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A PDE approach to fractional diffusion: a space-fractional wave equation

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Abstract

We study solution techniques for an evolution equation involving second order derivative in time and the spectral fractional powers, of order $s \in (0, 1)$, of symmetric, coercive, linear, elliptic, second-order operators in bounded domains Ω. We realize fractional diffusion as the Dirichlet-to-Neumann map for a nonuniformly elliptic problem posed on the semi-infinite cylinder $\mathcal{C} = \Omega \times (0, \infty)$. We thus rewrite our evolution problem as a quasi-stationary elliptic problem with a dynamic boundary condition and derive space, time, and space–time regularity estimates for its solution. The latter problem exhibits an exponential decay in the extended dimension and thus suggests a truncation that is suitable for numerical approximation. We propose and analyze two fully discrete schemes. The discretization in time is based on finite difference discretization techniques: the trapezoidal and leapfrog schemes. The discretization in space relies on the tensorization of a first-degree FEM in Ω with a suitable hp-FEM in the extended variable. For both schemes we derive stability and error estimates. We consider a first-degree FEM in Ω with mesh refinement near corners and the aforementioned hp-FEM in the extended variable and extend the a priori error analysis of the trapezoidal scheme for open, bounded, polytopal but not necessarily convex domains $\Omega \subset \mathbb{R}^2$. We discuss implementation details and report several numerical examples.

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1 Introduction

We are interested in the numerical approximation of an initial boundary value problem for a space-fractional wave equation. Let Ω be an open and bounded domain in $\mathbb{R}^n$ ($n \geq 1$) with boundary $\partial\Omega$. Given $s \in (0, 1)$, a forcing function f, and initial data g and h, we seek u such that

$$\begin{cases} \partial_t^2 u + \mathcal{L}^s u = f & \text{in } \Omega \times (0, T), \\ u(0) = g \text{ in } \Omega, \quad \partial_t u(0) = h \text{ in } \Omega. \end{cases} \tag{1.1}$$

Here, $\mathcal{L}$ denotes the linear, elliptic, self-adjoint, second order, differential operator

$$\mathcal{L}w = -\text{div}_{x'}(A\nabla_{x'}w) + cw,$$

supplemented with homogeneous Dirichlet boundary conditions. The coefficient $A \in C^{0,1}(\bar{\Omega}, \mathsf{GL}(n, \mathbb{R}))$ is symmetric and uniformly positive definite and $0 \leq c \in L^\infty(\Omega)$. By $\mathcal{L}^s$, with $s \in (0, 1)$, we denote the *spectral* fractional powers of the operator $\mathcal{L}$.

One of the most common and studied physical processes is diffusion: the tendency of a substance to evenly spread into an available space. Classical models of diffusion lead to well-known models and even better studied equations. However, in recent times, it has become evident that many of the assumptions that lead to these models are not always satisfactory or even realistic: memory, heterogeneity, or a multiscale structure might violate them. In this setting, the assumption of locality does not hold and to describe diffusion one needs to resort to nonlocal operators. Different models of diffusion have been proposed, fractional diffusion being one of them. An incomplete list of problems where fractional diffusion appears includes finance [14,40,51], turbulent flow [15], quasi-geostrophic flows models [12,36], models of anomalous thermoviscous behaviors [16], peridynamics [21,56], and image science [28,29].

The design of efficient solution techniques for problems involving fractional diffusion is intricate, mainly due to the nonlocal character of $\mathcal{L}^s$ [8–10,13]. Recently, and in order to overcome such a nonlocal feature, the Caffarelli-Silvestre extension has been proved useful in numerical analysis [5,46,47]. When $\mathcal{L} = -\Delta$ and $\Omega = \mathbb{R}^n$, i.e., in the case of the Laplacian in the whole space, Caffarelli and Silvestre [10] showed that $\mathcal{L}^s$ can be realized as the Dirichlet-to-Neumann map for an extension problem to the upper half-space $\mathbb{R}^{n+1}_+$; the extension corresponds to a nonuniformly elliptic PDE. This result was later extended in [9,13,57] to bounded domains Ω and more general operators, thereby obtaining an extension problem posed on the semi-infinite cylinder $\mathcal{C} := \Omega \times (0, \infty)$. We shall thus rewrite problem (1.1) as the following quasi-stationary elliptic problem with a dynamic boundary condition [3,18,19,60]:

$$\begin{cases} -\text{div}\left(y^\alpha \mathbf{A}\nabla \mathscr{U}\right) + y^\alpha c\mathscr{U} = 0 & \text{in } \mathcal{C} \times (0, T), \\ \mathscr{U} = 0 & \text{on } \partial_L \mathcal{C} \times (0, T), \\ d_s \partial_t^2 \mathscr{U} + \partial_\nu^\alpha \mathscr{U} = d_s f & \text{on } (\Omega \times \{0\}) \times (0, T), \end{cases} \tag{1.2}$$

with the initial conditions

$$\mathscr{U} = g \text{ on } \Omega \times \{0\}, \ t = 0, \quad \partial_t \mathscr{U} = h \text{ on } \Omega \times \{0\}, \ t = 0, \tag{1.3}$$

where $\partial_L \mathcal{C} = \partial\Omega \times [0, \infty)$ corresponds to the lateral boundary of $\mathcal{C}$, $\alpha = 1 - 2s \in (-1, 1)$, $d_s = 2^\alpha \Gamma(1-s)/\Gamma(s)$, and the conormal exterior derivative of $\mathscr{U}$ at $\Omega \times \{0\}$ is

$$\partial_\nu^\alpha \mathscr{U} = -\lim_{y \to 0^+} y^\alpha \mathscr{U}_y;$$

the limit must be understood in the sense of distributions [10,13,57]. We will call y the *extended variable* and the dimension $n+1$ in $\mathbb{R}_+^{n+1}$ the *extended dimension* of problem (1.2), (1.3). Finally, $\mathbf{A} = \mathrm{diag}\{A, 1\} \in C^{0,1}(\mathcal{C}, \mathsf{GL}(n+1, \mathbb{R}))$. With the solution $\mathscr{U}$ to the extension problem (1.2), (1.3) at hand, we can find the solution to (1.1) via [3,9,10,13,18,19,60]:

$$u = \mathscr{U}|_{y=0}.$$

To the best of the authors knowledge this is the first work that provides a comprehensive treatment of efficient solution techniques for the space-fractional wave equation (1.1). In (1.1), $\mathcal{L}^s$ denotes the *spectral* fractional powers of the operator $\mathcal{L}$. Recently, problem (1.1) has been considered in [2] but with $\mathcal{L}^s = (-\Delta)^s$ being the *integral* fractional Laplace operator [39]. The authors of this work propose a discrete scheme that is based on standard Galerkin finite elements for space discretization and the convolution quadrature approach for the discretization in time. We immediately comment that the *spectral* and *integral* definitions of the fractional Laplace operator do not coincide. In fact, as shown in [45] their difference is positive and positivity preserving. This, in particular, implies that the boundary behavior of the respective solutions is quite different [11,32].

The outline of this paper is as follows. In Sect. 2 we introduce some terminology used throughout this work. We recall the definition of the fractional powers of elliptic operators via spectral theory in Sect. 2.1, and in Sect. 2.2 we briefly describe their localization via the Caffarelli–Silvestre extension and also introduce the functional framework that is suitable for studying problem (1.2), (1.3). In Sect. 3, we review existence and uniqueness results together with energy-estimates for problems (1.1) and (1.2), (1.3). In Sect. 4 we present space, time and space–time regularity results for the solution of problem (1.2), (1.3). The numerical analysis for problem (1.2), (1.3) begins in Sect. 5 where we introduce a truncated problem on the bounded cylinder $\mathcal{C}_{\mathscr{Y}} = \Omega \times (0, \mathscr{Y})$ and study some properties of its solution. In Sect. 6 we preset two fully discrete schemes for the truncated version of (1.2), (1.3) studied in Sect. 5. Both of them are based on the scheme of [5] for space discretization. For time discretization we consider an implicit finite difference discretization scheme and the so-called leapfrog scheme. We derive stability and a priori error estimates for the proposed schemes for all $s \in (0, 1)$. In Sect. 7, we extend the a priori error analysis of the trapezoidal scheme for open, bounded, polytopal but not necessarily convex domains $\Omega \subset \mathbb{R}^2$

on the basis of a continuous, piecewise linear, Lagrangian FEM in Ω with with mesh refinement near corners and a suitable hp-FEM in the extended variable. In Sect. 8 we comment on some implementation details pertinent to the problem at hand and present numerical experiments in one and two dimensional domains. An important conclusion is that due to the nonlocality of $(-\Delta)^s$, the explicit nature of the leapfrog scheme seems to be lost.

2 Notation and preliminaries

Throughout this work, with the exception of Sect. 7, Ω is an open, bounded, and convex polytopal subset of $\mathbb{R}^n$ ($n \geq 1$) with boundary $\partial\Omega$. In Sect. 7, instead, $\Omega \subset \mathbb{R}^2$ denotes an open and bounded, but not necessarily convex, polygon. We define the semi-infinite cylinder $\mathcal{C} := \Omega \times (0, \infty)$ and its lateral boundary $\partial_L \mathcal{C} := \partial\Omega \times [0, \infty)$. For $\mathcal{Y} > 0$, we define the truncated cylinder with base Ω and height $\mathcal{Y}$ as $\mathcal{C}_{\mathcal{Y}} := \Omega \times (0, \mathcal{Y})$; its lateral boundary is denoted by $\partial_L \mathcal{C}_{\mathcal{Y}} = \partial\Omega \times (0, \mathcal{Y})$. If $x \in \mathbb{R}^{n+1}$, we write $x = (x', y)$, with $x' \in \mathbb{R}^n$ and $y \in \mathbb{R}$.

For an open set $D \subset \mathbb{R}^n$ ($n \geq 1$), if ω is a weight and $p \in (1, \infty)$, we denote the Lebesgue space of p-integrable functions with respect to the measure $\omega \, dx$ by $L^p(\omega, D)$ [33,37,59]. Similar notation will be used for weighted Sobolev spaces. If $T > 0$ and $\phi : D \times (0, T) \to \mathbb{R}$, we consider ϕ as a function of t with values in a Banach space X, $\phi : (0, T) \ni t \mapsto \phi(t) \equiv \phi(\cdot, t) \in X$. For $1 \leq p \leq \infty$, $L^p(0, T; X)$ is the space of X-valued functions whose norm in X is in $L^p(0, T)$. This is a Banach space for the norm

$$\|\phi\|_{L^p(0,T;X)} = \left(\int_0^T \|\phi(t)\|_X^p \, dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$\|\phi\|_{L^\infty(0,T;X)} = \operatorname*{esssup}_{t \in (0,T)} \|\phi(t)\|_X.$$

Whenever X is a normed space, X' denotes its dual and $\|\cdot\|_X$ its norm. If, in addition, Y is a normed space, we write $X \hookrightarrow Y$ to indicate continuous embedding. The relation $a \lesssim b$ means $a \leq Cb$, with a constant C that neither depends on a or b. The value of C might change at each occurrence.

The next result, that follows from Young's inequality for convolutions, will be instrumental in the analysis that we will perform.

Lemma 1 (continuity) *If $g \in L^2(0, T)$ and $\phi \in L^1(0, T)$, then the operator*

$$g \mapsto \Phi, \quad \Phi(t) = \phi \star g(t) = \int_0^t \phi(t - r) g(r) \, dr$$

is continuous from $L^2(0, T)$ into itself and $\|\Phi\|_{L^2(0,T)} \leq \|\phi\|_{L^1(0,T)} \|g\|_{L^2(0,T)}$.

Finally, since we assume Ω to be convex, in what follows we will make use, without explicit mention, of the following regularity result [31]:

$$\|w\|_{H^2(\Omega)} \lesssim \|\mathcal{L}w\|_{L^2(\Omega)} \quad \forall w \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.1}$$

2.1 Fractional powers of second order elliptic operators

We adopt the *spectral* definition for the fractional powers of the operator $\mathcal{L}$. To define $\mathcal{L}^s$, we begin by noticing that $\mathcal{L}$ induces the following inner product $a_\Omega(\cdot, \cdot)$ on $H_0^1(\Omega)$

$$a_\Omega(w, v) = \int_\Omega (A\nabla_{x'}w \cdot \nabla_{x'}v + cwv) \, \mathrm{d}x', \tag{2.2}$$

and that $\mathcal{L}: H_0^1(\Omega) \ni u \mapsto a_\Omega(u, \cdot) \in H^{-1}(\Omega)$ is an isomorphism. The eigenvalue problem:

$$(\lambda, \varphi) \in \mathbb{R} \times H_0^1(\Omega)\backslash\{0\}: \quad a_\Omega(\varphi, v) = \lambda(\varphi, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega) \tag{2.3}$$

has a countable collection of solutions $\{\lambda_\ell, \varphi_\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{R}_+ \times H_0^1(\Omega)$ with the real eigenvalues enumerated in increasing order, counting multiplicities, and such that, $\{\varphi_\ell\}_{\ell=1}^\infty$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $(H_0^1(\Omega), a_\Omega(\cdot, \cdot))$ [6,35]. With these eigenpairs at hand, we introduce, for $s \geq 0$, the fractional Sobolev space

$$\mathbb{H}^s(\Omega) = \left\{ w = \sum_{\ell=1}^\infty w_\ell \varphi_\ell : \|w\|_{\mathbb{H}^s(\Omega)}^2 := \sum_{\ell=1}^\infty \lambda_\ell^s |w_\ell|^2 < \infty \right\}, \tag{2.4}$$

where, for $\ell \in \mathbb{N}$, $w_\ell = (w, \varphi_\ell)_{L^2(\Omega)}$. We denote by $\mathbb{H}^{-s}(\Omega)$ the dual space of $\mathbb{H}^s(\Omega)$. The duality pairing between the aforementioned spaces will be denoted by $\langle \cdot, \cdot \rangle$. We notice that, if $s \in (0, \frac{1}{2})$, $\mathbb{H}^s(\Omega) = H^s(\Omega) = H_0^s(\Omega)$, while, for $s \in (\frac{1}{2}, 1)$, $\mathbb{H}^s(\Omega)$ can be characterized by [41,42,58]

$$\mathbb{H}^s(\Omega) = \left\{ w \in H^s(\Omega) : w = 0 \text{ on } \partial\Omega \right\}.$$

If $s = \frac{1}{2}$, we have that $\mathbb{H}^{\frac{1}{2}}(\Omega)$ is the so-called Lions–Magenes space $H_{00}^{\frac{1}{2}}(\Omega)$ [41,58]. If $s \in (1, 2]$, owing to (2.1), we have that $\mathbb{H}^s(\Omega) = H^s(\Omega) \cap H_0^1(\Omega)$ [25].

The fractional powers of the operator $\mathcal{L}$ are thus defined by

$$\mathcal{L}^s : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega), \quad \mathcal{L}^s w := \sum_{\ell=1}^\infty \lambda_\ell^s w_\ell \varphi_\ell, \quad s \in (0, 1).$$

2.2 Weighted Sobolev spaces

Both extensions, the one by Caffarelli and Silvestre [10] and the ones in [9,13,57] for bounded domains Ω and general elliptic operators, require us to deal with a local but

nonuniformly elliptic problem. To provide an analysis for the latter it is thus suitable
to define the weighted Sobolev space

$$\mathring{H}_L^1(y^\alpha, \mathcal{C}) = \left\{ w \in H^1(y^\alpha, \mathcal{C}) : w = 0 \text{ on } \partial_L \mathcal{C} \right\}. \tag{2.5}$$

Since $\alpha \in (-1, 1)$, $|y|^\alpha$ belongs to the Muckenhoupt class A_2 [22,43,59]. The follow-
ing important consequences thus follow immediately: $H^1(y^\alpha, \mathcal{C})$ is a Hilbert space
and $C^\infty(\Omega) \cap H^1(y^\alpha, \mathcal{C})$ is dense in $H^1(y^\alpha, \mathcal{C})$ (cf. [59, Proposition 2.1.2, Corol-
lary 2.1.6], [38] and [30, Theorem 1]). In addition, as [46, inequality (2.21)] shows,
the following *weighted Poincaré inequality* holds:

$$\|w\|_{L^2(y^\alpha, \mathcal{C})} \lesssim \|\nabla w\|_{L^2(y^\alpha, \mathcal{C})} \quad \forall w \in \mathring{H}_L^1(y^\alpha, \mathcal{C}). \tag{2.6}$$

Thus, $\|\nabla w\|_{L^2(y^\alpha, \mathcal{C})}$ is equivalent to the norm in $\mathring{H}_L^1(y^\alpha, \mathcal{C})$.

We define the bilinear form $a : \mathring{H}_L^1(y^\alpha, \mathcal{C}) \times \mathring{H}_L^1(y^\alpha, \mathcal{C}) \to \mathbb{R}$ by

$$a(w, \phi) := \frac{1}{d_s} \int_{\mathcal{C}} y^\alpha \left(\mathbf{A}(x) \nabla w \cdot \nabla \phi + c(x') w \phi \right) \, \mathrm{d}x, \tag{2.7}$$

which is continuous and, owing to (2.6), coercive on $\mathring{H}_L^1(y^\alpha, \mathcal{C})$. Consequently, it
induces an inner product on $\mathring{H}_L^1(y^\alpha, \mathcal{C})$ and the following energy norm:

$$\|w\|_{\mathcal{C}}^2 := a(w, w) \sim \|\nabla w\|_{L^2(y^\alpha, \mathcal{C})}^2. \tag{2.8}$$

For $w \in H^1(y^\alpha, \mathcal{C})$, tr w denotes its trace onto $\Omega \times \{0\}$. We recall that, for $\alpha = 1 - 2s$, [46, Proposition 2.5] yields

$$\text{tr } \mathring{H}_L^1(y^\alpha, \mathcal{C}) = \mathbb{H}^s(\Omega), \qquad \| \text{tr } w\|_{\mathbb{H}^s(\Omega)} \le C_{\text{tr}} \|w\|_{\mathcal{C}}. \tag{2.9}$$

The seminal work of Caffarelli and Silvestre [10] and its extensions to bounded
domains [9,13,57] showed that the operator $\mathcal{L}^s$ can be realized as the Dirichlet-to-
Neumann map for a nonuniformly elliptic boundary value problem. Namely, if $\mathfrak{U}$
solves

$$\begin{cases} -\text{div} \left(y^\alpha \mathbf{A} \nabla \mathfrak{U} \right) + c y^\alpha \mathfrak{U} = 0 & \text{in } \mathcal{C}, \\ \mathfrak{U} = 0 & \text{on } \partial_L \mathcal{C}, \\ \partial_\nu^\alpha \mathfrak{U} = d_s f & \text{on } \Omega \times \{0\}, \end{cases} \tag{2.10}$$

where $\alpha = 1 - 2s$, $\partial_\nu^\alpha \mathfrak{U} = -\lim_{y \downarrow 0} y^\alpha \mathfrak{U}_y$ and $d_s = 2^\alpha \Gamma(1 - s)/\Gamma(s)$ is a normal-
ization constant, then $\mathfrak{u} = \text{tr } \mathfrak{U} \in \mathbb{H}^s(\Omega)$ solves

$$\mathcal{L}^s \mathfrak{u} = f. \tag{2.11}$$

3 Well-posedness and energy estimates

In this section we briefly review the results of [49] regarding the existence and uniqueness of weak solutions for problems (1.1) and (1.2), (1.3). We also provide basic energy estimates.

3.1 The fractional wave equation

We assume that the data of problem (1.1) is such that $f \in L^2(0, T; L^2(\Omega))$, $g \in \mathbb{H}^s(\Omega)$, and $h \in L^2(\Omega)$ and define

$$\Lambda(f, g, h) := \|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{\mathbb{H}^s(\Omega)} + \|h\|_{L^2(\Omega)}. \tag{3.1}$$

Definition 1 (*weak solution for* (1.1)) We call $u \in L^2(0, T; \mathbb{H}^s(\Omega))$, with $\partial_t u \in L^2(0, T; L^2(\Omega))$ and $\partial_t^2 u \in L^2(0, T; \mathbb{H}^{-s}(\Omega))$, a weak solution of problem (1.1) if $u(0) = g$, $\partial_t u(0) = h$ and, a.e. $t \in (0, T)$,

$$\langle \partial_t^2 u, v \rangle + \langle \mathcal{L}^s u, v \rangle = \langle f, v \rangle \quad \forall v \in \mathbb{H}^s(\Omega),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathbb{H}^s(\Omega)$ and $\mathbb{H}^{-s}(\Omega)$.

The following remark is in order.

Remark 1 (*initial conditions*) Since a weak solution u of (1.1) satisfies that $u \in L^2(0, T; \mathbb{H}^s(\Omega))$, $\partial_t u \in L^2(0, T; L^2(\Omega))$, and $\partial_t^2 u \in L^2(0, T; \mathbb{H}^{-s}(\Omega))$, an application of [52, Lemma 7.3] reveals that $u \in C([0, T]; L^2(\Omega))$ and $\partial_t u \in C([0, T]; \mathbb{H}^{-s}(\Omega))$. The initial conditions involved in Definition 1 are thus appropriately defined.

Theorem 1 (well-posedness) *Given $s \in (0, 1)$, $f \in L^2(0, T; L^2(\Omega))$, $g \in \mathbb{H}^s(\Omega)$, and $h \in L^2(\Omega)$, problem (1.1) has a unique weak solution. In addition,*

$$\|u\|_{L^\infty(0,T;\mathbb{H}^s(\Omega))} + \|\partial_t u\|_{L^\infty(0,T;L^2(\Omega))} \lesssim \Lambda(f, g, h), \tag{3.2}$$

where the hidden constant is independent of the problem data.

Proof The desired results can be obtained by slightly modifying the arguments, based on a Galerkin technique, of [24,41,52]. □

3.2 The extended fractional wave equation

We consider the following notion of weak solution for problem (1.2), (1.3).

Definition 2 (*extended weak solution*) We call $\mathscr{U} \in L^\infty(0, T; \mathring{H}_L^1(y^\alpha, \mathcal{C}))$, with $\operatorname{tr} \partial_t \mathscr{U} \in L^\infty(0, T; L^2(\Omega))$ and $\operatorname{tr} \partial_t^2 \mathscr{U} \in L^2(0, T; \mathbb{H}^{-s}(\Omega))$, a weak solution of problem (1.2), (1.3) if $\operatorname{tr} \mathscr{U}(0) = g$, $\operatorname{tr} \partial_t \mathscr{U}(0) = h$ and, for a.e. $t \in (0, T)$,

$$\langle \operatorname{tr} \partial_t^2 \mathscr{U}, \operatorname{tr} \phi \rangle + a(\mathscr{U}, \phi) = \langle f, \operatorname{tr} \phi \rangle \quad \forall \phi \in \mathring{H}_L^1(y^\alpha, \mathcal{C}), \tag{3.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathbb{H}^s(\Omega)$ and $\mathbb{H}^{-s}(\Omega)$ and the bilinear form a is defined as in (2.7).

Remark 2 (*dynamic boundary condition*) Problem (3.3) is an elliptic problem with the following dynamic boundary condition: $\partial_\nu^\alpha \mathscr{U} = d_s(f - \operatorname{tr} \partial_t^2 \mathscr{U})$ on $\Omega \times \{0\}$.

We present the following important localization result [3,7,9,10,13,18,19,60].

Theorem 2 (Caffarelli–Silvestre extension property) *Let $s \in (0, 1)$. If f, g, and h are as in Theorem 1, then the unique weak solution of problem* (1.1)*, in the sense of Definition 1 satisfies that $u = \operatorname{tr} \mathscr{U}$, where $\mathscr{U}$ denotes the unique weak solution to problem* (1.2)*,* (1.3) *in the sense of Definition 2.*

We now present the well-posedness of problem (3.3) together with energy estimates for its solution.

Theorem 3 (well-posedness) *Given $s \in (0, 1)$, $f \in L^2(0, T; L^2(\Omega))$, $g \in \mathbb{H}^s(\Omega)$ and $h \in L^2(\Omega)$, then problem* (1.2)*,* (1.3) *has a unique weak solution in the sense of Definition 2. In addition,*

$$\|\nabla \mathscr{U}\|_{L^\infty(0,T;L^2(y^\alpha,\mathcal{C}))} + \|\operatorname{tr}\partial_t \mathscr{U}\|_{L^\infty(0,T;L^2(\Omega))} \lesssim \Lambda(f, g, h), \qquad (3.4)$$

where the hidden constant is independent of the problem data and $\Lambda(f, g, h)$ is defined as in (3.1)*.*

Proof See [49, Theorem 3.11]. □

Remark 3 (*initial data*) The initial data g and h of problem (3.3) determine $\mathscr{U}(0)$ and $\partial_t \mathscr{U}(0)$ only on $\Omega \times \{0\}$ in a trace sense. However, in the analysis that follows it will be necessary to consider their extension to the whole cylinder $\mathcal{C}$. We thus define $\mathscr{U}(0) = \mathcal{E}_\alpha g$ and $\mathscr{U}_t(0) = \mathcal{E}_\alpha h$, where the α-harmonic extension operator

$$\mathcal{E}_\alpha : \mathbb{H}^s(\Omega) \to \mathring{H}_L^1(y^\alpha, \mathcal{C}) \qquad (3.5)$$

is defined as follows: If $w \in \mathbb{H}^s(\Omega)$, then $\mathscr{W} = \mathcal{E}_\alpha w \in \mathring{H}_L^1(y^\alpha, \mathcal{C})$ solves

$$\begin{cases} -\operatorname{div}(y^\alpha \mathbf{A} \nabla \mathscr{W}) + y^\alpha c \mathscr{W} = 0 & \text{in } \mathcal{C}, \\ \mathscr{W} = 0 & \text{on } \partial_L \mathcal{C}, \\ \mathscr{W} = w & \text{on } \Omega \times \{0\}. \end{cases} \qquad (3.6)$$

References [9,13] provide, for $w \in \mathbb{H}^s(\Omega)$, the estimate $\|\nabla \mathcal{E}_\alpha w\|_{L^2(y^\alpha,\mathcal{C})} \lesssim \|w\|_{\mathbb{H}^s(\Omega)}$.

3.3 Solution representation

In this section we present a solution representation formula for the solution to problem (3.3). To accomplish this task, we first notice that the solution to problem (1.1) can

be written as $u(x', t) = \sum_{k \in \mathbb{N}} u_k(t)\varphi_k(x')$, where, for $k \in \mathbb{N}$, the coefficient $u_k(t)$ solves

$$\partial_t^2 u_k(t) + \lambda_k^s u_k(t) = f_k(t), \quad t > 0, \quad u_k(0) = g_k, \quad \partial_t u_k(0) = h_k, \qquad (3.7)$$

with $g_k = (g, \varphi_k)_{L^2(\Omega)}, h_k = (h, \varphi_k)_{L^2(\Omega)}$, and $f_k(t) = (f(\cdot, t), \varphi_k)_{L^2(\Omega)}$. We recall that the sequence $\{\lambda_k, \varphi_k\}_{k \in \mathbb{N}}$ corresponds to the eigenpairs of the operator $\mathcal{L}$ and are defined by (2.3). Basic computations reveal, for $k \in \mathbb{N}$, that

$$u_k(t) = g_k \cos\left(\lambda_k^{s/2} t\right) + h_k \lambda_k^{-\frac{s}{2}} \sin\left(\lambda_k^{s/2} t\right)$$
$$+ \lambda_k^{-\frac{s}{2}} \int_0^t f_k(r) \sin\left(\lambda_k^{s/2}(t - r)\right) dr. \qquad (3.8)$$

With these ingredients at hand, we can write the solution $\mathcal{U}$ of problem (3.3) as

$$\mathcal{U}(x, t) = \sum_{k \in \mathbb{N}} u_k(t)\varphi_k(x')\psi_k(y), \qquad (3.9)$$

where, for $\alpha = 1 - 2s$, the functions ψ_k solve

$$\begin{cases} \dfrac{d^2}{dy^2}\psi_k(y) + \dfrac{\alpha}{y}\dfrac{d}{dy}\psi_k(y) = \lambda_k \psi_k(y), & y \in (0, \infty), \\ \psi_k(0) = 1, & \lim_{y \to \infty} \psi_k(y) = 0. \end{cases} \qquad (3.10)$$

If $s = \frac{1}{2}$, we thus have $\psi_k(y) = \exp(-\sqrt{\lambda_k}y)$ [9, Lemma 2.10]. If $s \in (0, 1)\backslash\{\frac{1}{2}\}$, then [13, Proposition 2.1]

$$\psi_k(y) = c_s(\sqrt{\lambda_k}y)^s K_s(\sqrt{\lambda_k}y), \qquad (3.11)$$

where $c_s = 2^{1-s}/\Gamma(s)$ and K_s denotes the modified Bessel function of the second kind. We refer the reader to [1, Chapter 9.6] and [48, Chapter 7.8] for a comprehensive treatment of the Bessel function K_s. We immediately state the following property of the function ψ_k:

$$\lim_{s \to \frac{1}{2}} \psi_k(y) = \exp(-\sqrt{\lambda_k}y) \quad \forall y > 0.$$

In addition, for $a, b \in \mathbb{R}^+, a < b$, we have [46, formula (2.33)]

$$\int_a^b y^\alpha \left(\lambda_k \psi_k(y)^2 + \psi_k'(y)^2\right) dy = y^\alpha \psi_k(y)\psi_k'(y)|_a^b, \qquad (3.12)$$

[46, formula (2.32)]

$$|y^\alpha \psi_k(y)\psi_k'(y)| \lesssim \lambda_k^s e^{-\sqrt{\lambda_k}y}, \quad y \geq 1, \qquad (3.13)$$

and [46, formula (2.31)]

$$\lim_{y\downarrow 0^+} \frac{y^\alpha \psi_k'(y)}{d_s \lambda_k^s} = -1. \tag{3.14}$$

4 Regularity

In this section we review and derive space, time, and space–time regularity results for the solution $\mathscr{U}$ of problem (3.3).

4.1 Space regularity

To present the space regularity properties of $\mathscr{U}$, we introduce the weight

$$\omega_{\beta,\theta}(y) = y^\beta e^{\theta y}, \qquad 0 \le \theta < 2\sqrt{\lambda_1}, \tag{4.1}$$

where $\beta \in \mathbb{R}$ will be specified later. With this weight at hand, we define the norm

$$\|v\|_{L^2(\omega_{\beta,\theta},\mathcal{C})} := \left(\int_0^\infty \int_\Omega \omega_{\beta,\theta}(y) |v(x',y)|^2 \, \mathrm{d}x' \, \mathrm{d}y \right)^{\frac{1}{2}}. \tag{4.2}$$

We now present the following pointwise, in time, bounds for $\mathscr{U}$ [49, Theorem 4.2].

Proposition 1 (pointwise bounds) *Let $\mathscr{U}$ solve problem (1.2), (1.3) for $s \in (0,1)$. Let $0 \le \sigma < s$ and $0 \le v < 1+s$. Then, there exists $\kappa > 1$ such that the following estimates hold for all $\ell \in \mathbb{N}_0$:*

$$\|\partial_y^{\ell+1} \mathscr{U}(\cdot,t)\|_{L^2(\omega_{\alpha+2\ell-2\sigma,\theta},\mathcal{C})}^2 \lesssim \kappa^{2(\ell+1)} (\ell+1)!^2 \|u(\cdot,t)\|_{\mathbb{H}^{\sigma+s}(\Omega)}^2, \tag{4.3}$$

$$\|\nabla_{x'} \partial_y^{\ell+1} \mathscr{U}(\cdot,t)\|_{L^2(\omega_{\alpha+2(\ell+1)-2v,\theta},\mathcal{C})}^2 \lesssim \kappa^{2(\ell+1)} (\ell+1)!^2 \|u(\cdot,t)\|_{\mathbb{H}^{v+s}(\Omega)}^2, \tag{4.4}$$

$$\|\mathcal{L}\partial_y^{\ell+1} \mathscr{U}(\cdot,t)\|_{L^2(\omega_{\alpha+2(\ell+1)-2v,\theta},\mathcal{C})}^2 \lesssim \kappa^{2(\ell+1)} (\ell+1)!^2 \|u(\cdot,t)\|_{\mathbb{H}^{1+v+s}(\Omega)}^2. \tag{4.5}$$

In all inequalities the hidden constants are independent of $\mathscr{U}$, ℓ, and problem data.

The result below shows the spatial analyticity of the solution $\mathscr{U}$ with respect to the extended variable $y \in (0,\infty)$: $\mathscr{U}$ belongs to countably normed, power-exponentially weighted Bochner spaces of analytic functions with respect to y taking values in spaces $\mathbb{H}^r(\Omega)$.

Proposition 2 (space regularity) *Let $\mathscr{U}$ solve (1.2), (1.3) for $s \in (0,1)$. Let $0 \le \sigma < s$ and $0 \le v < 1+s$. Then, there exists $\kappa > 1$ such that the following regularity estimates hold for all $\ell \in \mathbb{N}_0$:*

$$\|\partial_y^{\ell+1}\mathscr{U}\|_{L^2(0,T;L^2(\omega_{\alpha+2\ell-2\sigma,\theta},\mathcal{C}))}^2 \lesssim \kappa^{2(\ell+1)}(\ell+1)!^2\big(\|g\|_{\mathbb{H}^{\sigma+s}(\Omega)}^2$$
$$+\|h\|_{\mathbb{H}^\sigma(\Omega)}^2 + \|f\|_{L^2(0,T;\mathbb{H}^\sigma(\Omega))}^2\big), \tag{4.6}$$
$$\|\nabla_{x'}\partial_y^{\ell+1}\mathscr{U}\|_{L^2(0,T;L^2(\omega_{\alpha+2(\ell+1)-2\nu,\theta},\mathcal{C}))}^2 \lesssim \kappa^{2(\ell+1)}(\ell+1)!^2\big(\|g\|_{\mathbb{H}^{\nu+s}(\Omega)}^2$$
$$+\|h\|_{\mathbb{H}^\nu(\Omega)}^2 + \|f\|_{L^2(0,T;\mathbb{H}^\nu(\Omega))}^2\big). \tag{4.7}$$

and

$$\|\mathcal{L}\partial_y^{\ell+1}\mathscr{U}\|_{L^2(0,T;L^2(\omega_{\alpha+2(\ell+1)-2\nu,\theta},\mathcal{C}))}^2 \lesssim \kappa^{2(\ell+1)}(\ell+1)!^2\big(\|g\|_{\mathbb{H}^{1+\nu+s}(\Omega)}^2$$
$$+\|h\|_{\mathbb{H}^{1+\nu}(\Omega)}^2 + \|f\|_{L^2(0,T;\mathbb{H}^{1+\nu}(\Omega))}^2\big). \tag{4.8}$$

The hidden constants are independent of $\mathscr{U}$, ℓ, and the problem data.

Proof In view of (3.8) and the continuity estimate of Lemma 1, we conclude, for $k \in \mathbb{N}$, that

$$\|u_k\|_{L^2(0,T)}^2 \lesssim Tg_k^2 + T\lambda_k^{-s}h_k^2 + T^2\lambda_k^{-s}\|f_k\|_{L^2(0,T)}^2.$$

The desired estimates (4.6)–(4.8) thus follow directly from (4.3)–(4.5). $\qquad\square$

4.2 Time regularity

We begin this section by defining, for $\ell \in \{1, \dots, 4\}$,

$$\Sigma_\ell(f, g, h) := \|g\|_{\mathbb{H}^{(\ell+1)s}(\Omega)} + \|h\|_{\mathbb{H}^{\ell s}(\Omega)} + \|f\|_{L^2(0,T;\mathbb{H}^{\ell s}(\Omega))}. \tag{4.9}$$

In addition, and to shorten notation, we define

$$\Xi(f, g, h) := \Sigma_4(f, g, h) + \|\partial_t^2 f\|_{L^2(0,T;\mathbb{H}^s(\Omega))}. \tag{4.10}$$

We now derive regularity estimates in time for the solution $\mathscr{U}$. These estimates will be needed in the analysis of the fully discrete schemes proposed in Sect. 6.

Theorem 4 (time-regularity) *Let $\mathscr{U}$ be the solution to problem* (1.2), (1.3) *for* $s \in (0, 1)$. *The following regularity estimates in time hold:*

$$\|\partial_t \nabla \mathscr{U}\|_{L^\infty(0,T;L^2(y^\alpha,\mathcal{C}))} \lesssim \Sigma_1(f, g, h), \tag{4.11}$$
$$\|\partial_t^2 \nabla \mathscr{U}\|_{L^\infty(0,T;L^2(y^\alpha,\mathcal{C}))} \lesssim \Sigma_2(f, g, h), \tag{4.12}$$
$$\|\partial_t^3 \nabla \mathscr{U}\|_{L^\infty(0,T;L^2(y^\alpha,\mathcal{C}))} \lesssim \Sigma_3(f, g, h) + \|\partial_t f\|_{L^2(0,T;\mathbb{H}^s(\Omega))}, \tag{4.13}$$
$$\|\partial_t^4 \nabla \mathscr{U}\|_{L^\infty(0,T;L^2(y^\alpha,\mathcal{C}))} \lesssim \Xi(f, g, h). \tag{4.14}$$

In all these inequalities the hidden constants do not depend either on $\mathscr{U}$ or the problem data.

Proof Since $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $(H_0^1(\Omega), a_\Omega(\cdot,\cdot))$, the definition of the energy norm $\|\cdot\|_\mathcal{C}$, given in (2.8), and the properties (3.12) and (3.14) allow us to conclude, for $\ell \in \mathbb{N}_0$, that

$$\|\partial_t^\ell \nabla \mathscr{U}(\cdot,t)\|_\mathcal{C}^2 = d_s^{-1} \int_\mathcal{C} y^\alpha \left[\mathbf{A}\nabla \partial_t^\ell \mathscr{U}(\cdot,t) \cdot \nabla \partial_t^\ell \mathscr{U}(\cdot,t) + c(\partial_t^\ell \mathscr{U}(\cdot,t))^2 \right] \mathrm{d}x$$
$$= d_s^{-1} \sum_{k\in\mathbb{N}} (\partial_t^\ell u_k(t))^2 \int_0^\infty y^\alpha \left[\lambda_k \psi_k(y)^2 + \psi_k'(y)^2 \right] \mathrm{d}y = \sum_{k\in\mathbb{N}} (\partial_t^\ell u_k(t))^2 \lambda_k^s.$$

We have thus arrived at the estimate $\|\partial_t^\ell \nabla \mathscr{U}(\cdot,t)\|_\mathcal{C}^2 = \|\partial_t^\ell u(\cdot,t)\|_{\mathbb{H}^s(\Omega)}^2$.

We now invoke the explicit representation of the coefficient $u_k(t)$, with $k \in \mathbb{N}$, which is provided in (3.8), to obtain

$$\partial_t u_k(t) = -g_k \lambda_k^{s/2} \sin\left(\lambda_k^{s/2} t\right) + h_k \cos\left(\lambda_k^{s/2} t\right) + \int_0^t f_k(r) \cos\left(\lambda_k^{s/2}(t-r)\right) \mathrm{d}r.$$

This, on the basis of the definition of the norm $\|\cdot\|_{\mathbb{H}^r(\Omega)}$, given in (2.4), and an application of Lemma 1, reveal that

$$\|\partial_t \nabla \mathscr{U}(\cdot,t)\|_\mathcal{C}^2 \lesssim \sum_{k\in\mathbb{N}} (\partial_t u_k(t))^2 \lambda_k^s$$
$$\lesssim \|g\|_{\mathbb{H}^{2s}(\Omega)}^2 + \|h\|_{\mathbb{H}^s(\Omega)}^2 + \sum_{k\in\mathbb{N}} \lambda_k^s \|f_k\|_{L^2(0,T)}^2,$$

which implies the desired estimate (4.11).

To derive (4.12) we invoke, again, the representation formula (3.8) and write

$$\partial_t^2 u_k(t) = -g_k \lambda_k^s \cos\left(\lambda_k^{s/2} t\right) - h_k \lambda_k^{s/2} \sin\left(\lambda_k^{s/2} t\right)$$
$$+ f_k(t) - \lambda_k^{s/2} \int_0^t f_k(r) \sin\left(\lambda_k^{s/2}(t-r)\right) \mathrm{d}r. \qquad (4.15)$$

We thus use the definition of the norm $\|\cdot\|_{\mathbb{H}^r(\Omega)}$ to arrive at (4.12).

The estimates (4.13) and (4.14) follow similar arguments upon taking derivatives to the explicit representation of the coefficient $u_k(t)$, with $k \in \mathbb{N}$, provided in (3.8). This concludes the proof. $\qquad\qquad\square$

4.3 Space-time regularity

We present the following regularity result in space and time.

Theorem 5 (space–time regularity) *Let $\mathscr{U}$ solve* (1.2), (1.3) *for $s \in (0,1)$. Let $0 \leq \sigma < s$ and $0 \leq \nu < 1 + s$. Then, there exists $\kappa > 1$ such that the following regularity*

estimates hold for all $\ell \in \mathbb{N}_0$:

$$\|\partial_t^2 \partial_y^{\ell+1} \mathscr{U}\|_{L^2(0,T;L^2(\omega_{\alpha+2\ell-2\sigma,\theta},\mathcal{C}))}^2 \lesssim \kappa^{2(\ell+1)}(\ell+1)!^2 \big(\|g\|_{\mathbb{H}^{\sigma+3s}(\Omega)}^2$$
$$+\|h\|_{\mathbb{H}^{\sigma+2s}(\Omega)}^2 + \|f\|_{L^2(0,T;\mathbb{H}^{\sigma+2s}(\Omega))}^2 \big), \tag{4.16}$$

$$\|\partial_t^2 \nabla_{x'} \partial_y^{\ell+1} \mathscr{U}\|_{L^2(0,T;L^2(\omega_{\alpha+2(\ell+1)-2\nu,\theta},\mathcal{C}))}^2 \lesssim \kappa^{2(\ell+1)}(\ell+1)!^2 \big(\|g\|_{\mathbb{H}^{\nu+3s}(\Omega)}^2$$
$$+\|h\|_{\mathbb{H}^{\nu+2s}(\Omega)}^2 + \|f\|_{L^2(0,T;\mathbb{H}^{\nu+2s}(\Omega))}^2 \big), \tag{4.17}$$

and

$$\|\partial_t^2 \mathcal{L} \partial_y^{\ell+1} \mathscr{U}\|_{L^2(0,T;L^2(\omega_{\alpha+2(\ell+1)-2\nu,\theta},\mathcal{C}))}^2 \lesssim \kappa^{2(\ell+1)}(\ell+1)!^2 \big(\|g\|_{\mathbb{H}^{1+\nu+3s}(\Omega)}^2$$
$$+\|h\|_{\mathbb{H}^{1+\nu+2s}(\Omega)}^2 + \|f\|_{L^2(0,T;\mathbb{H}^{1+\nu+2s}(\Omega))}^2 \big). \tag{4.18}$$

The hidden constants do not depend either on $\mathscr{U}$ or the problem data.

Proof Similar arguments to the ones used to derive (4.3) reveal that

$$\|\partial_t^2 \partial_y^{\ell+1} \mathscr{U}(\cdot,t)\|_{L^2(\omega_{\alpha+2\ell-2\sigma,\theta},\mathcal{C})}^2 \lesssim \kappa^{2(\ell+1)}(\ell+1)!^2 \|\partial_t^2 u(\cdot,t)\|_{\mathbb{H}^{\sigma+s}(\Omega)}^2.$$

To control the right-hand side of the previous inequality we invoke formula (4.15) and the definition of the $\mathbb{H}^r(\Omega)$-norm, given in (2.4). These arguments reveal the estimate

$$\|\partial_t^2 u(\cdot,t)\|_{\mathbb{H}^{\sigma+s}(\Omega)}^2 \lesssim \|g\|_{\mathbb{H}^{\sigma+3s}(\Omega)}^2 + \|h\|_{\mathbb{H}^{\sigma+2s}(\Omega)}^2$$
$$+ \sum_{k\in\mathbb{N}} \lambda_k^{\sigma+s} \left(f_k^2(t) + \lambda_k^s \|f_k\|_{L^2(0,T)}^2 \right). \tag{4.19}$$

This yields (4.16). Similar arguments allow us to derive the regularity estimates (4.17) and (4.18). □

As it will be used in the analysis that follows, we introduce

$$\mathfrak{A}(f,g,h) = \|g\|_{\mathbb{H}^{1+3s}(\Omega)} + \|h\|_{\mathbb{H}^{1+2s}(\Omega)} + \|f\|_{L^2(0,T;\mathbb{H}^{1+2s}(\Omega))}, \tag{4.20}$$

and notice that, if $0 \le \sigma < s$, then

$$\|\partial_t^2 \partial_y^{\ell+1} \mathscr{U}\|_{L^2(0,T;L^2(\omega_{\alpha+2\ell-2\sigma,\theta},\mathcal{C}))}^2 + \|\partial_t^2 \nabla_{x'} \partial_y^{\ell+1} \mathscr{U}\|_{L^2(0,T;L^2(\omega_{\alpha+2(\ell+1),\theta},\mathcal{C}))}^2$$
$$+\|\partial_t^2 \mathcal{L} \partial_y^{\ell+1} \mathscr{U}\|_{L^2(0,T;L^2(\omega_{\alpha+2(\ell+1),\theta},\mathcal{C}))}^2 \lesssim \kappa^{2(\ell+1)}(\ell+1)!^2 \mathfrak{A}(f,g,h)^2. \tag{4.21}$$

As a consequence of the estimate (4.19) and the previous definition, we can immediately arrive at the following regularity estimate.

Corollary 1 (space–time regularity) *Let u solve* (1.1) *for $s \in (0,1)$. Then*

$$\|\partial_t^2 u(t)\|_{\mathbb{H}^{1+s}(\Omega)} \lesssim \mathfrak{A}(f,g,h). \tag{4.22}$$

The hidden constant does not depend either on u or the problem data.

5 Truncation

A first step towards space-discretization is to truncate the semi-infinite cylinder $\mathcal{C}$. In the next result we show that the solution $\mathscr{U}$ to problem (1.2), (1.3) decays exponentially in the extended variable y for a.e. $t \in (0, T)$. This suggests to truncate $\mathcal{C}$ to $\mathcal{C}_{\mathcal{Y}} = \Omega \times (0, \mathcal{Y})$, with a suitable truncation parameter $\mathcal{Y}$, and seek solutions in this bounded domain.

Proposition 3 (exponential decay) *Let $s \in (0, 1)$ and $\mathscr{U}$ be the solution to (3.3). Then, for every $\mathcal{Y} \geq 1$, we have that*

$$\|\nabla \mathscr{U}\|_{L^2(0,T;L^2(y^\alpha, \Omega \times (\mathcal{Y}, \infty)))} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2} \Lambda(f, g, h), \tag{5.1}$$

where λ_1 denotes the first eigenvalue of $\mathcal{L}$ and $\Lambda(f, g, h)$ is defined in (3.1).

Proof We invoke (3.9) and the fact that $\{\varphi_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $(H_0^1(\Omega), a_\Omega(\cdot, \cdot))$ to conclude that

$$\int_0^T \int_{\mathcal{C} \setminus \mathcal{C}_{\mathcal{Y}}} y^\alpha \left(\mathbf{A} \nabla \mathscr{U} \cdot \nabla \mathscr{U} + c \mathscr{U}^2 \right) \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^T \sum_{k \in \mathbb{N}} u_k^2(t) \int_{\mathcal{Y}}^\infty y^\alpha \left(\lambda_k \psi_k^2 + \psi_k'(y)^2 \right) \, \mathrm{d}y \, \mathrm{d}t.$$

We now apply formulas (3.12) and (3.13) to obtain that

$$\int_0^T \int_{\mathcal{C} \setminus \mathcal{C}_{\mathcal{Y}}} y^\alpha \left(\mathbf{A} \nabla \mathscr{U} \cdot \nabla \mathscr{U} + c \mathscr{U}^2 \right) \, \mathrm{d}x \, \mathrm{d}t$$
$$= \sum_{k \in \mathbb{N}} |\mathcal{Y}^\alpha \psi_k(\mathcal{Y}) \psi_k'(\mathcal{Y})| \int_0^T u_k^2(t) \, \mathrm{d}t$$
$$\lesssim \sum_{k \in \mathbb{N}} e^{-\sqrt{\lambda_k}\mathcal{Y}} \lambda_k^s \|u_k\|_{L^2(0,T)}^2 \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}} \|u\|_{L^2(0,T;\mathbb{H}^s(\Omega))}^2.$$

The desired estimate (5.1) is thus a consequence of the energy estimate (3.2) for the solution u to problem (1.1). $\qquad\square$

To describe the truncated version of (3.3), we define the weighted Sobolev space

$$\mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathcal{Y}}) = \left\{ w \in H^1(y^\alpha, \mathcal{C}_{\mathcal{Y}}) : w = 0 \text{ on } \partial_L \mathcal{C}_{\mathcal{Y}} \cup \Omega_{\mathcal{Y}} \right\},$$

and the bilinear form $a_\mathcal{Y} : \mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y}) \times \mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y})$ as

$$a_\mathcal{Y}(w, \phi) = \frac{1}{d_s} \int_{\mathcal{C}_\mathcal{Y}} y^\alpha \left(\mathbf{A}(x)\nabla w \cdot \nabla \phi + c(x')w\phi\right) \, \mathrm{d}x, \tag{5.2}$$

where $\mathcal{C}_\mathcal{Y} = \Omega \times (0, \mathcal{Y})$ and $\Omega_\mathcal{Y} = \Omega \times \{\mathcal{Y}\}$.

On the basis of the results of Proposition 3, we thus consider the following truncated problem: Find $\mathcal{U} \in L^\infty(0, T; \mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y}))$ with $\operatorname{tr} \partial_t \mathcal{U} \in L^\infty(0, T; L^2(\Omega))$ and $\operatorname{tr} \partial_t^2 \mathcal{U} \in L^2(0, T; \mathbb{H}^{-s}(\Omega))$ such that $\operatorname{tr} \mathcal{U}(0) = g$, $\operatorname{tr} \partial_t \mathcal{U}(0) = h$, and, for a.e. $t \in (0, T)$,

$$\langle \operatorname{tr} \partial_t^2 \mathcal{U}, \operatorname{tr} \phi \rangle + a_\mathcal{Y}(\mathcal{U}, \phi) = \langle f, \operatorname{tr} \phi \rangle \qquad \forall \phi \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y}). \tag{5.3}$$

We define $\mathcal{H}_\alpha : \mathbb{H}^s(\Omega) \to \mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y})$, the truncated α-harmonic extension operator, as follows: if $w \in \mathbb{H}^s(\Omega)$, then $\mathcal{W} = \mathcal{H}_\alpha w$ solves

$$\begin{cases} -\operatorname{div}(y^\alpha \mathbf{A} \nabla \mathcal{W}) + y^\alpha c \mathcal{W} = 0 & \text{in } \mathcal{C}_\mathcal{Y}, \\ \mathcal{W} = 0 & \text{on } \partial_L \mathcal{C}_\mathcal{Y} \cup \Omega_\mathcal{Y}, \\ \mathcal{W} = w & \text{on } \Omega \times \{0\}. \end{cases} \tag{5.4}$$

Remark 4 (*initial data*) As in Remark 3, we define $\mathcal{U}(0) = \mathcal{H}_\alpha g$ and $\mathcal{U}_t(0) = \mathcal{H}_\alpha h$, where $\mathcal{H}_\alpha$ is defined by (5.4). References [9,13] provide the estimates $\|\mathcal{U}(0)\|_{L^2(y^\alpha, \mathcal{C})} \lesssim \|g\|_{\mathbb{H}^s(\Omega)}$ and $\|\partial_t \mathcal{U}(0)\|_{L^2(y^\alpha, \mathcal{C})} \lesssim \|h\|_{\mathbb{H}^s(\Omega)}$.

The following result shows that by considering (5.3) instead of (3.3) we only incur an exponentially small error

Lemma 2 (exponential error estimate) *Let $\mathscr{U}$ and $\mathcal{U}$ be the solutions of problems (3.3) and (5.3), respectively. Then, for every $s \in (0, 1)$ and $\mathcal{Y} \geq 1$, we have*

$$\| \operatorname{tr} \partial_t (\mathscr{U} - \mathcal{U})\|_{L^\infty(0, T; L^2(\Omega))}^2 + \|\nabla(\mathscr{U} - \mathcal{U})\|_{L^\infty(0, T; L^2(y^\alpha, \mathcal{C}_\mathcal{Y}))}^2$$

$$\lesssim e^{-\sqrt{\lambda_1} \mathcal{Y}/2} \Sigma_1(f, g, h), \tag{5.5}$$

where Σ_1 is defined by (4.9) and the hidden constant does not depend on either $\mathscr{U}$, $\mathcal{U}$, or the problem data.

Proof We begin the proof by defining the cutoff function $\rho \in W^{1,\infty}(0, \infty)$ as

$$\rho(y) = 1, \ \ 0 \leq y \leq \frac{\mathcal{Y}}{2}, \quad \rho(y) = \frac{2}{\mathcal{Y}}(\mathcal{Y} - y), \ \ \frac{\mathcal{Y}}{2} < y < \mathcal{Y}, \quad \rho(y) = 0, \ \ y \geq \mathcal{Y}.$$

Notice that by a trivial zero extension we realize that $\mathcal{U} \in \mathring{H}_L^1(y^\alpha, \mathcal{C})$. We are thus allow to set $\phi = \partial_t(\mathcal{U} - \rho \mathscr{U})$ in problems (3.3) and (5.3). With these choices of test functions, we subtract the ensuing equalities and obtain that

$$\langle \operatorname{tr} \partial_t^2(\mathcal{U} - \mathscr{U}), \operatorname{tr} \partial_t(\mathcal{U} - \rho \mathscr{U}) \rangle + a_\mathcal{Y}(\mathcal{U} - \mathscr{U}, \partial_t(\mathcal{U} - \rho \mathscr{U})) = 0.$$

This expression yields

$$\frac{1}{2}\partial_t \| \operatorname{tr} \partial_t (\mathcal{U} - \mathscr{U}) \|^2_{L^2(\Omega)} + \frac{1}{2}\partial_t a_{\mathscr{Y}}(\mathcal{U} - \mathscr{U}, \mathcal{U} - \mathscr{U}) = a_{\mathscr{Y}}(\mathcal{U} - \mathscr{U}, \partial_t (\rho \mathscr{U} - \mathscr{U})).$$

We thus integrate over time and use that $\operatorname{tr} \partial_t (\mathcal{U} - \mathscr{U})|_{t=0} = 0$ to arrive at

$$\| \operatorname{tr} \partial_t (\mathcal{U} - \mathscr{U})(t) \|^2_{L^2(\Omega)} + \| \nabla (\mathcal{U} - \mathscr{U})(t) \|^2_{L^2(y^\alpha, \mathcal{C}_{\mathscr{Y}})}$$

$$\lesssim \| \nabla (\mathcal{U}(0) - \mathscr{U}(0)) \|^2_{L^2(y^\alpha, \mathcal{C}_{\mathscr{Y}})} + \int_0^t |a_{\mathscr{Y}}(\mathcal{U} - \mathscr{U}, \partial_\zeta (\rho \mathscr{U} - \mathscr{U}))| \, d\zeta \quad (5.6)$$

$$=: \mathrm{I} + \mathrm{II}.$$

It thus remains to bound the right-hand side of (5.6). First, in view of the fact that $\mathscr{U}(0) = \mathcal{E}_\alpha g$ and $\mathcal{U}(0) = \mathcal{H}_\alpha g$, with $\mathcal{E}_\alpha$ and $\mathcal{H}_\alpha$ being defined as in (3.5) and (5.4), respectively, the results of [46, Lemma 3.3] allow us to conclude the estimate

$$\mathrm{I} = \| \nabla (\mathcal{H}_\alpha - \mathcal{E}_\alpha) g \|_{L^2(y^\alpha, \mathcal{C}_{\mathscr{Y}})} \lesssim e^{-\sqrt{\lambda_1} \mathscr{Y}/4} \| g \|_{\mathbb{H}^s(\Omega)}. \quad (5.7)$$

To bound the term II, we notice that if $y \leq \mathscr{Y}/2$, $(\rho - 1)\mathscr{U} \equiv 0$. If $y > \mathscr{Y}/2$, then

$$|\nabla (\rho - 1)\partial_t \mathscr{U}|^2 \leq 2 \left(\frac{4}{\mathscr{Y}^2} |\partial_t \mathscr{U}|^2 + |\partial_t \nabla \mathscr{U}|^2 \right).$$

Consequently,

$$\| \nabla (\rho - 1)\partial_t \mathscr{U} \|^2_{L^2(y^\alpha, \mathcal{C}_{\mathscr{Y}})} \lesssim \frac{1}{\mathscr{Y}^2} \int_{\frac{\mathscr{Y}}{2}}^{\mathscr{Y}} \int_\Omega y^\alpha |\partial_t \mathscr{U}|^2 \, dx' \, dy$$

$$+ \int_{\frac{\mathscr{Y}}{2}}^{\mathscr{Y}} \int_\Omega y^\alpha |\partial_t \nabla \mathscr{U}|^2 \, dx' \, dy. \quad (5.8)$$

A weighted Poincaré inequality, an application of (3.12) and (3.13), as in the proof of Proposition 3, and the use of the estimate (5.1) allow us to conclude the estimate

$$\| \nabla (\rho - 1)\partial_t \mathscr{U} \|^2_{L^2(y^\alpha, \mathcal{C}_{\mathscr{Y}})} \lesssim \| \partial_t \nabla \mathscr{U} \|^2_{L^2(y^\alpha, \Omega \times (\mathscr{Y}/2, \mathscr{Y}))}$$

$$\leq \| \partial_t \nabla \mathscr{U} \|^2_{L^2(y^\alpha, \Omega \times (\mathscr{Y}/2, \infty))} \lesssim e^{-\sqrt{\lambda_1} \mathscr{Y}/2} \| \partial_t \nabla \mathscr{U} \|^2_{L^2(y^\alpha, \mathcal{C})}.$$

The regularity estimate (4.11) thus implies that

$$\mathrm{II} \leq C e^{-\sqrt{\lambda_1} \mathscr{Y}/2} \Sigma_1^2(f, g, h) + \frac{1}{2} \| \nabla (\mathcal{U} - \mathscr{U}) \|^2_{L^\infty(0,T;L^2(y^\alpha, \mathcal{C}_{\mathscr{Y}}))},$$

where C denotes a positive constant. Replacing the previous estimate for II and the one in (5.7) for I into (5.6) we obtain the desired exponential error estimate (5.5). $\qquad\square$

6 Space and time discretization

In this section we present two fully discrete schemes for approximating the solution to problem (1.1). In view of the localization results of Theorem 2 and the exponential error estimate (5.5) we shall thus discretize the truncated problem (5.3). We begin by setting notation on finite element spaces and introducing a finite element approximation in Ω.

6.1 Finite element methods

We follow [5] and present a scheme based on the tensorization of a first-degree FEM in Ω with a suitable hp-FEM in the extended variable. The scheme achieves log-linear complexity with respect to the number of degrees of freedom in Ω. To describe it, on the interval $[0, \mathcal{Y}]$, we consider geometric meshes $\mathcal{G}_\sigma^M = \{I_m \mid m = 1, \ldots M\}$ with M elements and grading factor $\sigma \in (0, 1)$:

$$I_1 = [0, \mathcal{Y}\sigma^{M-1}], \qquad I_i = [\mathcal{Y}\sigma^{M-m+1}, \mathcal{Y}\sigma^{M-m}], \quad m \in \{2, \ldots, M\}. \quad (6.1)$$

Notice that the meshes $\mathcal{G}_\sigma^M$ are refined towards $y = 0$ in order to capture the singular behavior exhibited by the solution $\mathscr{U}$ on the extended variable y as described in Propositions 1 and 2. On the aforementioned meshes, we consider a *linear degree vector* $\boldsymbol{r} = (r_1, \ldots, r_M) \in \mathbb{N}^M$ with slope $\mathfrak{s}$: $r_m := 1 + \lceil \mathfrak{s}(m-1) \rceil$, where $m = 1, 2, \ldots, M$. With these ingredients at hand, we define the finite element space

$$S^{\boldsymbol{r}}((0, \mathcal{Y}), \mathcal{G}_\sigma^M) = \left\{ v_M \in C[0, \mathcal{Y}] : v_M|_{I_m} \in \mathbb{P}_{r_m}(I_m), I_m \in \mathcal{G}_\sigma^M, m = 1, \ldots, M \right\}$$

and the subspace of $S^{\boldsymbol{r}}((0, \mathcal{Y}), \mathcal{G}_\sigma^M)$ containing functions that vanish at $y = \mathcal{Y}$:

$$S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0, \mathcal{Y}), \mathcal{G}_\sigma^M) = \left\{ v_M \in S^{\boldsymbol{r}}((0, \mathcal{Y}), \mathcal{G}_\sigma^M) : v_M(\mathcal{Y}) = 0 \right\}.$$

Let $\mathscr{T} = \{K\}$ be a conforming partition of $\bar{\Omega}$ into simplices K. We denote by $\mathbb{T}$ a collection of conforming and shape regular meshes that are refinements of an original mesh $\mathscr{T}_0$. For $\mathscr{T} \in \mathbb{T}$, we define $h_\mathscr{T} = \max\{\text{diam}(K) : K \in \mathscr{T}\}$ and $N = \#\mathscr{T}$, the number of degrees of freedom of $\mathscr{T}$. We introduce the finite element space:

$$S_0^1(\Omega, \mathscr{T}) = \left\{ v_h \in C(\bar{\Omega}) : v_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathscr{T}, \ v_h|_{\partial\Omega} = 0 \right\}. \quad (6.2)$$

With the meshes $\mathcal{G}_\sigma^M$ and $\mathscr{T}$ at hand, we define $\mathscr{T}_\mathcal{Y} = \mathscr{T} \otimes \mathcal{G}_\sigma^M$ and the finite-dimensional *tensor product space*

$$\mathbb{V}_{N,M}^{1,\boldsymbol{r}}(\mathscr{T}_\mathcal{Y}) := S_0^1(\Omega, \mathscr{T}) \otimes S^{\boldsymbol{r}}_{\{\mathcal{Y}\}}((0, \mathcal{Y}), \mathcal{G}_\sigma^M) \subset \mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y}). \quad (6.3)$$

We write $\mathbb{V}(\mathscr{T}_\mathcal{Y})$ if the arguments are clear from the context.

Finally, we recall the standard $L^2(\Omega)$-orthogonal projection operator $\Pi_{x'}$: $L^2(\Omega) \to \mathbb{S}_0^1(\Omega, \mathscr{T})$ which is such that

$$(\Pi_{x'}w, W)_{L^2(\Omega)} = (w, W)_{L^2(\Omega)} \quad \forall W \in \mathbb{S}_0^1(\Omega, \mathscr{T}). \tag{6.4}$$

If $\mathscr{T}$ is quasi-uniform, then [23, Lemma 1.131]

$$\|\Pi_{x'}w\|_{L^2(\Omega)} \le \|w\|_{L^2(\Omega)}, \quad \|\nabla \Pi_{x'}w\|_{L^2(\Omega)} \lesssim \|\nabla w\|_{L^2(\Omega)} \tag{6.5}$$

for all $w \in H^1(\Omega)$. If, in addition, $w \in H^2(\Omega)$, then [23, Proposition 1.134]

$$\|w - \Pi_{x'}w\|_{L^2(\Omega)} + h_{\mathscr{T}}\|\nabla(w - \Pi_{x'}w)\|_{L^2(\Omega)} \lesssim h_{\mathscr{T}}^2|w|_{H^2(\Omega)}. \tag{6.6}$$

6.2 Weighted elliptic projector

We define the *weighted elliptic projector*

$$G_{\mathscr{T}_{\mathscr{Y}}} : \mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathscr{Y}}) \to \mathbb{V}(\mathscr{T}_{\mathscr{Y}})$$

such that, for $w \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathscr{Y}})$, it is given by

$$a_{\mathscr{Y}}\left(G_{\mathscr{T}_{\mathscr{Y}}}w, W\right) = a_{\mathscr{Y}}(w, W) \quad \forall W \in \mathbb{V}(\mathscr{T}_{\mathscr{Y}}). \tag{6.7}$$

This operator is stable in $\mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathscr{Y}})$ [47, Proposition 26]:

$$\|\nabla G_{\mathscr{T}_{\mathscr{Y}}}w\|_{L^2(y^\alpha, \mathcal{C}_{\mathscr{Y}})} \lesssim \|\nabla w\|_{L^2(y^\alpha, \mathcal{C}_{\mathscr{Y}})} \quad \forall w \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathscr{Y}}). \tag{6.8}$$

In what follows we present approximation properties for $G_{\mathscr{T}_{\mathscr{Y}}}$.

Lemma 3 (error estimates for $G_{\mathscr{T}_{\mathscr{Y}}}$) *Fix* $0 < \sigma < 1$ *and, for each ℓ, denote by $\mathcal{G}_\sigma^M$ the geometric mesh defined in* (6.1) *where* $\mathscr{Y} \sim |\log h_{\mathscr{T}}|$ *with a sufficiently large constant. Assume that M is such that $c_1 M \le \mathscr{Y} \le c_2 M$ with absolute constants c_1 and c_2. Let $w \in \mathbb{H}^{1+s}(\Omega)$. If $\mathcal{W}$ denotes the truncated α-harmonic extension of w, then there exists a minimal slope $\mathfrak{s}_{min}$ such that for linear degree vectors $\mathbf{r}$ with slope $\mathfrak{s} \ge \mathfrak{s}_{min}$ there holds*

$$\|\nabla(\mathcal{W} - G_{\mathscr{T}_{\mathscr{Y}}}\mathcal{W})\|_{L^2(y^\alpha, \mathcal{C}_{\mathscr{Y}})} \lesssim h_{\mathscr{T}}\|w\|_{\mathbb{H}^{1+s}(\Omega)}. \tag{6.9}$$

In addition, if $\mathscr{W}$ denotes the α-harmonic extension of w, i.e., the solution to (3.6) *with $w \in \mathbb{H}^{1+s}(\Omega)$ as a datum, then*

$$\|\operatorname{tr}(\mathscr{W} - G_{\mathscr{T}_{\mathscr{Y}}}\mathscr{W})\|_{\mathbb{H}^s(\Omega)} \lesssim \|\nabla(\mathscr{W} - G_{\mathscr{T}_{\mathscr{Y}}}\mathscr{W})\|_{L^2(y^\alpha, \mathcal{C})}$$
$$\lesssim h_{\mathscr{T}}\|w\|_{\mathbb{H}^{1+s}(\Omega)}. \tag{6.10}$$

The hidden constants are independent of $\mathcal{W}$, $\mathscr{W}$, w, and $h_{\mathscr{T}}$.

Proof Let $\Pi^r_{y,\{\mathcal{Y}\}}$ and $\Pi_{x'}$ be the univariate hp-interpolation operator of [5, Section 5.5.1] and the $L^2(\Omega)$-projection operator defined in (6.4), respectively:

$$\Pi^r_{y,\{\mathcal{Y}\}} : C([0,\mathcal{Y}]) \to S^r_{\{\mathcal{Y}\}}((0,\mathcal{Y}), \mathcal{G}^M_\sigma), \quad \Pi_{x'} : L^2(\Omega) \to S^1_0(\Omega, \mathscr{T}). \quad (6.11)$$

Set $W = \Pi_{x'} \otimes \Pi^r_{y,\{\mathcal{Y}\}} \mathcal{W}$. Since $W \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$, Galerkin orthogonality and definition (6.7) yield

$$\|\nabla(\mathcal{W} - G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{W})\|^2_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})} \lesssim a_{\mathcal{Y}}(\mathcal{W} - G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{W}, \mathcal{W} - W).$$

It suffices to bound $\|\nabla(\mathcal{W}-W)\|_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})}$. The stability properties of $\Pi_{x'}$, as described in (6.5), reveal that

$$\|\nabla(\mathcal{W} - W)\|_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})} \lesssim \|\nabla(\mathcal{W} - \Pi_{x'}\mathcal{W})\|_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})}$$
$$+ \|\nabla(\mathcal{W} - \Pi^r_{y,\{\mathcal{Y}\}}\mathcal{W})\|_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})}.$$

The estimate (6.9) thus follows from the approximation properties of $\Pi_{x'}$ as described in (6.6), the exponential interpolation error estimates of [5, Lemma 11], and the regularity properties of $\mathcal{W}$ [5, Theorem 1]. The estimate (6.10) follows similar arguments by using first the exponential decay of $\mathscr{W}$ in the extended dimension [46, Proposition 3.1]:

$$\|\nabla(\mathscr{W} - G_{\mathscr{T}_{\mathcal{Y}}}\mathscr{W})\|_{L^2(y^\alpha, \mathcal{C})} \leq \|\nabla\mathscr{W}\|_{L^2(y^\alpha, \mathcal{C}\setminus\mathcal{C}_{\mathcal{Y}})} + \|\nabla(\mathscr{W} - G_{\mathscr{T}_{\mathcal{Y}}}\mathscr{W})\|_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})}$$
$$\lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2}\|w\|_{\mathbb{H}^s(\Omega)} + \|\nabla(\mathscr{W} - G_{\mathscr{T}_{\mathcal{Y}}}\mathscr{W})\|_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})}.$$

This concludes the proof. $\qquad\square$

The following improved estimate for the weighted elliptic projection $G_{\mathscr{T}_{\mathcal{Y}}}$ in the $L^2(\Omega)$-norm can be obtained by invoking the estimates of Lemma 3 and the arguments elaborated in the proof of [47, Proposition 28].

Lemma 4 ($L^2(\Omega)$-error estimates for $G_{\mathscr{T}_{\mathcal{Y}}}$) *Let $\mathcal{G}^M_\sigma$ be the geometric mesh defined in (6.1) where $\mathcal{Y} \sim |\log h_{\mathscr{T}}|$ with a sufficiently large constant. Let $w \in \mathbb{H}^{1+s}(\Omega)$. If $\mathcal{W}$ denotes the truncated α-harmonic extension of w, then there exists a minimal slope $\mathfrak{s}_{min}$ such that for linear degree vectors $\boldsymbol{r}$ with slope $\mathfrak{s} \geq \mathfrak{s}_{min}$ there holds*

$$\|\operatorname{tr}(\mathcal{W} - G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{W})\|_{L^2(\Omega)} \lesssim h^{1+s}_{\mathscr{T}}\|w\|_{\mathbb{H}^{1+s}(\Omega)}. \quad (6.12)$$

In addition, if $\mathscr{W}$ denotes the α-harmonic extension of w, i.e., the solution to (3.6) with $w \in \mathbb{H}^{1+s}(\Omega)$ as a datum, then

$$\|\operatorname{tr}(\mathscr{W} - G_{\mathscr{T}_{\mathcal{Y}}}\mathscr{W})\|_{L^2(\Omega)} \lesssim h^{1+s}_{\mathscr{T}}\|w\|_{\mathbb{H}^{1+s}(\Omega)}. \quad (6.13)$$

The hidden constants are independent of $\mathcal{W}$, $\mathscr{W}$, w, and $h_{\mathscr{T}}$.

6.3 Time discretization

Let $K \in \mathbb{N}$ be the number of time steps. We define the uniform time step as $\Delta t = T/K$, and we set $t_k = k\Delta t$, $k = 0, \ldots, K$. If $\mathcal{X}$ is a normed space with norm $\|\cdot\|_{\mathcal{X}}$, then for $w \in C([0, T]; \mathcal{X})$ we denote $w_k = w(t_k) \in \mathcal{X}$ and $w_{\Delta t} = \{w_k\}_{k=0}^{K} \subset \mathcal{X}$. In addition, for $w_{\Delta t} \subset \mathcal{X}$ and $p \in [1, \infty)$, we define

$$\|w_{\Delta t}\|_{\ell^p(\mathcal{X})} = \left(\sum_{k=1}^{K} \Delta t \|w_k\|_{\mathcal{X}}^p \right)^{\frac{1}{p}}, \quad \|w_{\Delta t}\|_{\ell^\infty(\mathcal{X})} = \max_{0 \leq k \leq K} \|w_k\|_{\mathcal{X}}. \quad (6.14)$$

For a sequence of time-discrete functions $w_{\Delta t} \subset \mathcal{X}$, we define, for $k = 0, \ldots, K-1$,

$$\mathfrak{d}w_{k+1} := (\Delta t)^{-1}(w_{k+1} - w_k), \qquad w_{k+1/2} := \tfrac{1}{2}(w_{k+1} + w_k). \quad (6.15)$$

We also define, for $k = 1, \ldots, K-1$,

$$\mathfrak{c}w_k := \tfrac{1}{4}(w_{k+1} + 2w_k + w_{k-1}) = \tfrac{1}{2}(w_{k+1/2} + w_{k-1/2}), \quad (6.16)$$

and

$$\mathfrak{d}^2 w_k := (\Delta t)^{-2}(w_{k+1} - 2w_k + w_{k-1}). \quad (6.17)$$

6.4 Trapezoidal multistep method

Let us now describe our first fully discrete numerical scheme to solve problem (5.3). The space discretization is based on the finite element method on the truncated cylinder $\mathcal{C}_{\mathscr{Y}}$ described in Sect. 6.1. The discretization in time is based on a trapezoidal multistep method.

The fully discrete scheme computes the sequence $V_{\Delta t} \subset \mathbb{V}(\mathscr{T}_{\mathscr{Y}})$, an approximation of the solution to (5.3) at each time step. We initialize the scheme by setting

$$V_0 = G_{\mathscr{T}_{\mathscr{Y}}} \mathcal{H}_\alpha g, \quad V_1 = G_{\mathscr{T}_{\mathscr{Y}}} \left(\mathcal{H}_\alpha g + \Delta t \mathcal{H}_\alpha h + \frac{1}{2}(\Delta t)^2 \partial_t^2 \mathcal{U}(0) \right), \quad (6.18)$$

where $\mathcal{H}_\alpha$ denotes the truncated α-harmonic extension and $\partial_t^2 \mathcal{U}(0) = \mathcal{H}_\alpha w$ with $w \in \mathbb{H}^s(\Omega)$ satisfying

$$\langle w, \operatorname{tr} \phi \rangle = -a_{\mathscr{Y}}(\mathcal{U}(0), \phi) + \langle f(0), \operatorname{tr} \phi \rangle \quad \forall \phi \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathscr{Y}}). \quad (6.19)$$

Note that, if $\operatorname{tr} \phi = 0$ the previous equation is satisfied for any w.

For $k = 1, \ldots, K-1$, let $V_{k+1} \in \mathbb{V}(\mathscr{T}_{\mathscr{Y}})$ solve

$$\frac{1}{\Delta t^2} \langle \operatorname{tr}(V_{k+1} - 2V_k + V_{k-1}), \operatorname{tr} W \rangle + a_{\mathscr{Y}}(\mathfrak{c}V_k, W) = \langle \mathfrak{c}f_k, \operatorname{tr} W \rangle \quad (6.20)$$

for all $W \in \mathbb{V}(\mathcal{T}_{\mathcal{Y}})$, where $\mathfrak{c} V_k$ and $\mathfrak{c} f_k$ are defined in (6.16). To obtain an approximate solution to the fractional wave equation (1.1), we define the sequence

$$U_{\Delta t} = \{U_k\}_{k=0}^K \subset S_0^1(\Omega, \mathcal{T}) : \quad U_{\Delta t} := \operatorname{tr} V_{\Delta t}. \tag{6.21}$$

Remark 5 (*locality*) The main advantage of problem (6.18)–(6.20) is that it provides an approximated solution to the fractional wave equation (1.1) based on the resolution of the *local* elliptic problem with a dynamic boundary condition (5.3).

6.4.1 Stability

To present the stability of the scheme we introduce, for $k = 1, \ldots, K$, the unconditionally nonnegative discrete energy

$$E_k(W_{\Delta t}) := \tfrac{1}{2} \| \operatorname{tr} \mathfrak{d} W_k \|_{L^2(\Omega)}^2 + \tfrac{1}{2} \| W_{k-1/2} \|_{\mathcal{C}}^2. \tag{6.22}$$

Lemma 5 (energy conservation) *If $f \equiv 0$, then the fully discrete scheme* (6.18)–(6.20) *conserves energy, i.e., for all $k \in \{1, \ldots, K\}$, we have that*

$$E_k(V_{\Delta t}) = E_1(V_{\Delta t}). \tag{6.23}$$

If $f \neq 0$, then, for $\ell \in \{1, \ldots, K\}$, we have that

$$E_\ell(V_{\Delta t})^{\frac{1}{2}} \leq E_1(V_{\Delta t})^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \sum_{k=1}^{\ell} \Delta t \| \mathfrak{c} f_k \|_{L^2(\Omega)}. \tag{6.24}$$

In particular, we have that $E_K(V_{\Delta t})^{\frac{1}{2}} \leq E_1(V_{\Delta t})^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \| \mathfrak{c} f \|_{\ell^1(L^2(\Omega))}$.

Proof Set $W = (2\Delta t)^{-1}(V_{k+1} - V_{k-1}) = 2^{-1}(\mathfrak{d} V_{k+1} + \mathfrak{d} V_k) = (\Delta t)^{-1}(V_{k+1/2} - V_{k-1/2})$ in (6.20). Basic computations reveal that

$$\frac{1}{\Delta t} (E_{k+1}(V_{\Delta t}) - E_k(V_{\Delta t})) = \frac{1}{2} \langle \mathfrak{c} f_k, \operatorname{tr}(\mathfrak{d} V_{k+1} + \mathfrak{d} V_k) \rangle. \tag{6.25}$$

If $f \equiv 0$, the previous relation immediately yields (6.23). If $f \neq 0$, an application of the Cauchy–Schwarz inequality allows us to conclude

$$E_{k+1}(V_{\Delta t}) - E_k(V_{\Delta t}) \leq \frac{\Delta t}{\sqrt{2}} \| \mathfrak{c} f_k \|_{L^2(\Omega)} \left(E_{k+1}(V_{\Delta t})^{\frac{1}{2}} + E_k(V_{\Delta t})^{\frac{1}{2}} \right),$$

which yields $E_{k+1}(V_{\Delta t})^{\frac{1}{2}} - E_k(V_{\Delta t})^{\frac{1}{2}} \leq \frac{\Delta t}{\sqrt{2}} \| \mathfrak{c} f_k \|_{L^2(\Omega)}$. Adding over ℓ we arrive at the desired estimate (6.24). This concludes the proof. $\square$

Let us now show the stability of the scheme.

Lemma 6 (stability) *The fully discrete scheme* (6.18)–(6.20) *is stable, namely, for* $\ell \in \{1, \ldots, K\}$, *we have that*

$$\| \operatorname{tr} \mathfrak{d} V_\ell \|_{L^2(\Omega)} + \| V_{\ell-1/2} \|_{\mathcal{C}} \lesssim \| \operatorname{tr} \mathfrak{d} V_1 \|_{L^2(\Omega)} + \| V_{1/2} \|_{\mathcal{C}} + \sum_{k=1}^{\ell} \Delta t \| \mathfrak{c} f_k \|_{L^2(\Omega)},$$

where the hidden constant is independent of $V_{\Delta t}$ *and* Δt.

Proof The proof follows immediately from (6.24).   □

6.4.2 Error analysis

Let us now present an error analysis for the fully discrete scheme (6.18)–(6.20). To accomplish this task, we introduce the error $e_{\Delta t} := V_{\Delta t} - \mathcal{U}_{\Delta t}$ and write, as usual,

$$e_{\Delta t} = (V_{\Delta t} - G_{\mathscr{T}_{\mathcal{Y}}} \mathcal{U}_{\Delta t}) + (G_{\mathscr{T}_{\mathcal{Y}}} \mathcal{U}_{\Delta t} - \mathcal{U}_{\Delta t}) =: \Theta_{\Delta t} + P_{\Delta t}. \tag{6.26}$$

The control of $P_{\Delta t}$ follows from (6.9) and (6.12): For $\ell \in \{0, 1, 2\}$, we have the error estimates

$$\| \partial_t^\ell \nabla P_{\Delta t} \|_{\ell^2(L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}}))} \lesssim h_{\mathscr{T}} \mathfrak{A}(f, g, h) \tag{6.27}$$

and

$$\| \partial_t^\ell \operatorname{tr} P_{\Delta t} \|_{\ell^2(L^2(\Omega))} \lesssim h_{\mathscr{T}}^{1+s} \mathfrak{A}(f, g, h), \tag{6.28}$$

where $\mathfrak{A}(f, g, h)$ is defined in (4.20). Notice that to obtain the estimates (6.27) and (6.28) the regularity estimates of Corollary 1 are essential.

In what follows we bound the sequence $\Theta_{\Delta t}$.

Lemma 7 (error estimate for $\Theta_{\Delta t}$) *Let* $\mathcal{U}$ *be the solution to* (5.3) *and let* $V_{\Delta t}$ *be its fully discrete approximation defined as the solution to* (6.18)–(6.20). *If* $\mathfrak{A}(f, g, h) < \infty$ *and* $\Xi(f, g, h) < \infty$, *then*

$$E_K(\Theta_{\Delta t})^{\frac{1}{2}} \lesssim h_{\mathscr{T}}^{1+s} \mathfrak{A}(f, g, h) + (\Delta t)^2 \Xi(f, g, h), \tag{6.29}$$

where $\Theta_{\Delta t} = V_{\Delta t} - G_{\mathscr{T}_{\mathcal{Y}}} \mathcal{U}_{\Delta t}$, $\mathfrak{A}(f, g, h)$ *and* $\Xi(f, g, h)$ *are defined in* (4.20) *and* (4.10), *respectively, and the hidden constant is independent of* $V_{\Delta t}$, $\mathcal{U}$, Δt, *and* $h_{\mathscr{T}}$.

Proof We proceed in three steps.

Step 1. We invoke the continuous problem (5.3), the discrete equation (6.20), and the definition of $G_{\mathscr{T}_{\mathcal{Y}}}$, given by (6.7), to arrive at the problem that controls the error: For $k = 1, \ldots, K-1$, $\Theta_{k+1} \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$ solves

$$\frac{1}{\Delta t^2} \langle \operatorname{tr}(\Theta_{k+1} - 2\Theta_k + \Theta_{k-1}), \operatorname{tr} W \rangle + a_{\mathcal{Y}}(\mathfrak{c}\Theta_k, W)$$
$$= \langle \operatorname{tr}[\mathfrak{c}\partial_t^2 \mathcal{U}(t_k) - \mathfrak{d}^2 G_{\mathscr{T}_{\mathcal{Y}}} \mathcal{U}(t_k)], \operatorname{tr} W \rangle \quad \forall W \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}}), \tag{6.30}$$

where $\mathfrak{c}\partial_t^2\mathcal{U}(t_k)$ and $\mathfrak{d}^2 G_{\mathscr{T}_y}\mathcal{U}(t_k)$ are defined by (6.16) and (6.17), respectively. On the other hand, in view of Remark 4 and (6.18), we have that

$$\Theta_0 = G_{\mathscr{T}_y}(\mathcal{H}_\alpha g - \mathcal{U}(0)) = 0,$$

and

$$\Theta_1 = G_{\mathscr{T}_y}\left(\mathcal{H}_\alpha g + \Delta t \mathcal{H}_\alpha h + \frac{1}{2}(\Delta t)^2 \partial_t^2 \mathcal{U}(0) - \mathcal{U}(t_1)\right).$$

Now, we write, for $k \geq 1$, the difference $\mathfrak{c}\partial_t^2\mathcal{U}(t_k) - \mathfrak{d}^2 G_{\mathscr{T}_y}\mathcal{U}(t_k)$ as follows:

$$\mathfrak{c}\partial_t^2\mathcal{U}(t_k) - \mathfrak{d}^2 G_{\mathscr{T}_y}\mathcal{U}(t_k) = \left[\partial_t^2\mathcal{U}(t_k) - \mathfrak{d}^2\mathcal{U}(t_k)\right] + \left[\mathfrak{d}^2\mathcal{U}(t_k) - \mathfrak{d}^2 G_{\mathscr{T}_y}\mathcal{U}(t_k)\right]$$
$$+ \frac{\Delta t}{4}\left[\mathfrak{d}\partial_t^2\mathcal{U}(t_{k+1}) - \mathfrak{d}\partial_t^2\mathcal{U}(t_k)\right] =: \mathrm{I}_k + \mathrm{II}_k + \mathrm{III}_k.$$

We thus apply the stability estimate (6.24) to (6.30) and obtain

$$E_K(\Theta_{\Delta t})^{\frac{1}{2}} \leq E_1(\Theta_{\Delta t})^{\frac{1}{2}} + \frac{1}{\sqrt{2}}\|\delta_{\Delta t}\|_{\ell^1(L^2(\Omega))}, \tag{6.31}$$

where $\delta_{\Delta t} = \{\delta_k\}_{k=1}^{K-1}$ and $\delta_k = \mathrm{tr}[\mathfrak{c}\partial_t^2\mathcal{U}(t_k) - \mathfrak{d}^2 G_{\mathscr{T}_y}\mathcal{U}(t_k)]$.

Step 2. We proceed to control the term $\|\delta_{\Delta t}\|_{\ell^1(L^2(\Omega))}$. First, notice that

$$\|\delta_{\Delta t}\|_{\ell^1(L^2(\Omega))} \leq \sum_{k=1}^{K-1} \Delta t\big(\|\operatorname{tr} \mathrm{I}_k\|_{L^2(\Omega)} + \|\operatorname{tr} \mathrm{II}_k\|_{L^2(\Omega)}$$
$$+ \|\operatorname{tr} \mathrm{III}_k\|_{L^2(\Omega)}\big). \tag{6.32}$$

To control $\|\operatorname{tr} \mathrm{I}_k\|_{L^2(\Omega)}$ we employ a basic result based on Taylor's Theorem. In fact, for $k \geq 1$, we have

$$\|\operatorname{tr} \mathrm{I}_k\|_{L^2(\Omega)} \lesssim (\Delta t)^2 \sup_z \|\operatorname{tr} \partial_t^4 \mathcal{U}(\cdot, z)\|_{L^2(\Omega)}. \tag{6.33}$$

Now, notice that in view of (6.26) we have that $\mathrm{II}_k = -\mathfrak{d}^2 P(t_k)$. The same argument that yields (6.33) allow us to conclude the estimate

$$\|\operatorname{tr} \mathrm{II}_k\|_{L^2(\Omega)} \lesssim \|\operatorname{tr} \partial_t^2 P(t_k)\|_{L^2(\Omega)} + (\Delta t)^2 \sup_z \|\operatorname{tr} \partial_t^4 P(\cdot, z)\|_{L^2(\Omega)}.$$

We invoke the trace estimate (2.9) and the stability estimate (6.8) of the weighted elliptic projection to conclude, for $z \in (0, T)$, that $\|\operatorname{tr} \partial_t^4 P(\cdot, z)\|_{L^2(\Omega)}$ is bounded by

$$\|\operatorname{tr} \partial_t^4(\mathcal{U} - G_{\mathscr{T}_y}\mathcal{U})(\cdot, z)\|_{L^2(\Omega)} \lesssim \|\operatorname{tr} \partial_t^4\mathcal{U}(\cdot, z)\|_{L^2(\Omega)} + \|\nabla\partial_t^4\mathcal{U}(\cdot, z)\|_{L^2(y^\alpha, \mathcal{C}_y)}.$$

Consequently, an application, again, of the trace estimate allows us to conclude an estimate for $\mathrm{tr}\, \mathrm{II}_k$:

$$\| \mathrm{tr}\, \mathrm{II}_k \|_{L^2(\Omega)} \lesssim h_{\mathscr{T}}^{1+s} \mathfrak{A}(f, g, h) + (\Delta t)^2 \|\nabla \partial_t^4 \mathcal{U}\|_{L^\infty(0,T;L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}}))}, \quad (6.34)$$

where we have used (6.28) with $\ell = 2$; $\mathfrak{A}(f, g, h)$ is defined in (4.20).

We finally bound III_k. To accomplish this task, we invoke an argument based on Taylor's Theorem. In fact, for $k \geq 1$, we have

$$\begin{aligned} \| \mathrm{tr}\, \mathrm{III}_k \|_{L^2(\Omega)} \\ = \tfrac{\Delta t}{4} \| \mathrm{tr}[\partial_t^3 \mathcal{U}(\cdot, t_{k+1}) + \tfrac{\Delta t}{2} \partial_t^4 \mathcal{U}(\cdot, \overline{z}) - \partial_t^3 \mathcal{U}(\cdot, t_k) - \tfrac{\Delta t}{2} \partial_t^4 \mathcal{U}(\cdot, \underline{z})] \|_{L^2(\Omega)} \\ \lesssim (\Delta t)^2 \| \mathrm{tr}\, \partial_t^4 \mathcal{U} \|_{L^\infty(0,T;L^2(\Omega))}, \end{aligned} \quad (6.35)$$

where $\underline{z}$ and $\overline{z}$ belong to (t_{k-1}, t_{k+1}).

Replacing the estimates (6.33), (6.34), and (6.35) into (6.32) we arrive at

$$\|\delta_{\Delta t}\|_{\ell^1(L^2(\Omega))} \lesssim h_{\mathscr{T}}^{1+s} \mathfrak{A}(f, g, h) + (\Delta t)^2 \|\partial_t^4 \nabla \mathcal{U}\|_{L^\infty(0,T;L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}}))}. \quad (6.36)$$

Step 3. We bound $E_1(\Theta_{\Delta t})$. Since $\Theta_0 = 0$, we utilize (6.15) and write

$$\begin{aligned} E_1(\Theta_{\Delta t}) &= \frac{1}{2} \| \mathrm{tr}\, \mathfrak{d}\Theta_1 \|_{L^2(\Omega)}^2 + \frac{1}{2} \|\Theta_{1/2}\|_{\mathcal{C}}^2 \\ &= \frac{1}{2(\Delta t)^2} \| \mathrm{tr}\, \Theta_1 \|_{L^2(\Omega)}^2 + \frac{1}{8} \|\Theta_1\|_{\mathcal{C}}^2. \end{aligned}$$

In view of the trace estimate (2.9) and the stability property of $G_{\mathscr{T}_{\mathscr{Y}}}$, given in (6.8), we can thus conclude that

$$\begin{aligned} \| \mathrm{tr}\, \Theta_1 \|_{L^2(\Omega)} &\lesssim \|\Theta_1\|_{\mathcal{C}} \\ &\lesssim \|\mathcal{U}(0) + \Delta t \mathcal{U}_t(0) + \tfrac{1}{2}(\Delta t)^2 \partial_t^2 \mathcal{U}(0) - \mathcal{U}(t_1)\|_{\mathcal{C}}. \end{aligned} \quad (6.37)$$

An application of Taylor's Theorem reveals that

$$\| \mathrm{tr}\, \Theta_1 \|_{L^2(\Omega)} \lesssim \|\Theta_1\|_{\mathcal{C}} \lesssim (\Delta t)^3 \|\partial_t^3 \nabla \mathcal{U}\|_{L^\infty(0,T;L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}}))},$$

which immediately yields

$$E_1(\Theta_{\Delta t})^{\frac{1}{2}} \lesssim (\Delta t)^2 \|\partial_t^3 \nabla \mathcal{U}\|_{L^\infty(0,T;L^2(y^\alpha,\mathcal{C}_{\mathscr{Y}}))}. \quad (6.38)$$

The desired estimate follows from replacing (6.36) and (6.38) into (6.31) and using the time-regularity results of Theorem 4. □

The exponential error estimate of Lemma 2 combined with the error estimate of Lemma 7 allow us to conclude the following error estimates.

Lemma 8 (error estimates for (6.18)–(6.20)) *Let $\mathcal{U}$ be the solution to (3.3) and let $V_{\Delta t}$ be the solution to the fully discrete problem (6.18)–(6.20). If $\mathfrak{A}(f, g, h) < \infty$ and $\Xi(f, g, h) < \infty$ then, we have the following error estimates*

$$
\| \operatorname{tr}(\partial_t \mathcal{U}(t_{K-1/2}) - \mathfrak{d}V_K) \|_{L^2(\Omega)} \lesssim h_{\mathcal{T}}^{1+s} \mathfrak{A}(f, g, h)
$$
$$
+ (\Delta t)^2 \Xi(f, g, h), \tag{6.39}
$$

and

$$
\| \mathcal{U}(t_{K-1/2}) - V_{K-1/2} \|_{\mathcal{C}} \lesssim h_{\mathcal{T}} \mathfrak{A}(f, g, h) + (\Delta t)^2 \Xi(f, g, h), \tag{6.40}
$$

where $\mathfrak{A}(f, g, h)$ and $\Xi(f, g, h)$ are defined by (4.20) and (4.10), respectively, and the hidden constants are independent of $V_{\Delta t}, \mathcal{U}, \mathcal{U}, \Delta t,$ and $h_{\mathcal{T}}$.

Proof We proceed in several steps.

Step 1. We begin with the following trivial application of the triangle inequality:

$$
\| \operatorname{tr}[\partial_t \mathcal{U}(t_{K-1/2}) - \mathfrak{d}V_K] \|_{L^2(\Omega)} \leq \| \operatorname{tr} \partial_t [\mathcal{U}(t_{K-1/2}) - \mathcal{U}(t_{K-1/2})] \|_{L^2(\Omega)}
$$
$$
+ \| \operatorname{tr}[\partial_t \mathcal{U}(t_{K-1/2}) - \mathfrak{d}V_K] \|_{L^2(\Omega)} =: \mathrm{I} + \mathrm{II}. \tag{6.41}
$$

To control the term I we invoke the exponential error estimate (5.5). The latter yields

$$
\mathrm{I} = \left\| \operatorname{tr} \partial_t [\mathcal{U}(t_{K-1/2}) - \mathcal{U}(t_{K-1/2})] \right\|_{L^2(\Omega)} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2} \Sigma_1(f, g, h).
$$

The control of the term II is as follows:

$$
\mathrm{II} \leq \left\| \operatorname{tr}[\partial_t \mathcal{U}(t_{K-1/2}) - \mathfrak{d}\mathcal{U}_K] \right\|_{L^2(\Omega)} + \| \operatorname{tr} \mathfrak{d}e_K \|_{L^2(\Omega)} =: \mathrm{II}_1 + \mathrm{II}_2,
$$

where we recall that $e_K = V_K - \mathcal{U}_K$. Replace the obtained estimates into (6.41). This yields

$$
\| \operatorname{tr}[\partial_t \mathcal{U}(t_{K-1/2}) - \mathfrak{d}V_K] \|_{L^2(\Omega)} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2} \Sigma_1(f, g, h) + \mathrm{II}_1 + \mathrm{II}_2. \tag{6.42}
$$

Step 2. The control of $\mathrm{II}_1 = \| \operatorname{tr}[\partial_t \mathcal{U}(t_{K-1/2}) - \mathfrak{d}\mathcal{U}_K] \|_{L^2(\Omega)}$ follows from a simple application of Taylor's Theorem. In fact, we have that

$$
\mathrm{II}_1 = \left\| \operatorname{tr} \left(\partial_t \mathcal{U}(t_{K-1/2}) - \frac{\mathcal{U}_K - \mathcal{U}_{K-1}}{\Delta t} \right) \right\|_{L^2(\Omega)} \lesssim (\Delta t)^2 \sup_z \| \operatorname{tr} \partial_t^3 \mathcal{U}(\cdot, z) \|_{L^2(\Omega)}.
$$

We now focus on the term $\mathrm{II}_2 = \| \operatorname{tr} \mathfrak{d}e_K \|_{L^2(\Omega)}$. The triangle inequality yields

$$
\mathrm{II}_2 \leq \| \operatorname{tr} \mathfrak{d}\Theta_K \|_{L^2(\Omega)} + \| \operatorname{tr} \mathfrak{d}P_K \|_{L^2(\Omega)}.
$$

The result of Lemma 7 implies that

$$
\begin{aligned}
\| \operatorname{tr} \eth \Theta_K \|_{L^2(\Omega)} &\lesssim h_{\mathscr{T}}^{1+s} \mathfrak{A}(f, g, h) \\
&\quad + (\Delta t)^2 \, \Xi(f, g, h).
\end{aligned}
\tag{6.43}
$$

The control of $\| \operatorname{tr} \eth P_K \|_{L^2(\Omega)}$ follows the same arguments used to bound II_1 :

$$
\begin{aligned}
\| \operatorname{tr} \eth P_K \|_{L^2(\Omega)} &= (\Delta t)^{-1} \| \operatorname{tr}(P_K - P_{K-1}) \|_{L^2(\Omega)} \\
&\lesssim \| \operatorname{tr} \partial_t P(t_{K-1/2}) \|_{L^2(\Omega)} + (\Delta t)^2 \| \operatorname{tr} \partial_t^3 P \|_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned}
$$

The stability property (6.8) and the estimate (6.28) imply the estimates

$$
\begin{aligned}
\| \operatorname{tr} \eth P_K \|_{L^2(\Omega)} &\lesssim \| \operatorname{tr} \partial_t P \|_{L^\infty(0,T;L^2(\Omega))} + (\Delta t)^2 \| \nabla \partial_t^3 \mathcal{U} \|_{L^\infty(0,T;L^2(y^\alpha, \mathcal{C}_\mathcal{Y}))} \\
&\lesssim h_{\mathscr{T}}^{1+s} \mathfrak{A}(f, g, h) + (\Delta t)^2 \| \nabla \partial_t^3 \mathcal{U} \|_{L^\infty(0,T;L^2(y^\alpha, \mathcal{C}_\mathcal{Y}))}.
\end{aligned}
$$

The previous estimate combined with (6.43) allow us to control II_2. Replacing the obtained estimates for II_1 and II_2 into (6.42) yield the desired estimate (6.39).

Step 3. To obtain (6.40) we invoke similar arguments upon using the estimate

$$
\| P_{K-1/2} \|_{\mathcal{C}} \lesssim \| P_K \|_{\mathcal{C}} + \| P_{K-1} \|_{\mathcal{C}} \lesssim \| \nabla P_{\Delta t} \|_{\ell^\infty(L^2(y^\alpha, \mathcal{C}_\mathcal{Y}))} \lesssim h_{\mathscr{T}} \mathfrak{A}(f, g, h).
$$

This concludes the proof.   □

The following error estimates follow immediately from Lemma 8 and show how the fully discrete approximation $U_{\Delta t}$ approximates u.

Theorem 6 (error estimates for (6.21)) *Let u be the solution to (1.1) and let $U_{\Delta t}$ be its fully discrete approximation defined by (6.21). If $\mathfrak{A}(f, g, h) < \infty$ and $\Xi(f, g, h) < \infty$, then*

$$
\| \partial_t u(t_{K-1/2}) - \eth U_K \|_{L^2(\Omega)} \lesssim h_{\mathscr{T}}^{1+s} \mathfrak{A}(f, g, h) + (\Delta t)^2 \Xi(f, g, h),
\tag{6.44}
$$

and

$$
\| u(t_{K-1/2}) - U_{K-1/2} \|_{\mathbb{H}^s(\Omega)} \lesssim h_{\mathscr{T}} \mathfrak{A}(f, g, h) + (\Delta t)^2 \Xi(f, g, h),
\tag{6.45}
$$

where $\mathfrak{A}(f, g, h)$ and $\Xi(f, g, h)$ are defined by (4.20) and (4.10), respectively, and the hidden constants are independent of $U_{\Delta t}$, u, Δt, and $h_{\mathscr{T}}$.

6.5 The leapfrog scheme

We now present a second fully discrete scheme to approximate the solution to (1.1). To advance in time we use the leapfrog time-stepping method while the discretization in space is based on the finite element method described in Sect. 6.1. The scheme

computes a sequence $V_{\Delta t} \subset \mathbb{V}(\mathcal{T}_{\mathcal{Y}})$, an approximation to the solution to (5.3) at each time step. To begin with the description of the scheme, we first initialize it by setting

$$V_0 = G_{\mathcal{T}_{\mathcal{Y}}} \mathcal{H}_\alpha g, \quad V_1 = G_{\mathcal{T}_{\mathcal{Y}}} \left(\mathcal{H}_\alpha g + \Delta t \mathcal{H}_\alpha h + \frac{1}{2} (\Delta t)^2 \partial_t^2 \mathcal{U}(0) \right), \quad (6.46)$$

where $\partial_t^2 \mathcal{U}(0) = \mathcal{H}_\alpha w$ and w solves (6.19). For $k = 1, \ldots, K - 1$, $V_{k+1} \in \mathbb{V}(\mathcal{T}_{\mathcal{Y}})$ solves

$$\frac{1}{\Delta t^2} \langle \mathrm{tr}(V_{k+1} - 2V_k + V_{k-1}), \mathrm{tr}\, W \rangle + a_{\mathcal{Y}}(V_k, W) = \langle f_k, \mathrm{tr}\, W \rangle \quad (6.47)$$

for all $W \in \mathbb{V}(\mathcal{T}_{\mathcal{Y}})$. As in the previous section, we define an approximated solution to problem (1.1) as

$$U_{\Delta t} = \{U_k\}_{k=0}^K \subset S_0^1(\Omega, \mathcal{T}), \quad U_{\Delta t} = \mathrm{tr}\, V_{\Delta t}. \quad (6.48)$$

Note that in the k-th step we can assume to have $\mathrm{tr}\, V_k$ and $\mathrm{tr}\, V_{k-1}$ already computed. However, to be able to compute $a_{\mathcal{Y}}(V_k, W)$ we need the discrete function V_k on the whole cylinder $\mathcal{C}_{\mathcal{Y}}$. The function V_k can be obtained by solving the following discrete problem: Find $V_k \in \mathbb{V}(\mathcal{T}_{\mathcal{Y}})$, an extension of $\mathrm{tr}\, V_k$, such that

$$a_{\mathcal{Y}}(V_k, W) = 0 \quad \forall W \in \mathbb{V}(\mathcal{T}_{\mathcal{Y}}) : \mathrm{tr}\, W = 0.$$

Hence, the leapfrog scheme when applied to our problem requires the resolution of a linear system at each time step even when using mass-lumping. As a consequence of the nonlocality of $(-\Delta)^s$, the explicit nature of the leapfrog scheme seems to be lost. Further details on the implementation of this scheme are given in Sect. 8.2.

In the analysis that follows, the following discrete inverse inequality will be instrumental.

Lemma 9 (discrete inverse inequality) *Let $\eta \in \mathbb{H}^{-s}(\Omega)$ and let $X \in \mathbb{V}(\mathcal{T}_{\mathcal{Y}})$ be the solution to*

$$a_{\mathcal{Y}}(X, W) = \langle \eta, \mathrm{tr}\, W \rangle \quad \forall W \in \mathbb{V}(\mathcal{T}_{\mathcal{Y}}). \quad (6.49)$$

We thus have that $\|X\|_{\mathcal{C}} \lesssim \|\mathrm{tr}\, X\|_{\mathbb{H}^s(\Omega)}$ and that

$$\|X\|_{\mathcal{C}} \leq C_{\mathrm{inv}} h_{\mathcal{T}}^{-s} \|\mathrm{tr}\, X\|_{L^2(\Omega)}, \quad (6.50)$$

for some constant $C_{\mathrm{inv}} > 0$

Proof Define $Z := \mathrm{tr}\, X \in S_0^1(\Omega, \mathcal{T})$. There exists $\chi \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathcal{Y}})$ such that $\mathrm{tr}\, \chi = Z$. In fact $\chi = \mathcal{H}_\alpha Z$, where $\mathcal{H}_\alpha$ is defined in (5.4).

Let us introduce the operator $\tilde{\Pi} = \Pi_{x'} \otimes \tilde{\Pi}_y^r$, where $\tilde{\Pi}_y^r$ is a slight modification of the operator of [5, Section 5.5.1]: on the first interval I_1, interpolation at the edge point 0 is used rather than in the middle point of I_1. The operator $\Pi_{x'}$ corresponds

to the $L^2(\Omega)$-orthogonal projection operator defined in (6.4). Define $\tilde{X} = \tilde{\Pi}\chi$ and notice that the stability properties of $\tilde{\Pi}$ and $\tilde{\Pi}_y^r$ yield

$$\|\tilde{X}\|_{\mathcal{C}} \lesssim \|\chi\|_{\mathcal{C}}.$$

This, in view of the fact that $\|\chi\|_{\mathcal{C}} \lesssim \|Z\|_{\mathbb{H}^s(\Omega)}$, implies $\|\tilde{X}\|_{\mathcal{C}} \lesssim \|Z\|_{\mathbb{H}^s(\Omega)}$.

Now, since $Z \in S_0^1(\Omega, \mathscr{T})$, we have that $\operatorname{tr} \tilde{X} = \operatorname{tr} \chi = Z = \operatorname{tr} X$ and then that $\operatorname{tr}(X - \tilde{X}) = 0$. Since $X - \tilde{X} \in \mathbb{V}(\mathscr{T}_{\mathscr{Y}})$, we can thus invoke problem (6.49) and conclude that

$$a_{\mathscr{Y}}(X, X - \tilde{X}) = \langle \eta, \operatorname{tr}(X - \tilde{X}) \rangle = 0,$$

which yields

$$\begin{aligned} a_{\mathscr{Y}}(\tilde{X}, \tilde{X}) &= a_{\mathscr{Y}}((\tilde{X} - X) + X, (\tilde{X} - X) + X) \\ &= a_{\mathscr{Y}}((\tilde{X} - X), (\tilde{X} - X)) + a_{\mathscr{Y}}(X, X) \geq a_{\mathscr{Y}}(X, X). \end{aligned}$$

This immediately implies that $\|X\|_{\mathcal{C}} \leq \|\tilde{X}\|_{\mathcal{C}}$, and thus, since $\|\tilde{X}\|_{\mathcal{C}} \lesssim \|Z\|_{\mathbb{H}^s(\Omega)}$, we can conclude that

$$\|X\|_{\mathcal{C}} \lesssim \|Z\|_{\mathbb{H}^s(\Omega)}.$$

Since $Z = \operatorname{tr} X$, we have thus obtained the desired estimate $\|X\|_{\mathcal{C}} \lesssim \|\operatorname{tr} X\|_{\mathbb{H}^s(\Omega)}$. The estimate (6.50) thus follows, for instance, from the results of [17]. □

6.5.1 Stability

To analyze the fully discrete scheme (6.46), (6.47), we define, for $k = 1, \ldots, K$, the discrete energy

$$\mathscr{E}_k(W_{\Delta t}) := \tfrac{1}{2} \|\operatorname{tr} \eth W_k\|_{L^2(\Omega)}^2 + \tfrac{1}{2} a_{\mathscr{Y}}(W_k, W_{k-1}), \tag{6.51}$$

where the bilinear form $a_{\mathscr{Y}}$ is defined in (5.2).

In the result that follows we show the nonnegativity of the discrete energy $\mathscr{E}_k$ under the following CFL condition: Δt is chosen to be sufficiently small such that

$$1 - C_{\text{inv}}^2 \frac{(\Delta t)^2}{2h_{\mathscr{T}}^{2s}} \geq \theta > 0, \quad \theta \in (0, 1). \tag{6.52}$$

The constant C_{inv} is as in (6.50).

Lemma 10 (CFL condition and nonnegativity of $\mathscr{E}_k$) *If* (6.52) *holds, then*

$$\mathscr{E}_k(V_{\Delta t}) \geq \frac{\theta}{2} \| \operatorname{tr} \eth V_k \|_{L^2(\Omega)}^2 + \frac{1}{4} \left[\|V_k\|_{\mathcal{C}}^2 + \|V_{k-1}\|_{\mathcal{C}}^2 \right] \geq 0 \qquad (6.53)$$

for all $k \in \{1, \cdots, K\}$.

Proof We invoke the inverse inequality (6.50) and the CFL condition (6.52) to arrive at

$$\begin{aligned}
2a_{\mathcal{Y}}(V_k, V_{k-1}) &= \|V_k\|_{\mathcal{C}}^2 + \|V_{k-1}\|_{\mathcal{C}}^2 - \|V_k - V_{k-1}\|_{\mathcal{C}}^2 \\
&\geq \|V_k\|_{\mathcal{C}}^2 + \|V_{k-1}\|_{\mathcal{C}}^2 - C_{\mathrm{inv}}^2 h_{\mathscr{T}}^{-2s} \| \operatorname{tr}(V_k - V_{k-1}) \|_{L^2(\Omega)}^2 \\
&\geq \|V_k\|_{\mathcal{C}}^2 + \|V_{k-1}\|_{\mathcal{C}}^2 + 2(\theta - 1) \| \operatorname{tr} \eth V_k \|_{L^2(\Omega)}^2,
\end{aligned}$$

where, in the last step, we have used definition (6.15). Consequently,

$$\mathscr{E}_k(V_{\Delta t}) \geq \frac{1}{2} \| \operatorname{tr} \eth V_k \|_{L^2(\Omega)}^2 + \frac{1}{4} \|V_k\|_{\mathcal{C}}^2 + \frac{1}{4} \|V_{k-1}\|_{\mathcal{C}}^2 + \frac{(\theta - 1)}{2} \| \operatorname{tr} \eth V_k \|_{L^2(\Omega)}^2,$$

which immediately yields (6.53). This concludes the proof.   □

Lemma 11 (energy conservation) *If* $f \equiv 0$, *then the fully discrete scheme* (6.46), (6.47) *conserves energy, i.e., for all* $k \in \{1, \ldots, K\}$, *we have that*

$$\mathscr{E}_k(V_{\Delta t}) = \mathscr{E}_1(V_{\Delta t}). \qquad (6.54)$$

If $f \neq 0$, *then, for* $\ell \in \{1, \ldots, K\}$, *we have that*

$$\mathscr{E}_\ell(V_{\Delta t})^{\frac{1}{2}} \leq \mathscr{E}_1(V_{\Delta t})^{\frac{1}{2}} + \frac{1}{\sqrt{2\theta}} \sum_{k=1}^{\ell} \Delta t \|f_k\|_{L^2(\Omega)}. \qquad (6.55)$$

In particular, we have that $\mathscr{E}_K(V_{\Delta t})^{\frac{1}{2}} \leq \mathscr{E}_1(V_{\Delta t})^{\frac{1}{2}} + \frac{1}{\sqrt{2\theta}} \|f\|_{\ell^1(L^2(\Omega))}$.

Proof Set $W = (2\Delta t)^{-1}(V_{k+1} - V_{k-1}) = 2^{-1}(\eth V_{k+1} + \eth V_k)$ in (6.47). This yields

$$\frac{1}{\Delta t} \left(\mathscr{E}_{k+1}(V_{\Delta t}) - \mathscr{E}_k(V_{\Delta t}) \right) = \frac{1}{2} \langle f_k, \operatorname{tr}(\eth V_{k+1} + \eth V_k) \rangle. \qquad (6.56)$$

In the case that $f \equiv 0$, (6.56) immediately yields (6.54). If $f \neq 0$, a trivial application of the Cauchy–Schwarz inequality reveals that

$$\mathscr{E}_{k+1}(V_{\Delta t}) - \mathscr{E}_k(V_{\Delta t}) \leq \frac{\Delta t}{2} \|f_k\|_{L^2(\Omega)} \left(\| \operatorname{tr} \eth V_{k+1} \|_{L^2(\Omega)} + \| \eth V_k \|_{L^2(\Omega)} \right).$$

Invoke the estimate (6.53) and conclude, for $k \in \{1, \ldots, K - 1\}$, that $\mathscr{E}_k(V_{\Delta t}) \geq (\theta/2)\| \operatorname{tr} \eth V_k\|^2_{L^2(\Omega)}$. Thus,

$$\mathscr{E}_{k+1}(V_{\Delta t}) - \mathscr{E}_k(V_{\Delta t}) \leq \frac{\Delta t}{\sqrt{2\theta}} \| f_k \|_{L^2(\Omega)} \left(\mathscr{E}_{k+1}^{\frac{1}{2}}(V_{\Delta t}) + \mathscr{E}_k^{\frac{1}{2}}(V_{\Delta t}) \right).$$

Consequently, we arrive at $\mathscr{E}_{k+1}(V_{\Delta t})^{\frac{1}{2}} - \mathscr{E}_k(V_{\Delta t})^{\frac{1}{2}} \leq (\Delta t/\sqrt{2\theta})\| f_k \|_{L^2(\Omega)}$ which, by adding over ℓ, yields (6.55). This concludes the proof. $\qquad \square$

Lemma 12 (stability) *The fully discrete scheme* (6.46)*,* (6.47) *is stable: for* $\ell \in \{1, \ldots, K\}$*, we have that*

$$\| \operatorname{tr} \eth V_\ell\|_{L^2(\Omega)} + \|V_\ell\|_{\mathcal{C}} \lesssim \| \operatorname{tr} \eth V_1\|_{L^2(\Omega)} + \|V_0\|_{\mathcal{C}}$$
$$+ \|V_1\|_{\mathcal{C}} + \sum_{k=1}^{\ell} \Delta t \| f_k \|_{L^2(\Omega)}, \tag{6.57}$$

where the hidden constant is independent of $V_{\Delta t}$*,* Δt *and* $h_{\mathscr{T}}$ *but depends on the parameter* θ*.*

Proof We begin by noticing that (6.53) yields

$$\| \operatorname{tr} \eth V_\ell\|_{L^2(\Omega)} + \|V_\ell\|_{\mathcal{C}} \lesssim (\theta^{-1/2} + 1)\mathscr{E}_l(V_{\Delta t})^{\frac{1}{2}}. \tag{6.58}$$

Now, since

$$2a_{\mathscr{Y}}(V_1, V_0) = a_{\mathscr{Y}}(V_1, V_1) + a_{\mathscr{Y}}(V_0, V_0) - a_{\mathscr{Y}}(V_1 - V_0, V_1 - V_0)$$
$$\leq a_{\mathscr{Y}}(V_1, V_1) + a_{\mathscr{Y}}(V_0, V_0),$$

an application of the estimate (6.55) allows us to conclude that

$$\mathscr{E}_\ell(V_{\Delta t})^{\frac{1}{2}} \lesssim \| \operatorname{tr} \eth V_1\|_{L^2(\Omega)} + \|V_0\|_{\mathcal{C}} + \|V_1\|_{\mathcal{C}} + \frac{1}{\sqrt{\theta}} \sum_{k=1}^{\ell} \Delta t \| f_k \|_{L^2(\Omega)}.$$

The desired estimate (6.57) thus follows from replacing the previous estimate into (6.58). This concludes the proof. $\qquad \square$

6.5.2 Error analysis

We now present error estimates for the fully discrete approximation $U_{\Delta t}$ defined in (6.48) that is based on the solution $V_{\Delta t}$ to the fully discrete scheme (6.46), (6.47). The arguments are similar to the ones used to prove the results in Lemma 7, Lemma 8, and Theorem 6. For brevity we leave details to the reader.

Theorem 7 (error estimates for (6.48)) *Let u be the solution to* (1.1) *and let* $U_{\Delta t}$ *be its fully discrete approximation defined by* (6.48). *If* $\mathfrak{A}(f, g, h) < \infty$ *and* $\Xi(f, g, h) < \infty$, *then*

$$\|\partial_t u(t_{K-1/2}) - \mathfrak{d}U_K\|_{L^2(\Omega)} \lesssim h_{\mathscr{T}}^{1+s}\mathfrak{A}(f, g, h) + (\Delta t)^2 \Xi(f, g, h), \quad (6.59)$$

and

$$\|u(t_K) - U_K\|_{\mathbb{H}^s(\Omega)} \lesssim h_{\mathscr{T}}\mathfrak{A}(f, g, h) + (\Delta t)^2 \Xi(f, g, h), \quad (6.60)$$

where $\mathfrak{A}(f, g, h)$ *and* $\Xi(f, g, h)$ *are defined by* (4.20) *and* (4.10), *respectively, and the hidden constants are independent of* $U_{\Delta t}$, *u*, Δt, *and* $h_{\mathscr{T}}$.

6.6 Computable data

In (6.18) we considered

$$V_0 = G_{\mathscr{T}_{\gamma}}\mathcal{H}_{\alpha}g, \quad V_1 = G_{\mathscr{T}_{\gamma}}\left(\mathcal{H}_{\alpha}g + \Delta t \mathcal{H}_{\alpha}h + \frac{1}{2}(\Delta t)^2\partial_t^2\mathcal{U}(0)\right) \quad (6.61)$$

as initial data for the fully discrete schemes of Sects. 6.4 and 6.5. Since the action of $\mathcal{H}_{\alpha}$ involves the resolution of a problem posed on an infinite dimensional space, we immediately conclude that the initial data V_0 and V_1 *are not computable*.

To overcome this deficiency, we introduce the discrete extension operator $\mathcal{H}_{\alpha}^{\mathscr{T}}$, which is defined as follows: if $e \in S_0^1(\Omega, \mathscr{T})$, then $\mathcal{H}_{\alpha}^{\mathscr{T}}e := E \in \mathbb{V}(\mathscr{T}_{\gamma})$ is such that

$$a_{\gamma}(E, W) = 0 \quad \forall W \in \mathbb{V}(\mathscr{T}_{\gamma}) : \text{tr } W = 0, \quad \text{tr } E = e.$$

With $\mathcal{H}_{\alpha}^{\mathscr{T}}$ at hand, we define the following computable initial data:

$$\widetilde{V}_0 = \mathcal{H}_{\alpha}^{\mathscr{T}}\Pi_{x'}g, \quad \widetilde{V}_1 = \mathcal{H}_{\alpha}^{\mathscr{T}}\left(\Pi_{x'}g + \Delta t \Pi_{x'}h + \frac{1}{2}(\Delta t)^2 Z\right), \quad (6.62)$$

where $Z \in S_0^1(\Omega, \mathscr{T})$ solves $\langle Z, \text{tr } W \rangle = -a_{\gamma}(\widetilde{V}_0, W) + \langle f(0), \text{tr } W \rangle$ for all $W \in \mathbb{V}(\mathscr{T}_{\gamma})$. Notice that Z corresponds to a finite element approximation of $\text{tr } \partial_t^2\mathcal{U}(0)$.

If we consider $\widetilde{V}_0$ and $\widetilde{V}_1$, instead of V_0 and V_1, as initial data for the schemes of Sects. 6.4 and 6.5, then, to provide an a priori error analysis, it is necessary to modify the first two elements of the sequence $\Theta_{\Delta t}$, defined in (6.26), as follows:

$$\Theta_0 = \widetilde{V}_0 - G_{\mathscr{T}_{\gamma}}\mathcal{U}(t_0), \quad \Theta_1 = \widetilde{V}_1 - G_{\mathscr{T}_{\gamma}}\mathcal{U}(t_1).$$

In particular, it suffices to estimate

$$E_1(\Theta_{\Delta t})^{\frac{1}{2}} = \left(\frac{1}{2}\|\text{tr } \mathfrak{d}\Theta_1\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\Theta_{1/2}\|_{\mathbb{C}}^2\right)^{\frac{1}{2}}.$$

We present the following error estimates.

Lemma 13 (error estimates for Θ_0 and Θ_1) *If (V_0, V_1) and $(\widetilde{V}_0, \widetilde{V}_1)$ are defined by* (6.61) *and* (6.62)*, respectively, then*

$$\|\Theta_0\|_{\mathcal{C}} \lesssim h_{\mathscr{T}} \|g\|_{\mathbb{H}^{1+s}(\Omega)}, \tag{6.63}$$

and

$$\|\Theta_1\|_{\mathcal{C}} \lesssim h_{\mathscr{T}} \mathfrak{A}(f, g, h) + (\Delta t)^2 \varXi(f, g, h), \tag{6.64}$$

where $\mathfrak{A}(f, g, h)$ and $\varXi(f, g, h)$ are defined by (4.20) *and* (4.10)*, respectively, and the hidden constants are independent of g, h, (V_0, V_1), $(\widetilde{V}_0, \widetilde{V}_1)$, and $h_{\mathscr{T}}$.*

Proof Since $V_0, \tilde{V}_0 \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$, we can invoke property (6.7) and conclude that

$$\begin{aligned}
\|\widetilde{V}_0 - V_0\|_{\mathcal{C}}^2 &= a_{\mathcal{Y}}(\widetilde{V}_0 - V_0, \widetilde{V}_0 - G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{H}_\alpha \Pi_{x'} g) + a_{\mathcal{Y}}(\widetilde{V}_0 - V_0, G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{H}_\alpha(\Pi_{x'} g - g)) \\
&= a_{\mathcal{Y}}(\widetilde{V}_0 - V_0, \widetilde{V}_0 - \mathcal{H}_\alpha \Pi_{x'} g) + a_{\mathcal{Y}}(\widetilde{V}_0 - V_0, \mathcal{H}_\alpha(\Pi_{x'} g - g)) = \mathrm{I} + \mathrm{II}.
\end{aligned}$$

To bound I, we notice that $\mathrm{tr}(\widetilde{V}_0 - \mathcal{H}_\alpha \Pi_{x'} g) = 0$. On the other hand, $V_0 - \widetilde{V}_0$ satisfies

$$a_{\mathcal{Y}}(V_0 - \widetilde{V}_0, W) = 0 \quad \forall W \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}}) : \mathrm{tr}\, W = 0$$

and $\mathrm{tr}(V_0 - \widetilde{V}_0) = \mathrm{tr}\, G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{H}_\alpha g - \Pi_{x'} g$. Consequently, $\mathrm{I} = 0$. Now, since $\mathcal{H}_\alpha$ satisfies $\|\mathcal{H}_\alpha w\|_{\mathcal{C}} \lesssim \|w\|_{\mathbb{H}^s(\Omega)}$ for all $w \in \mathbb{H}^s(\Omega)$, we arrive at

$$|\mathrm{II}| \lesssim \|\widetilde{V}_0 - V_0\|_{\mathcal{C}} \|g - \Pi_{x'} g\|_{\mathbb{H}^s(\Omega)} \lesssim h_{\mathscr{T}} \|V_0 - V_0\|_{\mathcal{C}} \|g\|_{\mathbb{H}^{1+s}(\Omega)}.$$

Since $\mathrm{I} = 0$, the estimate for II yields (6.63).

We now control $\|\Theta_1\|_{\mathcal{C}}$. A basic application of the triangle inequality together with estimate (6.37) reveal that

$$\begin{aligned}
\|\Theta_1\|_{\mathcal{C}} &\leq \|\widetilde{V}_1 - V_1\|_{\mathcal{C}} + \|V_1 - G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{U}(t_1)\|_{\mathcal{C}} \\
&\lesssim \|\widetilde{V}_1 - V_1\|_{\mathcal{C}} + (\Delta t)^3 \|\partial_t^3 \nabla \mathcal{U}\|_{L^\infty(0,T;L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}}))}.
\end{aligned}$$

It thus suffices to bound $\|\widetilde{V}_1 - V_1\|_{\mathcal{C}}$. To accomplish this task, we first notice that a simple application of Taylor's Theorem yields

$$\begin{aligned}
\widetilde{V}_1 &= \mathcal{H}_\alpha^{\mathscr{T}} \Pi_{x'} \left[\mathrm{tr}\, \mathcal{U}(0) + \Delta t \, \mathrm{tr}\, \partial_t \mathcal{U}(0) + \tfrac{(\Delta t)^2}{2} \mathrm{tr}\, \partial_t^2 \mathcal{U}(0) \right] \\
&\quad + \tfrac{(\Delta t)^2}{2} \mathcal{H}_\alpha^{\mathscr{T}}(Z - \Pi_{x'} \mathrm{tr}\, \partial_t^2 \mathcal{U}(0)) = \mathcal{H}_\alpha^{\mathscr{T}} \Pi_{x'} \left[\mathrm{tr}\, \mathcal{U}(t_1) - \tfrac{(\Delta t)^3}{6} \mathrm{tr}\, \partial_t^3 \mathcal{U}(\overline{\zeta}) \right] \\
&\quad + \tfrac{(\Delta t)^2}{2} \mathcal{H}_\alpha^{\mathscr{T}}(Z - \Pi_{x'} \mathrm{tr}\, \partial_t^2 \mathcal{U}(0)),
\end{aligned} \tag{6.65}$$

with $\overline{\zeta} \in (0, t_1)$. Similar arguments allow us to conclude that

$$V_1 = G_{\mathscr{T}_y}\left(\mathcal{U}(t_1) - \frac{(\varDelta t)^3}{6}\partial_t^3\mathcal{U}(\underline{\zeta})\right), \qquad (6.66)$$

with $\underline{\zeta} \in (0, t_1)$. Consequently,

$$\begin{aligned}
\|\widetilde{V}_1 - V_1\|_{\mathcal{C}} &\lesssim \|\mathcal{H}_\alpha^{\mathscr{T}} \varPi_{x'} \operatorname{tr}\mathcal{U}(t_1) - G_{\mathscr{T}_y}\mathcal{U}(t_1)\|_{\mathcal{C}} \\
&\quad + (\varDelta t)^3\|\partial_t^3\nabla\mathcal{U}\|_{L^\infty(0,T;L^2(y^\alpha,\mathcal{C}))} \\
&\quad + (\varDelta t)^2\|\mathcal{H}_\alpha^{\mathscr{T}}(Z - \varPi_{x'}\operatorname{tr}\partial_t^2\mathcal{U}(0))\|_{\mathcal{C}} = \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

To estimate I we invoke the same arguments that lead to (6.63):

$$\mathrm{I} \lesssim h_{\mathscr{T}}\|\operatorname{tr}\mathcal{U}(t_1)\|_{\mathbb{H}^{1+s}(\varOmega)} \lesssim h_{\mathscr{T}}\|\operatorname{tr}\mathcal{U}\|_{L^\infty(0,T;\mathbb{H}^{1+s}(\varOmega))}.$$

A bound for the term III follows from stability results. The collection of these estimates yield (6.64) □

Lemma 14 (estimate for $E_1(\varTheta_{\varDelta t})$) *If we consider $\widetilde{V}_0$ and $\widetilde{V}_1$ as initial data for the schemes of Sects. 6.4 and 6.5, we then have that*

$$\begin{aligned}
\|\operatorname{tr}\mathfrak{d}\varTheta_1\|_{L^2(\varOmega)} &= \frac{1}{\varDelta t}\|\operatorname{tr}(\varTheta_1 - \varTheta_0)\|_{L^2(\varOmega)} \\
&\lesssim h_{\mathscr{T}}^{1+s}\mathfrak{A}(f, g, h) + (\varDelta t)^2\varXi(f, g, h),
\end{aligned} \qquad (6.67)$$

and

$$\|\varTheta_{1/2}\|_{\mathcal{C}} \lesssim \|\varTheta_0\|_{\mathcal{C}} + \|\varTheta_1\|_{\mathcal{C}} \lesssim h_{\mathscr{T}}\mathfrak{A}(f, g, h) + (\varDelta t)^2\varXi(f, g, h), \qquad (6.68)$$

where $\mathfrak{A}(f, g, h)$ and $\varXi(f, g, h)$ are defined by (4.20) and (4.10), respectively, and the hidden constants are independent of g, h, (V_0, V_1), $(\widetilde{V}_0, \widetilde{V}_1)$, and $h_{\mathscr{T}}$

Proof The proof of (6.68) follows directly from the estimates (6.63) and (6.64). In what follows we derive (6.67). To accomplish this task and simplify notation, we define

$$\mathfrak{D}(\mathcal{U}) := \mathcal{U}(0) + \varDelta t\,\partial_t\mathcal{U}(0) + \frac{1}{2}(\varDelta t)^2\partial_t^2\mathcal{U}(0).$$

Now, notice that

$$\begin{aligned}
\|\operatorname{tr}(\varTheta_1 - \varTheta_0)\|_{L^2(\varOmega)} &= \|\operatorname{tr}(\widetilde{V}_1 - G_{\mathscr{T}_y}\mathcal{U}(t_1) - \widetilde{V}_0 + G_{\mathscr{T}_y}\mathcal{U}(0))\|_{L^2(\varOmega)} \\
&\leq \|\operatorname{tr}(\widetilde{V}_1 - \mathfrak{D}(G_{\mathscr{T}_y}\mathcal{U}) - \widetilde{V}_0 + G_{\mathscr{T}_y}\mathcal{U}(0))\|_{L^2(\varOmega)} \\
&\quad + \|\operatorname{tr}(\mathfrak{D}(G_{\mathscr{T}_y}\mathcal{U}) - G_{\mathscr{T}_y}\mathcal{U}(t_1))\|_{L^2(\varOmega)}.
\end{aligned}$$

The trace estimate (2.9), the stability property (6.8) and an application of Taylor's Theorem reveal that

$$\| \operatorname{tr}(\mathfrak{D}(G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{U}) - G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{U}(t_1))\|_{L^2(\Omega)} \lesssim \|\mathfrak{D}(G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{U}) - G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{U}(t_1)\|_{\mathcal{C}}$$
$$\lesssim \|\mathfrak{D}(\mathcal{U}) - \mathcal{U}(t_1)\|_{\mathcal{C}} \lesssim (\Delta t)^3 \|\partial_t^3 \nabla \mathcal{U}(\cdot, z)\|_{L^\infty(0,T;L^2(y^\alpha,\mathcal{C}))}.$$

We now use the definitions of $\widetilde{V}_0$ and $\widetilde{V}_1$ to arrive at

$$\| \operatorname{tr}(\widetilde{V}_1 - \mathfrak{D}(G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{U}) - \widetilde{V}_0 + G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{U}(0))\|_{L^2(\Omega)}$$
$$\leq (\Delta t)\|\Pi_{x'}h - \operatorname{tr} G_{\mathscr{T}_{\mathcal{Y}}}\partial_t \mathcal{U}(0)\|_{L^2(\Omega)} + (\Delta t)^2 \|Z - \operatorname{tr} G_{\mathscr{T}_{\mathcal{Y}}}\partial_t^2 \mathcal{U}(0)\|_{L^2(\Omega)}.$$

To bound $\|\Pi_{x'}h - \operatorname{tr} G_{\mathscr{T}_{\mathcal{Y}}}\partial_t \mathcal{U}(0)\|_{L^2(\Omega)}$ we proceed as follows:

$$\|\Pi_{x'}h - \operatorname{tr} G_{\mathscr{T}_{\mathcal{Y}}}\partial_t \mathcal{U}(0)\|_{L^2(\Omega)} \leq \|\Pi_{x'}h - h\|_{L^2(\Omega)}$$
$$+ \| \operatorname{tr}\partial_t \mathcal{U}(0) - \operatorname{tr} G_{\mathscr{T}_{\mathcal{Y}}}\partial_t \mathcal{U}(0)\|_{L^2(\Omega)} \lesssim h_{\mathscr{T}}^{1+s}(\|h\|_{\mathbb{H}^{1+s}(\Omega)} + \mathfrak{A}(f, g, h)),$$

where we have used (6.28) with $\ell = 1$. Similar arguments allow us to control $\|Z - \operatorname{tr} G_{\mathscr{T}_{\mathcal{Y}}}\partial_t^2 \mathcal{U}(0)\|_{L^2(\Omega)}$. This concludes the proof. □

Remark 6 (*influence of computable data in error estimates*) If $(\widetilde{V}_0, \widetilde{V}_1)$ are used as initial data for the trapezoidal multistep method of Sect. 6.4 and the leapfrog scheme of Sect. 6.5, then the error estimates of Theorems 6 and 7 hold with no modifications.

7 Polygonal domains $\subset \mathbb{R}^2$

In this section, we let $\Omega \subset \mathbb{R}^2$ be an open and bounded, but not necessarily convex, polygon and provide a priori error estimates for the trapezoidal multistep method (6.18)–(6.21). Notice that the error estimates that we have previously derived for this scheme, i.e., estimates (6.44) and (6.45), rely on the convexity of the domain Ω. Given $g \in H^{-1}(\Omega)$, we consider the Dirichlet problem

$$\mathcal{L}w = g \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega. \tag{7.1}$$

It is immediate that problem (7.1) has a unique solution $w \in H_0^1(\Omega)$. However, in general the solution $w \notin H^2(\Omega)$, even if $g \in C^\infty(\bar{\Omega})$ [31]. More precisely, for every $\epsilon > 0$, there exists a polygon Ω_ϵ and a smooth function g_ϵ such that the corresponding solution w_ϵ does not belong to $H^{3/2+\epsilon}(\Omega_\epsilon)$ [53]. However, under additional regularity assumptions on A, c, and g, a regularity analysis can be carried out in weighted function spaces related to the geometry of $\partial\Omega$; see Proposition 4 below.

7.1 Regularity estimates in weighted spaces

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded, but not necessarily convex, polygon with straight sides and corners $\{\boldsymbol{c}\}$ and $\beta \geq 0$. We define the weight and the weighted Lebesgue space

$$\Omega \ni x \mapsto \Phi(x) = \prod_{c} |x - c|, \quad L_\beta^2(\Omega) = L^2(\Phi^{2\beta}, \Omega).$$

We also define $H_\beta^2(\Omega)$ as the closure of $H^2(\Omega) \cap H_0^1(\Omega)$ with respect to the norm

$$\|v\|_{H_\beta^2(\Omega)} = \|v\|_{H^1(\Omega)} + \|D^2 v\|_{L^2(\Phi^{2\beta}, \Omega)}. \tag{7.2}$$

With this setting at hand, we present the following regularity result for (7.1).

Proposition 4 (weighted regularity estimate) *Let $A \in W^{1,\infty}(\Omega, GL(\mathbb{R}^2))$ be uniformly positive definite and $c \in W^{1,\infty}(\Omega, \mathbb{R})$ with $c \geq 0$. Then, there exists $\beta \in [0, 1)$, depending only on Ω, A, and c, such that for $g \in L_\beta^2(\Omega)$ the solution w of (7.1) belongs to $H_\beta^2(\Omega)$. In addition, the solution w satisfies the estimate*

$$\|w\|_{H_\beta^2(\Omega)} \lesssim \|\mathcal{L}w\|_{L_\beta^2(\Omega)} = \|g\|_{L_\beta^2(\Omega)}, \tag{7.3}$$

where the hidden constant is independent of g.

Proof The result follows from [4, Theorem 1.1]. To observe this, it suffices to set, in the notation of that reference, $m = 1$, $b_j = 0$, and $\beta = 1 - a$. □

The following space regularity estimates follow from Propositions 2 and 4.

Theorem 8 (global regularity of $\mathscr{U}$: weighted estimates in Ω) *Let A and c satisfy the assumptions of Proposition 4. Let $\mathscr{U}$ solve (1.2), (1.3) for $s \in (0, 1)$. Then, there exists $\beta \in [0, 1)$, depending only on Ω, A, and c, such that the following regularity assertions hold:*

(i) For $0 \leq \nu' < 1 - s$, we have that

$$\|\mathscr{U}\|_{L^2(0,T;L^2(\omega_{\alpha-2\nu',\theta},(0,\infty);H_\beta^2(\Omega)))}^2 \lesssim \left(\|g\|_{\mathbb{H}^{1+\nu'+s}(\Omega)}^2 \right.$$
$$\left. + \|h\|_{\mathbb{H}^{1+\nu'}(\Omega)}^2 + \|f\|_{L^2(0,T;\mathbb{H}^{1+\nu'}(\Omega))}^2 \right). \tag{7.4}$$

(ii) For $0 \leq \tilde{\nu} < 1 + s$, there exists $\kappa > 1$ such that

$$\|\partial_y^{\ell+1} \mathscr{U}\|_{L^2(0,T;L^2(\omega_{\alpha+2(\ell+1)-2\tilde{\nu},\theta},(0,\infty);H_\beta^2(\Omega)))}^2 \lesssim \kappa^{2(\ell+1)}(\ell+1)!^2$$
$$\cdot \left(\|g\|_{\mathbb{H}^{1+\tilde{\nu}+s}(\Omega)}^2 + \|h\|_{\mathbb{H}^{1+\tilde{\nu}}(\Omega)}^2 + \|f\|_{L^2(0,T;\mathbb{H}^{1+\tilde{\nu}}(\Omega))}^2 \right), \tag{7.5}$$

for all $\ell \in \mathbb{N}_0$.

In both estimates the hidden constants are independent of $\mathscr{U}$ and the problem data. In addition, in (7.5) the hidden constant is also independent of ℓ.

Proof The proof of (7.5) follows from (4.8) and (7.3). In fact, for fixed $y > 0$ and $t > 0$, set $w = \partial_y^{\ell+1}\mathscr{U}(\cdot, y, t)$ in (7.1) and hence $g = \partial_y^{\ell+1}\mathcal{L}\mathscr{U}(\cdot, y, t)$. Since $\beta \geq 0$, we have that $g \in L_\beta^2(\Omega)$. We can thus apply the estimate (7.3) to arrive at

$$\|\partial_y^{\ell+1}\mathscr{U}(\cdot, y, t)\|_{H_\beta^2(\Omega)} \lesssim \|\partial_y^{\ell+1}\mathcal{L}\mathscr{U}(\cdot, y, t)\|_{L_\beta^2(\Omega)}.$$

Square the previous estimate, multiply by the weight $\omega_{\alpha+2(\ell+1)-2\tilde{\nu},\gamma}$ and integrate, first, with respect to y over $(0, \infty)$ and then with respect to t over $(0, T)$. We thus utilize (4.8) to arrive at the desired estimate (7.5).

Similar arguments allow us to derive (7.4). For brevity, we skip details. $\square$

We conclude this section by mentioning that, on the basis of the estimate (4.18), the following space–time regularity estimate can be derived. Let $0 \leq \sigma < s$ and $0 \leq \tilde{\nu} < 1 + s$. Then, there exists $\kappa > 1$ such that the following estimate holds for all $\ell \in \mathbb{N}_0$:

$$\|\partial_t^2 \partial_y^{\ell+1}\mathscr{U}\|^2_{L^2(0,T;L^2(\omega_{\alpha+2(\ell+1)-2\tilde{\nu},\theta},(0,\infty);H_\beta^2(\Omega)))} \lesssim \kappa^{2(\ell+1)}(\ell+1)!^2$$
$$\cdot \left(\|g\|^2_{\mathbb{H}^{1+\tilde{\nu}+3s}(\Omega)} + \|h\|^2_{\mathbb{H}^{1+\tilde{\nu}+2s}(\Omega)} + \|f\|^2_{L^2(0,T;\mathbb{H}^{1+\tilde{\nu}+2s}(\Omega))} \right). \tag{7.6}$$

The hidden constant is independent of $\mathscr{U}$, ℓ, and the problem data. Consequently,

$$\|\partial_t^2 \partial_y^{\ell+1}\mathscr{U}\|^2_{L^2(0,T;L^2(\omega_{\alpha+2(\ell+1),\theta},(0,\infty);H_\beta^2(\Omega)))} \lesssim \kappa^{2(\ell+1)}(\ell+1)!^2\mathfrak{A}(f, g, h)^2,$$

where $\mathfrak{A}$ is defined in (4.20); compare with (4.21).

7.2 Meshes, finite element spaces, and quasi-interpolation operators

In the next section we will present error estimates for the trapezoidal scheme on the basis of a piecewise linear finite element discretization on properly refined meshes in Ω. We stress that $\Omega \subset \mathbb{R}^2$ is an open and bounded, but not necessarily convex, polygon. The analysis requires meshes and approximation operators suitable for the approximation of functions with $H_\beta^2(\Omega)$ regularity. This is achieved with appropriate refinement toward the vertices of Ω [27].

In what follows, we will consider nested sequences $\{\mathscr{T}^\ell\}_{\ell \geq 0}$ of triangulations of Ω that are generated by bisection-tree refinement of a coarse, regular initial triangulation $\mathscr{T}^0$ of Ω. We denote by $h_\ell = \max\{\text{diam}(K) : K \in \mathscr{T}^\ell\}$ the meshwidth of $\mathscr{T}^\ell$. On these meshes we will consider the finite element spaces $S_0^1(\Omega, \mathscr{T}^\ell)$ defined in (6.2).

We present the existence of a sequence of meshes $\{\mathscr{T}_\beta^\ell\}_{\ell \geq 0}$ and operators Π_β^ℓ that satisfy a series of properties on which our theory rests. In what follows, $\beta \in (0, 1]$.

Lemma 15 (meshes $\{\mathscr{T}_\beta^\ell\}_{\ell \geq 0}$ and operators Π_β^ℓ) *There is a sequence $\{\mathscr{T}_\beta^\ell\}_{\ell \geq 0}$ of nested, regular bisection-tree meshes with corresponding quasi-interpolation opera-tors $\Pi_\beta^\ell : L^2(\Omega) \to S_0^1(\Omega, \mathscr{T}_\beta^\ell)$ such that the following properties hold:*

(i) $N_\ell := \dim S_0^1(\Omega, \mathcal{T}_\beta^\ell) \lesssim h_\ell^{-2}$.
(ii) Simultaneous stability:

$$\|\Pi_\beta^\ell v\|_{L^2(\Omega)} \lesssim \|v\|_{L^2(\Omega)} \qquad \forall v \in L^2(\Omega),$$
$$\|\Pi_\beta^\ell v\|_{H^1(\Omega)} \lesssim \|v\|_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$

(iii) Projection property: $\Pi_\beta^\ell v = v$ *for all* $v \in S_0^1(\Omega, \mathcal{T}_\beta^\ell)$.
(iv) Optimal approximation rates for $H_0^1(\Omega)$ *and* $H_\beta^2(\Omega)$*-functions:*

$$N_\ell \|w - \Pi_\beta^\ell w\|_{L^2(\Omega)}^2 \lesssim \|w\|_{H^1(\Omega)}^2 \tag{7.7}$$
$$N_\ell \|w - \Pi_\beta^\ell w\|_{L^2(\Omega)}^2 + \|\nabla_{x'}(w - \Pi_\beta^\ell w)\|_{L^2(\Omega)}^2 \lesssim N_\ell^{-1} \|w\|_{H_\beta^2(\Omega)}^2, \tag{7.8}$$

for all $w \in H_0^1(\Omega)$ *and all* $w \in H_0^1(\Omega) \cap H_\beta^2(\Omega)$*, respectively.*

In (ii) and (iv), constants hidden in $\lesssim$ *are independent of* ℓ.

Proof The meshes $\{\mathcal{T}_\beta^\ell\}_{\ell \geq 0}$ are constructed as described in [27]. By construction, property (i) follows. Approximation properties follow from [27] and [44, Section 5]:

$$h_\ell^{-1}\|w - I_{\mathcal{T}}w\|_{L^2(\Omega)} + \|\nabla_{x'}(w - I_{\mathcal{T}}w)\|_{L^2(\Omega)} \lesssim h_\ell |w|_{H_\beta^2(\Omega)}, \tag{7.9}$$

where $I_{\mathcal{T}}$ denotes the nodal interpolant. The operator Π_β^ℓ can be taken as the $L^2(\Omega)$-orthogonal projection. The results of [26] thus guarantee the stability estimates stated in item (ii). The estimates (7.7) and (7.8) follow from the fact that Π_β^ℓ preserves the finite element space combined with (7.9). We refer the reader to the proof of [5, Lemma 8] for further details. □

7.3 Approximation properties for the elliptic projector

We derive approximation properties for the elliptic projector $G_{\mathcal{T}_{\mathcal{Y}}}$, defined in (6.7), within the finite element framework described in Sect. 7.2.

Let $\beta \in (0, 1]$ be such that (7.3) holds. Let $(\mathcal{T}_\beta^\ell)_{\ell \geq 0}$ be a sequence of uniformly shape-regular meshes of meshwidth h_ℓ such that the properties stated in Lemma 15 hold. Within this framework, the following error estimates can be derived; see also [5, Theorem 5].

Lemma 16 (error estimates) *Fix* $0 < \sigma < 1$ *and, for each* ℓ*, denote by* $\mathcal{G}_\sigma^M$ *the geometric mesh on* $(0, \mathcal{Y})$ *defined in* (6.1)*, where* $\mathcal{Y} \sim |\log h_{\mathcal{T}}|$ *with a sufficiently large constant. Assume that* M *is such that* $c_1 M \leq \mathcal{Y} \leq c_2 M$ *with absolute constants* c_1 *and* c_2*. Let* $w \in \mathbb{H}^{1+s}(\Omega)$*. If* $\mathcal{W} = \mathcal{H}_\alpha w$ *denotes the truncated* α*-harmonic extension of* w*, then there exists a minimal slope* $\mathfrak{s}_{min}$ *such that for linear degree vectors* $\boldsymbol{r}$ *with slope* $\mathfrak{s} \geq \mathfrak{s}_{min}$ *there holds*

$$\|\nabla(\mathcal{W} - G_{\mathcal{T}_{\mathcal{Y}}}\mathcal{W})\|_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})} \lesssim h_\ell \|w\|_{\mathbb{H}^{1+s}(\Omega)}, \tag{7.10}$$

and

$$\|\operatorname{tr}(\mathcal{W} - G_{\mathscr{T}_{\mathcal{Y}}}\mathcal{W})\|_{L^2(\Omega)} \lesssim h_\ell^{1+s}\|w\|_{\mathbb{H}^{1+s}(\Omega)}. \tag{7.11}$$

The hidden constants are independent of $\mathcal{W}$, w, and h_ℓ.

Proof Replace, in the proof of Lemma 3, the operator $\Pi_{x'}$ by Π_β^ℓ. Invoke the stability and approximation properties of the latter, as described in Lemma 15, items (ii) and (iv), respectively, and utilize the weighted regularity estimate of [5, Theorem 2]. This concludes the proof. □

7.4 Error estimates

In previous sections we have elaborated all the ingredients that can allow us to conclude the following error estimates for the trapezoidal scheme.

Theorem 9 (error estimates for (6.21)) *Let u be the solution to (1.1) and let $U_{\Delta t}$ be its fully discrete approximation defined by (6.21). In the framework of Lemma 16, we have the following error estimates for the trapezoidal scheme*

$$\|\partial_t u(t_{K-1/2}) - \mathfrak{d}U_K\|_{L^2(\Omega)} \lesssim h_\ell^{1+s} + (\Delta t)^2, \tag{7.12}$$

and

$$\|u(t_{K-1/2}) - U_{K-1/2}\|_{\mathbb{H}^s(\Omega)} \lesssim h_\ell + (\Delta t)^2. \tag{7.13}$$

The hidden constants are independent of $U_{\Delta t}$, u, Δt, and h_ℓ.

Proof Invoke the arguments developed in the proof of Lemmas 7 and 8 upon utilizing the error estimates (7.10) and (7.11) for the weighted elliptic projector. □

We conclude this section with the following remark.

Remark 7 (*Leapfrog scheme*) The extension of the error estimates (6.59) and (6.60) to the setting described in Sect. 7.2 would require a stability result as (6.57). This, in turns, requires a CFL condition that limits the choice of the size of the time step to be bounded by the minimal meshsize in the spatial finite element mesh. In view of the nature of the meshes $\{\mathscr{T}_\beta^\ell\}_{\ell \geq 0}$ this would lead to an impractical algorithm. Several techniques have been developed in order to remove such a critical restriction. These include local time-stepping [20,34] and the incorporation of a subspace projection step inspired by numerical homogenisation [50].

8 Numerical results and implementation

Let $\{\phi_1, \ldots, \phi_{\mathcal{M}}\}$ denote a basis of $S^r((0, \mathscr{Y}), \mathcal{G}_\sigma^M)$ such that $\phi_1(0) = 1$ and $\phi_j(0) = 0$ for $j > 1$. The corresponding mass and stiffness matrices are denoted by $B_{\mathscr{Y}}$ and $A_{\mathscr{Y}}$:

$$\left(B_{\mathscr{Y}}\right)_{ij} = \int_0^{\mathscr{Y}} y^\alpha \phi_i(y)\phi_j(y)dy, \qquad \left(A_{\mathscr{Y}}\right)_{ij} = \int_0^{\mathscr{Y}} y^\alpha \phi_i'(y)\phi_j'(y)dy.$$

We denote by B_Ω and A_Ω the standard mass and stiffness matrices corresponding to the finite element space $S_0^1(\Omega, \mathscr{T})$. In what follows, we describe the implementation of a discrete Dirichlet-to-Neumann map. Once this operation is available the time-stepping methods that are proposed in this work can be implemented in a standard way; in the case of the implicit method some further steps may be needed in order to obtain an efficient algorithm.

8.1 Discrete Dirichlet-to-Neumann map

Given $U \in S_0^1(\Omega, \mathscr{T})$, we consider the problem: Find $V \in \mathbb{V}(\mathscr{T}_{\mathscr{Y}})$ and $\eta \in S_0^1(\Omega, \mathscr{T})$ such that

$$\begin{aligned} a_{\mathscr{Y}}(V, W) &= \langle \eta, \text{tr } W \rangle \qquad \forall W \in \mathbb{V}(\mathscr{T}_{\mathscr{Y}}), \\ \text{tr } V &= U. \end{aligned} \tag{8.1}$$

Let us denote by $\mathbf{U}$, $\mathbf{V}$, and $\boldsymbol{\eta}$ the coefficient vectors associated with the discrete functions U, V, and η, respectively. Note that the first N components of $\mathbf{V}$ and $\mathbf{U}$ are equal; we recall that $N = \#\mathscr{T}$, the number of degrees of freedom of $\mathscr{T}$. We denote the remaining components of $\mathbf{V}$ by $\widetilde{\mathbf{V}} = (\mathbf{V})_i$, $i = N + 1, \ldots, N\mathcal{M}$. With this notation at hand, the matrix system (8.1) takes the form

$$(B_{\mathscr{Y}} \otimes A_\Omega + A_{\mathscr{Y}} \otimes B_\Omega) \begin{pmatrix} \mathbf{U} \\ \widetilde{\mathbf{V}} \end{pmatrix} = \begin{pmatrix} B_\Omega \boldsymbol{\eta} \\ \mathbf{0} \end{pmatrix}.$$

We denote by $\widetilde{B}_{\mathscr{Y}}$ and $\widetilde{A}_{\mathscr{Y}}$ the matrices that are obtained by removing the first row and first column from $B_{\mathscr{Y}}$ and $A_{\mathscr{Y}}$, respectively. Let $\tilde{b}_{\mathscr{Y}}$ and $\tilde{a}_{\mathscr{Y}}$ denote the vectors containing the first components of the rows $i \in \{2, \ldots, \mathcal{M}\}$ of the matrices $B_{\mathscr{Y}}$ and $A_{\mathscr{Y}}$.

The vector $\widetilde{\mathbf{V}}$ is the solution to

$$(\widetilde{B}_{\mathscr{Y}} \otimes A_\Omega + \widetilde{A}_{\mathscr{Y}} \otimes B_\Omega)\widetilde{\mathbf{V}} = -\left(\tilde{b}_{\mathscr{Y}} \otimes A_\Omega + \tilde{a}_{\mathscr{Y}} \otimes B_\Omega\right)\mathbf{U}. \tag{8.2}$$

Once this system is solved we can obtain $\boldsymbol{\eta}$ by solving

$$B_\Omega \boldsymbol{\eta} = (bA_\Omega + aB_\Omega)\mathbf{U} + (\tilde{b}_{\mathscr{Y}}^T \otimes A_\Omega + \tilde{a}_{\mathscr{Y}}^T \otimes B_\Omega)\widetilde{\mathbf{V}},$$

where $b = (B_{\mathscr{Y}})_{11}$ and $a = (A_{\mathscr{Y}})_{11}$. We denote by L_h^s the matrix that describes the linear map $u \mapsto \eta$:

$$L_h^s \mathbf{U} = B_\Omega \boldsymbol{\eta}. \tag{8.3}$$

The main computational cost when solving (8.1) is the resolution of (8.2). This can be done efficiently by diagonalizing the small matrices $\widetilde{B}_{\mathscr{Y}}$ and $\widetilde{A}_{\mathscr{Y}}$. In fact, since both matrices are symmetric and positive definite, we can find a matrix of generalized eigenvectors X such that

$$X^T \widetilde{B}_{\mathscr{Y}} X = \mathrm{diag}(\mu_1, \ldots, \mu_{\mathcal{M}}), \qquad X^T \widetilde{A}_{\mathscr{Y}} X = \mathrm{diag}(1, \ldots, 1),$$

where $\mu_j > 0$ denote the eigenvalues of the generalized eigenvalue problem $\widetilde{B}_{\mathscr{Y}} X = \widetilde{A}_{\mathscr{Y}} X \mathrm{diag}(\mu_1, \ldots, \mu_{\mathcal{M}})$. The system (8.2) can thus be transformed to $\mathcal{M}$ independent linear systems

$$(\mu_j A_\Omega + B_\Omega)\hat{\mathbf{V}}_j = \left((X^T \tilde{b}_{\mathscr{Y}})_j A_\Omega + (X^T \tilde{a}_{\mathscr{Y}})_j B_\Omega \right) \mathbf{U},$$

where $\hat{\mathbf{V}}^T = (\hat{\mathbf{V}}_1^T, \ldots, \hat{\mathbf{V}}_{\mathcal{M}}^T)$,

$$\widetilde{\mathbf{V}} = (X \otimes I)\hat{\mathbf{V}},$$

and $I \in \mathbb{R}^{N \times N}$ is the identity matrix.

8.2 Leapfrog time-stepping scheme

Using the previously defined operator L_h^s, the leapfrog scheme can be now written in the familiar form

$$\frac{1}{\Delta t^2} B_\Omega (\mathbf{U}_{k+1} - 2\mathbf{U}_k + \mathbf{U}_{k-1}) + L_h^s \mathbf{U}_k = B_\Omega \mathbf{f}_k,$$

where $\mathbf{f}_k$ is the coefficient vector containing the L^2 projection of $f(t_k)$ onto the space $S_0^1(\Omega, \mathscr{T})$. The main cost is the application of $L_h^s \mathbf{U}_k$ in each step followed by the inversion of the mass matrix B_Ω; the latter being usually cheap.

8.3 Trapezoidal time-stepping scheme

The matrix system can be written, in a familiar form, involving only functions in $S_0^1(\Omega, \mathscr{T})$:

$$\frac{1}{\Delta t^2} B_\Omega (\mathbf{U}_{k+1} - 2\mathbf{U}_k + \mathbf{U}_{k-1}) + \frac{1}{4} L_h^s (\mathbf{U}_{k+1} + 2\mathbf{U}_k + \mathbf{U}_{k-1}) = \mathfrak{c} B_\Omega \mathbf{f}_k.$$

The difficulty now is that at each step we need to solve the system

$$\left(B_\Omega + \tfrac{\Delta t^2}{4} L_h^s\right) \mathbf{U}_{k+1} = \tilde{\mathbf{f}}_k,$$

where $\tilde{\mathbf{f}}_k$ contains known terms. While we could solve this system iteratively, it is more efficient to unwrap again the operator L_h^s to see that $\mathbf{U}_{k+1}$ satisfies the system

$$(B_\gamma \otimes A_\Omega + A_\gamma \otimes B_\Omega) \begin{pmatrix} \mathbf{U}_{k+1} \\ \tilde{\mathbf{V}}_{k+1} \end{pmatrix} = \begin{pmatrix} B_\Omega \eta_{k+1} \\ 0 \end{pmatrix}$$

and

$$B_\Omega \mathbf{U}_{k+1} + \tfrac{\Delta t^2}{4} B_\Omega \eta_{k+1} = \tilde{\mathbf{f}}_k.$$

Denoting by $E_1 = \mathrm{diag}(1, 0, \ldots, 0)$, we can write this as a single system

$$\left(\tfrac{\Delta t^2}{4} B_\gamma \otimes A_\Omega + (E_1 + \tfrac{\Delta t^2}{4} A_\gamma) \otimes B_\Omega\right) \begin{pmatrix} \mathbf{U}_{k+1} \\ \tilde{\mathbf{V}}_{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{f}}_k \\ 0 \end{pmatrix}.$$

As the matrices B_γ and $E_1 + (\Delta t^2/4)A_\gamma$ are again symmetric and positive definite a similar diagonalization procedure results in a decoupled system of linear systems that can be solved in parallel.

Remark 8 (*comparison of the two schemes*) When compared with the trapezoidal scheme the leapfrog scheme is easier to implement. However, it seems to lose one of the main advantages that it has for the resolution of the standard wave equation. As the diffusion operator is nonlocal, the explicit nature of the scheme is no longer an advantage; it requires the resolution of a linear system at each time step even when using mass-lumping. We would also like to mention that, in each time-step and for both schemes, the costliest operation is the solution of the decoupled linear systems. However, if the spatial mesh is graded, for instance, as in Sect. 7, the CFL condition for the leapfrog scheme requires a time-step that is much smaller. In conclusion, there seems to be little to recommend leapfrog over trapezoidal rule for the problem that we are considering.

8.4 Numerical results

8.4.1 A 1D example

Let us first perform a numerical example for $n = 1$. Let $\Omega = (0, 1)$ and consider the space–fractional wave equation

$$\partial_t^2 u + (-\Delta)^s u = f,$$

with $f(x', t) = (\pi^{2s} - 1) \sin(t) \sin(\pi x')$. The initial data are such that the exact solution is $u(x', t) = \sin(t) \sin(\pi x')$. The initial values for both time-stepping schemes
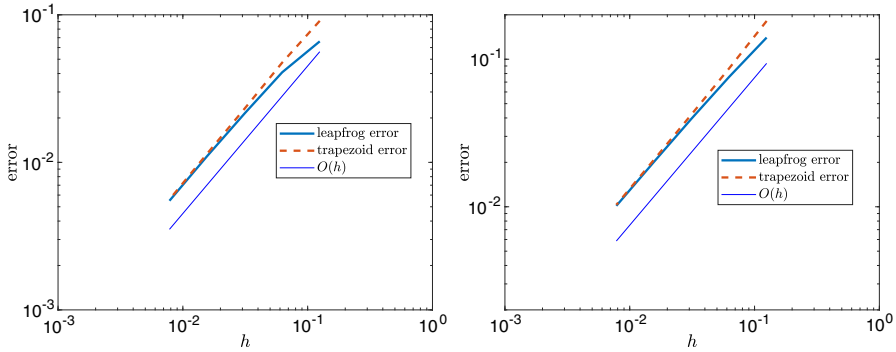
Fig. 1 Experimental rates of convergence for the error (8.4) for a one dimensional example. The experiment for $s = 1/4$ is shown on the left and the one for $s = 3/4$ on the right

can be taken as $U_0 = 0$ and $U_1 = \Delta t \, \Pi_{x'} \sin(\pi x')$. Note that the additional $O(\Delta t^2)$ term that is needed in the definition of U_1 is zero because $u''(0) = 0$.

We set the final time as $T = \pi/2$ and perform numerical experiments for the following choices of the parameter s: $s = 1/4$ and $s = 3/4$. For space discretization in Ω, we consider a uniform mesh with meshwidth $h_{\mathscr{T}}$. For the trapezoidal scheme, and to obtain linear convergence, we set $\Delta t = (0.5 h_{\mathscr{T}})^{1/2}$. For the leapfrog scheme, in order to ensure the stability of the scheme, we choose $\Delta t = (0.5 h_{\mathscr{T}})^{\max(1/2,s)}$. Notice that, the aforementioned choices of the parameter s would render the leapfrog scheme unstable as a solution technique for the standard wave equation.

In Fig. 1 we show the experimental rate of convergence for

$$\text{error} = \|U_K - u(T)\|_{\mathbb{H}^s(\Omega)}. \tag{8.4}$$

We observe that, as expected, the error decays linearly with respect to $h_{\mathscr{T}}$. We also notice that the errors for both discretization schemes are almost identical.

8.4.2 A 2D example

We let $n = 2$, and consider the square domain $\Omega = (-1, 1)^2 \subset \mathbb{R}^2$ and the space-fractional wave equation

$$\partial_t^2 u + (-\Delta)^s u = f.$$

The data of the problem is such that the exact solution is given by

$$u(x', t) = \cos(2^{s/2}\pi^s t) \sin(\pi x_1') \sin(\pi x_2').$$

We set $T = 1.5$ and $\Delta t = 0.5 h_{\mathscr{T}}$. In this experiment we measure, in the $L^2(\Omega)$-norm, the error committed in the approximation of the time-derivative:

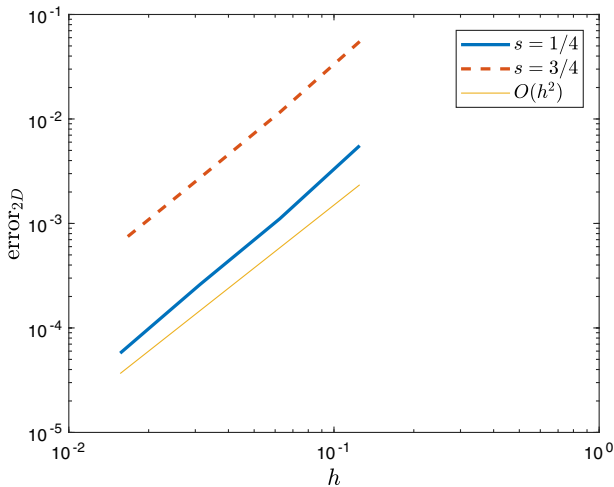$$\text{error}_{2D} = \|\eth U_K - \partial_t u(t_{K-1/2})\|_{L^2(\Omega)}. \tag{8.5}$$

Fig. 2 Experimental rates of convergence for the error (8.5) when $\Omega = (-1, 1)^2$ and for different values of s. Leapfrog time-discretization is used

The computations were done using the NGSolve/Netgen software package [54,55].

The convergence properties of the leapfrog scheme are presented in Fig. 2. It can be observed that, for both values of the parameter s considered, the experimental rate of convergence for the error$_{2D}$ decays quadratically with respect to $h_\mathcal{T}$. We notice that the observed rates are better than the ones derived in Theorem 7, but are in agreement with approximation theory since, in this case, u is smooth. The improvement of the estimate (6.59) of Theorem 7 from $h_\mathcal{T}^{1+s}$ to $h_\mathcal{T}^2$ is an open problem.

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