \mathrm{Res}_g(\tilde{u} - v_h) \\ &= (\tilde{u} - v_h, \partial^m \eta)_{\Omega_h} + \mathrm{Res}_g(\tilde{u} - v_h).\end{aligned}$$

Combining the two relations above yields the third equality. □

By the Hölder inequality, $|I_1| \le \|\tilde{u} - v_h\|_{W^{1,\infty}(\Omega_h)} \|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)}$. The other terms are estimated in the following three lemmas. There, boundary-skin estimates for g will be frequently exploited, which are collected in the appendix.

Lemma 3.3 $|I_2| \le C(h|\log h|)^{1-m} \|\tilde{u} - v_h\|_{L^\infty(\Omega_h)}$.

Proof By the Hölder inequality,

$$|\mathrm{Res}_g(\tilde{u} - v_h)| \le \|\tilde{u} - v_h\|_{L^\infty(\Omega_h)} (\|\tilde{g}\|_{W^{2,1}(\Gamma(\delta))} + \|\partial_{n_h}\tilde{g}\|_{L^1(\Gamma_h)}),$$

where $\|\tilde{g}\|_{W^{2,1}(\Gamma(\delta))} \le Ch^{1-m}$ as a result of Corollary B.1. Since $(\nabla g) \circ \pi \cdot n \circ \pi = 0$ on Γ_h, it follows again from Corollary B.1 that

$$\|\partial_{n_h}\tilde{g}\|_{L^1(\Gamma_h)} \le \|\nabla\tilde{g} \cdot (n_h - n \circ \pi)\|_{L^1(\Gamma_h)} + \|\big(\nabla\tilde{g} - (\nabla\tilde{g}) \circ \pi\big) \cdot n \circ \pi\|_{L^1(\Gamma_h)}$$

$$\leq Ch\|\nabla\tilde{g}\|_{L^1(\Gamma_h)} + C\|\nabla^2\tilde{g}\|_{L^1(\Gamma(\delta))} \leq C(h|\log h|)^{1-m} + Ch^{1-m},$$

which completes the proof. □

Lemma 3.4 $|I_3| \leq Ch\|u\|_{W^{2,\infty}(\Omega)}\|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)}$.

Proof By the Hölder inequality and stability of extensions,

$$|\operatorname{Res}_u(\tilde{g}-g_h)| \leq C\|u\|_{W^{2,\infty}(\Omega)}\|\tilde{g}-g_h\|_{L^1(\Omega_h\setminus\Omega)} + \|\partial_{n_h}\tilde{u}-\tilde{\tau}\|_{L^\infty(\Gamma_h)}\|\tilde{g}-g_h\|_{L^1(\Gamma_h)}.$$

From (2.2) and the trace theorem one has

$$\begin{aligned}\|\tilde{g} - g_h\|_{L^1(\Omega_h\setminus\Omega)} &\leq C\delta(\|\tilde{g} - g_h\|_{L^1(\Gamma_h)} + \|\nabla(\tilde{g} - g_h)\|_{L^1(\Omega_h\setminus\Omega)}) \\ &\leq Ch^2\|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)}.\end{aligned}$$

From $(\nabla u)\circ\pi\cdot n\circ\pi = \tau\circ\pi$ on Γ_h, (2.1), and the stability of extensions, it follows that

$$\begin{aligned}\|\partial_{n_h}\tilde{u} - \tilde{\tau}\|_{L^\infty(\Gamma_h)} &\leq \|\nabla\tilde{u}\cdot(n_h - n\circ\pi)\|_{L^\infty(\Gamma_h)} + \|\big(\nabla\tilde{u} - (\nabla\tilde{u})\circ\pi\big)\cdot n\circ\pi\|_{L^\infty(\Gamma_h)} \\ &\quad + \|\tau\circ\pi - \tilde{\tau}\|_{L^\infty(\Gamma_h)} \\ &\leq Ch\|\nabla\tilde{u}\|_{L^\infty(\Gamma_h)} + C\delta\|\nabla^2\tilde{u}\|_{L^\infty(\Gamma(\delta))} \\ &\quad + C\delta\|\nabla\tilde{\tau}\|_{L^\infty(\Gamma(\delta))} \leq Ch\|u\|_{W^{2,\infty}(\Omega_h)}.\end{aligned}$$

Combining the estimates above and using the trace theorem once again, we conclude

$$\begin{aligned}|\operatorname{Res}_u(\tilde{g} - g_h)| &\leq Ch^2\|u\|_{W^{2,\infty}(\Omega)}\|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)} + Ch\|u\|_{W^{2,\infty}(\Omega_h)}\|\tilde{g} - g_h\|_{L^1(\Gamma_h)} \\ &\leq Ch\|u\|_{W^{2,\infty}(\Omega)}\|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)}.\end{aligned}$$

This completes the proof. □

Lemma 3.5 $|I_4| \leq Ch(h|\log h|)^{1-m}\|u\|_{W^{2,\infty}(\Omega)}$.

Proof We recall from Lemma 3.1 that

$$\operatorname{Res}_u(\tilde{g}) = -(\tilde{f}, \tilde{g})'_{\Omega_h\triangle\Omega} - (\tilde{\tau}, \tilde{g})'_{\Gamma_h\cup\Gamma} + a'_{\Omega_h\triangle\Omega}(\tilde{u}, \tilde{g}).$$

Let us estimate each term in the right-hand side. By $(2.1)_2$ we obtain

$$|(\tilde{f}, \tilde{g})'_{\Omega_h\triangle\Omega}| \leq \|\tilde{f}\|_{L^\infty(\Gamma(\delta))}\|\tilde{g}\|_{L^1(\Gamma(\delta))} \leq C\delta|\log h|^{1-m}\|u\|_{W^{2,\infty}(\Omega)},$$

where $\delta = C_{0E}h^2$. Next, from (2.1) and Corollary B.1 we find that

$$\begin{aligned}(\tilde{\tau}, \tilde{g})'_{\Gamma_h\cup\Gamma} &= |(\tau, g)_\Gamma - (\tilde{\tau}, \tilde{g})_{\Gamma_h}| \leq |(\tau, g)_\Gamma - (\tau\circ\pi, g\circ\pi)_{\Gamma_h}| \\ &\quad + |(\tau\circ\pi, g\circ\pi - \tilde{g})_{\Gamma_h}| + |(\tau\circ\pi - \tilde{\tau}, \tilde{g})_{\Gamma_h}| \\ &\leq C\delta\|\tau\|_{L^\infty(\Gamma)}\|g\|_{L^1(\Gamma)} + C\|\tau\|_{L^\infty(\Gamma)}\|\nabla\tilde{g}\|_{L^1(\Gamma(\delta))} \\ &\quad + C\delta\|\nabla\tilde{\tau}\|_{L^\infty(\Gamma(\delta))}\|\tilde{g}\|_{L^1(\Gamma_h)}\end{aligned}$$

$$\le C\delta\|\nabla u\|_{L^\infty(\Omega)}|\log h|^m + C\|\nabla u\|_{L^\infty(\Omega)}\delta h^{-m}|\log h|^{1-m} + C\delta\|u\|_{W^{2,\infty}(\Omega)}|\log h|^{1-m} \le C\delta h^{-m}|\log h|^{1-m}\|u\|_{W^{2,\infty}(\Omega)}.$$

Finally, for the last term we obtain

$$|a'_{\Omega_h\triangle\Omega}(\tilde{u},\tilde{g})| \le \|\tilde{u}\|_{W^{1,\infty}(\Gamma(\delta))}\|\tilde{g}\|_{W^{1,1}(\Gamma(\delta))} \le C\|u\|_{W^{1,\infty}(\Omega)}\delta h^{-m}|\log h|^{1-m}.$$

Collecting the above estimates proves the lemma. □

Proposition 3.1 is now an immediate consequence of Lemmas 3.2–3.5.

4 Weighted H^1-estimates

As a consequence of the previous section, we need to estimate $\|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)}$, where we keep dropping the subscript m (either 0 or 1) of g_m and g_{mh}. To this end we introduce a dyadic decomposition $\mathcal{A}_{\Omega_h}(x_0, d_0) = \{\Omega_h \cap A_j\}_{j=0}^J$ of Ω_h, and observe from (2.3) that

$$\|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)} \le C\sum_{j=0}^J d_j^{N/2}\|\tilde{g} - g_h\|_{H^1(\Omega_h\cap A_j)}. \tag{4.1}$$

Then the weighted H^1-norm in the right-hand side is bounded as follows:

Proposition 4.1 *There exists $K_0 > 0$ such that, for any dyadic decomposition $\mathcal{A}_{\Omega_h}(x_0, d_0) = \{\Omega_h \cap A_j\}_{j=0}^J$ of Ω_h with $d_0 = Kh$, $K \ge K_0$, we obtain*

$$\sum_{j=0}^J d_j^{N/2}\|\tilde{g} - g_h\|_{H^1(\Omega_h\cap A_j)} \le CK^{m+N/2}h^{1-m} + C(h|\log h|)^{1-m} + C\sum_{j=0}^J d_j^{-1+N/2}\|\tilde{g} - g_h\|_{L^2(\Omega_h\cap A_j)}. \tag{4.2}$$

Here the constants K_0 and C are independent of h, x_0, ∂, and K.

The rest of this section is devoted to the proof of the proposition above. In order to estimate $\|\tilde{g} - g_h\|_{H^1(\Omega_h\cap A_j)}$ for $j = 0, \dots, J$, we use a cut off function $\omega_j \in C_0^\infty(\mathbb{R}^N)$, $\omega_j \ge 0$ such that

$$\omega_j \equiv 1 \quad \text{in} \quad A_j, \quad \operatorname{supp}\omega_j \subset A_j^{(1/4)}, \quad \|\nabla^k\omega_j\|_{L^\infty(\mathbb{R}^N)} \le Cd_j^{-k} \quad (k = 0, 1, 2). \tag{4.3}$$

Then we find that

$$\|\tilde{g} - g_h\|^2_{H^1(\Omega_h\cap A_j)} \le \big(\omega_j(\tilde{g} - g_h), \tilde{g} - g_h\big)_{\Omega_h} + \big(\omega_j\nabla(\tilde{g} - g_h), \nabla(\tilde{g} - g_h)\big)_{\Omega_h}$$

$$
\begin{aligned}
&= a_h\big(\omega_j(\tilde{g}-g_h), \tilde{g}-g_h\big) - \big(\nabla\omega_j(\tilde{g}-g_h), \nabla(\tilde{g}-g_h)\big)_{\Omega_h} \\
&= a_h\big(\omega_j(\tilde{g}-g_h) - v_h, \tilde{g}-g_h\big) \\
&\quad - \big((\nabla\omega_j)(\tilde{g}-g_h), \nabla(\tilde{g}-g_h)\big)_{\Omega_h} + \mathrm{Res}_g(v_h) \\
&=: I_1 + I_2 + I_3,
\end{aligned}
$$

where $v_h \in V_h$ is arbitrary and we have used Lemma 3.2.

Substituting $v_h = \mathcal{I}_h(\omega_j(\tilde{g}-g_h))$, where $\mathcal{I}_h$ is the interpolation operator given in Sect. 2.1, we estimate I_1, I_2, and I_3 in the following.

Lemma 4.1 *I_1 is bounded as*

$$
\begin{aligned}
|I_1| &\le Chd_j^{-2}\|\tilde{g}-g_h\|_{L^2(\Omega_h\cap A_j^{(1/2)})}\|\tilde{g}-g_h\|_{H^1(\Omega_h\cap A_j^{(1/2)})} \\
&\quad + Chd_j^{-1}\|\tilde{g}-g_h\|^2_{H^1(\Omega_h\cap A_j^{(1/2)})} \\
&\quad + C_j h d_j^{-m-N/2}\|\tilde{g}-g_h\|_{H^1(\Omega_h\cap A_j^{(1/2)})},
\end{aligned} \tag{4.4}
$$

where $C_0 = CK^{m+N/2}$ and $C_j = C$ for $1 \le j \le J$.

Proof By Corollary 2.1 we have $\operatorname{supp} v_h \subset \Omega_h \cap A_j^{(1/2)}$, and hence

$$
|I_1| \le \|\omega_j(\tilde{g}-g_h) - v_h\|_{H^1(\Omega_h)}\|\tilde{g}-g_h\|_{H^1(\Omega_h\cap A_j^{(1/2)})}.
$$

It follows from the interpolation error estimate, together with (4.3), that

$$
\begin{aligned}
&\|\omega_j(\tilde{g}-g_h) - v_h\|^2_{H^1(\Omega_h)} \le Ch^2 \sum_{T\in\mathcal{T}_h} \|\nabla^2\big(\omega_j(\tilde{g}-g_h)\big)\|^2_{L^2(T)} \\
&\quad \le Ch^2 \sum_{T\in\mathcal{T}_h} \Big(\|(\nabla^2\omega_j)(\tilde{g}-g_h)\|^2_{L^2(T)} + \|(\nabla\omega_j)\otimes\nabla(\tilde{g}-g_h)\|^2_{L^2(T)} + \|\nabla^2\tilde{g}\|^2_{L^2(T)}\Big) \\
&\quad \le Ch^2 \sum_{T\cap A_j^{(1/4)}\neq\emptyset} \Big(d_j^{-4}\|\tilde{g}-g_h\|^2_{L^2(T)} + d_j^{-2}\|\tilde{g}-g_h\|^2_{L^2(T)} + \|\nabla^2\tilde{g}\|^2_{L^2(T)}\Big),
\end{aligned}
$$

where we made use of the fact that $\nabla^2 g_h|_T \equiv 0$ for $T \in \mathcal{T}_h$. This combined with Lemma 2.3(i) implies

$$
\begin{aligned}
\|\omega_j(\tilde{g}-g_h) - v_h\|_{H^1(\Omega_h)} \le C(&d_j^{-2}\|\tilde{g}-g_h\|_{L^2(\Omega_h\cap A_j^{(1/2)})} \\
&+ d_j^{-1}\|\tilde{g}-g_h\|_{L^2(\Omega_h\cap A_j^{(1/2)})} \\
&+ \|\nabla^2\tilde{g}\|_{L^2(\Omega_h\cap A_j^{(1/2)})}).
\end{aligned}
$$

When $j = 0$, by the stability of extension and the H^2-regularity theory, we deduce that

$$\|\nabla^2 \tilde{g}\|_{L^2(\Omega_h \cap A_0^{(1/2)})} \le C\|g\|_{H^2(\Omega)} \le C\|\partial^m \eta\|_{L^2(\Omega)} \le Ch^{-m-N/2} = CK^{m+N/2} d_0^{-m-N/2}.$$

When $j \ge 1$, it follows from Lemma B.2 that $\|\nabla^2 \tilde{g}\|_{L^2(\Omega_h \cap A_0^{(1/2)})} \le Cd_j^{-m-N/2}$. Collecting the estimates above, we conclude (4.4). □

For I_2 we have

$$|I_2| \le Cd_j^{-1}\|\tilde{g} - g_h\|_{L^2(\Omega_h \cap A_j^{(1/2)})}\|\tilde{g} - g_h\|_{H^1(\Omega_h \cap A_j^{(1/2)})},$$

which dominates the first term in the right-hand side of (4.4) because $hd_j^{-1} \le 1$.

Lemma 4.2 $|I_3| \le Chd_j^{1/2-m-N/2}(\|\tilde{g}-g_h\|_{H^1(\Omega_h \cap A_j^{(1/4)})} + d_j^{-1}\|\tilde{g}-g_h\|_{L^2(\Omega_h \cap A_j^{(1/4)})})$.

Proof Since $I_3 = (v_h, -\Delta\tilde{g} + \tilde{g})_{\Omega_h \setminus \Omega} + (v_h, \partial_{n_h}\tilde{g})_{\Gamma_h}$, we observe that

$$\begin{aligned}|(v_h, -\Delta\tilde{g} + \tilde{g})_{\Omega_h \setminus \Omega}| &\le C\delta^{1/2}\|v_h\|_{H^1(\Omega_h)}(\delta d_j^{N-1})^{1/2} d_j^{-m-N} \\ &\le C\delta d_j^{-1/2-m-N/2}\|v_h\|_{H^1(\Omega_h)},\end{aligned}$$

and that

$$\begin{aligned}|(v_h, \partial_{n_h}\tilde{g})_{\Gamma_h}| &\le \|v_h\|_{L^2(\Gamma_h)}\|\partial_{n_h}\tilde{g}\|_{L^2(\Gamma_h \cap A_j^{(1/4)})} \\ &\le C\|v_h\|_{H^1(\Omega_h)}\big(\|\nabla\tilde{g} \cdot (n_h - n \circ \pi)\|_{L^2(\Gamma_h \cap A_j^{(1/4)})} \\ &\quad + \|(\nabla\tilde{g} - (\nabla\tilde{g}) \circ \pi) \cdot n \circ \pi\|_{L^2(\Gamma_h \cap A_j^{(1/4)})}\big) \\ &\le C\|v_h\|_{H^1(\Omega_h)}\big(h\|\nabla\tilde{g}\|_{L^2(\Gamma_h \cap A_j^{(1/4)})} \\ &\quad + |\Gamma_h \cap A_j^{(1/4)}|^{1/2}\delta\|\nabla^2\tilde{g}\|_{L^\infty(\Gamma_h \cap A_j^{(1/4)})}\big) \\ &\le C\|v_h\|_{H^1(\Omega_h)}(hd_j^{1/2-m-N/2} + h^2 d_j^{-1/2-m-N/2}) \\ &\le Chd_j^{1/2-m-N/2}\|v_h\|_{H^1(\Omega_h)}.\end{aligned}$$

Therefore, by the H^1-stability of $\mathcal{I}_h$ and by $d_j \le 2\,\mathrm{diam}\ \Omega$,

$$\begin{aligned}|I_3| &\le Chd_j^{1/2-m-N/2}\|\omega_j(\tilde{g} - g_h)\|_{H^1(\Omega_h)} \\ &\le Chd_j^{1/2-m-N/2}(\|\tilde{g} - g_h\|_{H^1(\Omega_h \cap A_j^{(1/4)})} + d_j^{-1}\|\tilde{g} - g_h\|_{L^2(\Omega_h \cap A_j^{(1/4)})}),\end{aligned}$$

which completes the proof. □

Collecting the estimates for I_1, I_2, and I_3 we deduce that

$$\begin{aligned} d_j^{N/2}&\|\tilde{g}-g_h\|_{H^1(\Omega_h\cap A_j)} \\ &\le C(hd_j^{-1})^{1/2}d_j^{N/2}\|\tilde{g}-g_h\|_{H^1(\Omega_h\cap A_j^{(1)})} \\ &\quad + C\left(d_j^{N/2}\|\tilde{g}-g_h\|_{H^1(\Omega_h\cap A_j^{(1)})}\right)^{1/2}\left(d_j^{-1+N/2}\|\tilde{g}-g_h\|_{L^2(\Omega_h\cap A_j^{(1)})}\right)^{1/2} \\ &\quad + \left(C_j hd_j^{-m}(1+d_j^{1/2})\right)^{1/2}(d_j^{N/2}\|\tilde{g}-g_h\|_{H^1(\Omega_h\cap A_j^{(1)})})^{1/2} \\ &\quad + C(hd_j^{1/2-m})^{1/2}\left(d_j^{-1+N/2}\|\tilde{g}-g_h\|_{L^2(\Omega_h\cap A_j^{(1)})}\right)^{1/2}. \end{aligned}$$

We now take the summation for $j=0,1,\dots,J$ and apply (2.4) to have

$$\begin{aligned} \sum_{j=0}^{J} d_j^{N/2}\|\tilde{g}-g_h\|_{H^1(\Omega_h\cap A_j)} \le{}& C'(hd_0^{-1})^{1/2}\sum_{j=0}^{J} d_j^{N/2}\|\tilde{g}-g_h\|_{H^1(\Omega_h\cap A_j)} \\ &+ \frac{1}{4}\sum_{j=0}^{J} d_j^{N/2}\|\tilde{g}-g_h\|_{H^1(\Omega_h\cap A_j)} \\ &+ \sum_{j=0}^{J} C_j hd_j^{-m}(1+d_j^{1/2}) + Ch\sum_{j=0}^{J} d_j^{1/2-m} \\ &+ C\sum_{j=0}^{J} d_j^{-1+N/2}\|\tilde{g}-g_h\|_{L^2(\Omega_h\cap A_j)}. \end{aligned}$$

If $hd_0^{-1}=K^{-1}\le 1/(4C')^2$, then one can absorb the first two terms into the left-hand side to conclude (4.2). This completes the proof of Proposition 4.1.

Thus we are left to deal with $\sum_{j=0}^{J} d_j^{-1+N/2}\|\tilde{g}-g_h\|_{L^2(\Omega_h\cap A_j)}$, which will be the scope of the next section.

5 Weighted L^2-estimates

Let us give estimation of the weighted L^2-norm appearing in the last term of (4.2).

Proposition 5.1 *There exists $K_0>0$ such that, for any dyadic decomposition $\mathcal{A}_{\Omega_h}(x_0,d_0)=\{\Omega_h\cap A_j\}_{j=0}^{J}$ of Ω_h with $d_0=Kh$, $K_0\le K\le h^{-1}$, we obtain*

$$\sum_{j=0}^{J} d_j^{-1+N/2} \|\tilde{g} - g_h\|_{L^2(\Omega_h \cap A_j)} \leq C(hd_0^{-1}) \left(\sum_{j=0}^{J} d_j^{N/2} \|\tilde{g} - g_h\|_{H^1(\Omega_h \cap A_j)} + \|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)} \right) + Ch^{3/2-m}, \tag{5.1}$$

where the constants K_0 and C are independent of h, x_0, ∂, and K.

To prove this, first we fix $j = 0, \ldots, J$ and estimate $\|\tilde{g} - g_h\|_{L^2(\Omega_h \cap A_j)}$ based on a localized version of the Aubin–Nitsche trick. In fact, since

$$\|\tilde{g} - g_h\|_{L^2(\Omega_h \cap A_j)} = \sup_{\substack{\varphi \in C_0^\infty(\Omega_h \cap A_j) \\ \|\varphi\|_{L^2(\Omega_h \cap A_j)} = 1}} (\varphi, \tilde{g} - g_h)_{\Omega_h},$$

it suffices to examine $(\varphi, \tilde{g} - g_h)_{\Omega_h}$ for such φ. To express this quantity with a solution of a dual problem, we consider

$$-\Delta w + w = \varphi \quad \text{in} \quad \Omega, \quad \partial_n w = 0 \quad \text{on} \quad \Gamma, \tag{5.2}$$

where φ is extended by 0 to the outside of $\Omega_h \cap A_j$. From the elliptic regularity theory we know that the solution w is smooth enough. We then obtain the following:

Lemma 5.1 *For all $w_h \in V_h$ we have*

$$\begin{aligned}(\varphi, \tilde{g} - g_h)_{\Omega_h} &= a_h(\tilde{w} - w_h, \tilde{g} - g_h) - \mathrm{Res}_w(\tilde{g} - g_h) - \mathrm{Res}_g(\tilde{w} - w_h) + \mathrm{Res}_g(\tilde{w}) \\ &=: I_1 + I_2 + I_3 + I_4,\end{aligned} \tag{5.3}$$

where $\tilde{w} := Pw$ and $\mathrm{Res}_w : H^1(\Omega_h) \to \mathbb{R}$ is given by

$$\mathrm{Res}_w(v) := (-\Delta\tilde{w} + \tilde{w} - \varphi, v)_{\Omega_h \setminus \Omega} + (\partial_{n_h}\tilde{w}, v)_{\Gamma_h}.$$

Proof We see that

$$\begin{aligned}(\varphi, \tilde{g} - g_h)_{\Omega_h} &= (\varphi, g - g_h)_{\Omega_h \cap \Omega} + (\varphi, \tilde{g} - g_h)_{\Omega_h \setminus \Omega} \\ &= (-\Delta\tilde{w} + \tilde{w}, \tilde{g} - g_h)_{\Omega_h} + (\Delta\tilde{w} - \tilde{w} + \varphi, \tilde{g} - g_h)_{\Omega_h \setminus \Omega} \\ &= a_h(\tilde{w}, \tilde{g} - g_h) - (\partial_{n_h}\tilde{w}, \tilde{g} - g_h)_{\Gamma_h} + (\Delta\tilde{w} - \tilde{w} + \varphi, \tilde{g} - g_h)_{\Omega_h \setminus \Omega} \\ &= a_h(\tilde{w} - w_h, \tilde{g} - g_h) + \mathrm{Res}_g(w_h) - \mathrm{Res}_w(g - g_h),\end{aligned}$$

where we have used $a_h(w_h, \tilde{g} - g_h) = \mathrm{Res}_g(w_h)$ from Lemma 3.2. This yields the desired equality. □

Remark 5.1 In a similar way to Lemma 3.1, one can derive another expression for $\mathrm{Res}_g(v)$ if $v \in H^1(\tilde{\Omega})$:

$$\mathrm{Res}_g(v) = a'_{\Omega_h \triangle \Omega}(v, \tilde{g}).$$

In the following four lemmas, taking $w_h = \mathcal{I}_h \tilde{w}$, we estimate I_1, I_2, I_3, and I_4 by dividing the integrals over Ω_h, Γ_h, or boundary-skin layers, into those defined near A_j and away from A_j. The former will be bounded, e.g., by the Hölder inequality of the form $\|\phi\|_{L^2(\Omega_h)} \|\psi\|_{L^2(\Omega_h \cap A_j^{(1/2)})}$ together with H^2-regularity estimates for w, whereas the latter will be bounded by $\|\phi\|_{L^\infty(\Omega_h \setminus A_j^{(1/2)})} \|\psi\|_{L^1(\Omega_h)}$ together with Green's function estimates for w (see Lemma B.4).

Lemma 5.2 $|I_1| \le Ch \|\tilde{g} - g_h\|_{H^1(\Omega_h \cap A_j^{(1/2)})} + Chd_j^{-N/2} \|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)}$.

Proof By the Hölder inequality mentioned above,

$$\begin{aligned} |I_1| &\le \|\tilde{w} - w_h\|_{H^1(\Omega_h)} \|\tilde{g} - g_h\|_{H^1(\Omega_h \cap A_j^{(1/2)})} \\ &\quad + \|\tilde{w} - w_h\|_{W^{1,\infty}(\Omega_h \setminus A_j^{(1/2)})} \|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)}, \end{aligned}$$

where we notice that

$$\|\tilde{w} - w_h\|_{H^1(\Omega_h)} \le Ch \|w\|_{H^2(\Omega)} \le Ch \|\varphi\|_{L^2(\mathbb{R}^N)} = Ch,$$

and from Lemma B.4 that

$$\|\tilde{w} - w_h\|_{W^{1,\infty}(\Omega_h \setminus A_j^{(1/2)})} \le Ch \|\nabla^2 \tilde{w}\|_{L^\infty(\Omega_h \setminus A_j^{(1/4)})} \le Chd_j^{-N/2}.$$

This completes the proof. □

Lemma 5.3 *I_2 is bounded as*

$$\begin{aligned} |I_2| &\le Ch^{1/2} \|\tilde{g} - g_h\|_{L^2(\Omega_h \cap A_j^{(3/4)})} + Ch \|\tilde{g} - g_h\|_{H^1(\Omega_h \cap A_j^{(3/4)})} \\ &\quad + Chd_j^{-N/2} \|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)}. \end{aligned}$$

Proof Recall that $I_2 = (\Delta \tilde{w} - \tilde{w} + \varphi, \tilde{g} - g_h)_{\Omega_h \setminus \Omega} - (\partial_{n_h} \tilde{w}, \tilde{g} - g_h)_{\Gamma_h} =: I_{21} + I_{22}$. Noting that $\varphi = 0$ in $\Omega_h \setminus A_j^{(1/2)}$ we estimate I_{21} by

$$\begin{aligned} |I_{21}| &\le C \left(\|w\|_{H^2(\Omega)} + \|\varphi\|_{L^2(\mathbb{R}^N)} \right) \|\tilde{g} - g_h\|_{L^2\left((\Omega_h \setminus \Omega) \cap A_j^{(1/2)}\right)} \\ &\quad + \|\tilde{w}\|_{W^{2,\infty}(\Omega_h \setminus A_j^{(1/2)})} \|\tilde{g} - g_h\|_{L^1(\Omega_h \setminus \Omega)} \\ &\le C \|\tilde{g} - g_h\|_{L^2((\Omega_h \setminus \Omega) \cap A_j^{(1/2)})} + Cd_j^{-N/2} \|\tilde{g} - g_h\|_{L^1(\Omega_h \setminus \Omega)}. \end{aligned}$$

To address the first term we introduce $\omega'_j \in C_0^\infty(\mathbb{R}^N)$, $\omega'_j \ge 0$ such that

$$\omega'_j \equiv 1 \quad \text{in} \quad A_j^{(1/2)}, \quad \mathrm{supp}\, \omega'_j \subset A_j^{(3/4)}, \quad \|\nabla^k \omega'_j\|_{L^\infty(\mathbb{R}^N)} \le Cd_j^{-k} \ (k = 0, 1, 2).$$

Then it follows from (2.2) and the trace estimate that

$$
\begin{aligned}
&\|\tilde{g} - g_h\|_{L^2((\Omega_h\setminus\Omega)\cap A_j^{(1/2)})} \\
&\quad \le \|\omega_j'(\tilde{g} - g_h)\|_{L^2(\Omega_h\setminus\Omega)} \\
&\quad \le C\delta^{1/2}\|\omega_j'(\tilde{g} - g_h)\|_{L^2(\Gamma_h)} + C\delta\|\nabla\big(\omega_j'(\tilde{g} - g_h)\big)\|_{L^2(\Omega_h\setminus\Omega)} \\
&\quad \le Ch\|\omega_j'(\tilde{g} - g_h)\|_{L^2(\Omega_h)}^{1/2}\|\omega_j'(\tilde{g} - g_h)\|_{H^1(\Omega_h)}^{1/2} \\
&\qquad + Ch^2 d_j^{-1}\|\tilde{g} - g_h\|_{L^2(\Omega_h\cap A_j^{(3/4)})} + Ch^2\|\nabla(\tilde{g} - g_h)\|_{L^2(\Omega_h\cap A_j^{(3/4)})} \\
&\quad \le Ch(1 + d_j^{-1/2})\|\tilde{g} - g_h\|_{L^2(\Omega_h\cap A_j^{(3/4)})} + Ch\|\tilde{g} - g_h\|_{H^1(\Omega_h\cap A_j^{(3/4)})} \\
&\quad \le Ch^{1/2}\|\tilde{g} - g_h\|_{L^2(\Omega_h\cap A_j^{(3/4)})} + Ch\|\tilde{g} - g_h\|_{H^1(\Omega_h\cap A_j^{(3/4)})},
\end{aligned}
\tag{5.4}
$$

where we have used $hd_j^{-1} \le 1$ and $h \le 1$. Again by (2.2) we also have

$$
\begin{aligned}
\|\tilde{g} - g_h\|_{L^1(\Omega_h\setminus\Omega)} &\le C\delta(\|\tilde{g} - g_h\|_{L^1(\Gamma_h)} + \|\nabla(\tilde{g} - g_h)\|_{L^1(\Omega_h\setminus\Omega)}) \\
&\le Ch^2\|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)}.
\end{aligned}
$$

Combining the estimates above now gives

$$
\begin{aligned}
|I_{21}| \le Ch^{1/2}\|\tilde{g} - g_h\|_{L^2(\Omega_h\cap A_j^{(3/4)})} &+ Ch\|\tilde{g} - g_h\|_{H^1(\Omega_h\cap A_j^{(3/4)})} \\
&+ Ch^2 d_j^{-N/2}\|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)}.
\end{aligned}
\tag{5.5}
$$

Next we estimate I_{22} by

$$
|I_{22}| \le \|\partial_{n_h}\tilde{w}\|_{L^2(\Gamma_h)}\|\tilde{g} - g_h\|_{L^2(\Gamma_h\cap A_j^{(1/2)})} + \|\partial_{n_h}\tilde{w}\|_{L^\infty(\Gamma_h\setminus A_j^{(1/2)})}\|\tilde{g} - g_h\|_{L^1(\Gamma_h)}.
$$

For the first term we see that

$$
\begin{aligned}
\|\partial_{n_h}\tilde{w}\|_{L^2(\Gamma_h)} &\le \|\nabla\tilde{w}\cdot(n_h - n\circ\pi)\|_{L^2(\Gamma_h)} + \|\big(\nabla\tilde{w} - (\nabla\tilde{w})\circ\pi\big)\cdot n\circ\pi\|_{L^2(\Gamma_h)} \\
&\le Ch\|\nabla\tilde{w}\|_{L^2(\Gamma_h)} + C\delta^{1/2}\|\nabla^2\tilde{w}\|_{L^2(\Gamma(\delta))} \le Ch\|w\|_{H^2(\Omega)} \le Ch,
\end{aligned}
$$

and, in a similar way as we derived (5.4), that

$$
\|\tilde{g} - g_h\|_{L^2(\Gamma_h\cap A_j^{(1/2)})} \le Cd_j^{-1/2}\|\tilde{g} - g_h\|_{L^2(\Omega_h\cap A_j^{(3/4)})} + C\|\tilde{g} - g_h\|_{H^1(\Omega_h\cap A_j^{(3/4)})}.
$$

For the second term, observe that

$$
\begin{aligned}
\|\partial_{n_h}\tilde{w}\|_{L^\infty(\Gamma_h\setminus A_j^{(1/2)})} &\le \|\nabla\tilde{w}\cdot(n_h - n\circ\pi)\|_{L^\infty(\Gamma_h\setminus A_j^{(1/2)})} \\
&\quad + \|\big(\nabla\tilde{w} - (\nabla\tilde{w})\circ\pi\big)\cdot n\circ\pi\|_{L^\infty(\Gamma_h\setminus A_j^{(1/2)})}
\end{aligned}
$$

$$\le Ch\|\nabla \tilde{w}\|_{L^\infty(\Gamma(\delta)\setminus A_j^{(1/2)})} + C\delta\|\nabla^2 \tilde{w}\|_{L^\infty(\Gamma(\delta)\setminus A_j^{(1/4)})}$$
$$\le Chd_j^{1-N/2} + Ch^2 d_j^{-N/2} \le Chd_j^{1-N/2},$$

and that $\|\tilde{g} - g_h\|_{L^1(\Gamma_h)} \le C\|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)}$. Combining these estimates, we deduce

$$|I_{22}| \le Chd_j^{-1/2}\|\tilde{g} - g_h\|_{L^2(\Omega_h \cap A_j^{(3/4)})} + Ch\|\tilde{g} - g_h\|_{H^1(\Omega_h \cap A_j^{(3/4)})}$$
$$+ Chd_j^{1-N/2}\|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)}. \tag{5.6}$$

From (5.5) and (5.6), together with $h \le d_j \le 2\,\mathrm{diam}\ \Omega$, we conclude the desired estimate. □

Lemma 5.4 $|I_3| \le Ch^{5/2-m} d_j^{-N/2}$.

Proof Recall that $I_3 = (\tilde{w} - w_h, \Delta\tilde{g} - \tilde{g})_{\Omega_h\setminus\Omega} - (\tilde{w} - w_h, \partial_{n_h}\tilde{g})_{\Gamma_h} =: I_{31} + I_{32}$. We estimate I_{31} by

$$|I_{31}| \le \|\tilde{w} - w_h\|_{L^2(\Omega_h)}\|\tilde{g}\|_{H^2(\Gamma(\delta)\cap A_j^{(1/2)})} + \|\tilde{w} - w_h\|_{L^\infty(\Omega_h\setminus A_j^{(1/2)})}\|\tilde{g}\|_{W^{2,1}(\Gamma(\delta))}$$
$$\le Ch^2\|\nabla^2\tilde{w}\|_{L^2(\Omega_h)}(\delta d_j^{N-1})^{1/2} d_j^{-m-N} + Ch^2\|\nabla^2\tilde{w}\|_{L^\infty(\Omega_h\setminus A_j^{1/4})}\delta d_0^{-1-m}$$
$$\le Ch^3 d_j^{-1/2-m-N/2} + Ch^2 d_j^{-N/2} h^{1-m} \le Ch^{5/2-m} d_j^{-N/2},$$

where we have used $h \le d_j$.

It remains to consider I_{32}; we estimate it by

$$\|\tilde{w} - w_h\|_{L^2(\Gamma_h)}\|\partial_{n_h}\tilde{g}\|_{L^2(\Gamma_h\cap A_j^{(1/2)})} + \|\tilde{w} - w_h\|_{L^\infty(\Gamma_h\setminus A_j^{(1/2)})}\|\partial_{n_h}\tilde{g}\|_{L^1(\Gamma_h)}.$$

For the first term, we have $\|\tilde{w} - w_h\|_{L^2(\Gamma_h)} \le Ch^{3/2}\|\nabla^2\tilde{w}\|_{L^2(\Omega_h)} \le Ch^{3/2}$ and

$$\|\partial_{n_h}\tilde{g}\|_{L^2(\Gamma_h\cap A_j^{(1/2)})}$$
$$\le |\Gamma_h \cap A_j^{(1/2)}|^{1/2}\big(\|\nabla\tilde{g}\cdot(n_h - n\circ\pi)\|_{L^\infty(\Gamma_h\cap A_j^{(1/2)})}$$
$$+ \|\nabla\tilde{g} - (\nabla\tilde{g})\circ\pi\|_{L^\infty(\Gamma_h\cap A_j^{(1/2)})}\big)$$
$$\le Cd_j^{(N-1)/2}(h\|\nabla\tilde{g}\|_{L^\infty(\Gamma_h\cap A_j^{(1/2)})} + \delta\|\nabla^2\tilde{g}\|_{L^\infty(\Gamma(\delta)\cap A_j^{(3/4)})})$$
$$\le Cd_j^{(N-1)/2}(hd_j^{1-m-N} + h^2 d_j^{-m-N}) \le Chd_j^{1/2-m-N/2}.$$

For the second term, we have $\|\tilde{w} - w_h\|_{L^\infty(\Gamma_h\setminus A_j^{(1/2)})} \le Ch^2\|\nabla^2\tilde{w}\|_{L^\infty(\Gamma_h\setminus A_j^{(1/4)})} \le Ch^2 d_j^{-N/2}$ and we find from Corollary B.1 that $\|\partial_{n_h}\tilde{g}\|_{L^1(\Gamma_h)} \le C(h|\log h|)^{1-m} \le Ch^{(1-m)/2}$. Therefore,

$$|I_{32}| \le Ch^{5/2} d_j^{1/2-m-N/2} + Ch^{5/2-m/2} d_j^{-N/2} \le Ch^{5/2-m} d_j^{-N/2},$$

which completes the proof. □

Lemma 5.5 $|I_4| \le Ch^2 d_j^{1/2-m-N/2} + Ch^{2-m}|\log h|^{1-m} d_j^{1-N/2}$.

Proof We estimate $I_4 = a'_{\Omega_h \triangle \Omega}(\tilde{w}, \tilde{g})$ by

$$|I_4| \le \|\tilde{w}\|_{H^1(\Gamma(\delta))} \|\tilde{g}\|_{H^1(\Gamma(\delta)\cap A_j^{(1/2)})} + \|\tilde{w}\|_{W^{1,\infty}(\Gamma(\delta)\setminus A_j^{(1/2)})} \|\tilde{g}\|_{W^{1,1}(\Gamma(\delta))}.$$

The first term of the right-hand side is bounded, using $(2.1)_2$ and Lemma B.3, by

$$C\delta^{1/2}\|w\|_{H^2(\Omega)}(\delta d_j^{N-1})^{1/2} d_j^{1-m-N} \le Ch^2 d_j^{1/2-m-N/2}.$$

The second term is bounded, in view of Lemma B.4 and Corollary B.1, by $Cd_j^{1-N/2}\delta h^{-m}|\log h|^{1-m}$. This completes the proof. □

Now we substitute the results of Lemmas 5.2–5.5 into (5.3) and multiply by $d_j^{-1+N/2}$ to obtain

$$\begin{aligned}
&d_j^{-1+N/2}\|\tilde{g} - g_h\|_{L^2(\Omega_h \cap A_j)} \\
&\quad \le C(hd_j^{-1})d_j^{N/2}\|\tilde{g} - g_h\|_{H^1(\Omega_h \cap A_j^{(1)})} + C(hd_j^{-1})\|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)} \\
&\quad + Ch^{1/2}d_j^{-1+N/2}\|\tilde{g} - g_h\|_{L^2(\Omega_h \cap A_j^{(1)})} \\
&\quad + Ch^{5/2-m}d_j^{-1} + Ch^2 d_j^{-1/2-m} + Ch^{2-m}|\log h|^{1-m}.
\end{aligned} \tag{5.7}$$

Taking the summation for $j = 0, \dots, J$, assuming h is sufficiently small and using (2.4), we are able to absorb the third term in the right-hand side of (5.7) and then arrive at

$$\begin{aligned}
&\sum_{j=0}^{J} d_j^{-1+N/2}\|\tilde{g} - g_h\|_{L^2(\Omega_h \cap A_j)} \\
&\quad \le C(hd_0^{-1})\left(\sum_{j=0}^{J} d_j^{N/2}\|\tilde{g} - g_h\|_{H^1(\Omega_h \cap A_j)} + \|\tilde{g} - g_h\|_{W^{1,1}(\Omega_h)}\right) \\
&\quad\quad + Ch^{5/2-m}d_0^{-1} + Ch^2 d_0^{-1/2-m} + Ch^{2-m}|\log h|^{1-m}|\log d_0|,
\end{aligned}$$

where we note that the last three terms can be estimated by $Ch^{3/2-m}$ because $d_0 = Kh \le 1$ and $K > 1$. This completes the proof of Proposition 5.1.

6 End of the proof of the main theorem

Substituting (5.1) into (4.2) we obtain

$$\begin{aligned}&\sum_{j=0}^{J} d_j^{N/2}\|\tilde{g}-g_h\|_{H^1(\Omega_h\cap A_j)}\\&\quad\le C''K^{-1}\left(\sum_{j=0}^{J} d_j^{N/2}\|\tilde{g}-g_h\|_{H^1(\Omega_h\cap A_j)}+\|\tilde{g}-g_h\|_{W^{1,1}(\Omega_h)}\right)\\&\qquad+CK^{m+N/2}h^{1-m}+C(h|\log h|)^{1-m}.\end{aligned}$$

If $K\ge 2C''$, then it follows that

$$\begin{aligned}&\sum_{j=0}^{J} d_j^{N/2}\|\tilde{g}-g_h\|_{H^1(\Omega_h\cap A_j)}\\&\quad\le CK^{-1}\|\tilde{g}-g_h\|_{W^{1,1}(\Omega_h)}+CK^{m+N/2}h^{1-m}+C(h|\log h|)^{1-m},\end{aligned}$$

which combined with (4.1) yields

$$\|\tilde{g}-g_h\|_{W^{1,1}(\Omega_h)}\le C'''K^{-1}\|\tilde{g}-g_h\|_{W^{1,1}(\Omega_h)}+CK^{m+N/2}h^{1-m}+C(h|\log h|)^{1-m}.$$

If $K\ge 2C'''$, then this implies the desired estimate (3.2), which together with Proposition 3.1 completes the proof of Theorem 3.1.

7 Numerical example

Letting $\Omega=\{(x,y)\in\mathbb{R}^2\colon \frac{(x-0.12)^2}{4}+\frac{(y+0.2)^2}{9}<1,\ (x-0.7)^2+(y-0.1)^2>0.5^2\}$, which is non-convex, we set an exact solution to be $u(x,y)=x^2$. We define f and τ so that (1.1) holds. They have natural extensions to $\mathbb{R}^2$, which are exploited as $\tilde{f}$ and $\tilde{\tau}$. Then we compute approximate solutions u_h^k of (1.2) based on the P_k-finite elements ($k=1,2,3$), using the software `FreeFEM++` [11]. The errors $\|u-u_h^k\|_{L^\infty(\Omega_h)}$ and $\|\nabla(u-u_h^k)\|_{L^\infty(\Omega_h)}$, which are calculated with the use of P_4-finite element spaces, are reported in Tables 1 and 2, respectively.

We see that the result for $k=1$ is in accordance with Theorem 3.1. The one for $k=3$ (although it is not covered by our theory) is also consistent with our theoretical expectation made in Remark 3.1(iii). When $k=2$, the L^∞-error remains sub-optimal convergence as expected. However, the $W^{1,\infty}$-error seems to be $O(h^2)$, which is significantly better than in the P_3-case. We remark that such behavior was also observed for different (and apparently more complicated) choices of Ω and u. There might be a super-convergence phenomenon in the P_2-approximation for Neumann problems in 2D smooth domains.

Table 1 Behavior of the L^∞-errors for the P_k-approximation ($k = 1, 2, 3$)

h	$\|u - u_h^1\|_{L^\infty(\Omega_h)}$	Rate	$\|u - u_h^2\|_{L^\infty(\Omega_h)}$	Rate	$\|u - u_h^3\|_{L^\infty(\Omega_h)}$	Rate
0.617	5.72e−2	–	1.89e−2	–	2.08e−2	–
0.314	1.75e−2	1.8	4.39e−3	2.2	5.07e−3	2.1
0.165	4.64e−3	2.1	1.05e−3	2.2	1.30e−3	2.1
0.085	1.42e−3	1.8	2.55e−4	2.1	3.33e−4	2.1
0.043	3.92e−4	1.9	6.28e−5	2.1	8.31e−5	2.1

Table 2 Behavior of the $W^{1,\infty}$-errors for the P_k-approximation ($k = 1, 2, 3$)

h	$\|\nabla(u - u_h^1)\|_{L^\infty(\Omega_h)}$	Rate	$\|\nabla(u - u_h^2)\|_{L^\infty(\Omega_h)}$	Rate	$\|\nabla(u - u_h^3)\|_{L^\infty(\Omega_h)}$	Rate
0.617	6.24e−1	–	9.98e−2	–	3.91e−1	–
0.314	3.21e−1	1.0	2.68e−2	1.9	2.15e−1	0.9
0.165	1.58e−1	1.1	6.85e−3	2.1	1.04e−1	1.1
0.085	9.18e−2	0.8	1.58e−3	2.2	5.47e−2	1.0
0.043	4.63e−2	1.0	4.42e−4	1.9	2.77e−2	1.0

Remark 7.1 If $k \geq 2$ and $\tilde{\tau}$ is chosen as $\nabla u \cdot n_h$, then u_h^k agrees with u (note that the above u is quadratic), because this amounts to assuming that the original problem (1.1) is given in a polygon Ω_h. This was observed in our numerical experiment as well (up to rounding errors). However, since such $\tilde{\tau}$ is unavailable without knowing an exact solution, one cannot expect it in a practical computation.

Appendix A: Auxiliary boundary-skin estimates

Local coordinate representation

We exploit the notations and observations given in [12, Section 8], which we briefly describe here. Since Ω is a bounded C^∞-domain, there exist a system of local coordinates $\{(U_r, y_r, \varphi_r)\}_{r=1}^M$ such that $\{U_r\}_{r=1}^M$ forms an open covering of Γ, $y_r = (y_r', y_{rN})$ is a rotated coordinate of x, and $\varphi_r \colon \Delta_r \to \mathbb{R}$ gives a graph representation $\Phi_r(y_r') := (y_r', \varphi_r(y_r'))$ of $\Gamma \cap U_r$, where Δ_r is an open cube in $\mathbb{R}^{N-1}_{y_r'}$.

For $S \in \mathcal{S}_h$, we may assume that $S \cup \pi(S)$ is contained in some U_r, where $\pi \colon \Gamma(\delta_0) \to \Gamma$ is the projection to Γ given in Sect. 2.2. Let $b_r \colon \mathbb{R}^N \to \mathbb{R}^{N-1}$; $y_r \mapsto y_r'$ be a projection to the base set and let $S' := b_r(\pi(S))$. Then Φ_r and $\Phi_{hr} := \pi^* \circ \Phi_r$, where $\pi^* \colon \Gamma \to \Gamma_h$ is the inverse map of $\pi|_{\Gamma_h}$, give smooth parameterizations of $\pi(S)$ and S respectively, with the domain S'. We also recall that π^* is also written as $\pi^*(\Phi_r(y_r')) = \Phi_r(y_r') + t^*(\Phi_r(y_r'))n(\Phi_r(y_r'))$.

Let us represent integrals associated with S in terms of local coordinates. In what follows, we omit the subscript r for simplicity. First, surface integrals along $\pi(S)$ and S are expressed as

$$\int_{\pi(S)} f\,d\gamma = \int_{S'} f(\Phi(y'))\sqrt{\det G(y')}\,dy',$$
$$\int_{S} f\,d\gamma_h = \int_{S'} f(\Phi_h(y'))\sqrt{\det G_h(y')}\,dy',$$

where G and G_h denote the Riemannian metric tensors obtained from the parameterizations Φ and Φ_h, respectively. Next, let $\pi(S,\delta) := \{\bar{x} + tn(\bar{x}) : \bar{x} \in S,\ -\delta \le t \le \delta\}$ be a tubular neighborhood with the base $\pi(S)$, where $\delta = C_{0E}h^2$, and consider volume integrals over $\pi(S,\delta)$. For this we introduce a one-to-one transformation $\Psi: S' \times [-\delta,\delta] \to \pi(S,\delta)$ by

$$y = \Psi(z',t) := \Phi(z') + tn(\Phi(z')) \iff z' = b(\pi(y)),\ t = d(y).$$

Then, by change of variables, we obtain

$$\int_{\pi(S,\delta)} f(y)\,dy = \int_{S'\times[-\delta,\delta]} f(\Psi(z',t)) \det J(z',t)\,dz'dt,$$

where $J := \nabla_{(z',t)}\Psi$ denotes the Jacobi matrix of Ψ. In the formulas above, $\det G$, $\det G_h$, and $\det J$ can be bounded, from above and below, by positive constants depending on the $C^{1,1}$-regularity of Ω, provided h is sufficiently small (for the proof, see [12, Section 8]).

Proof of (2.2)

In [12, Theorem 8.3], we estimated the L^p-norm of a function in the full layer $\Gamma(\delta)$. By slightly modifying the proof there, we can estimate it in $\Omega_h\setminus\Omega$, which is important to dispense with extensions from Ω_h to $\tilde{\Omega}$.

Lemma A.1 *Let $f \in W^{1,p}(\Omega_h)$ $(1 \le p \le \infty)$ and $\delta = C_{0E}h^2$. Then we have*

$$\|f\|_{L^p(\Omega_h\setminus\Omega)} \le C(\delta^{1/p}\|f\|_{L^p(\Gamma_h)} + \delta\|(n\circ\pi)\cdot\nabla f\|_{L^p(\Omega_h\setminus\Omega)}),$$

where C is independent of δ and f.

Proof To simplify the notation we use the abbreviation $t^*(z')$ to imply $t^*(\Phi(z'))$. For each $S \in \mathcal{S}_h$ we observe that

$$\begin{aligned}
&\int_{(\Omega_h\setminus\Omega)\cap\pi(S,\delta)} |f(y)|^p\,dy \\
&= \int_{S'}\int_0^{\max\{0,t^*(z')\}} |f(\Psi(z',t))|^p \det J\,dt\,dz' \\
&\le C\int_{S'}\int_0^{\max\{0,t^*(z')\}} \Big(|f(\Phi_h(z'))|^p + |f(\Psi(z',t)) - f(\Phi_h(z'))|^p\Big)\,dt\,dz' \\
&=: I_1 + I_2,
\end{aligned}$$

and that for $z' \in S'$ and $0 \le t \le t^*(z')$

$$
\begin{aligned}
|f(\Psi(z', t)) - f(\Phi_h(z'))| &= \left| \int_t^{t^*(z')} n(\Phi(z')) \cdot \nabla f(\Psi(z', s))\, ds \right| \\
&\le \int_0^{t^*(z')} |n(\Phi(z')) \cdot \nabla f(\Psi(z', s))|\, ds \\
&\le t^*(z')^{1-1/p} \left(\int_0^{t^*(z')} |n(\Phi(z')) \cdot \nabla f(\Psi(z', s))|^p\, ds \right)^{1/p}.
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
I_1 &\le C\|t^*\|_{L^\infty(S)} \int_{S'} |f(\Phi_h(z'))|^p\, dz' \\
&\le C\delta \int_{S'} |f(\Phi_h(z'))|^p \sqrt{\det G_h}\, dz' = C\delta \|f\|_{L^p(S)}^p
\end{aligned}
$$

and that

$$
\begin{aligned}
I_2 &\le C\|t^*\|_{L^\infty(S)}^p \int_{S'} \int_0^{\max\{0, t^*(z')\}} |n(\Phi(z')) \cdot \nabla f(\Psi(z', s))|^p \det J\, dt\, dz' \\
&\le C\delta^p \|n \circ \pi \cdot \nabla f\|_{L^p(\pi(S,\delta))}^p.
\end{aligned}
$$

Adding up the above estimates for $S \in \mathcal{S}_h$ gives the conclusion. □

Lemma A.2 *For a measurable set $D \subset \mathbb{R}^N$ and $f \in W^{1,\infty}(\Gamma(\delta))$ we have*

$$
\|f - f \circ \pi\|_{L^\infty(\Gamma(\delta) \cap D)} \le \delta \|\nabla f\|_{L^\infty(\Gamma(\delta) \cap D_{2\delta})},
$$

where $D_{2\delta} = \{x \in \mathbb{R}^N \colon \operatorname{dist}(x, D) \le 2\delta\}$.

Proof This is an easy consequence of the Lipschitz continuity of f. □

Proof of Proposition 2.1

Let us prove stability properties of the extension operator P defined in Sect. 2.3.

Theorem A.1 *Let $f \in W^{k,p}(\Omega)$ with $k = 0, 1, 2$, and $p \in [1, \infty]$. Then we have*

$$
\|Pf\|_{W^{k,p}(\Gamma(\delta))} \le C\|f\|_{W^{k,p}(\Omega \cap \Gamma(2\delta))},
$$

where C is independent of δ and f.

Proof First, for each $S \in \mathcal{S}_h$ we show

$$
\|Pf\|_{L^p(\pi(S,\delta) \setminus \Omega)}^p \le C\|f\|_{L^p(\pi(S,2\delta) \cap \Omega)}^p.
$$

In fact we have

$$\int_{\pi(S,\delta)\setminus\Omega} |Pf(y)|^p\,dy \le C\int_{S'\times[0,\delta]} |3f(z'-tn(z'))-2f(z'-2tn(z'))|^p\,dz'dt$$
$$\le C\int_{S'\times[0,\delta]} \Big(|f(z'-tn(z'))|^p+|f(z'-2tn(z'))|^p\Big)\,dz'dt$$
$$\le C\int_{\pi(S,\delta)\cap\Omega} |f(y)|^p\,dy + C\int_{\pi(S,2\delta)\cap\Omega} |f(y)|^p\,dy.$$

Next we show

$$\|\nabla Pf\|^p_{L^p(\pi(S,\delta)\setminus\Omega)} \le C\|\nabla f\|^p_{L^p(\pi(S,2\delta)\cap\Omega)}. \tag{A.1}$$

Since by the chain rule $\nabla_y = \nabla_y(b\circ\pi)\nabla_{z'} + (\nabla_y d)\partial_t$ and since $Pf(y) = 3f\circ\Psi(z',-t) - 2f\circ\Psi(z',-2t)$, it follows that

$$\nabla Pf(y) = \nabla_y(b\circ\pi)\Big(3\nabla_{z'}(f\circ\Psi)|_{(z',-t)} - 2\nabla_{z'}(f\circ\Psi)|_{(z',-2t)}\Big)$$
$$+\ \nabla_y d\Big(-3\partial_t(f\circ\Psi)|_{(z',-t)} + 4\partial_t(f\circ\Psi)|_{(z',-2t)}\Big),\quad y\in\pi(S,\delta)\setminus\Omega. \tag{A.2}$$

In particular, if $y\in\Gamma$ i.e. $t=0$, then

$$\nabla Pf(y) = \nabla_y(b\circ\pi)\nabla_{z'}(f\circ\Psi)|_{(z',0)} + (\nabla_y d)\partial_t(f\circ\Psi)|_{(z',0)}$$
$$= J^{-1}(z',0)J(z',0)\nabla_y f(y) = \nabla f(y),$$

which ensures that $Pf(y)\in W^{2,p}(\pi(S,\delta))$. Now, noting that $\nabla_y\binom{b\circ\pi}{d} = J^{-1}(z',t)$ and that $\nabla_{(z',t)}(f\circ\Psi)|_{(z',-it)} = J(z',-it)(\nabla_y f)|_{\Psi(z',-it)}$ $(i=1,2)$ where J and J^{-1} depend on the $C^{1,1}$-regularity of Ω, we deduce that

$$\int_{\pi(S,\delta)\setminus\Omega} |\nabla Pf(y)|^p\,dy \le C\int_{S'\times[0,\delta]} \Big(\big|(\nabla_y f)|_{\Psi(z',-t)}\big|^p + \big|(\nabla_y f)|_{\Psi(z',-2t)}\big|^p\Big)\,dz'dt,$$

from which (A.1) follows.

Finally we show

$$\|\nabla^2 Pf\|^p_{L^p(\pi(S,\delta)\setminus\Omega)} \le C(\|\nabla^2 f\|^p_{L^p(\pi(S,2\delta)\cap\Omega)} + \|\nabla f\|^p_{L^p(\pi(S,2\delta)\cap\Omega)}). \tag{A.3}$$

By differentiating (A.2) we find that for $y\in\pi(S,\delta)\setminus\Omega$

$$\nabla^2 Pf(y) = \sum_{i=1}^{2}\Big(A_i(z',t)\nabla^2_{(z',t)}(f\circ\Psi)_{(z',-it)} + B_i(z',t)\nabla_{(z',t)}(f\circ\Psi)_{(z',-it)}\Big),$$

where the coefficient tensors A_i, B_i depend on the $C^{1,1}$-regularity of Ω. Then the L^p-norm of the above quantity can be estimated similarly as before and one obtains (A.3).

Adding up the above estimates for $S \in \mathcal{S}_h$ deduces the desired stability properties. □

We also need local stability of the extension operator as follows.

Corollary A.1 *For a measurable set $D \subset \mathbb{R}^N$ and $\delta = C_{0E}h^2$ we have*

$$\|Pf\|_{W^{k,\infty}(\Gamma(\delta)\cap D)} \le C\|f\|_{W^{k,\infty}(\Omega\cap\Gamma(2\delta)\cap D_{3\delta})} \quad (k = 0, 1, 2),$$

where $D_{3\delta} = \{x \in \mathbb{R}^N\colon \operatorname{dist}(x, D) \le 3\delta\}$ and C is independent of δ, f, and D.

Proof We address the L^∞-norm of ∇Pf; the treatment of Pf and $\nabla^2 Pf$ is similar. For each $S \in \mathcal{S}_h$, we find from the analysis of Theorem A.1 that $\nabla Pf(y)$ for $y \in \pi(S, \delta)\backslash\Omega$ can be expressed as

$$\nabla Pf(y) = \sum_{i=1}^{2} A_i(z', t)(\nabla_y f)|_{\Psi(z',-it)},$$

where the matrices A_i depend on the $C^{0,1}$-regularity of Ω. Then the desired estimate follows from the observation that if $y = \Psi(z', t) \in \pi(S, \delta) \cap D\backslash\Omega$ then $\Psi(z', -it) \in \pi(S, i\delta) \cap D_{3\delta} \cap \Omega$ for $i = 1, 2$. □

Appendix B: Analysis of regularized Green's functions

Estimates for $\tilde{g}$

Recall that for arbitrarily fixed $x_0 \in \Omega_h$ we have introduced $\eta \in C_0^\infty(\Omega_h \cap \Omega)$ and $g_m \in C^\infty(\overline{\Omega})$ $(m = 0, 1)$ in Sect. 3. Using the Green's function $G(x, y)$ for the operator $-\Delta + 1$ in Ω with the homogeneous Neumann boundary condition, one can represent g_m as

$$g_0(x) = \int_{\operatorname{supp}\eta} G(x, y)\eta(y)\,dy, \quad g_1(x) = -\int_{\operatorname{supp}\eta} \partial_y G(x, y)\eta(y)\,dy, \quad x \in \Omega.$$

The following derivative estimates for G are well known (see e.g. [13, p. 965]):

$$|\nabla_x^k \nabla_y^l G(x, y)| \le \begin{cases} C(1 + |x - y|^{2-l-k-N}) & (l + k + N > 2), \\ C(1 + \big|\log|x - y|\big|) & (N = 2, l = k = 0). \end{cases}$$

From this, combined with a dyadic decomposition of Ω, we derive some local and global estimates for g_m and its extension $\tilde{g}_m := Pg_m$. Below the subscript m will be dropped for simplicity.

Lemma B.1 *Let $\mathcal{A}_{\Omega_h}(x_0, d_0) = \{\Omega_h \cap A_j\}_{j=0}^J$ be a dyadic decomposition of Ω_h with $d_0 \in [4h, 1]$. Then, for $j = 1, \ldots, J$ and $k \geq 0$ we have*

$$\|\nabla^k g\|_{L^\infty(\Omega \cap A_j)} \leq \begin{cases} C(1 + d_j^{2-m-k-N}) & (m + k + N > 2), \\ C(1 + |\log d_j|) & (N = 2, m = k = 0), \end{cases}$$

where C is independent of x_0, d_0, h, j, and ∂.

Proof We only consider $m+k+N > 2$ because the other case can be treated similarly. Notice that if $x \in \Omega \cap A_j$ ($j \geq 1$) and $y \in \operatorname{supp} \eta$ then $|x - y| \geq \frac{3}{4} d_{j-1}$, which is obtained from $|x - x_0| \geq d_{j-1}$ and $|y - x_0| \leq h$. It then follows that

$$\begin{aligned} \|\nabla^k g\|_{L^\infty(\Omega \cap A_j)} &= \sup_{x \in \Omega \cap A_j} \left| \int_{\operatorname{supp} \eta} \partial_y^m \nabla_x^k G(x, y) \eta(y)\, dy \right| \\ &\leq \sup_{|x-y| \geq \frac{3}{4} d_{j-1} |\partial_y^m \nabla_x^k G(x,y)|} \\ &\leq C(1 + d_j^{2-m-k-N}), \end{aligned}$$

which completes the proof. □

We transfer these estimates in Ω to those in $\tilde{\Omega} = \Omega \cup \Gamma(\delta)$ using an extension operator and its stability.

Lemma B.2 *Let $\mathcal{A}_{\Omega_h}(x_0, d_0) = \{\Omega_h \cap A_j\}_{j=0}^J$ be a dyadic decomposition of Ω_h with $d_0 \in [h, 1]$, $\delta = C_{0E} h^2$. For $p \in [1, \infty]$, $j = 1, \ldots, J$, and $m = 0, 1$, we have*

$$\|\nabla^2 \tilde{g}\|_{L^p(\tilde{\Omega} \cap A_j)} \leq C d_j^{-m-N/p'},$$

where $p' = p/(p-1)$ and C is independent of x_0, d_0, h, j, and ∂.

Proof By the Hölder inequality and Lemma B.1 we see that

$$\begin{aligned} \|\nabla^2 \tilde{g}\|_{L^p(\tilde{\Omega} \cap A_j)} &\leq C |\Omega_h \cap A_j|^{1/p} \|\nabla^2 \tilde{g}\|_{L^\infty(\tilde{\Omega} \cap A_j)} \leq C d_j^{N/p} \|g\|_{W^{2,\infty}(\Omega \cap A_j^{(1/4)})} \\ &\leq C d_j^{N/p} (1 + d_j^{2-m-N} + d_j^{1-m-N} + d_j^{-m-N}) \leq C d_j^{-m-N/p'}, \end{aligned}$$

where we have used $d_j \leq 2 \operatorname{diam} \Omega$ in the last inequality. □

We also need local estimates in intersections of annuli and boundary-skins (or boundaries).

Lemma B.3 *Under the assumptions in* Lemma B.2, *let $k = 0, 1, 2$. Then we have*

$$\begin{aligned} \|\nabla^k \tilde{g}\|_{L^p(\Gamma(\delta) \cap A_j)} &\leq C(\delta d_j^{N-1})^{1/p} (1 + d_j^{2-m-k-N}), \\ \|\nabla^k g\|_{L^p(\Gamma \cap A_j)} + \|\nabla^k \tilde{g}\|_{L^p(\Gamma_h \cap A_j)} &\leq C d_j^{(N-1)/p} (1 + d_j^{2-m-k-N}), \end{aligned}$$

provided $m+k+N>2$. Even when $N=2$ and $m=k=0$, the above estimates hold with the factor $d_j^{2-m-k-N}$ replaced by $|\log d_j|$. The constants C are independent of x_0, d_0, h, j, and ∂.

Proof We only consider $m+k+N>2$ since the other case may be treated similarly. From Corollary A.1 and Lemma B.1 we deduce that (note that $(A_j)_{3\delta} \subset A_j^{(1/4)}$ for small h)

$$\begin{aligned}\|\nabla^k \tilde{g}\|_{L^p(\Gamma(\delta)\cap A_j)} &\le |\Gamma(\delta)\cap A_j|^{1/p}\|\nabla^k \tilde{g}\|_{L^\infty(\Gamma(\delta)\cap A_j)} \\ &\le C(\delta d_j^{N-1})^{1/p}\|g\|_{W^{k,\infty}(\Omega\cap\Gamma(2\delta)\cap A_j^{(1/4)})} \\ &\le C(\delta d_j^{N-1})^{1/p}(1+d_j^{2-m-k-N}),\end{aligned}$$

where we have used $d_j \le 2\,\mathrm{diam}\ \Omega$ in the second line. Similarly,

$$\begin{aligned}\|\nabla^k \tilde{g}\|_{L^p(\Gamma_h\cap A_j)} &\le |\Gamma_h\cap A_j|^{1/p}\|\nabla^k \tilde{g}\|_{L^\infty(\Gamma_h\cap A_j)} \\ &\le C d_j^{(N-1)/p}\|g\|_{W^{k,\infty}(\Omega\cap\Gamma(2\delta)\cap A_j^{(1/4)})} \\ &\le C d_j^{(N-1)/p}(1+d_j^{2-m-k-N}).\end{aligned}$$

One sees that $\|\nabla^k g\|_{L^p(\Gamma\cap A_j)}$ obeys the same estimate. □

Remark B.1 The three lemmas above remain true with A_j replaced by $A_j^{(s)}$ $(0 \le s < 1)$, where the constants C become dependent on the choice of s.

Especially when $p=1$, the following global estimate in a boundary-skin layer holds.

Corollary B.1 *Let $\delta = C_{0E}h^2$ with sufficiently small h. Then we have*

$$\|\tilde{g}_0\|_{W^{k,1}(\Gamma(\delta))} \le \begin{cases} C\delta & (k=0), \\ C\delta|\log h| & (k=1), \\ C\delta h^{-1} & (k=2), \end{cases}$$

$$\|\nabla^k g_0\|_{L^1(\Gamma)} + \|\nabla^k \tilde{g}_0\|_{L^1(\Gamma_h)} \le \begin{cases} C & (k=0), \\ C|\log h| & (k=1), \\ Ch^{-1} & (k=2), \end{cases}$$

and

$$\|\tilde{g}_1\|_{W^{k,1}(\Gamma(\delta))} \le \begin{cases} C\delta|\log h| & (k=0), \\ C\delta h^{-1} & (k=1), \\ C\delta h^{-2} & (k=2), \end{cases}$$

$$\|\nabla^k g_1\|_{L^1(\Gamma)} + \|\nabla^k \tilde{g}_1\|_{L^1(\Gamma_h)} \le \begin{cases} C|\log h| & (k=0), \\ Ch^{-1} & (k=1), \\ Ch^{-2} & (k=2), \end{cases}$$

where C is independent of x_0, h, and ∂.

Proof We only consider the estimates in $W^{k,1}(\Gamma(\delta))$ because the boundary estimates can be derived similarly. With a dyadic decomposition $\mathcal{A}_{\Omega_h}(x_0, 4h) = \{\Omega_h \cap A_j\}_{j=0}^J$, we compute $\sum_{j=0}^J \|\tilde{g}\|_{W^{k,1}(\Gamma(\delta)\cap A_j)}$. When $j \ge 1$, it follows from Lemma B.3 that

$$\|\tilde{g}\|_{W^{k,1}(\Gamma(\delta)\cap A_j)} \le \begin{cases} C(\delta d_j^{N-1}) d_j^{2-m-k-N} & (m+k+N>2), \\ C(\delta d_j^{N-1})|\log d_j| & (N=2, m=k=0). \end{cases} \tag{B.1}$$

When $j=0$, notice that $\operatorname{dist}(\operatorname{supp}\eta, \Gamma(2\delta)) \ge Ch = \frac{C}{4} d_0$ for sufficiently small h, which results from (3.1). Then, calculating in the same way as above, we find that (B.1) holds for $j=0$ as well. Adding up the above estimate for $j=0,\dots,J$ and using (2.5), we obtain the desired result. □

Remark B.2 We could improve the above estimates for g_0 when $k=1$ if the Dirichlet boundary condition were considered. In fact, the Green's function $G_D(x,y)$ in this case is known to satisfy $|\nabla_x G_D(x,y)| \le C \operatorname{dist}(y, \partial\Omega)|x-y|^{-N}$ (see [10, Theorem 3.3(v)]). Then, taking a dyadic decomposition with $d_0 = \operatorname{dist}(\operatorname{supp}\eta, \partial\Omega) \ge Ch$, we see that

$$\begin{aligned} \|\nabla \tilde{g}_0\|_{L^1(\Gamma_h)} &\le C \sum_{j=0}^J d_j^{N-1} \|\nabla \tilde{g}_0\|_{L^\infty(\Gamma_h \cap A_j)} \\ &\le C \operatorname{dist}(\operatorname{supp}\eta, \partial\Omega) \sum_{j=0}^J d_j^{-1} \\ &\le C d_0 d_0^{-1} = C, \end{aligned}$$

and that $\|\nabla \tilde{g}_0\|_{L^1(\Gamma(\delta))} \le C\delta$. However, such an auxiliary Green's function estimate is not available in the case of the Neumann boundary condition. A similar inequality is proved in [17, eq. (5.8)] by a different method using the maximum principle, but its extension to the Neumann case seems non-trivial.

Estimates for $\tilde{w}$

Let us recall the situation of Sect. 5: fixing a dyadic decomposition $\mathcal{A}_{\Omega_h}(x_0, d_0)$ and an annulus A_j $(0 \le j \le J)$, we have introduced the solution $w \in C^\infty(\overline{\Omega})$ of (5.2) for arbitrary $\varphi \in C_0^\infty(\Omega_h \cap A_j)$ such that $\|\varphi\|_{L^2(\Omega_h \cap A_j)} = 1$. Hence w is represented, using the Green's function $G(x,y)$, as

$$w(x) = \int_{\Omega\cap\Omega_h\cap A_j} G(x, y)\varphi(y)\, dy \quad (x \in \Omega).$$

Then we obtain the following local L^∞-estimates away from A_j:

Lemma B.4 *For $k = 0, 1, 2$ and $\delta = C_{0E}h^2$, we have*

$$\|\tilde{w}\|_{W^{k,\infty}(\tilde{\Omega}\setminus A_j^{(1/2)})} \le \begin{cases} Cd_j^{2-k-N/2} & (N+k>2),\\ Cd_j(1+|\log d_j|) & (N=2, k=0),\end{cases}$$

where $\tilde{\Omega} := \Omega \cup \Gamma(\delta)$, $\tilde{w} := Pw$, and C is independent of h, x_0, d_0, and j.

Proof We focus on the case $N + k > 2$; the other case is similar. We find that

$$\begin{aligned}
&\|\tilde{w}\|_{W^{k,\infty}(\tilde{\Omega}\setminus A_j^{(1/2)})}\\
&\le C\|w\|_{W^{k,\infty}(\Omega\setminus A_j^{(1/4)})} = C\sum_{l=0}^{k} \sup_{x\in\Omega\setminus A_j^{(1/4)}} \left| \int_{\Omega\cap\Omega_h\cap A_j} \nabla_x^l G(x, y)\varphi(y)\, dy \right|\\
&\le C\sum_{l=0}^{k} |\Omega\cap\Omega_h\cap A_j|^{1/2} \sup_{|x-y|\ge d_{j-1}/8} |\nabla_x^l G(x, y)|\, \|\varphi\|_{L^2(\Omega\cap\Omega_h\cap A_j)}\\
&\le Cd_j^{N/2}\left(1 + d_j^{2-N} + \cdots + d_j^{2-k-N}\right) \le Cd_j^{2-k-N/2},
\end{aligned}$$

where we have used $d_j \le 2\,\mathrm{diam}\ \Omega$ in the last inequality. □

Remark B.3 The lemma remains true with $A_j^{(1/2)}$ replaced by $A_j^{(s)}$ $(0 < s \le 1)$, where the constant C becomes dependent on the choice of s.

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