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Error estimates for variational normal derivatives and Dirichlet control problems with energy regularization

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Abstract
This article deals with error estimates for the finite element approximation of variational normal derivatives and, as a consequence, error estimates for the finite element approximation of Dirichlet boundary control problems with energy regularization. The regularity of the solution is carefully carved out exploiting weighted Sobolev and Hölder spaces. This allows to derive a sharp relation between the convergence rates for the approximation and the structure of the geometry, more precisely, the largest opening angle at the vertices of polygonal domains. Numerical experiments confirm that the derived convergence rates are sharp.

Mathematics Subject Classification 49J20 · 65M60 · 65N15 · 35L67

1 Introduction

The problem investigated in this article is the optimal Dirichlet control problem

$$\min_{z\in H^{1/2}(\Gamma)} \left\{ \frac{1}{2}\, \|u(z) - u_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\, |z|^2_{H^{1/2}(\Gamma)} \right\}, \tag{1}$$

where $u(z) \in H^1(\Omega)$ is the solution of the boundary value problem

$$-\Delta u = f \text{ in } \Omega, \qquad u = z \text{ on } \Gamma. \tag{2}$$

The domain $\Omega \in \mathbb{R}^2$ is assumed to have a polygonal boundary Γ. The function $u_d \in L^2(\Omega)$ is referred to as desired state and $f \in L^2(\Omega)$ is a given source term. The parameter $\nu > 0$ is a regularization parameter and the corresponding term in the objective guarantees the existence of a solution in the space $H^{1/2}(\Gamma)$.

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This optimal control problem has first been formulated by Lions [19]. Later, a regularization using the $L^2(\Gamma)$-norm of the control became more attention [3,8,20,21]. From the modeling point of view, the $L^2(\Gamma)$ regularization is reasonable as the regularization term can be interpreted as a measure for control costs, but the disadvantage is that the control has a rather unexpected behavior near the corners. In the general case the control tends to 0 at convex and to infinity at reentrant corners [2]. Thus, the idea of using a regularization in stronger norms was revealed e.g. in [14] where Dirichlet control problems using an $H^1(\Gamma)$-regularization are studied and [9,27] where $H^{1/2}(\Gamma)$-regularization is considered. The latter idea, also referred to as energy regularization and is also studied in the present paper. It has to be noted that the behavior near the corners is in this approach just shifted to the tangential derivatives of the control. The physical interpretation of the regularization term using the $H^{1/2}(\Gamma)$-norm of the control is, that in case of $f \equiv 0$ it is equivalent to the energy norm of the corresponding state $u(z)$, which might be, depending on the concrete application, a measure for control costs as well. This becomes clear when defining the seminorm in $H^{1/2}(\Gamma)$ by

$$|z|^2_{H^{1/2}(\Gamma)} := \int_\Gamma \partial_n u(z)\, z = \|\nabla u(z)\|^2_{L^2(\Omega)}.$$

Closely related are the investigations for the Neumann control problem with an $H^{-1/2}(\Gamma)$-regularization [6,32]. Note, that the optimal state is in both approaches equivalent.

Error estimates for approximate solutions of the Dirichlet control problem are discussed already in [27] where all variables are approximated by piecewise linear finite elements. For this approach, and in case of convex computational domains, the convergence rate of 1 for the control in the $H^{1/2}(\Gamma)$-norm was proved, but in the numerical experiments a higher convergence rate is observed. The results in the present article will show that the rate 1 is only a worst-case estimate for convex domains, meaning, that if an opening angle of a corner tends to 180°, the convergence rate will tend to 1. The same convergence rate is proved in [17] for arbitrary polygonal domains for a discretization using the energy corrected finite element method.

It is the aim of the present paper to prove sharp convergence rates. Depending on the opening angle at the corners one can prove a convergence rate up to $3/2$ for the control in the $H^{1/2}(\Gamma)$-norm. It turns out that this is in general only possible when the opening angles are all less than 120° as the corresponding singularities are mild enough to guarantee $H^2(\Gamma)$-regularity of the control.

The difficult part of the convergence proof is to derive an error estimate for a variational normal derivative of the finite element solution of the Poisson and the Laplace equation in the $H^{-1/2}(\Gamma)$-norm. Such an error term appears due do the approximation of the Steklov–Poincaré operator $z \mapsto \partial_n u(z)$ used to realize the $H^{1/2}(\Gamma)$-norm, and the approximation for the normal derivative of the adjoint state variable which appears in the optimality condition. A worst-case estimate for variational normal derivatives in the $H^{-1/2}(\Gamma)$-norm, as used in [27], can be easily derived when using a trace theorem and standard finite element error estimates. Sharp error estimates require some more effort and will be discussed intensively in the present article. Closely related are the error estimates in the $L^2(\Gamma)$-norm for the exact normal derivative of the finite element approximation from [15,29]. In the latter reference the variational normal derivative used in the present paper is discussed as well. In the present article we consider estimates for the variational normal derivative in $H^{-1/2}(\Gamma)$. The convergence rate we prove will be related to $\omega_{\max}$ denoting the largest opening angle of the corners of the domain Ω. Moreover, $u \in H^1(\Omega)$ and $u_h \in V_h$ are the solution of the Poisson or Laplace equation and its finite element approximation, respectively. Throughout this article V_h is the space of linear and globally continuous finite elements. Under the assumption that the input data are sufficiently smooth, and the normal derivative is continuous in the corners (this is guaranteed e. g. if $z \equiv 0$ or if u is the optimal state of the Dirichlet control problem (1)–(2)) we show that the variational normal derivative $\partial_n^h u_h \in V_h^\partial := \mathrm{tr} V_h$ defined by

$$(\partial_n^h u_h, v_h)_{L^2(\Gamma)} = (\nabla u_h, \nabla v_h)_{L^2(\Omega)^2} - (f, v_h)_{L^2(\Omega)}$$

for all $v_h \in V_h$ satisfies the estimate

$$\|\partial_n u - \partial_n^h u_h\|_{H^{-1/2}(\Gamma)} \le c\, h^{\min\{3/2, \pi/\omega_{\max} - \varepsilon\}}.$$

Here, $c > 0$ is a constant independent of the mesh size h and $\varepsilon > 0$ is an arbitrary but sufficiently small number. The proof is based on an idea developed in [29] where estimates in the $L^2(\Gamma)$-norm on a sequence of boundary concentrated meshes is proved.

As an application, we use this result to derive sharp discretization error estimates for the optimal control problem (1)–(2). Therefore, we approximate the control, state and adjoint state by a linear finite element discretization. Under the assumption that u_d is Hölder continuous in case of convex Ω, or belongs to $L^2(\Omega)$ in case of non-convex Ω, we show the same convergence rate for the control approximation in the $H^{1/2}(\Gamma)$-norm, this is,

$$\|z - z_h\|_{H^{1/2}(\Gamma)} \le c\, h^{\min\{3/2, \pi/\omega_{\max} - \varepsilon\}},$$

where z and z_h are the continuous and discrete optimal control. This confirms the behavior figured out in the numerical experiments from [27] on the unit square, where the rate $3/2$ was predicted numerically. The conjecture that this rate is achieved on arbitrary convex polygonal domains is obviously wrong. Our theory promises that this rate is obtained unless all opening angles of corners are less than $2\pi/3$ which is also confirmed by numerical experiments. The worst-case convergence rate of 1 is indeed achieved unless the domain remains convex. If the largest angle tends to 2π, the convergence rate will tend to $1/2$.

As a further application of estimates for variational normal derivatives we mention Steklov–Poincaré operators that are frequently used for parallel finite element methods relying on domain decomposition [1,30,34]. Closely related are the error estimates from [24]. Therein, the authors derive optimal error estimates for discrete Lagrange multipliers in $H^{-1/2}(\Gamma)$ defined on the interfaces of the subdomains. The approximation of the multipliers corresponds to some variational approximation of a normal derivative as well.

The article is structured as follows. In Sect. 2 we collect a priori estimates for solutions of the Poisson and Laplace equation in weighted norms involving a regularized boundary distance function. Moreover, we have to carve out the singular behavior near corners of the domain which is done by weighted Sobolev and Hölder spaces. To this end, we provide the required shift theorems. Error estimates for the solution of the Dirichlet boundary value problem in the $L^2(\Omega)$- and $H^1(\Omega)$-norm as well as for the discrete normal derivatives in the $H^{-1/2}(\Gamma)$-norm are derived in Sect. 3. These estimates are applied to the discretization of our optimal control problem in Sect. 4. The results derived therein are confirmed by the numerical experiments in Sect. 5.

2 Auxiliary results

Let us first explain the notation we will use in this paper. The computational domain is denoted by $\Omega \subset \mathbb{R}^2$ and is always assumed to have a polygonal boundary Γ. By $W^{k,p}(\Omega)$, $k \in \mathbb{N}_0$, $p \in [1, \infty]$ we denote the usual Sobolev spaces and write $H^k(\Omega) := W^{k,2}(\Omega)$, $L^2(\Omega) := H^0(\Omega)$. Frequently, we use the space $H_0^1(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the $H^1(\Omega)$-norm. For the corresponding norms and inner products we write $\|\cdot\|_X$ and $(\cdot,\cdot)_X$, respectively. The subscript X indicates the related space.

The fractional-order Sobolev space $H^{1/2}(\Gamma)$ is defined as the set of measurable functions $u\colon \Gamma \to \mathbb{R}$ with finite Sobolev-Slobodeckij-norm

$$\|u\|^2_{H^{1/2}(\Gamma)} := \|u\|^2_{L^2(\Gamma)} + \iint_\Gamma \frac{|u(x) - x(y)|^2}{|x-y|^2}\,\mathrm{d}s_x\,\mathrm{d}s_y.$$

Later, we will use equivalent definitions, e. g. as the natural trace space of $H^1(\Omega)$, see [33, §8], or as the interpolation space $[L^2(\Gamma), H^1(\Gamma)]_{1/2,2}$ [23, Theorem B.11].

Moreover, $\langle\cdot,\cdot\rangle$ stands for the dual pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

The aim of this section is to collect some regularity results for the solution of the Laplace and Poisson equation. The weak form reads: Find $y \in H^1(\Omega)$ satisfying

$$u|_\Gamma = z, \qquad (\nabla u, \nabla v)_{L^2(\Omega)^2} = (f, v)_{L^2(\Omega)} \qquad \forall v \in H_0^1(\Omega). \tag{3}$$

The functions $f \in L^2(\Omega)$ and $z \in H^{1/2}(\Gamma)$ are given input data.

2.1 Weighted regularity

For technical reasons we recall some a priori estimates in weighted norms involving the weight function $\sigma(x) := \kappa h + \operatorname{dist}(x, \Gamma)$ with arbitrary $\kappa > 0$. This is a regularized distance function with respect to the boundary of the domain Ω. The following result is proved already in [29, Lemma 1].

Lemma 1 *Let $w \in H_0^1(\Omega)$ be the weak solution of $-\Delta w = f$ in Ω. Then, the a priori estimate*

$$\|\sigma^{-1}\, w\|_{L^2(\Omega)} \le c\,\|\nabla w\|_{L^2(\Omega)} \le c\,\|\sigma\, f\|_{L^2(\Omega)}$$

holds.

Furthermore, we will need an interior regularity result:

Lemma 2 *Let $w \in H^1(\Omega)$ satisfy*

$$(\nabla w, \nabla v)_{L^2(\Omega)^2} = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

with some function $f \in L^2(\Omega)$. Moreover, let be given $\Omega_0 \subset\subset \Omega_1 \subset \Omega$ and denote by $d := \operatorname{dist}(\partial\Omega_1, \partial\Omega_0)$ the distance between the boundaries of Ω_0 and Ω_1. Then, the estimate

$$\|\nabla^2 w\|_{L^2(\Omega_0)} \leq c \left(\|f\|_{L^2(\Omega_1)} + d^{-1} \|\nabla w\|_{L^2(\Omega_1)}\right)$$

is valid.

Proof The estimate (i) can be concluded from the proof of [12, Theorem 8.8] where this assertion is stated with a generic constant depending on the quantity d that we want to carve out exactly. Thus, we repeat the proof for the convenience of the reader. The proof basically relies on [12, Lemma 7.24] which states that a function $u \in L^2(\Omega)$ belongs to $H^1(\Omega_0)$ if its difference quotients $D_k^h u(x) := \frac{1}{h}(u(x + h\mathbf{e}_k) - u(x))$, $k \in \{1, 2\}$, are bounded in the $L^2(\Omega_0)$-norm for all $h \in \mathbb{R}$ with $|h|$ sufficiently small such that D_k^h is well-defined in Ω_1. Moreover, the inclusion

$$\|D_k^h w\|_{L^2(\Omega_0)} \leq K \quad \Rightarrow \quad \|\partial_k w\|_{L^2(\Omega_0)} \leq K \tag{4}$$

is valid. To conclude the desired estimate we thus have to confirm that $\|D_k^h \nabla w\|_{L^2(\Omega_0)}$ is bounded. For technical reasons we introduce a further set $\tilde{\Omega}$ satisfying $\Omega_0 \subset\subset \tilde{\Omega} \subset\subset \Omega_1$ and $\operatorname{dist}(\Omega_0, \partial\tilde{\Omega}) \sim d$. For an arbitrary test function $v \in H_0^1(\Omega)$ with $\operatorname{dist}(\operatorname{supp} v, \partial\tilde{\Omega}) > 2h$ we obtain

$$\begin{aligned}\int_\Omega (\nabla D_k^h w) \cdot \nabla v &= -\int_\Omega \nabla w \cdot (\nabla D_k^{-h} v) = -\int_\Omega f\, D_k^{-h} v \\ &\leq \|f\|_{L^2(\Omega_1)} \|\nabla v\|_{L^2(\Omega)}.\end{aligned} \tag{5}$$

In the last step we bounded the difference quotient by the first derivative of v. Such an estimate is proved in [12, Lemma 7.23]. Next, we introduce a smooth cut-off function $\eta \in C_0^\infty(\Omega)$ satisfying $\eta \equiv 1$ in Ω_0 and $\operatorname{supp} \eta \subset \overline{\tilde{\Omega}}$. Moreover, η is constructed in such a way that $|\nabla\eta| \leq c\, d^{-1}$. For sufficiently small h we obtain from the product rule and (5) for $v = \eta^2 D_k^h w$

$$\begin{aligned}\|\eta\, D_k^h \nabla w\|^2_{L^2(\Omega)} &= \int_\Omega (\nabla D_k^h w) \cdot (\eta^2\, (\nabla D_k^h w)) \\ &= \int_\Omega (\nabla D_k^h w) \cdot (\nabla(\eta^2\, D_k^h w) - 2\,\eta\, \nabla\eta D_k^h w) \\ &\leq \|f\|_{L^2(\Omega_1)} \|\nabla(\eta^2\, D_k^h w)\|_{L^2(\Omega)} + c\, d^{-1} \|\eta\, D_k^h \nabla w\|_{L^2(\tilde{\Omega})} \|D_k^h w\|_{L^2(\tilde{\Omega})}.\end{aligned} \tag{6}$$

Again, we apply [12, Lemma 7.23] to obtain $\|D_k^h w\|_{L^2(\tilde{\Omega})} \le c\,\|\nabla w\|_{L^2(\Omega_1)}$. Moreover, with the product rule we obtain

$$\begin{aligned}\|\nabla(\eta^2\, D_k^h w)\|_{L^2(\Omega)} &\le 2\,\|\eta\,\nabla\eta\, D_k^h w\|_{L^2(\Omega)} + \|\eta^2\,\nabla D_k^h w\|_{L^2(\Omega)}\\ &\le c\,\left(d^{-1}\,\|\nabla w\|_{L^2(\Omega_1)} + \|\eta\, D_k^h \nabla w\|_{L^2(\Omega)}\right).\end{aligned}$$

Insertion of this estimate into (6) yields with Young's inequality and a kick-back argument for the latter term on the right-hand side

$$\|\nabla D_k^h w\|_{L^2(\Omega_0)} \le \|\eta\,\nabla D_k^h w\|_{L^2(\Omega)} \le c\,\left(\|f\|_{L^2(\Omega_1)} + d^{-1}\,\|\nabla w\|_{L^2(\Omega_1)}\right).$$

The desired estimate then follows from (4). □

2.2 Weighted Sobolev and Hölder spaces

In order to describe the regularity of the solution of boundary value problems in an accurate way we exploit regularity results in weighted Sobolev spaces. These spaces capture the corner singularities contained in the solution and allow us to derive sharp interpolation error estimates. Throughout the paper we denote the corners of Ω by $\boldsymbol{c}_j$, $j \in \mathcal{C} := 1, \ldots, d$. Moreover, denote by Γ_j the boundary edge having endpoints $\boldsymbol{c}_j$ and $\boldsymbol{c}_{j+1}$ or $\boldsymbol{c}_1$ in case of $j = d$. The interior angle between the edges intersecting in $\boldsymbol{c}_j$ is $\omega_j \in (0, 2\pi)$.

In order to introduce the weighted Sobolev spaces used for the analysis, we divide the domain into circular sectors $\Omega_R^j := \{x \in \Omega\colon |x - \boldsymbol{c}_j| < R\}$, $j \in \mathcal{C}$, with sufficiently small R such that these sectors do not overlap. The remaining sets are denoted by $\hat{\Omega}_R := \Omega \setminus \cup\{\Omega_R^j\colon j \in \mathcal{C}\}$. For each $k \in \mathbb{N}_0$, $p \in [1, \infty)$ and some weight $\beta \in \mathbb{R}_+$ ($\mathbb{R}_+ := [0, \infty)$), we introduce the local norms

$$\begin{aligned}\|u\|^p_{V_\beta^{k,p}(\Omega_R^j)} &:= \sum_{|\alpha|\le k} \|r_j^{\beta-k+|\alpha|}\, D^\alpha u\|^p_{L^p(\Omega_R^j)},\\ \|u\|^p_{W_\beta^{k,p}(\Omega_R^j)} &:= \sum_{|\alpha|\le k} \|r_j^{\beta}\, D^\alpha u\|^p_{L^p(\Omega_R^j)},\end{aligned}$$

and for an analogous definition in case of $p = \infty$, the sum has to be replaced by the maximum over $|\alpha| \le k$. For some $\boldsymbol{\beta} \in \mathbb{R}_+^d$ the global norms are defined by

$$\|u\|_{V_{\boldsymbol{\beta}}^{k,p}(\Omega)} := \left(\sum_{j\in\mathcal{C}} \|u\|^p_{V_{\beta_j}^{k,p}(\Omega_R^j)} + \|u\|^p_{W^{k,p}(\hat{\Omega}_{R/2})}\right)^{1/p},$$

in case of $p \in [1, \infty)$ and with the obvious modification for $p = \infty$. When replacing V by W in the definition above, we obtain the global norm $\|\cdot\|_{W_{\boldsymbol{\beta}}^{k,p}(\Omega)}$. The weighted Sobolev spaces $V_{\boldsymbol{\beta}}^{k,p}(\Omega)$ and $W_{\boldsymbol{\beta}}^{k,p}(\Omega)$ are defined as the set of functions whose norms introduced above are finite. The trace spaces are denoted by $V_{\boldsymbol{\beta}}^{k-1/p,p}(\Gamma)$ and $W_{\boldsymbol{\beta}}^{k-1/p,p}(\Gamma)$, respectively. Later, we want to derive error estimates using regularity results in the $V_{\boldsymbol{\beta}}^{k-1/p,p}(\Gamma)$-norm. To this end, we equip the trace spaces with the equivalent weighted Sobolev-Slobodeckij-norms. Therefore, introduce the boundary segments $\Gamma_R^j := \Gamma \cap \partial\Omega_R^j$, $j \in \mathcal{C}$, and $\hat{\Gamma}_R = \Gamma \cap \hat{\Omega}_R$. Moreover, $\Gamma_R^{j,1}$ and $\Gamma_R^{j,2}$ are the two legs of Γ_R^j. For $k \in \mathbb{N}$, $p \in [1, \infty)$ and $\boldsymbol{\beta} \in \mathbb{R}_+^d$ we then define

$$\|u\|^p_{V_{\boldsymbol{\beta}}^{k-1/p,p}(\Gamma)} := \|u\|^p_{W^{k-1/p,p}(\hat{\Gamma}_{R/2})} + \sum_{j\in\mathcal{C}} \sum_{i=1}^{2} \|u\|^p_{V_{\beta_j}^{k-1/p,p}(\Gamma_R^{j,i})}.$$

with

$$\begin{aligned}\|u\|_{V_{\beta_j}^{k-1/p,p}(\Gamma_R^{j,i})} := &\sum_{|\alpha|\le k-1} \|r_j^{\beta_j-k+1/p+|\alpha|} D_t^\alpha u\|^p_{L^p(\Gamma_R^{j,i})}\\ &+ \sum_{|\alpha|=k-1} \iint_{\Gamma_R^{j,i}} \frac{|r_j(x)^{\beta_j} D_t^\alpha u(x) - r_j(y)^{\beta_j} D_t^\alpha u(y)|^p}{|x-y|^p}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where D_t^α denotes the tangential derivative of order α. In the obvious way, a norm for the case $p = \infty$ can be defined as well.

The previous definitions and an intensive discussion on the relation between V- and W-spaces can be found in [26, Chapter 4, §5], [22, Section 6.2].

Later, we will frequently derive error estimates where the convergence rate will depend on the largest weight. Thus, we define

$$\overline{\boldsymbol{\beta}} := \max_{j\in\mathcal{C}} \beta_j.$$

In the next chapter, we will frequently exploit regularity results in these space with $p = \infty$, but for this case a shift theorem is not valid. As a remedy, weighted Hölder spaces are used and we take the definition from [22, Section 6.7.1]. Again, we define some local norms with parameters $k \in \mathbb{N}_0$, $\sigma \in (0, 1]$ and $\delta \ge \sigma$ by

$$\begin{aligned}\|u\|_{\Lambda_\delta^{k,\sigma}(\Omega_R^j)} &:= \sup_{x\in\Omega_R^j} \sum_{|\alpha|\le k} r_j(x)^{\delta-k-\sigma+|\alpha|}\,|D^\alpha u(x)| + \langle u\rangle_{k,\sigma,\beta,\Omega_R^j},\\ \|u\|_{C_\delta^{k,\sigma}(\Omega_R^j)} &:= \sup_{x\in\Omega_R^j} \sum_{|\alpha|\le k} r_j(x)^{\max\{0,\delta-k-\sigma+|\alpha|\}}\,|D^\alpha u(x)| + \langle u\rangle_{k,\sigma,\delta,\Omega_R^j},\end{aligned}$$

where the seminorm is defined by

$$\langle u\rangle_{k,\sigma,\delta,\Omega_R^j} := \sup_{x,y\in\Omega_R^j} \sum_{|\alpha|=k} \frac{|r_j(x)^\delta\, D^\alpha u(x) - r_j(y)^\delta\, D^\alpha u(y)|}{|x-y|^\sigma}.$$

The global norm is then given by

$$\|u\|_{\Lambda_{\boldsymbol{\delta}}^{k,\sigma}(\Omega)} := \sum_{j\in\mathcal{C}} \|u\|_{\Lambda_{\delta_j}^{k,\sigma}(\Omega_R^j)} + \|u\|_{C^{k,\sigma}(\hat{\Omega}_{R/2})}$$

with some vector $\boldsymbol{\delta} \in [\sigma,\infty)^d$. Analogously, the norm $\|\cdot\|_{C_{\boldsymbol{\delta}}^{k,\sigma}(\Omega)}$ is defined. The corresponding function spaces are defined by

$$\Lambda_{\boldsymbol{\delta}}^{k,\sigma}(\Omega) := \overline{C_0^\infty(\overline{\Omega}\setminus\mathcal{S})}^{\|\cdot\|_{\Lambda_{\boldsymbol{\delta}}^{k,\sigma}(\Omega)}}, \qquad C_{\boldsymbol{\delta}}^{k,\sigma}(\Omega) := \overline{C_0^\infty(\overline{\Omega})}^{\|\cdot\|_{C_{\boldsymbol{\delta}}^{k,\sigma}(\Omega)}},$$

where $\mathcal{S} := \{\boldsymbol{c}_j : j \in \mathcal{C}\}$. The corresponding trace spaces are endowed with the norm

$$\|u\|_{\Lambda_{\boldsymbol{\delta}}^{k,\sigma}(\Gamma)} := \inf\{\|\tilde{u}\|_{\Lambda_{\boldsymbol{\delta}}^{k,\sigma}(\Omega)} : \tilde{u}|_{\Gamma\setminus\mathcal{S}} \equiv u\}, \tag{7}$$

and analogously for $C_{\boldsymbol{\delta}}^{k,\sigma}(\Gamma)$.

Next, we establish a regularity result for weighted Sobolev spaces.

Lemma 3 *Let $f \in W_{\boldsymbol{\beta}}^{0,2}(\Omega)$ and $z \in W_{\boldsymbol{\beta}}^{3/2,2}(\Gamma)$ with $\boldsymbol{\beta} \in [0,1)^d$ satisfying $\beta_j > 1-\lambda_j$ for all $j \in \mathcal{C}$. Then, the solution u of (3) belongs to $W_{\boldsymbol{\beta}}^{2,2}(\Omega)$. In case of $z \in V_{\boldsymbol{\beta}}^{3/2,2}(\Gamma)$ the function u belongs to $V_{\boldsymbol{\beta}}^{2,2}(\Omega)$.*

Proof The regularity result for V-spaces can be deduced from [18, Theorem 1.4.3]. Note that this result holds even for a larger range of the weights, this is, $\beta_j \in (1-\lambda_j, 1+\lambda_j)$. From this result we infer the solvability in W-spaces as each function $y \in W_{\beta_j}^{2,2}(\Omega_R^j)$ with $\beta_j \in (0,1)$ can be decomposed into $u_0 + p$ with a constant $p = z(\boldsymbol{c}_j)$ and $u_0 \in V_{\beta_j}^{2,2}(\Omega_R^j)$. This is basically the idea which leads to [26, Theorem 4.§5.11] from which we could conclude the same result. □

An analogue of this result is true for the weighted Hölder spaces introduced above. This is used to show boundedness of the solution of (3) in a weighted $W^{2,\infty}$-space.

Lemma 4 *Assume that $f \in \Lambda_{\boldsymbol{\delta}}^{0,\sigma}(\Omega)$ and $z \in \Lambda_{\boldsymbol{\delta}}^{2,\sigma}(\Gamma)$ with $\sigma \in (0,1]$ and weights $\boldsymbol{\delta} \in (\sigma, 2+\sigma)^d$ satisfying $2-\lambda_j > \delta_j - \sigma$ for $j \in \mathcal{C}$. Moreover, we exclude the case $\delta_j - \sigma = 1$. Then, the solution u of (3) belongs to $\Lambda_{\boldsymbol{\delta}}^{2,\sigma}(\Omega)$ and depends continuously on the input data. This result remains true when replacing Λ by C.*

Proof The proof for the regularity in Λ-spaces can be deduced from [18, Theorem 1.4.5]. In order to show the regularity result in the weighted C-spaces we basically follow the ideas used in [28, Lemma 3.13]. First, introduce the numbers $\nu_j \in \mathbb{Z}$, $j \in \mathcal{C}$, such that $\nu_j < \delta_j - \sigma < \nu_j + 1$. Then, we split the solution into

$$u = w + \sum_{j=1}^{d} \eta_j\, p_j,$$

with smooth cut-off functions η_j satisfying $\eta_j \equiv 1$ in $\Omega^j_{R/2}$ and $\operatorname{supp} \eta_j \subset \overline{\Omega}^j_R$ for all $j \in \mathcal{C}$, and polynomials p_j of order not greater than $1 - \nu_j$. The key idea is to reuse the regularity results shown in the weighted Λ-spaces for the function w. A direct application of these results to u would not be possible as the homogeneous weights $r_j(x)^{\delta-k-\sigma+|\alpha|}$, $j \in \mathcal{C}$, could, depending on the choice of δ and σ, tend to infinity as $r_j \to 0$. As a remedy, the polynomials p_j have to be chosen in such a way that $w(c_j) = 0$ if $\nu_j = 1$ and $w(c_j) = D^\alpha w(c_j) = 0$ for all $|\alpha| = 1$ if $\nu_j = 0$.

Once regularity for w in a weighted Λ-space is shown, the desired result follows from certain relations between C- and Λ-spaces. First, we observe that w solves the boundary value problem

$$
\begin{aligned}
-\Delta w &= f + \sum_{j=1}^{d} \left(\Delta \eta_j \, p_j + 2 \, \nabla \eta_j \cdot \nabla p_j \right) := F \qquad && \text{in } \Omega, \\
w &= z - \sum_{j=1}^{d} \eta_j \, p_j := G && \text{on } \Gamma.
\end{aligned}
$$

Our aim is to show that w belongs to $\Lambda^{2,\sigma}_{\boldsymbol{\delta}}(\Omega)$ which would follow under the assumption $F \in \Lambda^{0,\sigma}_{\boldsymbol{\delta}}(\Omega)$ and $G \in \Lambda^{2,\sigma}_{\boldsymbol{\delta}}(\Gamma)$. To achieve this, we have to construct the polynomials p_j appropriately and therefore, we define the projection

$$
q_k(v; \boldsymbol{c}_j)(x) := \sum_{|\alpha|=0}^{k} \frac{1}{|\alpha|!} (D^\alpha v)(\boldsymbol{c}_j) \, (x - \boldsymbol{c}_j)^\alpha \tag{8}
$$

for $j \in \mathcal{C}$ and $k \in \mathbb{N}_0$. In a similar way we construct a projection for functions defined on the boundary by means of $q^\partial_k(v, \boldsymbol{c}_j) := \gamma_0 q_k(\tilde{v}; \boldsymbol{c}_j)$, where $\tilde{v}$ is an arbitrary extension of v and γ_0 is the trace operator. The polynomial $q^\partial_k(v; \boldsymbol{c}_j)$ is independent of the extension $\tilde{v}$, and hence, there holds $\gamma_0 q_k(u; \boldsymbol{c}_j) = q^\partial_k(z; \boldsymbol{c}_j)$ as $u \equiv z$ on Γ. In the following we use the choice $p_j := q_{1-\nu_j}(u; \boldsymbol{c}_j)$. That F belongs to $\Lambda^{0,\sigma}_{\boldsymbol{\delta}}(\Omega)$ is obvious, as f is assumed to be contained in $C^{0,\sigma}_{\boldsymbol{\delta}}(\Omega)$ and this space is equivalent to $\Lambda^{0,\sigma}_{\boldsymbol{\delta}}(\Omega)$ if $\boldsymbol{\delta} \geq \sigma$, see the arguments before Lemma 6.7.1 in [22]. Moreover, the cut-off functions η_j are constant in the neighborhood of the corners and thus, the products $\nabla \eta_j \cdot \nabla p_j$ and $\Delta \eta_j p_j$ belong trivially to that space. Consequently, we get

$$
\|F\|_{\Lambda^{0,\sigma}_{\boldsymbol{\delta}}(\Omega)} \leq c \left(\|f\|_{C^{0,\sigma}_{\boldsymbol{\delta}}(\Omega)} + \sum_{j=1}^{d} \|p_j\|_{C^1(\Omega^j_R)} \right). \tag{9}
$$

With the definition (8) and the imposed Dirichlet boundary conditions, taking into account $p_j|_\Gamma = q^\partial_{1-\nu_j}(z; \boldsymbol{c}_j)$, $j \in \mathcal{C}$, we deduce

$$
\begin{aligned}
\|p_j\|_{C^1(\Omega^j_R)} &\leq c \sum_{|\alpha|=0}^{1-\nu_j} |D^\alpha z(\boldsymbol{c}_j)| \leq c \, \|\tilde{z}\|_{C^{1-\nu_j,\varepsilon}_0(\Omega^j_R)} \\
&\leq c \, \|\tilde{z}\|_{C^{2,\sigma}_{1+\nu_j+\sigma-\varepsilon}(\Omega^j_R)} \leq c \, \|\tilde{z}\|_{C^{2,\sigma}_{\delta_j}(\Omega^j_R)} = c \, \|z\|_{C^{2,\sigma}_{\delta_j}(\Gamma^j_R)}
\end{aligned} \tag{10}
$$

for $\delta_j - \sigma < 1 + \nu_j$ and sufficiently small $\varepsilon > 0$, and $\tilde{z}$ is a suitable extension of z, see (7). The second and third step follow from the equivalence of $C_0^{k,\sigma}(\Gamma_R^j)$ and $C^{k,\sigma}(\Gamma_R^j)$ stated in [22, Lemma 6.7.2] and an embedding theorem for weighted Hölder spaces, see [22, Lemma 6.7.1]. The embedding used in the second to the last step is trivial.

The property $G \in \Lambda_{\boldsymbol{\delta}}^{2,\sigma}(\Gamma)$ follows from [22, Theorem 6.7.6] which provides the a priori estimate

$$\|G\|_{\Lambda_{\boldsymbol{\delta}}^{2,\sigma}(\Gamma)} = \|z - \sum_{j=1}^{d} \eta_j\, p_j\|_{\Lambda_{\boldsymbol{\delta}}^{2,\sigma}(\Gamma)} \le c\, \|z\|_{C_{\boldsymbol{\delta}}^{2,\sigma}(\Gamma)}. \tag{11}$$

The regularity result proved in [18, Theorem 1.4.5(2)] then guarantees $w \in \Lambda_{\boldsymbol{\delta}}^{2,\sigma}(\Omega)$, and with the triangle inequality, the trivial estimate $\|v\|_{C_{\boldsymbol{\delta}}^{2,\sigma}(\Omega)} \le c\, \|v\|_{\Lambda_{\boldsymbol{\delta}}^{2,\sigma}(\Omega)}$ for $v \in \Lambda_{\boldsymbol{\delta}}^{2,\sigma}(\Omega)$ and (10) we infer

$$\begin{aligned} \|u\|_{C_{\boldsymbol{\delta}}^{2,\sigma}(\Omega)} &\le \|w\|_{C_{\boldsymbol{\delta}}^{2,\sigma}(\Omega)} + \sum_{j=1}^{d} \|\eta_j\, p_j\|_{C^1(\Omega)} \\ &\le c\left(\|w\|_{\Lambda_{\boldsymbol{\delta}}^{2,\sigma}(\Omega)} + \|z\|_{C_{\boldsymbol{\delta}}^{2,\sigma}(\Gamma)}\right). \end{aligned}$$

An a priori estimate for the weighted Λ-norm of w can be concluded from [18, Theorem 1.4.5(1)], which leads to

$$\begin{aligned} \|u\|_{C_{\boldsymbol{\delta}}^{2,\sigma}(\Omega)} &\le c\left(\|F\|_{\Lambda_{\boldsymbol{\delta}}^{0,\sigma}(\Omega)} + \|G\|_{\Lambda_{\boldsymbol{\delta}}^{2,\sigma}(\Gamma)} + \|w\|_{L^1(\Omega)} + \|z\|_{C_{\boldsymbol{\delta}}^{2,\sigma}(\Gamma)}\right) \\ &\le c\left(\|f\|_{C_{\boldsymbol{\delta}}^{0,\sigma}(\Omega)} + \|z\|_{C_{\boldsymbol{\delta}}^{2,\sigma}(\Gamma)} + \|w\|_{L^1(\Omega)}\right), \end{aligned} \tag{12}$$

where the second step follows from the estimates (9) and (11). The $L^1(\Omega)$ norm of u can be bounded by the $V_{\boldsymbol{\beta}}^{2,2}(\Omega)$-norm with weights $\beta_j = \max\{0, \delta_j - \sigma - 1\} + \varepsilon$, $j \in \mathcal{C}$, and $\varepsilon > 0$ sufficiently small. Using the norm equivalence from [26, Theorem 5.6] or [22, Lemma 6.2.12] we arrive at

$$\|w\|_{L^1(\Omega)} \le c\, \|w\|_{V_{\boldsymbol{\beta}}^{2,2}(\Omega)} \le c\, \|u\|_{W_{\boldsymbol{\beta}}^{2,2}(\Omega)}. \tag{13}$$

With Lemma 3 and embeddings of W- into C-spaces, see e. g. [28, Lemma 2.39], we deduce

$$\|u\|_{W_{\boldsymbol{\beta}}^{2,2}(\Omega)} \le c\left(\|f\|_{C_{\boldsymbol{\delta}}^{0,\sigma}(\Omega)} + \|z\|_{C_{\boldsymbol{\delta}}^{2,\sigma}(\Gamma)}\right). \tag{14}$$

The desired a priori estimate follows after insertion of (13) and (14) into (12). □

The regularity results in weighted Hölder spaces allow us to extend the assertion of Lemma 3 to L^∞ based norms. This is a simple conclusion from the definition of the spaces V and Λ as well as W and C.

Corollary 1 *Assume that* $\boldsymbol{\delta} \in (\sigma, 2+\sigma)^d$ *satisfies the assumptions of Lemma 4. Let* $\boldsymbol{\gamma} \in (0,2)^d$ *be a weight vector defined by* $\gamma_j := \delta_j - \sigma$ *for* $j \in \mathcal{C}$.

(i) *If* $f \in \Lambda^{0,\sigma}_{\boldsymbol{\delta}}(\Omega)$ *and* $z \in \Lambda^{2,\sigma}_{\boldsymbol{\delta}}(\Gamma)$*, the solution* u *of* (3) *belongs to* $V^{2,\infty}_{\boldsymbol{\gamma}}(\Omega)$.

(ii) *If* $f \in C^{0,\sigma}_{\boldsymbol{\delta}}(\Omega)$ *and* $z \in C^{2,\sigma}_{\boldsymbol{\delta}}(\Gamma)$*, the solution* u *of* (3) *belongs to* $W^{2,\infty}_{\boldsymbol{\gamma}}(\Omega)$.

3 Error estimates for normal derivatives

In this section we consider a finite element discretization for the weak form of the boundary value problem (3) which reads

$$u|_\Gamma \equiv z, \quad (\nabla u, \nabla v)_{L^2(\Omega)^2} = (f, v)_{L^2(\Omega)} \qquad \forall v \in H^1_0(\Omega).$$

Therefore, let $\{\mathcal{T}_h\}_{h>0}$ be a quasi-uniform family of shape-regular triangulations of Ω, which are feasible in the sense of [10, Section 5]. The parameter h denotes the maximal diameter of all elements from $\mathcal{T}_h$ and is always assumed to be sufficiently small. The trial and test spaces are defined by

$$V_h := \{v_h \in C(\overline{\Omega}) \colon v_h|_T \in \mathcal{P}_1 \text{ for all } T \in \mathcal{T}_h\}, \quad V_{0h} := V_h \cap H^1_0(\Omega).$$

Moreover, the traces of function from V_h belong to the space

$$V_h^\partial := \{w_h \in C(\Gamma) \colon w_h = v_h|_\Gamma \text{ for some } v_h \in V_h\}.$$

The finite-element approximation $u_h \in V_h$ of u is defined by

$$u_h|_\Gamma \equiv z_h, \quad (\nabla u_h, \nabla v_h)_{L^2(\Omega)^2} = (f, v_h)_{L^2(\Omega)} \qquad \forall v_h \in V_{0h}, \tag{15}$$

where $z_h \in V_h^\partial$ is some appropriate interpolation or projection of z. In the following, z_h will be the $L^2(\Gamma)$-projection of z onto V_h^∂, this is, $z_h := Q_h(z)$. Moreover, we denote by

$$I_h \colon C(\overline{\Omega}) \to V_h, \qquad [I_h u](x) = \sum_{i=1}^N u(x_i)\,\varphi_i(x) \tag{16}$$

the nodal interpolant. Here, $x_i \in \overline{\Omega}$, $i = 1, \ldots, N$, denote the nodes of $\mathcal{T}_h$ and $\{\varphi_i\}_{i=1}^N$ the nodal basis of V_h. Moreover, we will use a slightly modified interpolant defined by

$$\tilde{I}_h u = I_h u + \tilde{S}_h(Q_h z - I_h z), \tag{17}$$

where $\tilde{S}_h \colon V_h^\partial \to V_h$ is the zero extension which vanishes in the interior nodes of $\mathcal{T}_h$. For functions $u \in C(\overline{\Omega})$ with $u|_\Gamma = z$, the interpolant fulfills the essential property $[\tilde{I}_h u]|_\Gamma \equiv Q_h z$ that is needed for instance in the proof of a Céa-Lemma. As the local interpolation error estimates will frequently depend on the distance to the corners, we introduce the notation

$$r_{j,T} = \inf_{x \in T} |x - \boldsymbol{c}_j| \qquad j \in \mathcal{C},\ T \in \mathcal{T}_h.$$

We start our investigations with an interpolation error estimate for the boundary datum z.

Lemma 5 *Let be given weight vectors $\boldsymbol{\alpha} \in [0, 1/2)^d$ and $\boldsymbol{\gamma} \in [0, 3/2)^d$. Then, the interpolation error estimates*

$$\|z - I_h z\|_{L^2(\Gamma)} + h^{1/2} \|z - I_h z\|_{H^{1/2}(\Gamma)} \le c\, h^{2-\overline{\boldsymbol{\gamma}}} |z|_{W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)},$$
$$\|z - I_h z\|_{L^2(\Gamma)} + h^{1/2} \|z - I_h z\|_{H^{1/2}(\Gamma)} \le c\, h^{3/2-\overline{\boldsymbol{\alpha}}} \|z\|_{W^{3/2,2}_{\boldsymbol{\alpha}}(\Gamma)},$$

are valid, provided that z possesses the regularity demanded by the right-hand side.

Proof The first estimate can be deduced from [32, Lemma 3.2.4]. There, the desired error estimate in the $L^2(\Gamma)$- and $H^1(\Gamma)$-norm is proved. The estimate in $H^{1/2}(\Gamma)$ follows from an interpolation argument.

To show the second estimate we will reuse existing interpolation error estimates exploiting regularity in weighted V-spaces. To this end, we split up the function z by means of $z = z_0 + \eta_j\, p_j$ with $z_0 \in V^{3/2,2}_{\boldsymbol{\alpha}}(\Gamma)$, certain constants $p_j \in \mathbb{R}$, $j \in \mathcal{C}$, and smooth cut-off functions $\eta_j = \eta_j(|x - \boldsymbol{c}_j|)$ satisfying

$$\eta_j|_{\Omega^j_{R/2}} \equiv 1, \quad \operatorname{supp}(\eta_j) \subset \overline{\Omega}^j_R \quad \text{and} \quad \|D^\alpha \eta_j\|_{L^\infty(\Omega)} \le c \;\forall |\alpha| \le 2.$$

Note that the nodal interpolant preserves the functions $\eta_j\, p_j$ near the corners provided that $h > 0$ is sufficiently small. Hence, it suffices to prove an estimate for z_0. In order to derive local interpolation error estimates we denote by $\hat{E} := (0, 1)$ the reference interval, and by $F_E : \hat{E} \to E$ the affine reference transformation. Moreover, we write $\hat{v}(\hat{x}) = v(F_E(\hat{x}))$ for all $\hat{x} \in \hat{E}$. The norms of the weighted Sobolev spaces on the reference element, $V^{k-1/2,2}_\alpha(\hat{E})$, are defined analogous to the global norms introduced in Sect. 2.2 but the weight function is defined by $\hat{r}(\hat{x}) := |\hat{x}|$. For elements $E \in \mathcal{E}_h$ touching the corner $\boldsymbol{c}_j$, $j \in \mathcal{C}$, there holds the property $r_j(F_E(\hat{x})) \sim h\, \hat{r}(\hat{x})$.

For all elements $E \in \mathcal{E}_h$ with $r_{j,E} = 0$ for some $j \in \mathcal{C}$, we obtain the estimate

$$\begin{aligned}\|z_0 - I_h z_0\|_{L^2(E)} &\le c\, |E|^{1/2} \|\hat{z}_0\|_{L^\infty(\hat{E})} \le c\, |E|^{1/2} \|\hat{z}_0\|_{V^{3/2,2}_{\alpha_j}(\hat{E})} \\ &\le c\, h^{3/2-\alpha_j} \|z_0\|_{V^{3/2,2}_{\alpha_j}(E)},\end{aligned}$$

which follows from the arguments used in the proof of [4, Lemma 4.5]. In case of $E \subset \Omega^j_{R/2}$ for some $j \in \mathcal{C}$ and $r_{j,E} > 0$ we deduce

$$\|z_0 - I_h z_0\|_{L^2(E)} \le c\, h^{3/2} |z_0|_{H^{3/2}(E)} \le c\, h^{3/2-\alpha_j} \|z_0\|_{V^{3/2,2}_{\alpha_j}(E)},$$

where the argument used in the last step can also be found in [4, Lemma 4.5]. Far away from the corners, i. e., $r_{j,E} > 1/4$ for all $j \in \mathcal{C}$, we can use a standard estimate to get $\|z_0 - I_h z_0\|_{L^2(E)} \le c\, h^{3/2} |z_0|_{H^{3/2}(E)}$. Next, we sum up over all elements $E \in \mathcal{E}_h$ (note that for Sobolev-Slobodeckij-norms there holds $\sum_{E \in \mathcal{E}_h} |u|^2_{H^{3/2}(E)} \le C\, |u|^2_{H^{3/2}(\Gamma)}$, but not the reverse estimate) and together with a standard estimate for the error terms $p_j\,\eta_j - I_h(p_j\,\eta_j)$ we obtain

$$\begin{aligned}\|z - I_h z\|_{L^2(\Gamma)} &\le c\left(\|z_0 - I_h z_0\|_{L^2(\Gamma)} + h^2\sum_{j\in\mathcal{C}}|p_j|\right)\\ &\le ch^{3/2-\overline{\boldsymbol{\alpha}}}\left(\|z_0\|_{V^{3/2,2}_{\boldsymbol{\alpha}}(\Gamma)} + \sum_{j\in\mathcal{C}}|p_j|\right) \le c\,h^{3/2-\overline{\boldsymbol{\alpha}}}\,\|z\|_{W^{3/2,2}_{\boldsymbol{\alpha}}(\Gamma)}.\end{aligned} \tag{18}$$

The last step is a consequence of the norm equivalence stated in [26, Ch. 4, Theorem 5.7].

The estimate in the $H^{1/2}(\Gamma)$-norm follows from an interpolation argument between estimates in $L^2(\Gamma)$ and $H^1(\Gamma)$. Note that $H^{1/2}(\Gamma)$ is equivalent to the interpolation space $[L^2(\Gamma), H^1(\Gamma)]_{1/2,2}$, see [23, Theorem B.11]. The space $H^1(\Gamma)$ can be defined in the usual way via local Lipschitz continuous parametrizations of Γ. To show an estimate in $H^1(\Gamma)$ we derive local estimates first. For all elements $E \in \mathcal{E}_h$ with $r_{j,E} = 0$ we obtain

$$\begin{aligned}\|z_0 - I_h z_0\|_{H^1(E)} &\le c\,h^{-1}\,|E|^{1/2}\,\|\hat{z}_0\|_{H^1(\hat{E})} \le c\,h^{-1}\,|E|^{1/2}\,\|\hat{z}_0\|_{V^{3/2,2}_{\alpha_j}(\hat{E})}\\ &\le c\,h^{1/2-\alpha_j}\,\|z_0\|_{V^{3/2,2}_{\alpha_j}(E)},\end{aligned}$$

where the second step is an application of the embedding $V^{3/2,2}_{\alpha_j}(\hat{E}) \hookrightarrow H^1(\hat{E})$, which is valid for $\alpha_j < 1/2$. Otherwise, if $E \subset \Omega^j_{R/2}$ and $r_{j,E} > 0$, we obtain with similar arguments

$$\|z_0 - I_h z_0\|_{H^1(E)} \le c\,h^{1/2}\,|z_0|_{H^{3/2}(E)} \le c\,h^{1/2-\alpha_j}\,\|z_0\|_{V^{3/2,2}_{\alpha_j}(E)}.$$

In the far interior, i.e., for $E \in \mathcal{E}_h$ with $r_{j,E} > 1/4$ for all $j \in \mathcal{C}$, we can use a standard estimate exploiting $H^{3/2}$-regularity. Summation over all elements $E \in \mathcal{E}_h$ and an interpolation argument lead to the desired estimate for $z_0 - I_h z_0$ in the $H^{1/2}(\Gamma)$-norm. With the splitting $z = z_0 + \sum_{j\in\mathcal{C}} p_j\,\eta_j$ we get an estimate for $z - I_h z$ when using the arguments from (18). □

Using the ideas of [7] we can derive error estimates for the approximate solutions u_h in the norms $H^1(\Omega)$ and $L^2(\Omega)$. However, in this reference $H^2(\Gamma_j)$-regularity ($j \in \mathcal{C}$) for the Dirichlet datum z is assumed. As we deal with optimal Dirichlet control problems, the boundary datum for the state is the control function which might be less regular. Thus, we repeat the proof assuming less regularity for z in some weighted Sobolev space.

Lemma 6 *Assume that $u \in W^{2,2}_{\boldsymbol{\alpha}}(\Omega)$ and $z \in W^{3/2,2}_{\boldsymbol{\alpha}}(\Gamma)$ with a weight vector $\boldsymbol{\alpha} \in [0, 1/2)^d$. Moreover, let $z_h := Q_h z$. Then, the solution $u_h \in V_h$ of (15) satisfies the error estimates*

$$\begin{aligned}\|u-u_h\|_{H^1(\Omega)} &\le c\,h^{1-\bar{\alpha}}\left(|u|_{W^{2,2}_{\boldsymbol\alpha}(\Omega)}+\|z\|_{W^{3/2,2}_{\boldsymbol\alpha}(\Gamma)}\right),\\ \|u-u_h\|_{L^2(\Omega)} &\le c\,h^{3/2-\overline{\boldsymbol\alpha}+\varepsilon(\Omega)}\left(|u|_{W^{2,2}_{\boldsymbol\alpha}(\Omega)}+\|z\|_{W^{3/2,2}_{\boldsymbol\alpha}(\Gamma)}\right),\end{aligned} \tag{19}$$

with some sufficiently small $\varepsilon(\Omega)\in(0,1/2]$ *depending on the opening angles of the corners of* Ω. *For convex domains, the choice* $\varepsilon(\Omega)=1/2$ *is possible.*

Proof First, we derive the error estimate in $H^1(\Omega)$. We apply the error equation $(\nabla(u-u_h),\nabla(\tilde{I}_hu-u_h))_{L^2(\Omega)^2}=0$ and obtain

$$\begin{aligned}\|\nabla(u-u_h)\|^2_{L^2(\Omega)} &= (\nabla(u-u_h),\nabla(u-I_hu))_{L^2(\Omega)^2}\\ &\quad+(\nabla(u-u_h),\nabla\tilde{S}_h(I_hz-Q_hz))_{L^2(\Omega)^2}.\end{aligned}$$

With a standard interpolation error estimate exploiting weighted regularity, see e. g [28, Lemma 3.31], we get

$$(\nabla(u-u_h),\nabla(u-I_hu))_{L^2(\Omega)^2}\le c\,h^{1-\overline{\boldsymbol\alpha}}\,|u|_{W^{2,2}_{\boldsymbol\alpha}(\Omega)}\,\|\nabla(u-u_h)\|_{L^2(\Omega)}.$$

Note that the zero extension satisfies $\|\nabla\tilde{S}_h\phi_h\|_{L^2(\Omega)}\le c\,h^{-1/2}\,\|\phi_h\|_{L^2(\Gamma)}$, see e. g. [21, Lemma 3.3]. Thus, together with Lemma 5 we obtain an estimate for the second term

$$\begin{aligned}(\nabla(u-u_h),\nabla\tilde{S}_h(I_hz-Q_hz))_{L^2(\Omega)^2} &\le c\,h^{-1/2}\,\|z-I_hz\|_{L^2(\Gamma)}\,\|\nabla(u-u_h)\|_{L^2(\Omega)}\\ &\le c\,h^{1-\overline{\boldsymbol\alpha}}\,\|z\|_{W^{3/2,2}_{\boldsymbol\alpha}(\Gamma)}\,\|\nabla(u-u_h)\|_{L^2(\Omega)}.\end{aligned}$$

In order to derive an estimate in the $L^2(\Omega)$-norm we use a duality argument. Let $w\in H^1_0(\Omega)$ be the weak solution of $-\Delta w=u-u_h$ in Ω. With partial integration, the orthogonality of the $L^2(\Gamma)$-projection Q_h, the estimate in the $H^1(\Omega)$-norm and Lemma 5, we obtain for sufficiently small $\varepsilon(\Omega)\in(0,1/2]$

$$\begin{aligned}&\|u-u_h\|^2_{L^2(\Omega)}=(u-u_h,-\Delta w)_{L^2(\Omega)}\\ &\quad=(\nabla(u-u_h),\nabla(w-I_hw))_{L^2(\Omega)^2}-(z-z_h,\partial_nw-Q_h(\partial_nw))_{L^2(\Gamma)}\\ &\quad\le c\,h^{3/2-\overline{\boldsymbol\alpha}+\varepsilon(\Omega)}\left(|u|_{W^{2,2}_{\boldsymbol\alpha}(\Omega)}\,\|w\|_{H^{3/2+\varepsilon(\Omega)}(\Omega)}+\|z\|_{W^{3/2,2}_{\boldsymbol\alpha}(\Gamma)}\,\|\partial_nw\|_{H^{\varepsilon(\Omega)}(\Omega)}\right).\end{aligned}$$

The assertion follows from the a priori estimate

$$\|\partial_nw\|_{H^{\varepsilon(\Omega)}(\Gamma)}\le c\,\|w\|_{H^{3/2+\varepsilon(\Omega)}(\Omega)}\le c\,\|u-u_h\|_{L^2(\Omega)}.$$

□

The aim in the remainder of this section is to derive error estimates for the variational normal derivative of the approximate solution u_h. Motivated by Green's identity this is defined by

$$\partial^h_nu_h\in V^\partial_h:\quad(\partial^h_nu_h,v_h)_{L^2(\Gamma)}=(\nabla u_h,\nabla v_h)_{L^2(\Omega)^2}-(f,v_h)_{L^2(\Omega)}\quad\forall v_h\in V_h. \tag{20}$$

Note that both the left- and right-hand side are zero for test functions from V_{0h}. Hence, in order to compute $\partial_n^h u_h$, it suffices to test the equation (20) with the nodal basis functions that belong to the boundary nodes.

We start our considerations with an existence and stability result.

Lemma 7 *For arbitrary input data $f \in L^2(\Omega)$, $z_h \in V_h^\partial$, the variational normal derivative $\partial_n^h u_h \in V_h^\partial$ defined by (20) exists, is unique, and satisfies the estimate*

$$\|\partial_n^h u_h\|_{H^{-1/2}(\Gamma)} \le c \left(\|f\|_{L^2(\Omega)} + \|z_h\|_{H^{1/2}(\Gamma)}\right).$$

Proof In the following $S_h : V_h^\partial \to V_h$ is the discrete harmonic extension which satisfies the estimate $\|S_h v_h\|_{H^1(\Omega)} \le c\,\|v_h\|_{H^{1/2}(\Gamma)}$, see [21, Lemma 3.2]. Together with the discrete stability of functions from V_h^∂ in $H^{1/2}(\Gamma)$ as well as (15) we obtain

$$\begin{aligned}
\|\partial_n^h u_h\|_{H^{-1/2}(\Gamma)} &\le c \sup_{\substack{v_h \in V_h^\partial \\ v_h \ne 0}} \frac{(\partial_n^h u_h, v_h)_{L^2(\Gamma)}}{\|v_h\|_{H^{1/2}(\Gamma)}} \\
&\le c \sup_{\substack{v_h \in V_h^\partial \\ v_h \ne 0}} \frac{(\nabla u_h, \nabla S_h v_h)_{L^2(\Omega)^2} - (f, S_h v_h)_{L^2(\Omega)}}{\|S_h v_h\|_{H^1(\Omega)}} \\
&\le c \left(\|\nabla u_h\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right).
\end{aligned} \tag{21}$$

The a priori estimate $\|u_h\|_{H^1(\Omega)} \le c\,(\|f\|_{L^2(\Omega)} + \|z_h\|_{H^{1/2}(\Gamma)})$ implies the assertion. □

Next, we show an error estimate for the variational normal derivative, for which we exploit the $W^{2,2}_{\boldsymbol{\alpha}}(\Omega)$-regularity of the solution. The result of the following theorem is sharp for non-convex domains Ω, and also for convex domains when the solution is not more regular than $H^2(\Omega)$ (this happens e. g. when the right-hand side belongs to $L^2(\Omega)$, but not to $L^p(\Omega)$ with $p > 2$). Later, we prove an estimate which promises a higher convergence rate for convex domains, provided that the solution belongs to $W^{2,\infty}_{\boldsymbol{\beta}}(\Omega)$.

Theorem 1 *Let Ω be an arbitrary polygonal domain. Moreover, let $z_h = Q_h z$. Under the assumptions $u \in W^{2,2}_{\boldsymbol{\alpha}}(\Omega)$ and $z \in W^{3/2,2}_{\boldsymbol{\alpha}}(\Gamma)$ with $\boldsymbol{\alpha} \in [0, 1/2)^d$, there holds the error estimate*

$$\|\partial_n u - \partial_n^h u_h\|_{H^{-1/2}(\Gamma)} \le c\, h^{1-\overline{\boldsymbol{\alpha}}} \left(\|u\|_{W^{2,2}_{\boldsymbol{\alpha}}(\Omega)} + \|z\|_{W^{3/2,2}_{\boldsymbol{\alpha}}(\Gamma)}\right).$$

Proof Using the triangle inequality we split up the norm into an error term for the $L^2(\Gamma)$-projection onto V_h^∂, and a fully discrete term, this is

$$\|\partial_n u - \partial_n^h u_h\|_{H^{-1/2}(\Gamma)} \le \|\partial_n u - Q_h(\partial_n u)\|_{H^{-1/2}(\Gamma)} + \|Q_h(\partial_n u) - \partial_n^h u_h\|_{H^{-1/2}(\Gamma)}.$$

With a standard duality argument we obtain for the first term

$$\|\partial_n u - Q_h(\partial_n u)\|_{H^{-1/2}(\Gamma)} \le c\, h^{1/2}\, \|\partial_n u - Q_h(\partial_n u)\|_{L^2(\Gamma)}. \tag{22}$$

Next, we show a best-approximation error estimate in the $L^2(\Gamma)$-norm. To this end, we use the splitting splitting $u = u_0 + p_j\, \eta_j$, see e. g. [26, Theorem 5.6(2)], with a function $u_0 \in V^{2,2}_{\boldsymbol{\alpha}}(\Omega)$, certain constants p_j, $j \in \mathcal{C}$, and smooth cut-off functions $\eta_j = \eta_j(|x - \boldsymbol{c}_j|)$ which are equal to one near $\boldsymbol{c}_j$ and have support in $\overline{\Omega}^j_R$. A similar argument has been already used in the proof of Lemma 4. For functions belonging to $V^{2,2}_{\boldsymbol{\alpha}}(\Omega)$ the estimate

$$\|\partial_n u_0 - C_h(\partial_n u_0)\|_{L^2(\Gamma)} \le c\, h^{1/2-\overline{\boldsymbol{\alpha}}}\, \|u_0\|_{V^{2,2}_{\boldsymbol{\alpha}}(\Omega)}$$

can be found in the proof of Theorem 9 in [29] for some Clément-type interpolation operator $C_h\colon L^1(\Gamma) \to V_h^\partial$. Note that $\partial_n(p_j\,\eta_j)$ and its interpolant vanish and thus, we easily deduce an estimate for the function $u \in W^{2,2}_{\boldsymbol{\alpha}}(\Omega)$. Moreover, due to norm equivalences of V- and W-spaces [26, Theorem 5.6(2)], we obtain $\|u_0\|_{V^{2,2}_{\boldsymbol{\alpha}}(\Omega)} + \sum_{j\in\mathcal{C}} |u(\boldsymbol{c}_j)| \sim \|u\|_{W^{2,2}_{\boldsymbol{\alpha}}(\Omega)}$, which leads together with the previous estimate and (22) to

$$\|\partial_n u - Q_h(\partial_n u)\|_{H^{-1/2}(\Gamma)} \le c\, h^{1-\overline{\boldsymbol{\alpha}}}\, \|u\|_{W^{2,2}_{\boldsymbol{\alpha}}(\Omega)}.$$

With the discrete stability used already in (21), the definition of ∂_n^h from (20), orthogonality of the $L^2(\Gamma)$-projection Q_h and Greens identity, we deduce

$$\begin{aligned}
\|Q_h(\partial_n u) - \partial_n^h u_h\|_{H^{-1/2}(\Gamma)} &\le c \sup_{\substack{\varphi_h\in V_h^\partial\\ \varphi_h\not\equiv 0}} \frac{(Q_h(\partial_n u) - \partial_n^h u_h, \varphi_h)_{L^2(\Gamma)}}{\|\varphi_h\|_{H^{1/2}(\Gamma)}}\\
&\le c \sup_{\substack{\varphi_h\in V_h^\partial\\ \varphi_h\not\equiv 0}} \frac{(\nabla(u-u_h), \nabla S_h\varphi_h)_{L^2(\Omega)^2}}{\|\varphi_h\|_{H^{1/2}(\Gamma)}}
\end{aligned} \tag{23}$$

where $S_h\colon V_h^\partial \to V_h$ is the discrete harmonic extension. This operator satisfies the estimate $\|\nabla S_h\varphi_h\|_{L^2(\Omega)} \le c\,\|\varphi_h\|_{H^{1/2}(\Gamma)}$. Together with the $H^1(\Omega)$-error estimate from Lemma 6 applied to $\|\nabla(u-u_h)\|_{L^2(\Omega)}$ we conclude the assertion. □

As already mentioned before the previous theorem, we expect a convergence rate higher than one for convex domains, provided that the input data are more regular. The proof of sharp convergence rates in this case is more complicated and we start with some notation required in the following. As in [29] we introduce a dyadic decomposition towards the boundary of Ω, namely

$$\Omega_J := \{x \in \Omega\colon \rho(x) \in (d_{J+1}, d_J)\} \quad \text{for } J = -1, \ldots, I, \tag{24}$$

where $\rho(x) := \operatorname{dist}(x, \Gamma)$. We set $d_J := 2^{-J}$ for $J = 0, \ldots, I$ and use modifications for the interior domain by $d_{-1} := \operatorname{diam}(\Omega)$, and the outermost domain by $d_{I+1} := 0$. Note that this forms a complete decomposition of Ω, i. e.,

$$\overline{\Omega} = \bigcup_{J=-1}^{I} \overline{\Omega}_J. \tag{25}$$

In the following we will frequently exploit the following two properties that can be directly concluded from the definition:

$$|\Omega_J| \sim d_J, \qquad \inf_{x\in\Omega_J} \operatorname{dist}(x,\Gamma) \sim \sup_{x\in\Omega_J} \operatorname{dist}(x,\Gamma) \sim d_J \ (J \neq I). \tag{26}$$

The termination index I is chosen such that $d_I = c_I h$ with some mesh-independent constant $c_I > 1$ specified later. This implies that $I \sim |\ln h|$. Moreover, we introduce the patches with the adjacent subsets given by

$$\Omega'_J := \Omega_{\min\{I,J+1\}} \cup \overline{\Omega}_J \cup \Omega_{\max\{-1,J-1\}},$$
$$\Omega''_J := \Omega'_{\min\{I,J+1\}} \cup \overline{\Omega}_J \cup \Omega'_{\max\{-1,J-1\}}.$$

Note that the patches satisfy the properties (26) as well due to $d_{J+1} \sim d_J$ for $J = -1,\ldots,I-1$.

We start the proof of the desired finite element error estimate with some local error estimates for the nodal interpolant defined in (17).

Lemma 8 *Assume that Ω is convex and $u \in W^{2,\infty}_{\boldsymbol{\beta}}(\Omega)$, $z \in W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)$ with $\boldsymbol{\beta} \in [0,2)^d$, $\boldsymbol{\gamma} \in [0,3/2)^d$. Then, there holds the estimate*

$$\begin{aligned}&\|u - \tilde{I}_h u\|_{L^2(\Omega_J)} + h\,\|\nabla(u - \tilde{I}_h u)\|_{H^1(\Omega_J)}\\ &\quad\le c\,h^2 d_J^{\min\{1/2,1-\overline{\boldsymbol{\beta}}\}}\,|\ln h|^{s/2}\,|u|_{W^{2,\infty}_{\boldsymbol{\beta}}(\Omega'_J)} + \delta_{J,I}\,h^{5/2-\overline{\boldsymbol{\gamma}}}\,|z|_{W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)},\end{aligned}$$

with $s = 1$ if $\overline{\boldsymbol{\beta}} = 1/2$, and $s = 0$ if $\overline{\boldsymbol{\beta}} \neq 1/2$.

Proof Throughout the proof we will hide the constant c_I in the generic constant c as it is not needed for the terms considered here. For elements $T \in \mathcal{T}_h$ touching a corner, i. e., $r_{j,T} = 0$ for some $j \in \mathcal{C}$, we directly deduce the estimate

$$\begin{aligned}&\|u - I_h u\|_{L^2(T)} + h\,\|\nabla(u - I_h u)\|_{L^2(T)}\\ &\quad\le c\,h^{3-\beta_j}\,|u|_{W^{2,\infty}_{\beta_j}(T)} \le c\,h^2\,d_I^{1-\beta_j}\,|u|_{W^{2,\infty}_{\beta_j}(T)},\end{aligned} \tag{27}$$

which follows from the estimate from [28, Corollary 3.33] and the property $d_I \sim h$. On that part of Ω_J excluding the elements touching a corner we obtain for $J = -1,\ldots,I$ with a standard estimate

$$\|u - I_h u\|_{L^2(\Omega_J\setminus S_h)} + h\,\|\nabla(u - I_h u)\|_{L^2(\Omega_J\setminus S_h)} \le c\,h^2\,\|\nabla^2 u\|_{L^2(\Omega'_J\setminus S_h)}, \tag{28}$$

where $S_h := \cup\{T \in \mathcal{T}_h\colon r_{j,T} = 0,\ j \in \mathcal{C}\}$. It remains to bound the term on the right-hand side of (28). Therefore, we bound $\|\nabla^2 u\|_{L^2(\Omega_{\tilde{J}}\setminus S_h)}$ for $\tilde{J} = \max\{-1, J-1\},\ldots,\min\{J+1, I\}$ by some weighted $W^{2,\infty}(\Omega)$-norm of u. This is done by an application of the Hölder inequality on a further dyadic decomposition of $\Omega_{\tilde{J}}$ with respect to the corners. A similar technique is used e. g. in [5,32] where error estimates in $L^2(\Gamma)$ for the Neumann problem in three-dimensional polyhedral domains are derived. Therein, the domain is decomposed twice into dyadic subsets to resolve both edge and corner singularities. Following these ideas we introduce

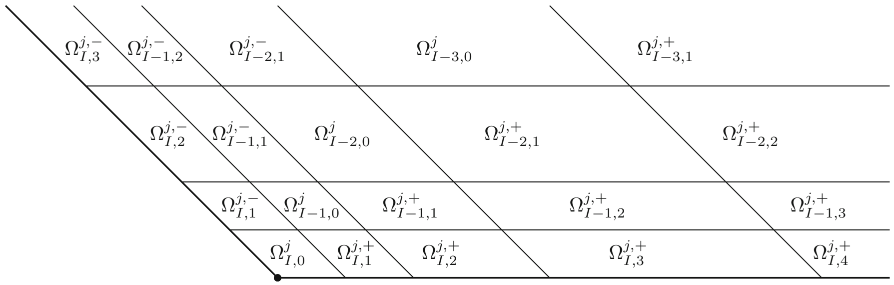

Fig. 1 Definition of the domains $\Omega^{j,\pm}_{J,K}$

$$d_{J,K} := 2^K\, d_J = 2^{K-J},$$

and define the subdomains

$$\Omega^{j,+}_{J,K} := \{x \in \Omega\colon d_{J+1} < \operatorname{dist}(x, \tilde{\Gamma}_j) \le d_J,\ d_{J,K} < \operatorname{dist}(x, \tilde{\Gamma}_{j-1}) \le d_{J,K+1}\}, \tag{29}$$

for $J = 0, \dots, I$, $K = 0, \dots, J-1$ and $j \in \mathcal{C}$. Here $\tilde{\Gamma}_j$ stands for the straight line which coincides with the boundary edge Γ_j. Each domain $\Omega^{j,+}_{J,K}$ is a parallelogram bounded by that parallels to $\tilde{\Gamma}_j$ having distance d_{J+1} and d_J from $\tilde{\Gamma}_j$, and by that parallels to $\tilde{\Gamma}_{j-1}$ having distance $d_{J,K}$ and $d_{J,K+1}$ from $\tilde{\Gamma}_{j-1}$. In a similar way we define the subdomains $\Omega^{j,-}_{J,K}$ by simply changing the roles of $\tilde{\Gamma}_j$ and $\tilde{\Gamma}_{j-1}$ in the definition (29). Note that $\Omega^j_{J,0} := \Omega^{j,+}_{J,0} = \Omega^{j,-}_{J,0}$. These subdomains are illustrated in Fig. 1.

By construction we have the property

$$|\Omega^{j,\pm}_{J,K}| \sim d_J\, d_{J,K} = d_J^2\, 2^K. \tag{30}$$

Moreover, we will exploit the property

$$\inf_{x \in \Omega^{\pm,j}_{J,K} \setminus S_h} r_j(x) \sim \sup_{x \in \Omega^{\pm,j}_{J,K} \setminus S_h} r_j(x) \sim d_{J,K}, \tag{31}$$

for all $J = 0, \dots, I$, $K = 0, \dots, J-1$ and $j \in \mathcal{C}$, which follows directly from the definition of the sets $\Omega^{\pm,j}_{J,K}$. This allows us to locally trade the quantities $d_{J,K}$ by the weights $r_j(x)$ contained in the weighted Sobolev spaces. The union of the domains introduced in (29) leads to a covering of our initial decomposition (25) near a ball of radius 1 around the corner $\boldsymbol{c}_j$, $j \in \mathcal{C}$, i.e.,

$$\Omega_J \cap \Omega^j_R \subset \bigcup_{K=0}^{J-1} \Omega^{j,\pm}_{J,K}, \qquad J = 0, \dots, I. \tag{32}$$

In order to bound the term on the right-hand side of (28), we apply the Hölder inequality on each subset $\Omega^{j,\pm}_{\tilde{J},K}$ using the property (30), and insert appropriate weights taking (31) into account. This implies

$$\|\nabla^2 u\|^2_{L^2(\Omega_{\tilde J}\cap\Omega^j_R\setminus S_h)} \le \sum_{K=0}^{\tilde J-1} d_{\tilde J}\, d^{1-2\beta_j}_{\tilde J,K}\, \|r_j^{\beta_j}\,\nabla^2 u\|^2_{L^\infty(\Omega^{j,\pm}_{\tilde J,K})}$$
$$\le c\, d_{\tilde J}^{\min\{1,2-2\beta_j\}}\,|\ln h|^s \max_{K=0,\dots,\tilde J-1} |u|^2_{W^{2,\infty}_{\beta_j}(\Omega^{j,\pm}_{\tilde J,K})},$$

where the last step follows from the limit value of the geometric series and the property $\tilde J \le I \sim |\ln h|$, i. e.,

$$\sum_{K=0}^{\tilde J-1} d^t_{\tilde J,K} = d^t_{\tilde J}\sum_{K=0}^{\tilde J-1}(2^K)^t \le c\cdot\begin{cases} d^t_{\tilde J}, & \text{if } t<0,\\ 1, & \text{if } t>0,\\ |\ln h|, & \text{if } t=0,\end{cases} \qquad t := 1-2\beta_j. \tag{33}$$

With the Hölder inequality we obtain a similar estimate on the set $\Omega_{\tilde J}\setminus\cup_{j\in\mathcal C}\Omega^j_R$, this is,

$$\|\nabla^2 u\|^2_{L^2(\Omega_{\tilde J}\setminus\cup_{j\in\mathcal C}\Omega^j_R)} \le c\, d_J\,\|\nabla^2 u\|^2_{L^\infty(\Omega_{\tilde J}\setminus\cup_{j\in\mathcal C}\Omega^j_R)} \le c\, d_J\, |u|^2_{W^{2,\infty}_{\boldsymbol\beta}(\Omega_{\tilde J})}.$$

Combining the previous estimates and summing up over the indices $\tilde J = \max\{-1, J-1\},\dots,\min\{J+1, I\}$ finally yields together with (27)

$$\|u - I_h u\|_{L^2(\Omega_J)} + h\,\|\nabla(u - I_h u)\|_{L^2(\Omega_J)}$$
$$\le c\, h^2\, d_J^{\min\{1/2,1-\overline{\boldsymbol\beta}\}}\,|\ln h|^{s/2}\,|u|_{W^{2,\infty}_{\boldsymbol\beta}(\Omega'_J)}. \tag{34}$$

In case of $J = I$, we still have to discuss the boundary terms to obtain an estimate for $\tilde I_h$. This follows from

$$\|\tilde S_h v_h\|_{L^2(\Omega)} + h\,\|\nabla(\tilde S_h v_h)\|_{L^2(\Omega)} \le c\, h^{1/2}\,\|v_h\|_{L^2(\Gamma)}, \qquad v_h\in V_h^\partial,$$

and the estimate derived in Lemma 5. □

As an intermediate result we prove a weighted $L^2(\Omega)$-error estimate. The weight function we use is defined by

$$\sigma(x) = d_I + \operatorname{dist}(x,\Gamma).$$

Note that such a weight function has been discussed already in Sect. 2.1. The regularizer in the present situation is the width of the outermost subset Ω_I. Here, the relation between the weight function σ and the dyadic decomposition (25) becomes clear, as the definition directly implies

$$\sigma(x)\sim d_J \qquad \text{for } x\in\Omega_J,\ J=-1,\dots,I. \tag{35}$$

Lemma 9 *Assume that $\Omega\subset\mathbb R^2$ is a convex polygonal domain. Let $u\in W^{2,\infty}_{\boldsymbol\beta}(\Omega)$ and $z\in W^{2,2}_{\boldsymbol\gamma}(\Gamma)$ with $\boldsymbol\beta\in[0,2)^d$ and $\boldsymbol\gamma\in[0,3/2)^d$. Moreover, let $z_h := Q_h z$. Then the solutions of* (15) *fulfill the error estimate*

$$\|\sigma^{-2}(u-u_h)\|_{L^2(\Omega)} \le c\left(h^{\min\{1/2,1-\overline{\beta}\}}\,|\ln h|^{s/2}\,|u|_{W^{2,\infty}_{\boldsymbol{\beta}}(\Omega)} + h^{1/2-\overline{\gamma}}\,|z|_{W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)}\right), \tag{36}$$

provided that c_I is sufficiently large.

Proof We follow the arguments of the Nitsche trick using the slightly modified dual problem

$$-\Delta w = \sigma^{-2}\psi \quad \text{in} \quad \Omega, \qquad w = 0 \quad \text{on} \quad \Gamma, \tag{37}$$

with $\psi = \sigma^{-2}(u-u_h)/\|\sigma^{-2}(u-u_h)\|_{L^2(\Omega)}$. Note that $\|\psi\|_{L^2(\Omega)} = 1$. With partial integration and the Galerkin orthogonality we conclude

$$\begin{aligned}\|\sigma^{-2}(u-u_h)\|_{L^2(\Omega)} &= (u-u_h, \sigma^{-2}\psi)_{L^2(\Omega)} \\ &= (\nabla(u-u_h), \nabla(w-I_h w))_{L^2(\Omega)^2} - (z-z_h, \partial_n w)_{L^2(\Gamma)}.\end{aligned} \tag{38}$$

First, we consider the second term on the right-hand side of (38). With the orthogonality of the $L^2(\Gamma)$-projection we obtain

$$(z-z_h, \partial_n w)_{L^2(\Gamma)} \le c\,h^{1/2}\,\|z-I_h z\|_{L^2(\Gamma)}\,\|\partial_n w\|_{H^{1/2}(\Gamma)} \le c\,h^{1/2-\overline{\gamma}}\,|z|_{W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)}, \tag{39}$$

where the second step follows from Lemma 5 and the estimate $\|\partial_n w\|_{H^{1/2}(\Gamma)} \le c\,\|w\|_{H^2(\Omega)} \le c\,\|\sigma^{-2}\psi\|_{L^2(\Gamma)} \le c\,h^{-2}$ which is a consequence of a trace theorem, an a priori estimate, and $\sigma(x) \ge d_I \sim h$ for all $x \in \Omega$.

Next, we discuss the first term on the right-hand side of (38). A subset-wise application of the Cauchy-Schwarz inequality with respect to the dyadic decomposition (25) yields

$$(\nabla(u-u_h), \nabla(w-I_h w))_{L^2(\Omega)^2} \le \sum_{J=-1}^{I} \|\nabla(u-u_h)\|_{L^2(\Omega_J)}\,\|\nabla(w-I_h w)\|_{L^2(\Omega_J)}. \tag{40}$$

Moreover, with the local finite-element error estimates from [11] we obtain

$$\begin{aligned}&\|\nabla(u-u_h)\|_{L^2(\Omega_J)} \\ &\quad\le c\left(\|\nabla(u-\tilde{I}_h u)\|_{L^2(\Omega'_J)} + d_J^{-1}\,\|u-\tilde{I}_h u\|_{L^2(\Omega'_J)} + d_J^{-1}\,\|u-u_h\|_{L^2(\Omega'_J)}\right)\end{aligned} \tag{41}$$

for all $J = -1, \ldots, I$. Note that this estimate would not hold for I_h as the boundary traces of u_h and the used interpolant must coincide.

Next, we insert (41) into (40) and discuss the resulting terms separately. First, consider the product of the interpolation terms. For the interpolation error of the dual solution we apply a standard estimate and Lemma 2 in case of $J = -1, \ldots, I-2$. As we can locally trade σ by d_J, see (35), we obtain

$$\begin{aligned}\|\nabla(w-I_h w)\|_{L^2(\Omega_J)} &\le c\,h\,\|\nabla^2 w\|_{L^2(\Omega'_J)} \\ &\le c\,h\,d_J^{-1}\left(\|\nabla w\|_{L^2(\Omega''_J)} + \|\sigma^{-1}\psi\|_{L^2(\Omega''_J)}\right).\end{aligned} \tag{42}$$

In case of $J = I - 1, I$ we use a global a priori estimate to arrive at

$$\|\nabla(w - I_h w)\|_{L^2(\Omega_J)} \leq c\, h\, \|\nabla^2 w\|_{L^2(\Omega)} \leq c\, h\, d_I^{-1}\, \|\sigma^{-1}\,\psi\|_{L^2(\Omega)}, \qquad (43)$$

where the last step follows from the property $\sigma(x) \geq d_I$ for $x \in \Omega$. Together with the interpolation error estimates from Lemma 8 and the discrete Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&\sum_{J=-1}^{I} \left(\|\nabla(u - \tilde{I}_h u)\|_{L^2(\Omega'_J)} + d_J^{-1}\, \|u - \tilde{I}_h u\|_{L^2(\Omega'_J)}\right) \|\nabla(w - I_h w)\|_{L^2(\Omega_J)} \\
&\quad \leq c\, h^2 \sum_{J=-1}^{I} \left(|\ln h|^{s/2}\, d_J^{\min\{-1/2,-\overline{\boldsymbol{\beta}}\}}\, |u|_{W^{2,\infty}_{\boldsymbol{\beta}}(\Omega''_J)} + \delta_{J,I}\, d_I^{-1} h^{1/2-\overline{\boldsymbol{\gamma}}}\, |z|_{W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)}\right) \\
&\qquad \times \left(\|\nabla w\|_{L^2(\Omega''_J)} + \|\sigma^{-1}\,\psi\|_{L^2(\Omega''_J)} + (\delta_{J,I-1} + \delta_{J,I})\, \|\sigma^{-1}\,\psi\|_{L^2(\Omega)}\right) \\
&\quad \leq c\, h^2 \left(|\ln h|^{s/2} \left(\sum_{J=-1}^{I} d_J^{2\min\{-1/2,-\overline{\boldsymbol{\beta}}\}}\right)^{1/2} |u|_{W^{2,\infty}_{\boldsymbol{\beta}}(\Omega)} + h^{-1/2-\overline{\boldsymbol{\gamma}}}\, |z|_{W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)}\right) \\
&\qquad \times \left(\|\nabla w\|_{L^2(\Omega)} + \|\sigma^{-1}\,\psi\|_{L^2(\Omega)}\right) \\
&\quad \leq c \left(h^{1/2+\min\{0,1/2-\overline{\boldsymbol{\beta}}\}}\, |\ln h|^{s/2}\, |u|_{W^{2,\infty}_{\boldsymbol{\beta}}(\Omega)} + h^{1/2-\overline{\boldsymbol{\gamma}}}\, |z|_{W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)}\right). \qquad (44)
\end{aligned}$$

The last step follows from the limit value of the geometric series. Analogous to (33) this can be calculated by means of

$$\sum_{J=0}^{I-1} d_J^t = \sum_{J=0}^{I-1} (2^{-t})^J \leq c\,(1 + (2^{-t})^I) \leq c\,(1 + d_I^t), \qquad (45)$$

with $t = 2\min\{-1/2, -\overline{\boldsymbol{\beta}}\}$. Moreover, we exploited the property $d_I \sim h$ and the estimates from Lemma 1 taking into account $\|\sigma^{-1}\,\psi\| \leq c\, d_I^{-1} \leq c\, h^{-1}$. Note that the constant c_I vanishes in c as it is not needed here.

Next, we discuss the product of the pollution term for the primal problem from (41) and the interpolation error for the dual problem. With similar arguments as in (44) we get

$$\begin{aligned}
&\sum_{J=-1}^{I} d_J^{-1}\, \|u - u_h\|_{L^2(\Omega'_J)}\, \|\nabla(w - I_h w)\|_{L^2(\Omega_J)} \\
&\quad \leq c\, h \sum_{J=-1}^{I} d_J^{-2}\, \|u - u_h\|_{L^2(\Omega'_J)} \Big(\|\nabla w\|_{L^2(\Omega''_J)} + \|\sigma^{-1}\,\psi\|_{L^2(\Omega''_J)} \\
&\qquad\qquad + (\delta_{J,I-1} + \delta_{J,I})\, \|\sigma^{-1}\,\psi\|_{L^2(\Omega)}\Big) \\
&\quad \leq c\, h\, \|\sigma^{-2}\,(u - u_h)\|_{L^2(\Omega)} \left(\|\nabla w\|_{L^2(\Omega)} + \|\sigma^{-1}\,\psi\|_{L^2(\Omega)}\right)
\end{aligned}$$

$$\leq c\, c_I^{-1}\, \|\sigma^{-2}\,(u-u_h)\|_{L^2(\Omega)}, \tag{46}$$

and in the last step we applied Lemma 1 and $\|\sigma^{-1}\,\psi\|_{L^2(\Omega)} \leq c\, d_I^{-1} = c\, c_I^{-1}\, h^{-1}$. Finally, insertion of (44) and (46) into (40) and the resulting estimate together with (39) into (38) leads to

$$\begin{aligned}\|\sigma^{-2}\,(u-u_h)\|_{L^2(\Omega)} &\leq c\, h^{1/2+\min\{0,1/2-\overline{\boldsymbol{\beta}}\}}\, |\ln h|^{s/2}\, |u|_{W^{2,\infty}_{\boldsymbol{\beta}}(\Omega)}\\ &\quad + ch^{1/2-\overline{\boldsymbol{\gamma}}}\, |z|_{W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)} + cc_I^{-1}\, \|\sigma^{-2}\,(u-u_h)\|_{L^2(\Omega)}.\end{aligned} \tag{47}$$

The last term on the right-hand side can be neglected when c_I is chosen sufficiently large such that $cc_I^{-1} \leq 1/2$. This implies the assertion. □

Now, we are in the position to show an improved convergence rate for the variational normal derivative in case of convex domains.

Theorem 2 *Let $z_h := Q_h z$. Assume that Ω is a convex polygonal domain. Let $u \in H^2(\Omega) \cap W^{2,\infty}_{\boldsymbol{\beta}}(\Omega)$ and $z \in W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)$ with $\boldsymbol{\beta} \in [0,1)^d$, $\boldsymbol{\gamma} \in [0,3/2)^d$. Moreover, it is assumed that $\partial_n u$ is continuous in the corners of Ω. Then, there holds the error estimate*

$$\begin{aligned}&\|\partial_n u - \partial_n^h u_h\|_{H^{-1/2}(\Gamma)}\\ &\quad \leq c\, h^{3/2}\, |\ln h|^{s/2} \left(h^{-\max\{0,\overline{\boldsymbol{\beta}}-1/2\}}\, |u|_{W^{2,\infty}_{\boldsymbol{\beta}}(\Omega)} + h^{-\overline{\boldsymbol{\gamma}}}\, |z|_{W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)}\right)\end{aligned}$$

with $s = 1$ if $\overline{\boldsymbol{\beta}} = 1/2$ and $s = 0$ otherwise.

Proof The beginning of the proof is analogous to the proof of Theorem 1. First, we derive an interpolation error estimates for some interpolant of $\partial_n u$ in the $L^2(\Gamma)$-norm. Therefore, we use the a Clément-type interpolant $C_h \colon C(\Gamma) \to V_h^\partial$ with a slight modification in the nodes located in a corner of Ω. In the following $\{x_i\}_{i=1}^{N_{\mathrm{bd}}}$ are the nodes of $\mathcal{E}_h$, and $\{\varphi_i\}_{i=1}^{N_{\mathrm{bd}}}$ are the nodal basis functions of $\mathcal{E}_h$. Each basis function is the boundary trace of a nodal basis function of V_h (the 2D "hat functions"). The precise definition of C_h is given by

$$[C_h v](x) = \sum_{i=1}^{N_{\mathrm{bd}}} a_i(v)\, \varphi_i(x), \quad a_i(v) := \begin{cases} v(x_i), & \text{if } x_i = \boldsymbol{c}_j \text{ for some } j \in \mathcal{C},\\ |\sigma_i|^{-1} \int_{\sigma_i} v, & \text{otherwise,}\end{cases}$$

where $\sigma_i := \cup\{E \in \mathcal{E}_h \colon x_i \in \overline{E}\}$ if $x_i \notin \{\boldsymbol{c}_j,\ j \in \mathcal{C}\}$. For the nodes x_i located in the vertices of Ω we simply set $\sigma_i := \emptyset$. For some $E \in \mathcal{E}_h$ we denote by T the corresponding triangle from $\mathcal{T}_h$, this is, $E \subset \bar{T}$, and by $F_T \colon \hat{T} \to T$ the affine mappings from the reference triangle $\hat{T} := \mathrm{conv}\{(0,0),\ (1,0),\ (0,1)\}$ to the world element T. Moreover, we will use the notation $\hat{v}(\hat{x}) := v(F_T(\hat{x}))$. In addition, we introduce the patches $S_E := \cup\{\sigma_i : x_i \in \bar{E}\}$ and $D_E := \cup\{T \in \mathcal{T}_h \colon \bar{T} \cap \bar{E} \neq \emptyset\}$, as well as the corresponding reference patches $S_{\hat{E}} := F_T^{-1}(S_E)$ and $D_{\hat{E}} := F_T^{-1}(D_E)$.

First, we easily see that the interpolant satisfies the stability estimate

$$\|C_h(v)\|_{L^2(E)} \le \sum_{i\,:\,x_i\in\bar{E}} a_i(v)\,\|\varphi_i\|_{L^2(E)} \le c\,|E|^{1/2}\,\|v\|_{L^\infty(S_E)}$$

for an arbitrary function $v \in L^\infty(E)$. For elements $E \in \mathcal{E}_h$ touching the corner $\boldsymbol{c}_j$, $j \in \mathcal{C}$, we insert an arbitrary first-order polynomial p and infer with the triangle inequality and the stability estimate for C_h

$$\begin{aligned}\|\partial_n u - C_h(\partial_n u)\|_{L^2(E)} &\le c\,\left(\|\partial_n u - \partial_n p\|_{L^2(E)} + \|C_h(\partial_n u - \partial_n p)\|_{L^2(E)}\right)\\ &\le c\,h^{-1}\,|E|^{1/2}\,\|\partial_{\hat{n}}\hat{u} - \partial_{\hat{n}}\hat{p}\|_{L^\infty(S_{\hat{E}})}\\ &\le c\,h^{-1}\,|E|^{1/2}\,\|\hat{u} - \hat{p}\|_{W^{1,\infty}(D_{\hat{E}})}.\end{aligned}$$

We proceed with the embedding $W^{2,2+\varepsilon}(D_{\hat{E}}) \hookrightarrow W^{1,\infty}(D_{\hat{E}})$, the Bramble-Hilbert Lemma, as well as the embedding $W^{0,\infty}_{\beta_j}(D_{\hat{E}}) \hookrightarrow L^{2+\varepsilon}(D_{\hat{E}})$, which holds for all $\beta_j < 1$, provided that $\varepsilon > 0$ is sufficiently small. The weighted Sobolev spaces in the reference setting are defined analogous to the spaces defined in Sect. 2.2 with the exception that the weight function is defined by $\hat{r} := |\hat{x}|$. When assuming w.l.o.g that $F_T(0) = \boldsymbol{c}_j$ we obtain the property $\hat{r}(\hat{x}) \sim r_j(F_T(\hat{x}))\,h^{-1}$. A transformation of variables then yields

$$\|\partial_n u - C_h(\partial_n u)\|_{L^2(E)} \le c\,h^{-1}\,|E|^{1/2}\,|\hat{u}|_{W^{2,\infty}_{\beta_j}(D_{\hat{E}})} \le c\,h^{3/2-\beta_j}\,|u|_{W^{2,\infty}_{\beta_j}(D_E)}. \tag{48}$$

For elements $E \in \mathcal{E}_h$ away from the corners we apply similar arguments, but use instead the stability estimate $\|C_h v\|_{L^2(E)} \le c\,\|v\|_{L^2(S_E)}$, to arrive at

$$\begin{aligned}\|\partial_n u - C_h(\partial_n u)\|_{L^2(E)} &\le c\,h^{-1}\,|E|^{1/2}\,\|\partial_{\hat{n}}\hat{u} - \partial_{\hat{n}}\hat{p}\|_{L^2(S_{\hat{E}})}\\ &\le c\,h^{-1}\,|E|^{1/2}\,\|\hat{u} - \hat{p}\|_{H^2(D_{\hat{E}})} \le c\,h^{-1}\,|E|^{1/2}\,|\hat{u}|_{H^2(D_{\hat{E}})}\\ &\le c\,h^{1/2}\,|u|_{H^2(D_E)}.\end{aligned}$$

Summation over all boundary elements $E \in \mathcal{E}_h$ with $E \not\subset S_h := \cup\{E \in \mathcal{E}_h : r_{j,E} = 0,\ j \in \mathcal{C}\}$ yields

$$\begin{aligned}\|\partial_n u - C_h(\partial_n u)\|_{L^2(\Gamma\setminus\overline{S}_h)} &\le c\,h^{1/2}\,|u|_{H^2(\Omega_I\setminus S_h)}\\ &\le c\,h^{1-\max\{0,\overline{\boldsymbol{\beta}}-1/2\}}\,|\ln h|^{s/2}\,|u|_{W^{2,\infty}_{\boldsymbol{\beta}}(\Omega)},\end{aligned}$$

where the last step is an application of the estimate (34) with $J = I$ taking into account $d_I \sim h$. Together with the estimates (22) and (48) we deduce

$$\|\partial_n u - Q_h(\partial_n u)\|_{H^{-1/2}(\Gamma)} \le c\,h^{3/2-\max\{0,\overline{\boldsymbol{\beta}}-1/2\}}\,|\ln h|^{s/2}\,|u|_{W^{2,\infty}_{\boldsymbol{\beta}}(\Omega)}.$$

The fully discrete part $Q_h(\partial_n u) - \partial_n^h u_h$ is treated as in (23) with the only difference that we use instead of $S_h\varphi_h$ the extension $I_h S\varphi_h$ (note that the choice of the discrete extension in (23) is arbitrary). Here, $S\colon H^{1/2}(\Gamma) \to H^1(\Omega)$ denotes the harmonic extension, i.e., $[S\varphi]|_\Gamma \equiv \varphi$ and $(\nabla S\varphi, \nabla v)_{L^2(\Omega)^2} = 0$ for all $v \in H^1_0(\Omega)$. We have to derive an upper bound for $(\nabla(u-u_h), \nabla I_h(S\varphi_h))_{L^2(\Omega)^2}$ which must depend linearly on $\|\varphi\|_{H^{1/2}(\Gamma)}$. Introducing $S\varphi_h$ as intermediate function yields

$$\begin{aligned}(\nabla(u-u_h), \nabla I_h(S\varphi_h))_{L^2(\Omega)^2} \\ = (\nabla(u-u_h), \nabla(I_h(S\varphi_h)-S\varphi_h))_{L^2(\Omega)^2} + (\nabla(u-u_h), \nabla S\varphi_h)_{L^2(\Omega)^2}. \quad (49)\end{aligned}$$

The latter term is the simpler one. With integration by parts and the trace theorem for normal derivatives from [25, Theorem 1.3.2] we obtain

$$\begin{aligned}(\nabla(u-u_h), \nabla S\varphi_h)_{L^2(\Omega)^2} &= (u-u_h, \partial_n(S\varphi_h))_{L^2(\Gamma)} \\ &= (z-Q_hz, \partial_n(S\varphi_h))_{L^2(\Gamma)} \\ &\le c\,\|z-Q_hz\|_{H^{1/2}(\Gamma)}\,\|S\varphi_h\|_{H^1(\Omega)} \\ &\le c\,h^{3/2-\overline{\gamma}}\,|z|_{W^{2,2}_{\gamma}(\Gamma)}\,\|\varphi_h\|_{H^{1/2}(\Gamma)}. \quad (50)\end{aligned}$$

In the last step we used the fact that the $L^2(\Gamma)$-projection is stable in $H^{1/2}(\Gamma)$, see [31], and fulfills thus a best-approximation property in the $H^{1/2}(\Gamma)$-norm. The desired estimate then follows from Lemma 5.

The first term on the right-hand side of (49) has the structure of the term (40) from the proof of Lemma 9. The only difference is that the dual solution w used in that lemma, has to be replaced by the function $S\varphi_h$. To this end, we first confirm the estimate

$$\begin{aligned}\|\nabla(S\phi_h)-I_h(S_{\phi_h}))\|_{L^2(\Omega)} &\le c\,h^{1/2-\varepsilon}\,\|S\varphi_h\|_{H^{3/2-\varepsilon}(\Omega)} \\ &\le c\,h^{1/2-\varepsilon}\,\|\varphi_h\|_{H^{1-\varepsilon}(\Gamma)} \le c\,\|\varphi_h\|_{H^{1/2}(\Gamma)}. \quad (51)\end{aligned}$$

provided that $\varepsilon\in(0,1/2)$. With the same arguments as in (44) and (46), taking into account (42) with w and ψ replaced by $S\varphi_h$ and 0 for $J=-1,\ldots,I-2$ and (51) for $J=I-1, I$, as well as the interpolation error estimates from Lemma 8 and the property $d_I\sim h$, we obtain

$$\begin{aligned}&(\nabla(u-u_h), \nabla(I_h(S\varphi_h)-S\varphi_h))_{L^2(\Omega)^2} \\ &\le c\,sum_{J=-1}^{I}\Big(\|\nabla(u-\tilde I_hu)\|_{L^2(\Omega'_J)} + d_J^{-1}\,\|u-\tilde I_hu\|_{L^2(\Omega'_J)} + d_J^{-1}\,\|u-u_h\|_{L^2(\Omega'_J)}\Big) \\ &\quad\times\|\nabla(I_h(S\varphi_h)-S\varphi_h)\|_{L^2(\Omega_J)} \\ &\le c\,h^2\sum_{J=-1}^{I}\Big(d_J^{\min\{-1/2,-\overline{\beta}\}}\,|\ln h|^{s/2}\,|u|_{W^{2,\infty}_{\beta}(\Omega)} + \delta_{J,I}\,h^{-1/2-\overline{\gamma}}\,|z|_{W^{2,2}_{\gamma}(\Gamma)} \\ &\quad + h^{-1}\,\|\sigma^{-2}\,(u-u_h)\|_{L^2(\Omega'_J)}\Big)\Big(\|\nabla(S\varphi_h)\|_{L^2(\Omega''_J)} + (\delta_{J,I-1}+\delta_{J,I})\|\varphi_h\|_{H^{1/2}(\Gamma)}\Big) \\ &\le c\,h^{3/2}\Big(h^{\min\{0,1/2-\overline{\beta}\}}|\ln h|^{s/2}|u|_{W^{2,\infty}_{\beta}(\Omega)} + h^{-\overline{\gamma}}|z|_{W^{2,2}_{\gamma}(\Gamma)} \\ &\quad + h^{-1/2}\|\sigma^{-2}(u-u_h)\|_{L^2(\Omega)}\Big)\,\|\varphi_h\|_{H^{1/2}(\Gamma)}.\end{aligned}$$

The last term in the parentheses on the right-hand side is discussed in Lemma 9 already and we can bound this term by the first two ones.

Insertion of the previous estimate, (50) and (49) into (23) and canceling out the terms $\|\varphi_h\|_{H^{1/2}(\Gamma)}$ leads to the desired estimate for the term $\|Q_h(\partial_n u) - \partial_n^h u_h\|_{H^{-1/2}(\Gamma)}$. □

Remark 1 The best possible convergence rate of $3/2$ is achieved when $z \in H^2(\Gamma)$ and $u \in W^{2,\infty}_{\boldsymbol{\beta}}(\Omega)$ with $\beta_j < 1/2$ for all $j \in \mathcal{C}$. In general, the latter assumption is only satisfied when the opening angles of the corners of Ω satisfy $\omega_j < 2\pi/3$, $j \in \mathcal{C}$, and when f is sufficiently smooth. As an example, assuming f to be Hölder continuous would be sufficient, compare Corollary 1. Otherwise, for angles larger than $2\pi/3$ we find a relation between the convergence rate and the exponent of the dominating singularity $\bar\lambda = \pi/\omega_{\max}$ by choosing $\bar\beta = 2 - \bar\lambda + \varepsilon$ if $\omega_{\max} \in (2\pi/3, \pi)$ and $\bar\alpha = 1 - \bar\lambda + \varepsilon$ if $\omega_{\max} \in (\pi, 2\pi)$ for arbitrary but sufficiently small $\varepsilon > 0$. Under the assumption that f and z are sufficiently smooth we then infer

$$\|\partial_n u - \partial_n^h u_h\|_{H^{-1/2}(\Gamma)} \le c\, h^{\min\{3/2, \bar\lambda - \varepsilon\}}\, |\ln h|^{s/2}.$$

4 Dirichlet control problems

This section is devoted to the numerical approximation of the optimal control problem

$$J(u, z) := \frac{1}{2}\,\|u - u_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\,\langle Nz, z\rangle \to \min! \tag{52}$$

subject to the constraints

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad u = z & \text{on } \Gamma. \end{cases} \tag{53}$$

Here, $f, u_d \in L^2(\Omega)$ are given functions and $\nu > 0$ is a regularization parameter. The operator $N\colon H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a Steklov–Poincaré operator which is used to realize an $H^{1/2}(\Gamma)$-seminorm.

We introduce the linear operators $S\colon H^{1/2}(\Gamma) \to H^1(\Omega)$ and $P\colon H^1(\Omega)^* \to H^1_0(\Omega)$ defined by

$$\begin{aligned} u_z = Sz \ &:\Longleftrightarrow\ u_z \text{ solves (53) for } f \equiv 0,\\ u_f = Pf \ &:\Longleftrightarrow\ u_f \text{ solves (53) for } z \equiv 0. \end{aligned}$$

We can express the operator N by means of $Nz := \partial_n(Sz)$. Note that the regularization term is equivalent to the square of the $H^{1/2}(\Gamma)$-seminorm of z.

Necessary optimality conditions, that are also sufficient due to the convexity of this optimization problem, can be found in [27]. Therein, it is shown that the pair $(u, z) \in H^1(\Omega) \times H^{1/2}(\Gamma)$ is the unique global minimizer of (52)–(53) if and only if an adjoint state $p \in H^1(\Omega)$ exists such that the optimality system

$$\begin{cases} -\Delta u = f & -\Delta p = u - u_d & \text{in } \Omega, \\ \quad u = z & \quad p = 0 & \text{on } \Gamma, \\ & \nu Nz + \partial_n p = 0 & \text{in } H^{-1/2}(\Gamma), \end{cases} \tag{54}$$

is fulfilled. One can reformulate the optimality system using the operators S and P introduced above. Taking also into account the relation $S^*u = \partial_n Pu$ leads to a compact form of the optimality system

$$u = Sz + Pf, \qquad \nu Nz + S^*(u - u_d) = 0.$$

Eliminating u leads to the variational problem

$$\langle Tz, v\rangle = \langle g, v\rangle \qquad \forall v \in H^{1/2}(\Gamma) \tag{55}$$

with

$$T := S^*S + \nu N, \qquad g := S^*(u_d - u_f),$$

where $u_f := Pf$. The existence of a unique solution z of (55) follows from the Lax-Milgram Lemma. Note that the operator T is coercive due to

$$\langle Tz, z\rangle = \|Sz\|^2_{L^2(\Omega)} + \nu|Sz|^2_{H^1(\Gamma)} \geq c\,\min\{1, \nu\}\|z\|^2_{H^{1/2}(\Gamma)}.$$

It remains to discuss the regularity of the optimal solution and the corresponding state and adjoint state. These results will be needed for sharp discretization error estimates.

Lemma 10 *Assume that $f, u_d \in L^2(\Omega)$. Let $\boldsymbol{\alpha} \in [0,1)^d$ be a weight vector satisfying $1 - \lambda_j < \alpha_j$, $j \in \mathcal{C}$. Then, the solution of (54) possesses the regularity*

$$Sz, Pf \in W^{2,2}_{\boldsymbol{\alpha}}(\Omega), \quad p \in V^{2,2}_{\boldsymbol{\alpha}}(\Omega), \quad z \in W^{3/2,2}_{\boldsymbol{\alpha}}(\Gamma). \tag{56}$$

Moreover, if $u_d \in C^{0,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$, there holds

$$Sz \in W^{2,\infty}_{\boldsymbol{\beta}}(\Omega), \quad p \in V^{2,\infty}_{\boldsymbol{\beta}}(\Omega), \quad z \in W^{2,\infty}_{\boldsymbol{\beta}}(\Gamma) \cap W^{2,2}_{\boldsymbol{\gamma}}(\Gamma), \tag{57}$$

with $\boldsymbol{\beta} \in [0,2)^d$, $\boldsymbol{\gamma} \in [0,3/2)^d$ satisfying $2 - \lambda_j < \beta_j$ and $3/2 - \lambda_j < \gamma_j$ for $j \in \mathcal{C}$.

Proof In order to transfer the regularity of the adjoint state to the state, we introduce the auxiliary function u_0 solving the boundary value problem

$$-\Delta u_0 = \frac{1}{\nu}(u - u_d) \quad \text{in } \Omega, \qquad \partial_n u_0 = 0 \quad \text{on } \Gamma.$$

Note that the function u_0 can be determined uniquely as the optimal state satisfies $\int_\Omega u = \int_\Omega u_d$, see e. g. [16, Section 3.2.3]. With (54) it is easy to confirm that the state can be decomposed by means of $Sz = u_0 - \frac{1}{\nu}p$.

The assertion then follows from bootstrapping arguments. Standard regularity results, and in particular [13, Theorem 4.4.3.7], immediately imply

$$\begin{aligned} z \in H^{1/2}(\Gamma) \;&\Rightarrow u \in H^1(\Omega) \hookrightarrow L^q(\Omega) \;\Rightarrow\; p, u_0 \in W^{2,q}(\Omega) \\ &\Rightarrow Sz \in W^{2,q}(\Omega) \;\Rightarrow\; z \in W^{2-1/q,q}(\Gamma), \end{aligned}$$

for arbitrary $q \in [1, \infty)$ satisfying $2/q > 2 - \lambda_j$ for all $j \in \mathcal{C}$. The regularity results collected in (56) then directly follow from Lemma 3.

From embedding theorems we moreover conclude that $u, p \in C^{0,\sigma'}(\overline{\Omega})$ for some $\sigma' \in (0, \min\{1, \bar{\lambda}\})$, and with Corollary 1 we directly infer (57). The assertion $z \in W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)$ follows from the $W^{2,\infty}_{\boldsymbol{\beta}}(\Gamma)$-regularity due to the Hölder inequality. □

In order to discretize the optimality system we replace S and P by the finite element solution operators $S_h \colon V_h^\partial \to V_h$ and $P_h \colon H^1(\Omega)^* \to V_h^\partial$ defined by

$$\begin{aligned} u_h = S_h z_h \quad &:\Longleftrightarrow \quad u_h|_\Gamma \equiv z_h \quad (\nabla u_h, \nabla v_h)_{L^2(\Omega)^2} = 0 && \forall v_h \in V_{0h}, \\ p_h = P_h u_h \quad &:\Longleftrightarrow \quad (\nabla p_h, \nabla v_h)_{L^2(\Omega)^2} = (u_h, v_h)_{L^2(\Omega)} && \forall v_h \in V_{0h}. \end{aligned}$$

Instead of S^* we use its discrete version $S_h^* := \partial_n^h P_h$ which is the adjoint operator to S_h. Then, we seek a state $u_h \in V_h$ and a control $z_h \in V_h^\partial$ as solution of the finite-dimensional optimization problem

$$J_h(u_h, z_h) := \frac{1}{2}\, \|u_h - u_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\, \langle N_h z_h, z_h\rangle \to \min! \tag{58}$$

subject to

$$u_h|_\Gamma \equiv z_h, \quad (\nabla u_h, \nabla v_h)_{L^2(\Omega)^2} = (f, v_h)_{L^2(\Omega)} \qquad \forall v_h \in V_{0h}. \tag{59}$$

In order to define an appropriate discrete Steklov–Poincaré operator we use the variational normal derivative introduced in (20), and define $N_h \colon V_h^\partial \to V_h^\partial$ by $N_h z_h := \partial_n^h(S_h z_h)$. Note that by this definition, the functional $\langle N_h \cdot, \cdot\rangle$ induces a mesh-independent $H^{1/2}(\Gamma)$-seminorm for functions in V_h^∂. Analogous to the continuous case we can derive the discrete optimality system

$$\begin{aligned} u_h|_\Gamma \equiv z_h, \quad (\nabla u_h, \nabla v_h)_{L^2(\Omega)^2} &= (f, v_h)_{L^2(\Omega)} && \forall v_h \in V_{0h}, \\ (\nabla p_h, \nabla v_h)_{L^2(\Omega)^2} &= (u_h - u_d, v_h)_{L^2(\Omega)} && \forall v_h \in V_{0h}, \\ (\nu\, \partial_n^h(S_h z_h) + \partial_n^h p_h, w_h)_{L^2(\Gamma)} &= 0 && \forall w_h \in V_h^\partial, \end{aligned} \tag{60}$$

with an adjoint state $p_h \in V_{0h}$. This system can be rewritten by means of

$$\langle T_h z_h, v_h\rangle = \langle g_h, v_h\rangle \qquad \forall v_h \in V_h^\partial \tag{61}$$

with $T_h = S_h^* S_h + \nu N_h$, $g_h := S_h^*(u_d - u_{f,h})$ and $u_{f,h} := P_h f$.

The remainder of this section is devoted to the proof of error estimates for the finite-element approximation (u_h, z_h, p_h). To this end, we introduce an auxiliary function $\tilde{z}_h \in V_h^\partial$ solving the variational formulation

$$\langle T \tilde{z}_h, v_h\rangle = \langle g, v_h\rangle \qquad \forall v_h \in V_h^\partial. \tag{62}$$

The Lax-Milgram-Lemma guarantees the existence and uniqueness of the solution $\tilde{z}_h$ of (62). Moreover, by the Céa-Lemma and the interpolation error estimates from Lemma 5 we obtain the following intermediate result:

Lemma 11 *Let $z \in H^{1/2}(\Gamma)$ be the optimal control solving (52)–(53). The approximate solutions $\tilde{z}_h$ of (62) satisfy the estimate*

$$\|z-\tilde{z}_h\|_{H^{1/2}(\Gamma)} \le c \begin{cases} h^{1-\overline{\boldsymbol{\alpha}}}\, \|z\|_{W^{3/2,2}_{\boldsymbol{\alpha}}(\Gamma)}, & \text{if } u_d \in L^2(\Omega), \\ h^{3/2-\overline{\boldsymbol{\gamma}}}\, |z|_{W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)}, & \text{if } u_d \in C^{0,\sigma}(\overline{\Omega}). \end{cases}$$

The weights $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are chosen as in Lemma 10.

It remains to derive an estimate for the error between the continuous and the discrete control z and z_h, respectively. In the following Lemma we present a general estimate. The idea of the proof is taken from [27].

Lemma 12 *The solutions z and z_h of (55) and (61), respectively, satisfy the general error estimate*

$$\begin{aligned} \|z-z_h\|_{H^{1/2}(\Gamma)} \le c\Big(&\|z-\tilde{z}_h\|_{H^{1/2}(\Gamma)} + \|\partial_n p - \partial_n^h p_h(u)\|_{H^{-1/2}(\Gamma)} \\ &+ \|u-u_h(Q_h z)\|_{L^2(\Omega)} + \|\partial_n(Sz) - \partial_n^h(S_h Q_h z)\|_{H^{-1/2}(\Gamma)}\Big), \end{aligned}$$

with $u_h(Q_h z) = S_h(Q_h z) + u_{f,h}$ and $p_h(u) = P_h(u-u_d)$.

Proof First, we confirm that the bilinear form $\langle T_h\cdot,\cdot\rangle$ is V_h^∂-elliptic and continuous, this is, for all $v_h, w_h \in V_h^\partial$ there holds

$$\begin{aligned} \gamma\ \langle T_h v_h, v_h\rangle &\ge \|v_h\|^2_{H^{1/2}(\Gamma)}, \\ \langle T_h v_h, w_h\rangle &\le c\,\|v_h\|_{H^{1/2}(\Gamma)}\,\|w_h\|_{H^{1/2}(\Gamma)}, \end{aligned}$$

with some constant $\gamma > 0$ independent of h. This follows directly from the mapping properties of N_h, S_h and S_h^* as well as Lemma 7. In the following we write $w_h := z_h - \tilde{z}_h$. With the ellipticity, the equations (61) and (62) and Young's inequality, we obtain

$$\begin{aligned} \gamma\,\|w_h\|^2_{H^{1/2}(\Gamma)} &\le \langle T_h(z_h-\tilde{z}_h), w_h\rangle \\ &= \langle g_h - g + (T-T_h)\tilde{z}_h, w_h\rangle \\ &\le \frac{\gamma}{2}\,\|w_h\|^2_{H^{1/2}(\Gamma)} + c\,\|g-g_h+(T-T_h)\tilde{z}_h\|^2_{H^{-1/2}(\Gamma)}. \end{aligned} \tag{63}$$

Insertion of the definitions of g and g_h yields

$$g_h - g = (S^*-S_h^*)(Pf-u_d) + S_h^*(P-P_h)f. \tag{64}$$

Rearrangement of the remaining terms and the definitions of T and T_h lead to

$$\begin{aligned} &(T-T_h)\tilde{z}_h \\ &\quad= T(\tilde{z}_h - z) + Tz - T_h Q_h z + T_h(Q_h z - \tilde{z}_h) \\ &\quad= T(\tilde{z}_h - z) + (S^*-S_h^*)Sz + S_h^*(Sz - S_h Q_h z) \\ &\qquad+ \nu\,(Nz - N_h Q_h z) + T_h(Q_h z - \tilde{z}_h) \end{aligned} \tag{65}$$

Next, we insert (64) and (65) into (63), apply the triangle inequality, and use the abbreviations

$$u = Sz + Pf, \quad \partial_n p = S^*(u-u_d), \quad \partial_n(Sz) = Nz,$$

as well as their discrete counterparts

$$u_h(Q_h z) = S_h Q_h z + P_h f, \quad \partial_n^h p_h(u) = S_h^*(u - u_d), \quad \partial_n^h (S_h Q_h z) = N_h Q_h z.$$

Insertion of (64) and (65) into (63), and exploiting the stability estimates

$$\|T v\|_{H^{-1/2}(\Gamma)} \le c\, \|v\|_{H^{1/2}(\Gamma)}, \qquad \|T_h v_h\|_{H^{-1/2}(\Gamma)} \le c\, \|v_h\|_{H^{1/2}(\Gamma)},$$
$$\|S_h^* v\|_{H^{-1/2}(\Gamma)} \le c\, \|v\|_{L^2(\Omega)},$$

that can be concluded from Lemma 7, as well as the stability of Q_h in $H^{1/2}(\Gamma)$ [31], leads to the estimate

$$\begin{aligned}\frac{\gamma}{2}\|w_h\|^2_{H^{1/2}(\Gamma)} \le c\, \|z - \tilde{z}_h\|^2_{H^{1/2}(\Gamma)} + \|\partial_n p - \partial_n^h p_h(u)\|^2_{H^{-1/2}(\Gamma)} \\ + c\, \|u - u_h(Q_h z)\|^2_{L^2(\Omega)} + \|\partial_n (Sz) - \partial_n^h (S_h Q_h z)\|^2_{H^{-1/2}(\Gamma)}.\end{aligned}$$

With the triangle inequality $\|z - z_h\|_{H^{1/2}(\Gamma)} \le \|z - \tilde{z}_h\|_{H^{1/2}(\Gamma)} + \|w_h\|_{H^{1/2}(\Gamma)}$ we conclude the assertion. □

This general estimate and the estimates presented in Lemma 6, Theorems 1, 2 and Lemma 11 lead to the main result of this section.

Theorem 3 *Let $\Omega \subset \mathbb{R}^2$ be an arbitrary polygonal domain and assume that $f, u_d \in L^2(\Omega)$. Let (u, z, p) be the solution of (54), and (u_h, z_h, p_h) the corresponding finite element approximation solving (60). Then, the error estimate*

$$\|z - z_h\|_{H^{1/2}(\Gamma)} \le c\, h^{\min\{1, \bar{\lambda} - \varepsilon\}} \tag{66}$$

is valid for arbitrary $\varepsilon > 0$.

Furthermore, if Ω is convex and $u_d \in C^{0,\sigma}(\overline{\Omega})$ for some $\sigma \in (0, 1)$, there holds the estimate

$$\|z - z_h\|_{H^{1/2}(\Gamma)} \le c\, h^{\min\{3/2, \bar{\lambda} - \varepsilon\}}. \tag{67}$$

Note that $\bar{\lambda} := \pi / \max_{j \in \mathcal{C}} \omega_j$.

The constant c depends linearly on the functions z, Sz, Pf and p, more precisely,

$$c = \begin{cases} c\left(\|z\|_{W^{3/2,2}_{\boldsymbol{\alpha}}(\Gamma)} + |Sz|_{W^{2,2}_{\boldsymbol{\alpha}}(\Omega)} + |Pf|_{W^{2,2}_{\boldsymbol{\alpha}}(\Omega)} + |p|_{W^{2,2}_{\boldsymbol{\alpha}}(\Omega)}\right), & \text{in (66)}, \\ c\left(|z|_{W^{2,2}_{\boldsymbol{\gamma}}(\Gamma)} + |Sz|_{W^{2,\infty}_{\boldsymbol{\beta}}(\Omega)} + |Pf|_{W^{2,2}_{\boldsymbol{\alpha}}(\Omega)} + |p|_{W^{2,\infty}_{\boldsymbol{\beta}}(\Omega)}\right), & \text{in (67)}. \end{cases}$$

The weights are defined by $\alpha_j := \max\{0, 1 - \lambda_j + \varepsilon\}$, $\beta_j := \max\{0, 2 - \lambda_j + \varepsilon\}$ and $\gamma_j := \max\{0, 3/2 - \lambda_j + \varepsilon\}$ for all $j \in \mathcal{C}$.

Proof The terms involving the approximation of the normal derivative require an application of Theorem 1 or Theorem 2. In the latter theorem continuity of the normal derivatives in the corners is assumed. This is trivially fulfilled for $\partial_n p$ as $p = 0$ on Γ and due to the optimality condition $Nz = -\frac{1}{\nu}\partial_n p$ this property is transferred to $\partial_n (Sz) = Nz$. □

As a simple conclusion we also obtain an error estimate for the state variable in the energy norm.

Corollary 2 *Assume that $f,\ u_d \in L^2(\Omega)$. Let $u \in H^1(\Omega)$ and $u_h \in V_h$ be the optimal states of* (52)–(53) *and* (58)–(59)*, respectively. Then, the error estimate*

$$\|u - u_h\|_{H^1(\Omega)} \le c\, h^{\min\{1, \overline{\lambda} - \varepsilon\}}$$

holds for arbitrary but sufficiently small $\varepsilon > 0$. The constant $c > 0$ is the same as in the previous theorem.

Proof With the triangle inequality we get

$$\|u - u_h\|_{H^1(\Omega)} \le \|Sz + u_f - (S_h Q_h z + u_{f,h})\|_{H^1(\Omega)} + \|S_h Q_h (z - z_h)\|_{H^1(\Omega)}.$$

Note that $S_h Q_h z + u_{f,h}$ is the finite element approximation of $u := Sz + u_f$. Thus, we infer with Lemma 6

$$\|Sz + u_f - (S_h Q_h z + u_{f,h})\|_{H^1(\Omega)} \le c\, h^{\min\{1, \overline{\lambda} - \varepsilon\}} \left(|u|_{W^{2,2}_{\boldsymbol{\alpha}}(\Omega)} + \|z\|_{W^{3/2,2}_{\boldsymbol{\alpha}}(\Gamma)} \right).$$

Moreover, with stability properties of S_h and Q_h we get

$$\|S_h Q_h (z - z_h)\|_{H^1(\Omega)} \le c\, \|Q_h (z - z_h)\|_{H^{1/2}(\Gamma)} \le c\, \|z - z_h\|_{H^{1/2}(\Gamma)}$$

and with (66) we conclude the assertion. □

5 Numerical experiments

In order to confirm the theoretically predicted convergence results we present some numerical experiments measuring the convergence rates. Thus, we computed the problem (52)–(53) in the domains

$$\begin{aligned}
\Omega_{90} &= (0, 1)^2, \\
\Omega_{135} &= (-1, 1)^2 \cap \{(r \cos\varphi, r \sin\varphi) \colon r \in (0, \infty), \varphi \in (0, 3\pi/4)\}, \\
\Omega_{270} &= (-1, 1)^2 \setminus [0, 1]^2,
\end{aligned}$$

with input data $\nu = 1$, $f \equiv 0$ and $u_d(x_1, x_2) = x_1 + x_2$.

We start with a structured grid consisting of 2, 3 or 6 triangles, respectively, and compute the discrete solutions solving (60) on a sequence of meshes obtained by bisection of each element so that new nodes of the grid are inserted at the midpoints of the longest edge of each element. The solution was computed by a GMRES method applied to the system (61) and in each iteration the linearized state and adjoint equation have to be solved. This was done by the parallel direct solver MUMPS which allows to reuse the factorization of the stiffness matrix. The implementation is written in C++ and the tests were performed on a Intel-Core-i7-4770 (4×3400 MHz) machine with 32 GB RAM.

As an explicit representation of the exact solution is not available for the given input data we measured the error by comparison with the solution on a very fine mesh with maximal element diameter $h_{ref} = 2^{-10}$. In Tables 1, 2 and 3 we report the error of the state in $H^1(\Omega)$, and the control in the $L^2(\Gamma)$-norm and the $H^{1/2}(\Gamma)$-seminorm, respectively. The latter norm is realized by the discrete harmonic extension S_h, this is,

Table 1 Results of the numerical experiment for the domain Ω_{90} showing finite element error and corresponding experimental convergence rates (in parentheses) for the state and control

$h \cdot \sqrt{2}$	$\#Dof$	$\#BdDof$	$\|u - u_h\|_{H^1(\Omega)}$	$\\|z - z_h\\|_{L^2(\Gamma)}$	$\|z - z_h\|_{H^{1/2}(\Gamma)}$
2^{-4}	961	128	3.37e−03 (0.98)	6.27e−05 (2.00)	4.93e−04 (1.55)
2^{-5}	3969	256	1.69e−03 (1.00)	1.57e−05 (2.00)	1.71e−04 (1.53)
2^{-6}	16129	512	8.43e−04 (1.00)	3.92e−06 (2.00)	6.04e−05 (1.50)
2^{-7}	65025	1024	4.19e−04 (1.01)	9.72e−07 (2.01)	2.22e−05 (1.44)
2^{-8}	261121	2048	2.04e−04 (1.04)	2.35e−07 (2.05)	8.64e−06 (1.36)
2^{-9}	1046530	4096	9.13e−05 (1.16)	5.07e−08 (2.22)	3.30e−06 (1.39)
Theory:			(1.00)	(2.00)	(1.50)

Table 2 Results of the numerical experiment for the domain Ω_{135} showing finite element error and corresponding experimental convergence rates (in parentheses) for the state and control

$h \cdot \sqrt{2}$	$\#Dof$	$\#BdDof$	$\|u - u_h\|_{H^1(\Omega)}$	$\\|z - z_h\\|_{L^2(\Gamma)}$	$\|z - z_h\|_{H^{1/2}(\Gamma)}$
2^{-4}	1457	160	5.93e−03 (0.98)	1.56e−04 (1.90)	1.06e−03 (1.40)
2^{-5}	5985	320	2.98e−03 (0.99)	4.19e−05 (1.89)	4.05e−04 (1.38)
2^{-6}	24257	640	1.49e−03 (1.00)	1.13e−05 (1.89)	1.58e−04 (1.36)
2^{-7}	97665	1280	7.44e−04 (1.01)	3.05e−06 (1.89)	6.22e−05 (1.34)
2^{-8}	391937	2560	3.63e−04 (1.03)	7.98e−07 (1.93)	2.48e−05 (1.33)
2^{-9}	1570300	5120	1.63e−04 (1.16)	1.80e−07 (2.15)	9.35e−06 (1.41)
Theory:			(1.00)	(1.83)	(1.33)

Table 3 Results of the numerical experiment for the domain Ω_{270} showing finite element error and corresponding experimental convergence rates (in parentheses) for the state and control

$h \cdot \sqrt{2}$	$\#Dof$	$\#BdDof$	$\|u - u_h\|_{H^1(\Omega)}$	$\\|z - z_h\\|_{L^2(\Gamma)}$	$\|z - z_h\|_{H^{1/2}(\Gamma)}$
2^{-4}	2945	256	1.71e−01 (0.70)	3.68e−02 (1.28)	9.83e−02 (0.71)
2^{-5}	12033	512	1.06e−01 (0.69)	1.51e−02 (1.29)	6.07e−02 (0.69)
2^{-6}	48641	1024	6.53e−02 (0.69)	6.11e−03 (1.30)	3.74e−02 (0.70)
2^{-7}	195585	2048	4.01e−02 (0.71)	2.43e−03 (1.33)	2.25e−02 (0.73)
2^{-8}	784385	4096	2.38e−02 (0.75)	9.03e−04 (1.43)	1.27e−02 (0.82)
2^{-9}	3141630	8192	1.27e−02 (0.91)	2.68e−04 (1.75)	6.05e−03 (1.07)
Theory:			(0.67)	(1.17)	(0.67)

$$|z - z_h|_{H^{1/2}(\Gamma)} \approx |z_{h_{ref}} - z_h|_{H^{1/2}(\Gamma)} \sim \|\nabla S_{h_{ref}}(z_{h_{ref}} - z_h)\|_{L^2(\Omega)}.$$

The convergence rates measured for the domain Ω_{90} are the same as in the experiments from [27]. These results confirm the rates predicted in Theorem 3 and Corollary 2. Note that the largest opening angle is $\bar{\omega} = \pi/2$ and thus, $\bar{\lambda} = 2$. Our theory moreover claims that the convergence rate for the discrete control is reduced when the largest opening angle exceeds the limiting case $2\pi/3$. This is the case for the domain Ω_{135}, where we have $\bar{\omega} = 3\pi/4$ and $\bar{\lambda} = 4/3$. The rate $4/3$ for the control in the $H^{1/2}(\Gamma)$-norm, which is predicted in Theorem 3, is the rate we also observe numerically. The convergence rate for the discrete states is still 1 as proved in Corollary 2. The fact that our error estimates are also valid and sharp for non-convex domains is confirmed by the experiment for the domain Ω_{270}. Here, the rate $\bar{\lambda} = 2/3$ is almost observed numerically for the discrete states and controls in $H^1(\Omega)$ and $H^{1/2}(\Gamma)$, respectively. Note that the convergence rates in the experiments are always slightly better than predicted which is due to the approximate computation of the error by comparison with a reference solution on a fine grid.

Moreover, we have to notice that we have not proved error estimates for the control in $L^2(\Gamma)$, but the experiments confirm in all cases that this convergence rate is higher by $1/2$ compared to the rate obtained in the $H^{1/2}(\Gamma)$-norm. In case of Dirichlet control problems with $L^2(\Gamma)$-regularization the convergence rate for the control in the $L^2(\Gamma)$-norm is $\min\{1, \bar{\lambda} - 1/2 - \varepsilon\}$ (up to logarithmic factors), see [3], or $\min\{1, \bar{\lambda}/2\} - \varepsilon$ in case of state-constraints, see [20]. Obviously, the rate is higher by 1 when $H^{1/2}(\Gamma)$-regularization is used which is due to the fact that the solutions are more regular. However, in order to prove estimates in $L^2(\Gamma)$ for the energy regularization approach, one has to establish a Nitsche trick for the non-conforming approximation (61) of (55). This will be subject of future research.

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