

# An unbiased Nitsche's approximation of the frictional contact between two elastic structures

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**Abstract** Most of the numerical methods dedicated to the contact problem involving two elastic bodies are based on the master/slave paradigm. It results in important detection difficulties in the case of self-contact and multi-body contact, where it may be impractical, if not impossible, to a priori nominate a master surface and a slave one. In this work we introduce an unbiased finite element method for the finite element approximation of frictional contact between two elastic bodies in the small deformation framework. In the proposed method the two bodies expected to come into contact are treated in the same way (no master and slave surfaces). The key ingredient is a Nitsche-based formulation of contact conditions, as in Chouly et al. (Math Comput 84:1089–1112, 2015). We carry out the numerical analysis of the method, and prove its well-posedness and optimal convergence in the  $H^1$ -norm. Numerical experiments are performed to illustrate the theoretical results and the performance of the method.

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## Introduction

Although being a very rich subject in the past, contact computational mechanics for deformable bodies in small or large strain is still the subject of intensive research. The most common paradigm to treat the problem of two deformable bodies in contact is known as the master/slave formulation. In this approach one distinguishes between a master surface and a slave one on which is prescribed the non-penetration condition. A presentation of this formulation and the contact problem can be found in Laursen's work [18, 19] (see also [20]) and a presentation of discretization schemes and numerical algorithms for mechanical contact is given in [28]. This approach is confronted with important difficulties especially in the case of self-contact and multi-body contact where it is impossible or impractical to a priori nominate a master surface and a slave one. Automating the detection and the separation between slave and master surfaces in these cases may generate a lack of robustness since it may create detection problems.

If the master/slave formulation consists in a natural extension of the contact treatment between a deformable body and a rigid ground, it has no complete theoretical justification. Consequently, to avoid these difficulties, we provide in this article an unbiased formulation of the two elastic bodies contact problem in the small strain framework. In this formulation we do not distinguish between a master surface and a slave one since we impose the non-penetration and the friction conditions on both of them. Unbiased contact and friction formulations have been considered before in [26] and references therein. There, the authors present a numerical study of the method and make use of a penalized formulation of contact and friction. The terms two-pass and two-half-pass are also used in literature to describe this type of methods.

This study can be seen as a first step in the construction of a method taking into account contact between two elastic solids and self-contact in large transformations in the same formalism. The present formulation, in small deformations, allows us to ensure the consistency, the convergence and the optimality of the method. In this context, the aim of this paper is to provide an unbiased description of the contact and Tresca friction conditions, that relies upon a Nitsche's treatment of contact conditions.

Nitsche's treatment of contact is an extension of the method proposed in 1971 by J. Nitsche to impose Dirichlet conditions within the variational formulation without adding Lagrange multipliers [23]. Nitsche's method has been widely applied on problems involving linear conditions on the boundary of a domain or at the interface between sub-domains: see, e.g. [27] for the Dirichlet problem or [1] for domain decomposition with non-matching meshes. More recently, in [13] and [15] it has been adapted for bilateral (persistent) contact, which still involves linear boundary conditions on the contact zone. A Nitsche-based formulation for the Finite Element discretization of the unilateral (non-linear) contact problem in linear elasticity was introduced in [6] and generalized in [8] to encompass symmetric and non-symmetric variants. A simple adaptation of the Nitsche-based Finite Element Method to Tresca's friction is proposed in [5]. Conversely to standard penalization techniques (see [7, 17]), the resulting method is consistent. Moreover, unlike mixed methods (see [14, 16]), no additional unknown (Lagrange multiplier) is needed.

Other possibilities for contact discretization are for instance node-to-segment techniques or the mortar method. Note that the mortar method is an efficient alternative that has been widely applied to contact problem (see [2, 21, 24]). The mortar technique allows to match independent discretizations of the contacting solids and takes into account the unilateral contact conditions in a convenient way. The procedure provides variationally consistent contact pressures. But mortar methods normally represent asymmetric formulations, by distinguishing between a master (or mortar) and a slave (or non-mortar) surface. Thus, the adaptation to an unbiased contact description is quite easier with Nitsche's method than a mortar one. In fact, since Nitsche's method uses the contact stress as a multiplier, it is very simple to divide this contact effort equitably on both of contact surfaces. A comparison between Nitsche's method and mortar-type ones for linear elasticity is provided in [12].

The formulation described in this paper uses an additional parameter  $\theta$  as in [8], allowing us to introduce some variants acting on the symmetry/skew-symmetry / non-symmetry of the discrete formulation. Moreover, a unified analysis of all these variants can be performed. We provide, as well, theoretical and numerical verifications of the proposed method. First, we prove the consistency of the method, its well-posedness and its optimal convergence. And then, a numerical verification is performed to confirm the theoretical results.

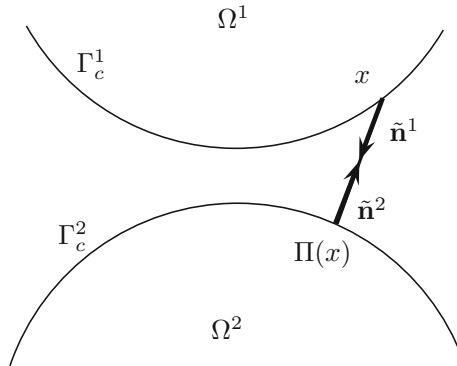
In Sect. 1 we build an unbiased formulation of the two elastic bodies frictional (Tresca) contact problem. This formulation will be based on Nitsche's method. To prove the efficiency of the method (15), we carry out some mathematical analysis in Sect. 2. In the last Sect. 3 of this paper, we present the results of several two/three-dimensional numerical tests. The tests cover a convergence study of the global relative error of displacement in  $H^1$ -norm and the contact pressure error in  $L^2$ -norm with different values of the parameter  $\theta$  and the Nitsche's parameter  $\gamma_0$ . The open source environment GetFEM++<sup>1</sup> is used to perform the tests.

## 1 Setting of the problem

### 1.1 Formal statement of the two bodies contact problem

We consider two elastic bodies expected to come into contact. To simplify notations, a general index  $i$  is used to represent indifferently the 1st or the 2nd body. Let  $\Omega^i$  be the domain in  $\mathbb{R}^d$  occupied by the reference configuration of the  $i$ -th body, with  $d = 2$  or 3. Small strain assumption is made, as well as plane strain when  $d = 2$ . We suppose that the boundary  $\partial\Omega^i$  of each body consists in three non-overlapping parts  $\Gamma_D^i$ ,  $\Gamma_N^i$  and  $\Gamma_C^i$ . On  $\Gamma_D^i$  (resp  $\Gamma_N^i$ ) displacements  $\mathbf{u}^i$  (resp. tractions  $\mathbf{t}^i$ ) are given. The body is clamped on  $\Gamma_D^i$  for the sake of simplicity. In addition each body can be subjected to a volumic force  $\mathbf{f}^i$  (such as gravity). We denote by  $\Gamma_C^i$  a portion of the boundary of the  $i$ -th body which is a candidate contact surface with an outward unit normal vector  $\mathbf{n}^i$ . The actual surface on which a body comes into contact with the other one is not known in advance, but is contained in the portion  $\Gamma_C^i$  of  $\partial\Omega^i$ .

<sup>1</sup> <http://getfem.org/>.



**Fig. 1** Example of definition of  $\tilde{\mathbf{n}}^i$

Furthermore let us suppose that  $\Gamma_c^i$  is smooth. For the contact surfaces, let us assume a sufficiently smooth one to one application (projection for instance) mapping each point of the first contact surface to a point of the second one:

$$\Pi^1 : \Gamma_c^1 \rightarrow \Gamma_c^2.$$

Let  $J^1$  be the Jacobian determinant of the transformation  $\Pi^1$  and  $J^2 = \frac{1}{J^1}$  the Jacobian determinant of  $\Pi^2 = (\Pi^1)^{-1}$ . We suppose in the following that  $J^1 > 0$ .

We define on each contact surface a normal vector  $\tilde{\mathbf{n}}^i$  (see Fig. 1) such that:

$$\tilde{\mathbf{n}}^i(\mathbf{x}) = \begin{cases} \frac{\Pi^i(\mathbf{x}) - \mathbf{x}}{\|\Pi^i(\mathbf{x}) - \mathbf{x}\|} & \text{if } \mathbf{x} \neq \Pi^i(\mathbf{x}), \\ \mathbf{n}^i & \text{if } \mathbf{x} = \Pi^i(\mathbf{x}). \end{cases}$$

Note that  $\tilde{\mathbf{n}}^1 = -\tilde{\mathbf{n}}^2 \circ \Pi^1$  and  $\tilde{\mathbf{n}}^2 = -\tilde{\mathbf{n}}^1 \circ \Pi^2$ .

The displacements of the bodies, relatively to the fixed spatial frame are represented by  $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2)$ , where  $\mathbf{u}^i$  is the displacement field of the  $i$ -th body.

The contact problem in linear elasticity consists in finding the displacement field  $\mathbf{u}$  satisfying the Eq. (1) and the contact conditions described hereafter:

$$\mathbf{div} \boldsymbol{\sigma}^i(\mathbf{u}^i) + \mathbf{f}^i = \mathbf{0} \text{ in } \Omega^i, \tag{1a}$$

$$\boldsymbol{\sigma}^i(\mathbf{u}^i) = A^i \boldsymbol{\varepsilon}(\mathbf{u}^i) \text{ in } \Omega^i, \tag{1b}$$

$$\mathbf{u}^i = \mathbf{0} \text{ on } \Gamma_D^i, \tag{1c}$$

$$\boldsymbol{\sigma}^i(\mathbf{u}^i)\mathbf{n}^i = \mathbf{t}^i \text{ on } \Gamma_N^i, \tag{1d}$$

where  $\boldsymbol{\sigma}^i = \sigma_{(j,k)}^i$ ,  $1 \leq j, k \leq d$ , stands for the stress tensor field and  $\mathbf{div}$  denotes the divergence operator of tensor valued functions. The notation  $\boldsymbol{\varepsilon}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$  represents the linearized strain tensor field and  $A^i$  is the fourth order symmetric elasticity tensor on  $\Omega^i$  having the usual uniform ellipticity and boundedness property.

For any displacement field  $\mathbf{v}^i$  and for any density of surface forces  $\boldsymbol{\sigma}^i(\mathbf{v}^i)\mathbf{n}^i$  defined on  $\partial\Omega_i$  we adopt the following notation:

$$\mathbf{v}^i = v_n^i \tilde{\mathbf{n}}^i + \mathbf{v}_t^i \text{ and } \boldsymbol{\sigma}^i(\mathbf{v}^i)\mathbf{n}^i = \sigma_n^i(\mathbf{v}^i)\tilde{\mathbf{n}}^i + \boldsymbol{\sigma}_t^i(\mathbf{v}^i),$$

where  $\mathbf{v}_t^i$  (resp  $\boldsymbol{\sigma}_t^i(\mathbf{v}^i)$ ) are the tangential components of  $\mathbf{v}^i$  (resp  $\boldsymbol{\sigma}^i(\mathbf{v}^i)\mathbf{n}^i$ ). We define an initial normal gap representing the normal distance between a point  $\mathbf{x}$  of  $\Gamma_C^i$  and its image on the other body:  $g_n^i = (\Pi^i(\mathbf{x}) - \mathbf{x}) \cdot \tilde{\mathbf{n}}^i$ .

We define, as well, the relative normal displacements  $[[u]]_n^1 = (\mathbf{u}^1 - \mathbf{u}^2 \circ \Pi^1) \cdot \tilde{\mathbf{n}}^1$  and  $[[u]]_n^2 = (\mathbf{u}^2 - \mathbf{u}^1 \circ \Pi^2) \cdot \tilde{\mathbf{n}}^2$ .

*Remark 1.1* Note that:  $g_n^1 \circ \Pi^2 = g_n^2$  and  $g_n^2 \circ \Pi^1 = g_n^1$ ;  $[[u]]_n^1 \circ \Pi^2 = [[u]]_n^2$  and  $[[u]]_n^2 \circ \Pi^1 = [[u]]_n^1$ .

In order to obtain an unbiased formulation of the contact problem we prescribe the contact conditions deduced from the Signorini problem conditions (see [17]) on the two surfaces in a symmetric way. Thus, the conditions describing contact on  $\Gamma_C^1$  and  $\Gamma_C^2$  are:

$$[[u]]_n^1 \leq g_n^1 \tag{2a}$$

$$\sigma_n^1(\mathbf{u}^1) \leq 0 \text{ on } \Gamma_C^1, \tag{2b}$$

$$\sigma_n^1(\mathbf{u}^1)([[u]]_n^1 - g_n^1) = 0 \tag{2c}$$

$$[[u]]_n^2 \leq g_n^2 \tag{3a}$$

$$\sigma_n^2(\mathbf{u}^2) \leq 0 \text{ on } \Gamma_C^2. \tag{3b}$$

$$\sigma_n^2(\mathbf{u}^2)([[u]]_n^2 - g_n^2) = 0 \tag{3c}$$

Let  $s^i \in L^2(\Gamma_C^i)$ ,  $s^i \geq 0$ , be the Tresca friction threshold associated to the physical properties of the i-th surface,  $[[\mathbf{u}]]_t^1 = \mathbf{u}_t^1 - \mathbf{u}_t^2 \circ \Pi^1$  and  $[[\mathbf{u}]]_t^2 = \mathbf{u}_t^2 - \mathbf{u}_t^1 \circ \Pi^2 = -[[\mathbf{u}]]_t^1 \circ \Pi^2$ .

The Tresca friction condition on  $\Gamma_C^1$  and  $\Gamma_C^2$  reads:

$$\begin{cases} \|\boldsymbol{\sigma}_t^i(\mathbf{u}^i)\| \leq s^i & \text{if } [[\mathbf{u}]]_t^i = 0, \\ \boldsymbol{\sigma}_t^i(\mathbf{u}^i) = -s^i \frac{[[\mathbf{u}]]_t^i}{\|[[\mathbf{u}]]_t^i\|} & \text{otherwise,} \end{cases} \tag{4}$$

where  $\|\cdot\|$  stands for the Euclidean norm in  $\mathbb{R}^{d-1}$ .

*Remark 1.2* In the frictionless contact case this condition is simply replaced by  $\boldsymbol{\sigma}_t^i = 0$ .

Finally, we need to consider the second Newton law between the two bodies:

$$\begin{cases} \int_{\gamma_C^1} \sigma_n^1(\mathbf{u}^1) ds - \int_{\gamma_C^2} \sigma_n^2(\mathbf{u}^2) ds = 0, \\ \int_{\gamma_C^1} \boldsymbol{\sigma}_t^1(\mathbf{u}^1) ds + \int_{\gamma_C^2} \boldsymbol{\sigma}_t^2(\mathbf{u}^2) ds = 0, \end{cases}$$

where  $\gamma_C^1$  is any subset of  $\Gamma_C^1$  and  $\gamma_C^2 = \Pi^1(\gamma_C^1)$ . Mapping all terms on  $\gamma_C^1$  allows writing:

$$\begin{cases} \int_{\gamma_C^1} \sigma_n^1(\mathbf{u}^1) - J^1 \sigma_n^2(\mathbf{u}^2 \circ \Pi^1) ds = 0, \\ \int_{\gamma_C^1} \sigma_t^1(\mathbf{u}^1) + J^1 \sigma_t^2(\mathbf{u}^2 \circ \Pi^1) ds = 0, \end{cases} \quad \forall \gamma_C^1 \subset \Gamma_C^1$$

so we obtain:

$$\begin{cases} \sigma_n^1(\mathbf{u}^1) - J^1 \sigma_n^2(\mathbf{u}^2 \circ \Pi^1) = 0, \\ \sigma_t^1(\mathbf{u}^1) + J^1 \sigma_t^2(\mathbf{u}^2 \circ \Pi^1) = 0, \end{cases} \quad \text{on } \Gamma_C^1. \tag{5}$$

*Remark 1.3* A similar condition holds on  $\Gamma_C^2$ :

$$\begin{cases} \sigma_n^2(\mathbf{u}^2) - J^2 \sigma_n^1(\mathbf{u}^1 \circ \Pi^2) = 0, \\ \sigma_t^2(\mathbf{u}^2) + J^2 \sigma_t^1(\mathbf{u}^1 \circ \Pi^2) = 0. \end{cases}$$

It is important to mention that, due to second Newton law, we need to fix  $s^1$  and  $s^2$  such that:  $-s^1 \frac{\llbracket \mathbf{u} \rrbracket_t^1}{\|\llbracket \mathbf{u} \rrbracket_t^1\|} = \sigma_t^1(\mathbf{u}^1) = -J^1 \sigma_t^2(\mathbf{u}^2 \circ \Pi^1) = J^1 s^2 \frac{\llbracket \mathbf{u} \rrbracket_t^2 \circ \Pi^1}{\|\llbracket \mathbf{u} \rrbracket_t^2 \circ \Pi^1\|} = -J^1 s^2 \frac{\llbracket \mathbf{u} \rrbracket_t^1}{\|\llbracket \mathbf{u} \rrbracket_t^1\|}$ .

And so:

$$s^1 = J^1 s^2. \tag{6}$$

### 1.2 Variational formulation using Nitsche’s method

In this section, we establish the weak formulation of problem (1)–(5) using Nitsche’s method and the unbiased form of the contact and the friction conditions given in Sect. 1.1.

As in [8], we introduce an additional parameter  $\theta$ . This generalization will allow several variants, depending on the value of  $\theta$ . The symmetric case is obtained when  $\theta = 1$ . The advantage of the symmetric formulation is that it derives from an energy potential (see 1.3). These features are lost when  $\theta \neq 1$ . Nevertheless the variants  $\theta = -1$  and  $0$  presents some other advantages, mostly from the numerical viewpoint. In particular, the case  $\theta = 0$  involves a reduced number of terms, which makes it easier to implement and to extend to contact problems involving non-linear elasticity. Also, for  $\theta = -1$ , the well-posedness of the discrete formulation and the optimal convergence are preserved irrespectively of the value of the Nitsche parameter  $\gamma^i$ . Some general guidelines on how to choose  $\gamma_0$  and  $\theta$  are provided in the Sect. 3.5. First, we introduce the Hilbert space

$$\mathbf{V} = \left\{ \mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in H^1(\Omega^1)^d \times H^1(\Omega^2)^d : \mathbf{v}^1 = \mathbf{0} \text{ on } \Gamma_D^1 \text{ and } \mathbf{v}^2 = \mathbf{0} \text{ on } \Gamma_D^2 \right\}.$$

Let  $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2)$  be the solution of the contact problem in its strong form (1)–(5). We assume that  $\mathbf{u}$  is sufficiently regular so that all the following calculations make sense. The derivation of a Nitsche-based method comes from a reformulation of the contact conditions (2a)–(2b)–(2c) (see for instance [6] and [8]). This reformulation is similar to the augmented Lagrangian formulation of contact problems. The contact conditions (2a)–(2b)–(2c) are equivalent to the Eq. (7) for a given positive function  $\gamma^i$ :

$$\sigma_n^i(\mathbf{u}^i) = -\frac{1}{\gamma^i} \left[ ([\![u]\!]_n^i - g_n^i) - \gamma^i \sigma_n^i(\mathbf{u}^i) \right]_+, \tag{7}$$

where the notation  $[\cdot]_+$  refers to the positive part of a scalar quantity. Similarly, as in [5], the Tresca friction condition is equivalent to the equation

$$\sigma_t(\mathbf{u}^i) = -\frac{1}{\gamma^i} \left[ [\![\mathbf{u}]\!]_t^i - \gamma^i \sigma(\mathbf{u}^i) \right]_{\gamma^i s_i}, \tag{8}$$

where, for any  $\alpha \in \mathbb{R}^+$ , the notation  $[\cdot]_\alpha$  refers to the orthogonal projection onto  $\mathcal{B}(0, \alpha) \subset \mathbb{R}^{d-1}$ , the closed ball centered at the origin and of radius  $\alpha$ . In what follows some properties of the positive part and the projection are mentioned. Those properties will be useful in the analysis of the method.

Since  $a \leq [a]_+$  and  $a[a]_+ = [a]_+^2 \forall a \in \mathbb{R}$ , we can write that for all  $a, b \in \mathbb{R}$ :

$$\begin{aligned} ([a]_+ - [b]_+)(a - b) &= a[a]_+ + b[b]_+ - b[a]_+ - a[b]_+ \\ &\geq [a]_+^2 + [b]_+^2 - 2[a]_+[b]_+ \\ &= ([a]_+ - [b]_+)^2. \end{aligned} \tag{9}$$

We note, also, the following classical property for a projection for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{d-1}$ :

$$(\mathbf{y} - \mathbf{x}) \cdot ([\mathbf{y}]_\alpha - [\mathbf{x}]_\alpha) \geq \|[\mathbf{y}]_\alpha - [\mathbf{x}]_\alpha\|^2. \tag{10}$$

From the Green formula and Eq. (1), we get for every  $\mathbf{v} \in \mathbf{V}$ :

$$\begin{aligned} \int_{\Omega^1} \sigma^1(\mathbf{u}^1) : \boldsymbol{\varepsilon}(\mathbf{v}^1) d\Omega + \int_{\Omega^2} \sigma^2(\mathbf{u}^2) : \boldsymbol{\varepsilon}(\mathbf{v}^2) d\Omega &= \int_{\Omega^1} \mathbf{f}^1 \cdot \mathbf{v}^1 d\Omega + \int_{\Omega^2} \mathbf{f}^2 \cdot \mathbf{v}^2 d\Omega \\ + \int_{\Gamma_N^1} \mathbf{t}^1 \cdot \mathbf{v}^1 d\Gamma + \int_{\Gamma_N^2} \mathbf{t}^2 \cdot \mathbf{v}^2 d\Gamma + \int_{\Gamma_c^1} \sigma^1(\mathbf{u}^1) \mathbf{n}^1 \cdot \mathbf{v}^1 d\Gamma &+ \int_{\Gamma_c^2} \sigma^2(\mathbf{u}^2) \mathbf{n}^2 \cdot \mathbf{v}^2 d\Gamma. \end{aligned}$$

We define

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega^1} \sigma^1(\mathbf{u}^1) : \boldsymbol{\varepsilon}(\mathbf{v}^1) d\Omega + \int_{\Omega^2} \sigma^2(\mathbf{u}^2) : \boldsymbol{\varepsilon}(\mathbf{v}^2) d\Omega,$$

and

$$L(\mathbf{v}) = \int_{\Omega^1} \mathbf{f}^1 \cdot \mathbf{v}^1 d\Omega + \int_{\Omega^2} \mathbf{f}^2 \cdot \mathbf{v}^2 d\Omega + \int_{\Gamma_N^1} \mathbf{t}^1 \cdot \mathbf{v}^1 d\Gamma + \int_{\Gamma_N^2} \mathbf{t}^2 \cdot \mathbf{v}^2 d\Gamma.$$

So, there holds:

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_{\Gamma_C^1} \sigma_n^1(\mathbf{u}^1) v_n^1 d\Gamma - \int_{\Gamma_C^2} \sigma_n^2(\mathbf{u}^2) v_n^2 d\Gamma - \int_{\Gamma_C^1} \sigma_t^1(\mathbf{u}^1) \cdot \mathbf{v}_t^1 d\Gamma \\ &\quad - \int_{\Gamma_C^2} \sigma_t^2(\mathbf{u}^2) \cdot \mathbf{v}_t^2 d\Gamma = L(\mathbf{v}). \end{aligned}$$

Using condition (5) we can write

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \frac{1}{2} \int_{\Gamma_C^1} (\sigma_n^1(\mathbf{u}^1) + J^1 \sigma_n^2(\mathbf{u}^2 \circ \Pi^1)) v_n^1 d\Gamma - \frac{1}{2} \int_{\Gamma_C^2} (\sigma_n^2(\mathbf{u}^2) \\ &\quad + J^2 \sigma_n^1(\mathbf{u}^1 \circ \Pi^2)) v_n^2 d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_C^1} (\sigma_t^1(\mathbf{u}^1) - J^1 \sigma_t^2(\mathbf{u}^2 \circ \Pi^1)) \cdot \mathbf{v}_t^1 d\Gamma - \frac{1}{2} \int_{\Gamma_C^2} (\sigma_t^2(\mathbf{u}^2) \\ &\quad - J^2 \sigma_t^1(\mathbf{u}^1 \circ \Pi^2)) \cdot \mathbf{v}_t^2 d\Gamma = L(\mathbf{v}). \end{aligned}$$

So, using the property  $\int_{\Gamma_C^1} J^1 f d\Gamma = \int_{\Gamma_C^2} f \circ \Pi^2 d\Gamma$ , we have

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \frac{1}{2} \int_{\Gamma_C^1} \sigma_n^1(\mathbf{u}^1) v_n^1 d\Gamma - \frac{1}{2} \int_{\Gamma_C^1} \sigma_n^1(\mathbf{u}^1) (v_n^2 \circ \Pi^1) d\Gamma - \frac{1}{2} \int_{\Gamma_C^2} \sigma_n^2(\mathbf{u}^2) v_n^2 d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_C^2} \sigma_n^2(\mathbf{u}^2) (v_n^1 \circ \Pi^2) d\Gamma - \frac{1}{2} \int_{\Gamma_C^1} \sigma_t^1(\mathbf{u}^1) \cdot \mathbf{v}_t^1 + \frac{1}{2} \int_{\Gamma_C^1} \sigma_t^1(\mathbf{u}^1) \cdot (\mathbf{v}_t^2 \circ \Pi^1) d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_C^2} \sigma_t^2(\mathbf{u}^2) \cdot \mathbf{v}_t^2 + \frac{1}{2} \int_{\Gamma_C^2} \sigma_t^2(\mathbf{u}^2) \cdot (\mathbf{v}_t^1 \circ \Pi^2) d\Gamma = L(\mathbf{v}). \end{aligned}$$

This leads to:

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \frac{1}{2} \int_{\Gamma_C^1} \sigma_n^1(\mathbf{u}^1) (v_n^1 + v_n^2 \circ \Pi^1) d\Gamma - \frac{1}{2} \int_{\Gamma_C^2} \sigma_n^2(\mathbf{u}^2) (v_n^2 + v_n^1 \circ \Pi^2) d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_C^1} \sigma_t^1(\mathbf{u}^1) \cdot (\mathbf{v}_t^1 - \mathbf{v}_t^2 \circ \Pi^1) d\Gamma - \frac{1}{2} \int_{\Gamma_C^2} \sigma_t^2(\mathbf{u}^2) \cdot (\mathbf{v}_t^2 - \mathbf{v}_t^1 \circ \Pi^2) d\Gamma = L(\mathbf{v}). \end{aligned}$$

Using the writings, for  $\theta \in \mathbb{R}$ ,

$$\begin{cases} v_n^1 + v_n^2 \circ \Pi^1 = (v_n^1 + v_n^2 \circ \Pi^1 - \theta \gamma^1 \sigma_n^1(\mathbf{v}^1)) + \theta \gamma^1 \sigma_n^1(\mathbf{v}^1) \\ v_n^2 + v_n^1 \circ \Pi^2 = (v_n^2 + v_n^1 \circ \Pi^2 - \theta \gamma^2 \sigma_n^2(\mathbf{v}^2)) + \theta \gamma^2 \sigma_n^2(\mathbf{v}^2) \end{cases}$$



$$\begin{cases} \mathbf{v}_t^1 - \mathbf{v}_t^2 \circ \Pi^1 = (\mathbf{v}_t^1 - \mathbf{v}_t^2 \circ \Pi^1 - \theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{v}^1)) + \theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{v}^1) \\ \mathbf{v}_t^2 - \mathbf{v}_t^1 \circ \Pi^2 = (\mathbf{v}_t^2 - \mathbf{v}_t^1 \circ \Pi^2 - \theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{v}^2)) + \theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{v}^2), \end{cases}$$

we obtain:

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \frac{1}{2} \int_{\Gamma_C^1} \theta\gamma^1\sigma_n^1(\mathbf{u}^1)\sigma_n^1(\mathbf{v}^1)d\Gamma - \frac{1}{2} \int_{\Gamma_C^2} \theta\gamma^2\sigma_n^2(\mathbf{u}^2)\sigma_n^2(\mathbf{v}^2)d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_C^1} \theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{u}^1) \cdot \boldsymbol{\sigma}_t^1(\mathbf{v}^1)d\Gamma - \frac{1}{2} \int_{\Gamma_C^2} \theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{u}^2) \cdot \boldsymbol{\sigma}_t^2(\mathbf{v}^2)d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_C^1} \sigma_n^1(\mathbf{u}^1)(v_n^1 + v_n^2 \circ \Pi^1 - \theta\gamma^1\sigma_n^1(\mathbf{v}^1))d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_C^2} \sigma_n^2(\mathbf{u}^2)(v_n^2 + v_n^1 \circ \Pi^2 - \theta\gamma^2\sigma_n^2(\mathbf{v}^2))d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_C^1} \boldsymbol{\sigma}_t^1(\mathbf{u}^1) \cdot (\mathbf{v}_t^1 - \mathbf{v}_t^2 \circ \Pi^1 - \theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{v}^1))d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_C^2} \boldsymbol{\sigma}_t^2(\mathbf{u}^2) \cdot (\mathbf{v}_t^2 - \mathbf{v}_t^1 \circ \Pi^2 - \theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{v}^2))d\Gamma \\ &= L(\mathbf{v}). \end{aligned} \tag{11}$$

Let us define:

$$\begin{aligned} P_{n,\gamma^i}^i(\mathbf{u}) &= \llbracket u \rrbracket_n^i - \gamma^i\sigma_n^i(\mathbf{u}^i) - g_n^i, \quad \mathbf{P}_{t,\gamma^i}^i(\mathbf{u}) = \llbracket \mathbf{u} \rrbracket_t^i - \gamma^i\boldsymbol{\sigma}_t^i(\mathbf{u}^i), \\ P_{n,\theta\gamma^i}^i(\mathbf{v}) &= \llbracket v \rrbracket_n^i - \theta\gamma^i\sigma_n^i(\mathbf{v}^i), \quad \mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{v}) = \llbracket \mathbf{v} \rrbracket_t^i - \theta\gamma^i\boldsymbol{\sigma}_t^i(\mathbf{v}^i) \end{aligned} \tag{12}$$

and

$$\begin{aligned} A_\theta(\mathbf{u}, \mathbf{v}) &= \mathbf{a}(\mathbf{u}, \mathbf{v}) - \frac{1}{2} \int_{\Gamma_C^1} \theta\gamma^1\sigma_n^1(\mathbf{u}^1)\sigma_n^1(\mathbf{v}^1)d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_C^2} \theta\gamma^2\sigma_n^2(\mathbf{u}^2)\sigma_n^2(\mathbf{v}^2)d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_C^1} \theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{u}^1) \cdot \boldsymbol{\sigma}_t^1(\mathbf{v}^1)d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_C^2} \theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{u}^2) \cdot \boldsymbol{\sigma}_t^2(\mathbf{v}^2)d\Gamma \\ &= \mathbf{a}(\mathbf{u}, \mathbf{v}) - \frac{1}{2} \int_{\Gamma_C^1} \theta\gamma^1\boldsymbol{\sigma}^1(\mathbf{u}^1)\mathbf{n} \cdot \boldsymbol{\sigma}^1(\mathbf{v}^1)\mathbf{n} d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_C^2} \theta\gamma^2\boldsymbol{\sigma}^2(\mathbf{u}^2)\mathbf{n} \cdot \boldsymbol{\sigma}^2(\mathbf{v}^2)\mathbf{n} d\Gamma. \end{aligned}$$

Now we insert the expressions (7) of  $\sigma_n^i(u^i)$  and (8) of  $\sigma_t^i(u^i)$  in (11) and the variational problem could be formally written as follows:

$$\left\{ \begin{array}{l} \text{Find a sufficiently regular } \mathbf{u} \in \mathbf{V} \text{ such that for all sufficiently regular } \mathbf{v} \in \mathbf{V}, \\ A_\theta(\mathbf{u}, \mathbf{v}) + \frac{1}{2} \int_{\Gamma_C^1} \frac{1}{\gamma^1} [P_{n,\gamma^1}^1(\mathbf{u})]_+ P_{n,\theta\gamma^1}^1(\mathbf{v}) d\Gamma \\ \quad + \frac{1}{2} \int_{\Gamma_C^2} \frac{1}{\gamma^2} [P_{n,\gamma^2}^2(\mathbf{u})]_+ P_{n,\theta\gamma^2}^2(\mathbf{v}) d\Gamma \\ + \frac{1}{2} \int_{\Gamma_C^1} \frac{1}{\gamma^1} [\mathbf{P}_{t,\gamma^1}^1(\mathbf{u})]_{\gamma^1 s^1} \cdot \mathbf{P}_{t,\theta\gamma^1}^1(\mathbf{v}) d\Gamma + \frac{1}{2} \int_{\Gamma_C^2} \frac{1}{\gamma^2} [\mathbf{P}_{t,\gamma^2}^2(\mathbf{u})]_{\gamma^2 s^2} \\ \quad \cdot \mathbf{P}_{t,\theta\gamma^2}^2(\mathbf{v}) d\Gamma = L(\mathbf{v}). \end{array} \right. \tag{13}$$

*Remark 1.4* In the frictionless contact case the formulation reads:

$$\left\{ \begin{array}{l} \text{Find a sufficiently regular } \mathbf{u} \in \mathbf{V} \text{ such that for all sufficiently regular } \mathbf{v} \in \mathbf{V} \\ A_\theta(\mathbf{u}, \mathbf{v}) + \frac{1}{2} \int_{\Gamma_C^1} \frac{1}{\gamma^1} [P_{n,\gamma^1}^1(\mathbf{u})]_+ P_{n,\theta\gamma^1}^1(\mathbf{v}) d\Gamma \\ \quad + \frac{1}{2} \int_{\Gamma_C^2} \frac{1}{\gamma^2} [P_{n,\gamma^2}^2(\mathbf{u})]_+ P_{n,\theta\gamma^2}^2(\mathbf{v}) d\Gamma = L(\mathbf{v}). \end{array} \right.$$

### 1.3 Derivation of the method from a potential

In this section we show, through a formal demonstration, that the method derives from a potential in the frictional symmetric ( $\theta = 1$ ) case. Let us define the potential:

$$J(\mathbf{u}) = \epsilon_\Omega(\mathbf{u}) + \sum_{i=1}^2 \left( \epsilon_n^i(\mathbf{u}) + \epsilon_t^i(\mathbf{u}) \right),$$

with:

$$\begin{aligned} \epsilon_\Omega(\mathbf{u}) &= \frac{1}{2} a(\mathbf{u}, \mathbf{u}) - \sum_{i=1}^2 \left( \frac{1}{4} \int_{\Gamma_C^i} \gamma^i (\sigma_n^i(\mathbf{u}^i))^2 + \frac{1}{4} \int_{\Gamma_C^i} \gamma^i \|\sigma_t^i(\mathbf{u}^i)\|^2 d\Gamma \right) - L(\mathbf{u}) \\ &= \frac{1}{2} A_1(\mathbf{u}, \mathbf{u}) - L(\mathbf{u}), \\ \epsilon_n^i(\mathbf{u}) &= \frac{1}{4} \int_{\Gamma_C^i} \frac{1}{\gamma^i} [P_{n,\gamma^i}^i(\mathbf{u})]_+^2 d\Gamma, \\ \epsilon_t^i(\mathbf{u}) &= \frac{1}{4} \int_{\Gamma_C^i} \frac{1}{\gamma^i} \|\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})\|^2 d\Gamma - \frac{1}{4} \int_{\Gamma_C^i} \frac{1}{\gamma^i} \|\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}) - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}\|^2 d\Gamma. \end{aligned}$$

We compute now the derivative of this potential. We have:

$$\begin{aligned}
 D\boldsymbol{\varepsilon}_\Omega(\mathbf{u})[\mathbf{v}] &= A_1(\mathbf{u}, \mathbf{v}) - L(\mathbf{v}) (L \text{ is linear and } A_\theta \text{ is bilinear}), \\
 D\boldsymbol{\varepsilon}_n^i(\mathbf{u})[\mathbf{v}] &= \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} \left[ P_{n,\gamma^i}^i(\mathbf{u}) \right]_+ D \left( [P_{n,\gamma^i}^i(\mathbf{u})]_+ \right) [\mathbf{v}] d\Gamma \\
 &= \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} \left[ P_{n,\gamma^i}^i(\mathbf{u}) \right]_+ H(P_{n,\gamma^i}^i(\mathbf{u})) \left( D(P_{n,\gamma^i}^i(\mathbf{u})) \right) [\mathbf{v}] d\Gamma,
 \end{aligned}$$

where  $H$  is the Heaviside step function. Using the equalities:  $H(\varphi(X))[\varphi(X)]_+ = [\varphi(X)]_+$  and  $D(P_{n,\gamma^i}^i(\mathbf{u}))[\mathbf{v}] = P_{n,\gamma^i}^i(\mathbf{v})$  (since  $P_{n,\gamma^i}^i$  is linear), we get:

$$D\boldsymbol{\varepsilon}_n^i(\mathbf{u})[\mathbf{v}] = \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} \left[ P_{n,\gamma^i}^i(\mathbf{u}) \right]_+ P_{n,\gamma^i}^i(\mathbf{v}) d\Gamma.$$

Finally:

$$\begin{aligned}
 D\boldsymbol{\varepsilon}_t^i(\mathbf{u})[\mathbf{v}] &= \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} \mathbf{P}_{t,\gamma^i}^i(\mathbf{u}) \cdot \mathbf{P}_{t,\gamma^i}^i(\mathbf{v}) d\Gamma \\
 &\quad - \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} \left( \mathbf{P}_{t,\gamma^i}^i(\mathbf{u}) - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i} \right) \cdot \left( \mathbf{P}_{t,\gamma^i}^i(\mathbf{v}) \right. \\
 &\quad \left. - D([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i})[\mathbf{v}] \right)
 \end{aligned}$$

$$\begin{cases}
 \text{if } \|\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})\| \leq \gamma^i s^i, \text{ then } \mathbf{P}_{t,\gamma^i}^i(\mathbf{u}) - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i} = 0 \\
 \text{if } \|\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})\| > \gamma^i s^i, \text{ then } D([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i})[\mathbf{v}] \text{ is tangential to } \mathcal{B}(0, \gamma^i s^i) \text{ and} \\
 D([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i})[\mathbf{v}] \cdot (\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}) - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}) = 0.
 \end{cases}$$

So, in both cases we have:

$$D\boldsymbol{\varepsilon}_t^i(\mathbf{u})[\mathbf{v}] = \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i} \cdot \mathbf{P}_{t,\gamma^i}^i(\mathbf{v}) d\Gamma$$

so, if we consider the first order optimality condition  $D\boldsymbol{\varepsilon}(\mathbf{u})[\mathbf{v}] = 0 \forall \mathbf{v} \in \mathbf{V}$ , we get:

$$\begin{aligned}
 A_1(\mathbf{u}, \mathbf{v}) + \sum_{i=1}^2 \left( \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} [P_{n,\gamma^i}^i(\mathbf{u})]_+ P_{n,\gamma^i}^i(\mathbf{v}) d\Gamma \right. \\
 \left. + \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i} \cdot \mathbf{P}_{t,\gamma^i}^i(\mathbf{v}) d\Gamma \right) = L(\mathbf{v}).
 \end{aligned}$$

This is exactly (13) when  $\theta = 1$ .

### 1.4 Strong–weak formulation equivalence

In this section, we are going to establish the formal equivalence between (13) and (1)–(5). Since the construction of (13) is quite elaborated and consists in particular in the splitting of the contact terms into two parts, this step is necessary to ensure the coherence of the formulation.

**Theorem 1.5** *Let  $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2)$  be a sufficiently regular solution to the problem (13), then  $\mathbf{u}$  solves the problem (1)–(5) for all  $\theta \in \mathbb{R}$ .*

*Proof* See “Appendix A” □

### 1.5 Discretization of the variational formulation

Let  $(\mathcal{T}_h^i)_{h>0}$  be a family of triangulations of the domain  $\Omega^i$  supposed regular and conformal to the subdivisions of the boundaries into  $\Gamma_D^i, \Gamma_N^i$  and  $\Gamma_C^i$ . We introduce

$$\begin{aligned} \mathbf{V}_h &= (\mathbf{V}_h^1 \times \mathbf{V}_h^2), \text{ with} \\ \mathbf{V}_h^i &= \left\{ \mathbf{v}_h^i \in \mathcal{C}^0(\overline{\Omega^i}) : \mathbf{v}_h^i|_T \in (\mathbb{P}_k(T))^d, \forall T \in \mathcal{T}_h^i, \mathbf{v}_h^i = \mathbf{0} \text{ on } \Gamma_D^i \right\}, \end{aligned}$$

the family of finite dimensional vector spaces indexed by  $h$  and coming from  $\mathcal{T}_h^i$ . We consider in what follows that  $\gamma^i$  is a positive piecewise constant function on the contact interface  $\Gamma_C^i$  which satisfies

$$\gamma^i|_{K^i \cap \Gamma_C^i} = \gamma_0 h_{K^i},$$

for every  $K^i \in \mathcal{T}_h^i$  that has a non-empty intersection of dimension  $d - 1$  with  $\Gamma_C^i$ , and where  $\gamma_0$  is a positive given constant. Note that the value of  $\gamma^i$  on element intersections has no influence. This allows to define a discrete counterpart of (13). Let us introduce for this purpose, with the same notation, the discrete linear operators:

$$\begin{aligned} P_{n,\gamma^i}^i(\mathbf{u}_h) &= \llbracket u_h \rrbracket_n - g_n^i - \gamma^i \sigma_n^i(\mathbf{u}_h^i), & \mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h) &= \llbracket \mathbf{u}_h \rrbracket_t^i - \gamma^i \boldsymbol{\sigma}_t^i(\mathbf{u}_h^i), \\ P_{n,\theta\gamma^i}^i(\mathbf{v}_h) &= \llbracket v_h \rrbracket_n^i - \theta \gamma^i \sigma_n^i(\mathbf{v}_h^i), & \mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{v}_h) &= \llbracket \mathbf{v}_h \rrbracket_t^i - \theta \gamma^i \boldsymbol{\sigma}_t^i(\mathbf{v}_h^i). \end{aligned} \tag{14}$$

Then the unbiased formulation of the two bodies contact in the discrete setting reads:

$$\left\{ \begin{array}{l} \text{Find } \mathbf{u}_h \in \mathbf{V}_h \text{ such that, for all } \mathbf{v}_h \in \mathbf{V}_h, \\ A_\theta(\mathbf{u}_h, \mathbf{v}_h) \\ + \frac{1}{2} \int_{\Gamma_C^1} \frac{1}{\gamma^1} P_{n,\theta\gamma^1}^1(\mathbf{v}_h) [P_{n,\gamma^1}^1(\mathbf{u}_h)]_+ d\Gamma + \frac{1}{2} \int_{\Gamma_C^2} \frac{1}{\gamma^2} P_{n,\theta\gamma^2}^2(\mathbf{v}_h) [P_{n,\gamma^2}^2(\mathbf{u}_h)]_+ d\Gamma \\ + \frac{1}{2} \int_{\Gamma_C^1} \frac{1}{\gamma^1} \mathbf{P}_{t,\theta\gamma^1}^1(\mathbf{v}_h) \cdot [\mathbf{P}_{t,\gamma^1}^1(\mathbf{u}_h)]_{\gamma^1 s^1} d\Gamma \\ + \frac{1}{2} \int_{\Gamma_C^2} \frac{1}{\gamma^2} \mathbf{P}_{t,\theta\gamma^2}^2(\mathbf{v}_h) \cdot [\mathbf{P}_{t,\gamma^2}^2(\mathbf{u}_h)]_{\gamma^2 s^2} d\Gamma = L(\mathbf{v}_h). \end{array} \right. \tag{15}$$

*Remark 1.6* Note that Nitsche’s method is not a standard penalty method, since it is consistent. In fact the Nitsche’s method is closer to Barbosa & Hughes stabilization (see [27] and [6, Section 2.3]), so the Nitsche parameter  $\gamma_0$  is in fact a stabilization parameter. As a result, making  $\gamma_0$  tend to 0 does not increase necessarily precision, conversely to standard penalty (see as well Figs. 16 and 17 in Sect. 3.5 for a numerical illustration here). The parameter  $\gamma_0$  must therefore be just smaller than a threshold value ensuring the coercivity so that the problem is well posed (and not too small not to cause ill-conditioning). This threshold value depends on the variant ( $\theta$ ).

## 2 Mathematical analysis of the method

A major difference between Nitsche’s method and standard penalty methods is the consistency demonstrated in 2.1. Using the same arguments as in [6] we prove the well-posedness and the optimal convergence of (15) when the mesh size  $h$  vanishes. To insure well-posedness and convergence of the method we need to impose  $\gamma_0$  to be sufficiently small when  $\theta \neq -1$ . This condition is avoided when  $\theta = -1$  which is a major advantage of this version.

### 2.1 Consistency

Similarly to Nitsche’s method for unilateral contact problems [6], our Nitsche-based formulation (15) is consistent:

**Lemma 2.1** *Suppose that the solution  $\mathbf{u}$  of (1)–(5) lies in  $(H^{\frac{3}{2}+v}(\Omega^1))^d \times (H^{\frac{3}{2}+v}(\Omega^2))^d$  with  $v > 0$ , then  $\mathbf{u}$  is also solution to:*

$$\begin{aligned}
 A_\theta(\mathbf{u}, \mathbf{v}_h) &+ \frac{1}{2} \int_{\Gamma_C^1} \frac{1}{\gamma^1} P_{n,\theta\gamma^1}^1(\mathbf{v}_h) [P_{n,\gamma^1}^1(\mathbf{u})]_+ d\Gamma \\
 &+ \frac{1}{2} \int_{\Gamma_C^2} \frac{1}{\gamma^2} P_{n,\theta\gamma^2}^2(\mathbf{v}_h) [P_{n,\gamma^2}^2(\mathbf{u})]_+ d\Gamma \\
 &+ \frac{1}{2} \int_{\Gamma_C^1} \frac{1}{\gamma^1} \mathbf{P}_{t,\theta\gamma^1}^1(\mathbf{v}_h) \cdot [\mathbf{P}_{t,\gamma^1}^1(\mathbf{u})]_{\gamma^1 s^1} d\Gamma
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \int_{\Gamma_C^2} \frac{1}{\gamma^2} \mathbf{P}_{t\theta\gamma^2}^2(\mathbf{v}_h) \cdot [\mathbf{P}_{t,\gamma^2}^2(\mathbf{u})]_{\gamma^2 s^2} d\Gamma \\
 & = L(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h.
 \end{aligned}
 \tag{16}$$

*Proof* Let  $\mathbf{u}$  be a solution of (1)–(5) and set  $\mathbf{v}_h \in \mathbf{V}_h$ . Since  $\mathbf{u}^i \in (H^{\frac{3}{2}+\nu}(\Omega^i))^d$ , we have  $\sigma_n^i(\mathbf{u}^i) \in (H^\nu(\Gamma_C^i))^d$  and  $P_{n\gamma^i}$  and  $\mathbf{P}_{t\gamma^i}$  are well-defined and belong to  $L^2(\Gamma_C^i)$ .

With Eqs. (1)–(4) and integration by parts, it holds:

$$\begin{aligned}
 \mathbf{a}(\mathbf{u}, \mathbf{v}_h) & - \int_{\Gamma_C^1} \sigma_n^1(\mathbf{u}^1) v_{hn}^1 d\Gamma - \int_{\Gamma_C^2} \sigma_n^2(\mathbf{u}^2) v_{hn}^2 d\Gamma - \int_{\Gamma_C^1} \sigma_t^1(\mathbf{u}^1) \cdot \mathbf{v}_{ht}^1 d\Gamma \\
 & - \int_{\Gamma_C^2} \sigma_t^2(\mathbf{u}^2) \cdot \mathbf{v}_{ht}^2 d\Gamma = L(\mathbf{v}_h).
 \end{aligned}$$

We use now (5) to write:

$$\begin{aligned}
 \mathbf{a}(\mathbf{u}, \mathbf{v}) & - \frac{1}{2} \int_{\Gamma_C^1} \sigma_n^1(\mathbf{u}^1)(v_{hn}^1 + v_{hn}^2 \circ \Pi^1) d\Gamma - \frac{1}{2} \int_{\Gamma_C^2} \sigma_n^2(\mathbf{u}^2)(v_{hn}^2 + v_{hn}^1 \circ \Pi^2) d\Gamma \\
 & - \frac{1}{2} \int_{\Gamma_C^1} \sigma_t^1(\mathbf{u}^1) \cdot (\mathbf{v}_{ht}^1 - \mathbf{v}_{ht}^2 \circ \Pi^1) d\Gamma \\
 & - \frac{1}{2} \int_{\Gamma_C^2} \sigma_t^2(\mathbf{u}^2) \cdot (\mathbf{v}_{ht}^2 - \mathbf{v}_{ht}^1 \circ \Pi^2) d\Gamma = L(\mathbf{v}_h).
 \end{aligned}$$

For any  $\theta \in \mathbb{R}$ , we can write:

$$\begin{cases}
 v_{hn}^1 + v_{hn}^2 \circ \Pi^1 = (v_{hn}^1 + v_{hn}^2 \circ \Pi^1 - \theta\gamma^1 \sigma_n^1(\mathbf{v}_h^1)) + \theta\gamma^1 \sigma_n^1(\mathbf{v}_h^1) \\
 v_{hn}^2 + v_{hn}^1 \circ \Pi^2 = (v_{hn}^2 + v_{hn}^1 \circ \Pi^2 - \theta\gamma^2 \sigma_n^2(\mathbf{v}_h^2)) + \theta\gamma^2 \sigma_n^2(\mathbf{v}_h^2) \\
 \mathbf{v}_{th}^1 - \mathbf{v}_{ht}^2 \circ \Pi^1 = (\mathbf{v}_{th}^1 - \mathbf{v}_{ht}^2 \circ \Pi^1 - \theta\gamma^1 \sigma_t^1(\mathbf{v}_h^1)) + \theta\gamma^1 \sigma_t^1(\mathbf{v}_h^1) \\
 \mathbf{v}_{th}^2 - \mathbf{v}_{ht}^1 \circ \Pi^2 = (\mathbf{v}_{th}^2 - \mathbf{v}_{ht}^1 \circ \Pi^2 - \theta\gamma^2 \sigma_t^2(\mathbf{v}_h^2)) + \theta\gamma^2 \sigma_t^2(\mathbf{v}_h^2).
 \end{cases}
 \tag{17}$$

Using (17), formulations (7) and (8) of the contact and friction conditions and the notations (12), we obtain (16). □

*Remark 2.2* The regularity assumption that we made in Lemma 2.1 is standard for Signorini contact. It was proved for an elliptic scalar problem in [22] and noted numerically for linear elasticity. In fact the singularities that appear with contact-non-contact transitions allow us, generally, to expect a Sobolev regularity between 3/2 and 5/2.

### 2.2 Well-posedness

To prove well-posedness of our formulation, we first need the following discrete trace inequality.

**Lemma 2.3** *There exists  $C > 0$ , independent of the parameter  $\gamma_0$  and of the mesh size  $h$ , such that:*

$$\|\gamma^{i\frac{1}{2}}\sigma_t^i(\mathbf{v}_h^i)\|_{0,\Gamma_c^i}^2 + \|\gamma^{i\frac{1}{2}}\sigma_n^i(\mathbf{v}_h^i)\|_{0,\Gamma_c^i}^2 \leq C\gamma_0\|\mathbf{v}_h^i\|_{1,\Omega^i}^2, \tag{18}$$

for all  $\mathbf{v}_h^i \in \mathbf{V}_h^i$ .

*Proof* The inequality (18) is obtained using a scaling argument as in [5, Lemma 3.2]. □

We then show in Theorem 2.4 that the problem (15) is well-posed using an argument from [4] for M-type and pseudo-monotone operators. In the proof of the well-posedness, two cases are discussed:  $\theta = 1$  and  $\theta \neq 1$ .

**Theorem 2.4** *Suppose that  $\gamma_0 > 0$  is sufficiently small or  $\theta = -1$ , then Problem (15) admits one unique solution  $\mathbf{u}_h$  in  $\mathbf{V}_h$ . When  $\theta = -1$  we do not need the assumption of smallness of  $\gamma_0$ .*

*Proof* Using the Riesz representation theorem, we define a (non-linear) operator  $\mathbf{B} : \mathbf{V}_h \rightarrow \mathbf{V}_h$ , by means of the formula:

$$\begin{aligned} (\mathbf{B}\mathbf{u}_h, \mathbf{v}_h)_1 &= A_\theta(\mathbf{u}_h, \mathbf{v}_h) + \frac{1}{2} \int_{\Gamma_C^1} \frac{1}{\gamma^1} P_{n,\theta\gamma^1}^1(\mathbf{v}_h)[P_{n,\gamma^1}^1(\mathbf{u}_h)]_+ d\Gamma \\ &+ \frac{1}{2} \int_{\Gamma_C^2} \frac{1}{\gamma^2} P_{n,\theta\gamma^2}^2(\mathbf{v}_h)[P_{n,\gamma^2}^2(\mathbf{u}_h)]_+ d\Gamma \\ &+ \frac{1}{2} \int_{\Gamma_C^1} \frac{1}{\gamma^1} \mathbf{P}_{t,\theta\gamma^1}^1(\mathbf{v}_h) \cdot [\mathbf{P}_{t,\gamma^1}^1(\mathbf{u}_h)]_{\gamma^1 s^1} d\Gamma \\ &+ \frac{1}{2} \int_{\Gamma_C^2} \frac{1}{\gamma^2} \mathbf{P}_{t,\theta\gamma^2}^2(\mathbf{v}_h) \cdot [\mathbf{P}_{t,\gamma^2}^2(\mathbf{u}_h)]_{\gamma^2 s^2} d\Gamma, \end{aligned}$$

for all  $\mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h$ , and where  $(\cdot, \cdot)_1$  stands for the scalar product in  $\mathbf{V}$  and the notations  $P_{n,\gamma^i}^i, \mathbf{P}_{t,\gamma^i}^i, P_{n,\theta\gamma^i}^i$  and  $\mathbf{P}_{t,\theta\gamma^i}^i$  are given by (12).

Note that Problem (15) is well-posed if and only if  $\mathbf{B}$  is a one-to-one operator. Let  $\mathbf{v}_h, \mathbf{w}_h \in \mathbf{V}_h$ , using the writings  $P_{n,\theta\gamma^i}^i(\cdot) = P_{n,\gamma^i}^i(\cdot) + g_n^i + (1 - \theta)\sigma_n^i(\cdot)$  and  $\mathbf{P}_{t,\theta\gamma^i}^i(\cdot) = \mathbf{P}_{t,\gamma^i}^i(\cdot) + (1 - \theta)\sigma_t^i(\cdot)$ , we have:

$$\begin{aligned} (\mathbf{B}\mathbf{v}_h - \mathbf{B}\mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h)_1 &= \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) \\ &+ \sum_{i=1}^2 \left( -\frac{\theta}{2} \|\gamma^{i\frac{1}{2}}\sigma(\mathbf{v}_h^i - \mathbf{w}_h^i)\mathbf{n}\|_{0,\Gamma_c^i}^2 \right. \\ &+ \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} P_{n,\gamma^i}^i(\mathbf{v}_h - \mathbf{w}_h)([P_{n,\gamma^i}^i(\mathbf{v}_h)]_+ - [P_{n,\gamma^i}^i(\mathbf{w}_h)]_+) d\Gamma \\ &\left. + \frac{(1 - \theta)}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} \gamma^i \sigma_n^i(\mathbf{v}_h^i - \mathbf{w}_h^i)([P_{n,\gamma^i}^i(\mathbf{v}_h)]_+ - [P_{n,\gamma^i}^i(\mathbf{w}_h)]_+) d\Gamma \right) \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} \mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h - \mathbf{w}_h) \cdot ([\mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h)]_{\gamma^i s^i} - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{w}_h)]_{\gamma^i s^i}) d\Gamma \\
 & + \frac{(1-\theta)}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} \gamma^i \boldsymbol{\sigma}_t^i(\mathbf{v}_h^i - \mathbf{w}_h^i) \cdot ([\mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h)]_{\gamma^i s^i} - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{w}_h)]_{\gamma^i s^i}) d\Gamma.
 \end{aligned}$$

We use Cauchy–Schwarz inequality and the proprieties (9) and (10) to get:

$$\begin{aligned}
 (\mathbf{B}\mathbf{v}_h - \mathbf{B}\mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h)_1 & \geq \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) \\
 & + \sum_{i=1}^2 \left( -\frac{\theta}{2} \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}_h^i - \mathbf{w}_h^i) \mathbf{n}\|_{0,\Gamma_C^i}^2 \right. \\
 & + \frac{1}{2} \|\gamma^{i-\frac{1}{2}} ([P_{n,\gamma^i}^i(\mathbf{v}_h)]_+ - [P_{n,\gamma^i}^i(\mathbf{w}_h)]_+) \|_{0,\Gamma_C^i}^2 \\
 & + \frac{1}{2} \|\gamma^{i-\frac{1}{2}} ([\mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h)]_{\gamma^i s^i} - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{w}_h)]_{\gamma^i s^i}) \|_{0,\Gamma_C^i}^2 \\
 & - \frac{|1-\theta|}{2} \|\gamma^{i-\frac{1}{2}} ([P_{n,\gamma^i}^i(\mathbf{v}_h)]_+ - [P_{n,\gamma^i}^i(\mathbf{w}_h)]_+) \|_{0,\Gamma_C^i} \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}_n^i(\mathbf{v}_h^i - \mathbf{w}_h^i) \|_{0,\Gamma_C^i} \\
 & \left. - \frac{|1-\theta|}{2} \|\gamma^{i-\frac{1}{2}} ([\mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h)]_{\gamma^i s^i} - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{w}_h)]_{\gamma^i s^i}) \|_{0,\Gamma_C^i} \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}_t^i(\mathbf{v}_h^i - \mathbf{w}_h^i) \|_{0,\Gamma_C^i} \right).
 \end{aligned}$$

If  $\theta = 1$ , we use the coercivity of  $a(\cdot, \cdot)$  and the property (18) to get:

$$\begin{aligned}
 (\mathbf{B}\mathbf{v}_h - \mathbf{B}\mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h)_1 & \geq \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) - \sum_{i=1}^2 \frac{1}{2} \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{w}_h^i) \mathbf{n}^i \|_{0,\Gamma_C^i}^2 \\
 & \geq \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) - \sum_{i=1}^2 \frac{1}{2} \left( \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}_n^i(\mathbf{v}_h^i - \mathbf{w}_h^i) \|_{0,\Gamma_C^i}^2 \right. \\
 & \quad \left. + \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}_t^i(\mathbf{v}_h^i - \mathbf{w}_h^i) \|_{0,\Gamma_C^i}^2 \right) \\
 & \geq C \|\mathbf{v}_h - \mathbf{w}_h\|_1^2
 \end{aligned}$$

when  $\gamma_0$  is sufficiently small.

We suppose now that  $\theta \neq 1$ ; let  $\beta > 0$ . Applying Young inequality yields:

$$\begin{aligned}
 (\mathbf{B}\mathbf{v}_h - \mathbf{B}\mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h)_1 & \geq \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) \\
 & + \sum_{i=1}^2 \left( -\frac{\theta}{2} \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{w}_h^i) \mathbf{n}^i \|_{0,\Gamma_C^i}^2 \right. \\
 & + \frac{1}{2} \|\gamma^{i-\frac{1}{2}} ([P_{n,\gamma^i}^i(\mathbf{v}_h)]_+ - [P_{n,\gamma^i}^i(\mathbf{w}_h)]_+) \|_{0,\Gamma_C^i}^2 \\
 & + \frac{1}{2} \|\gamma^{i-\frac{1}{2}} ([\mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h)]_{\gamma^i s^i} - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{w}_h)]_{\gamma^i s^i}) \|_{0,\Gamma_C^i}^2 \\
 & \left. - \frac{|1-\theta|}{4\beta} \|\gamma^{i-\frac{1}{2}} ([P_{n,\gamma^i}^i(\mathbf{v}_h)]_+ - [P_{n,\gamma^i}^i(\mathbf{w}_h)]_+) \|_{0,\Gamma_C^i}^2 \right)
 \end{aligned}$$



$$\begin{aligned}
 & -\frac{|1-\theta|\beta}{4} \|\gamma^{i\frac{1}{2}} \sigma_n^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|_{0,\Gamma_C^i}^2 \\
 & -\frac{|1-\theta|}{4\beta} \|\gamma^{i-\frac{1}{2}} (\mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h)]_{\gamma^i s^i} - [\mathbf{P}_{t,\gamma}^i(\mathbf{w}_h)]_{\gamma^i s^i})\|_{0,\Gamma_C^i}^2 \\
 & -\frac{|1-\theta|\beta}{4} \|\gamma^{i\frac{1}{2}} \sigma_t^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|_{0,\Gamma_C^i}^2 \\
 = & \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) + \sum_{i=1}^2 \left( -\frac{1}{2} \left( \theta + \frac{|1-\theta|\beta}{2} \right) (\|\gamma^{i\frac{1}{2}} \sigma_n^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|_{0,\Gamma_C^i}^2 \right. \\
 & \left. + \|\gamma^{i\frac{1}{2}} \sigma_t^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|_{0,\Gamma_C^i}^2) \right) \\
 & + \frac{1}{2} \left( 1 - \frac{|1-\theta|}{2\beta} \right) (\|\gamma^{i-\frac{1}{2}} ([P_{n,\gamma^i}^i(\mathbf{v}_h)]_+ - [P_{n,\gamma^i}^i(\mathbf{w}_h)]_+))\|_{0,\Gamma_C^i}^2 \\
 & + \|\gamma^{i-\frac{1}{2}} ([\mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h)]_{\gamma^i s^i} - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{w}_h)]_{\gamma^i s^i})\|_{0,\Gamma_C^i}^2).
 \end{aligned}$$

Choosing  $\beta = \frac{|1-\theta|}{2}$  and  $\gamma_0$  sufficiently small we get:

$$\begin{aligned}
 (\mathbf{B}\mathbf{v}_h - \mathbf{B}\mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h)_1 & \geq \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) \\
 & - \frac{(1+\theta)^2}{8} \sum_{i=1}^2 (\|\gamma^{i\frac{1}{2}} \sigma_n^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|_{0,\Gamma_C^i}^2 + \|\gamma^{i\frac{1}{2}} \sigma_t^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|_{0,\Gamma_C^i}^2). \\
 (\mathbf{B}\mathbf{v}_h - \mathbf{B}\mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h)_1 & \geq C \|\mathbf{v} - \mathbf{w}\|_1^2
 \end{aligned}$$

Note that, when  $\theta = -1$  we do not need the assumption of smallness of  $\gamma_0$ .

Let us show, now, that  $\mathbf{B}$  is hemicontinuous. Since  $\mathbf{V}^h$  is a vector space, it is sufficient to show that:

$$\begin{aligned}
 \varphi : [0, 1] & \rightarrow \mathbb{R} \\
 t & \mapsto (\mathbf{B}(\mathbf{v}_h - t\mathbf{w}_h), \mathbf{w}_h)_1
 \end{aligned}$$

is a continuous real function for all  $\mathbf{v}_h, \mathbf{w}_h \in \mathbf{V}_h$ . Let  $t, s \in [0, 1]$ , we compute:

$$\begin{aligned}
 & |\varphi(t) - \varphi(s)| \\
 = & \left| (\mathbf{B}(\mathbf{v}_h - t\mathbf{w}_h) - \mathbf{B}(\mathbf{v}_h - s\mathbf{w}_h), \mathbf{w}_h)_1 \right| \\
 = & \left| A_\theta((s-t)\mathbf{w}_h, \mathbf{w}_h) + \sum_{i=1}^2 \left( \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} P_{n,\theta\gamma^i}^i(\mathbf{w}_h) ([P_{n\gamma^i}^i(\mathbf{v}_h - t\mathbf{w}_h)]_+ \right. \right. \\
 & \left. \left. - [P_{n\gamma^i}^i(\mathbf{v}_h - s\mathbf{w}_h)]_+) d\Gamma \right. \right. \\
 & \left. \left. + \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} \mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{w}_h) ([\mathbf{P}_{t\gamma^i}^i(\mathbf{v}_h - t\mathbf{w}_h)]_{\gamma^i s^i} \right. \right. \\
 & \left. \left. - [\mathbf{P}_{t\gamma^i}^i(\mathbf{v}_h - s\mathbf{w}_h)]_{\gamma^i s^i}) d\Gamma \right) \right|
 \end{aligned}$$

$$\begin{aligned} &\leq |s - t|A_\theta(\mathbf{w}_h, \mathbf{w}_h) \\ &\quad + \sum_{i=1}^2 \left( \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} |P_{n,\theta\gamma^i}^i(\mathbf{w}_h)| \left| [P_{n\gamma^i}^i(\mathbf{v}_h - t\mathbf{w}_h)]_+ - [P_{n\gamma^i}^i(\mathbf{v}_h - s\mathbf{w}_h)]_+ \right| d\Gamma \right. \\ &\quad \left. + \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} \|\mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{w}_h)\| \left\| [P_{t\gamma^i}^i(\mathbf{v}_h - t\mathbf{w}_h)]_{\gamma^i s^i} - [P_{t\gamma^i}^i(\mathbf{v}_h - s\mathbf{w}_h)]_{\gamma^i s^i} \right\| d\Gamma \right). \end{aligned}$$

We use the bounds  $|[a]_+ - [b]_+| \leq |a - b|$  for all  $a, b \in \mathbb{R}$  and  $\|[\mathbf{a}]_{\gamma^i g^i} - [\mathbf{b}]_{\gamma^i g^i}\| \leq \|\mathbf{a} - \mathbf{b}\|$  for all  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{d-1}$  to deduce that:

$$\begin{aligned} &\int_{\Gamma_C^i} \frac{1}{\gamma^i} |P_{n,\theta\gamma^i}^i(\mathbf{w}_h)| \left| [P_{n\gamma^i}^i(\mathbf{v}_h - t\mathbf{w}_h)]_+ - [P_{n\gamma^i}^i(\mathbf{v}_h - s\mathbf{w}_h)]_+ \right| d\Gamma \\ &\quad + \int_{\Gamma_C^i} \frac{1}{\gamma^i} \|\mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{w}_h)\| \left\| [P_{t\gamma^i}^i(\mathbf{v}_h - t\mathbf{w}_h)]_{\gamma^i s^i} - [P_{t\gamma^i}^i(\mathbf{v}_h - s\mathbf{w}_h)]_{\gamma^i s^i} \right\| d\Gamma \\ &\leq \int_{\Gamma_C^i} \frac{1}{\gamma^i} |P_{n,\theta\gamma^i}^i(\mathbf{w}_h)| \left| P_{n\gamma^i}^i(\mathbf{v}_h - t\mathbf{w}_h) - P_{n\gamma^i}^i(\mathbf{v}_h - s\mathbf{w}_h) \right| d\Gamma \\ &\quad + \int_{\Gamma_C^i} \frac{1}{\gamma^i} \|\mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{w}_h)\| \left\| \mathbf{P}_{t\gamma^i}^i(\mathbf{v}_h - t\mathbf{w}_h) - \mathbf{P}_{t\gamma^i}^i(\mathbf{v}_h - s\mathbf{w}_h) \right\| d\Gamma \\ &\leq |s - t| \left( \int_{\Gamma_C^i} \frac{1}{\gamma^i} |P_{n,\theta\gamma^i}^i(\mathbf{w}_h)| |P_{n\gamma^i}^i(\mathbf{w}_h)| d\Gamma \right. \\ &\quad \left. + \int_{\Gamma_C^i} \frac{1}{\gamma^i} \|\mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{w}_h)\| \|\mathbf{P}_{t\gamma^i}^i(\mathbf{w}_h)\| d\Gamma \right). \end{aligned}$$

It results that:

$$\begin{aligned} |\varphi(t) - \varphi(s)| &\leq |s - t| \left( A_\theta(\mathbf{w}_h, \mathbf{w}_h) + \sum_{i=1}^2 \left( \int_{\Gamma_C^i} \frac{1}{2\gamma^i} |P_{n,\theta\gamma^i}^i(\mathbf{w}_h)| |P_{n\gamma^i}^i(\mathbf{w}_h)| d\Gamma \right. \right. \\ &\quad \left. \left. + \int_{\Gamma_C^i} \frac{1}{2\gamma^i} \|\mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{w}_h)\| \|\mathbf{P}_{t\gamma^i}^i(\mathbf{w}_h)\| d\Gamma \right) \right). \end{aligned}$$

Which means that  $\varphi$  is Lipschitz, so that  $\mathbf{B}$  is hemicontinuous. We finally apply the Corollary 15 (p.126) of [4] to conclude that  $\mathbf{B}$  is a one to one operator.  $\square$

### 2.3 A priori error analysis

Our Nitsche-based method (15) converges in a optimal way as the mesh parameter  $h$  vanishes. This is proved in the Theorem 2.6, where we provide an estimate of the displacement error in  $H^1$ -norm and of the contact error in  $L^2(\Gamma_C^i)$ -norm. We establish, first, the following abstract error estimate.

**Theorem 2.5** *Suppose that  $\mathbf{u}$  is a solution to (1–5) and belongs to  $(H^{\frac{3}{2}+v}(\Omega^1))^d \times (H^{\frac{3}{2}+v}(\Omega^2))^d$  with  $v > 0$ .*

1. We suppose  $\gamma_0$  sufficiently small. The solution  $\mathbf{u}_h$  to the discrete problem (15) satisfies the following error estimate:

$$\begin{aligned} & \sum_{i=1}^2 \left( \|\mathbf{u}^i - \mathbf{u}_h^i\|_{1,\Omega^i}^2 + \frac{1}{2} \|\gamma^{i\frac{1}{2}} (\sigma_n^i(\mathbf{u}^i) + \frac{1}{\gamma^i} [P_{n,\gamma}^i(\mathbf{u}_h)]_+ )\|_{0,\Gamma_C^i}^2 \right. \\ & \quad \left. + \frac{1}{2} \|\gamma^{i\frac{1}{2}} (\sigma_t^i(\mathbf{u}^i) + \frac{1}{\gamma^i} [P_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i})\|_{0,\Gamma_C^i}^2 \right) \\ & \leq C \inf_{\mathbf{v}_h \in \mathbf{V}_h} \left( \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{v}_h^i\|_{1,\Omega^i}^2 + \frac{1}{2} \|\gamma^{i-\frac{1}{2}} (\mathbf{u}^i - \mathbf{v}_h^i)\|_{0,\Gamma_C^i}^2 \right. \\ & \quad \left. + \frac{1}{2} \|\gamma^{i\frac{1}{2}} \sigma(\mathbf{u}^i - \mathbf{v}_h^i) \mathbf{n}^i\|_{0,\Gamma_C^i}^2 \right), \end{aligned} \tag{19}$$

where  $C > 0$  is a constant independent of  $h$ ,  $\mathbf{u}$  and  $\gamma_0$ .

2. If  $\theta = -1$ , for all  $\gamma_0 > 0$ , the solution  $\mathbf{u}_h$  to the problem (15) satisfies the estimate (19) with  $C > 0$  a constant independent of  $h$  and  $\mathbf{u}$ , but eventually dependent of  $\gamma_0$ .

*Proof* Let  $\mathbf{v}_h \in \mathbf{V}_h$ , using the coercivity and the continuity of the form  $a(\cdot, \cdot)$  as well as Young’s inequality, we obtain:

$$\begin{aligned} & \alpha \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{u}_h^i\|_{1,\Omega^i}^2 \leq a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) \\ & = a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) + a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \\ & \leq C \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{u}_h^i\|_{1,\Omega^i} \|\mathbf{u}^i - \mathbf{v}_h^i\|_{1,\Omega^i} + a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \\ & \leq \frac{\alpha}{2} \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{u}_h^i\|_{1,\Omega^i}^2 \\ & \quad + \frac{C^2}{2\alpha} \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{v}_h^i\|_{1,\Omega^i}^2 \\ & \quad + a(\mathbf{u}, \mathbf{v}_h - \mathbf{u}_h) - a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h). \end{aligned}$$

Therefore, we get:

$$\frac{\alpha}{2} \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{u}_h^i\|_{1,\Omega^i}^2 \leq \frac{C^2}{2\alpha} \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{v}_h^i\|_{1,\Omega^i}^2 + a(\mathbf{u}, \mathbf{v}_h - \mathbf{u}_h) - a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h).$$

Since  $\mathbf{u}$  solves (1-5) and  $\mathbf{u}_h$  solves (15), using the Lemma 2.1 yields:

$$\begin{aligned}
 \frac{\alpha}{2} \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{u}_h^i\|_{1,\Omega^i}^2 &\leq \frac{C^2}{2\alpha} \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{v}_h^i\|_{1,\Omega^i}^2 \\
 &+ \sum_{i=1}^2 \left( -\frac{\theta}{2} \int_{\Gamma_C^i} \gamma^i \boldsymbol{\sigma}^i(\mathbf{u}_h^i - \mathbf{u}^i) \mathbf{n}^i \cdot \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{u}_h^i) \mathbf{n}^i d\Gamma \right. \\
 &+ \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} \mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{v}_h - \mathbf{u}_h) \cdot ([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i} - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}) d\Gamma \\
 &\left. + \frac{1}{2} \int_{\Gamma_C^i} \frac{1}{\gamma^i} P_{n,\theta\gamma^i}^i(\mathbf{v}_h - \mathbf{u}_h) ([P_{n,\gamma^i}^i(\mathbf{u}_h)]_+ - [P_{n,\gamma^i}^i(\mathbf{u})]_+) d\Gamma \right). \tag{20}
 \end{aligned}$$

Let  $\beta_1 > 0$ . The first integral term in (20) is bounded, using Cauchy–Schwarz and Young’s inequalities, as follows:

$$\begin{aligned}
 &-\frac{\theta}{2} \int_{\Gamma_C^i} \gamma^i \boldsymbol{\sigma}^i(\mathbf{u}_h^i - \mathbf{u}^i) \mathbf{n}^i \cdot \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{u}_h^i) \mathbf{n}^i d\Gamma \\
 &= \frac{\theta}{2} \int_{\Gamma_C^i} \gamma^i \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{u}_h^i) \mathbf{n}^i \cdot \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{u}_h^i) \mathbf{n}^i d\Gamma \\
 &\quad - \frac{\theta}{2} \int_{\Gamma_C^i} \gamma^i \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{u}^i) \mathbf{n}^i \cdot \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{u}_h^i) \mathbf{n}^i d\Gamma \\
 &\leq \frac{\theta}{2} \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{u}_h^i) \mathbf{n}^i\|_{0,\Gamma_C^i}^2 \\
 &\quad + \frac{|\theta|}{2} \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{u}^i) \mathbf{n}^i\|_{0,\Gamma_C^i} \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{u}_h^i) \mathbf{n}^i\|_{0,\Gamma_C^i} \\
 &\leq \frac{\beta_1 \theta^2}{4} \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{u}^i) \mathbf{n}^i\|_{0,\Gamma_C^i}^2 \\
 &\quad + \frac{1}{2} \left( \theta + \frac{1}{2\beta_1} \right) \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{u}_h^i) \mathbf{n}^i\|_{0,\Gamma_C^i}^2. \tag{21}
 \end{aligned}$$

For the second integral term in (20), we can write:

$$\begin{aligned}
 &\int_{\Gamma_C^i} \frac{1}{\gamma^i} \mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{v}_h - \mathbf{u}_h) \cdot ([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i} - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}) d\Gamma \\
 &= \int_{\Gamma_C^i} \frac{1}{\gamma^i} \mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h - \mathbf{u}) \cdot ([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i} - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}) d\Gamma \\
 &\quad + \int_{\Gamma_C^i} \frac{1}{\gamma^i} \mathbf{P}_{t,\gamma^i}^i(\mathbf{u} - \mathbf{u}_h) \cdot ([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i} - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}) d\Gamma \\
 &\quad + \int_{\Gamma_C^i} (1 - \theta) \boldsymbol{\sigma}_t^i(\mathbf{v}_h^i - \mathbf{u}_h^i) \cdot ([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i} - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}) d\Gamma.
 \end{aligned}$$

Using the bound (10) and applying two times Cauchy–Schwarz and Young’s inequalities, we obtain for  $\beta_2 > 0$  and  $\beta_3 > 0$ :

$$\begin{aligned}
 & \int_{\Gamma_C^i} \frac{1}{\gamma^i} \mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{v}_h - \mathbf{u}_h) \cdot ([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i} - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}) d\Gamma \\
 & \leq \frac{1}{2\beta_2} \left\| \gamma^{i\frac{1}{2}} \left( \boldsymbol{\sigma}_t^i(\mathbf{u}^i) + \frac{1}{\gamma^i} [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i} \right) \right\|_{0,\Gamma_C^i}^2 \\
 & \quad + \frac{\beta_2}{2} \left\| \gamma^{i-\frac{1}{2}} [\mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h - \mathbf{u})]_{\gamma^i s^i} \right\|_{0,\Gamma_C^i}^2 \\
 & \quad - \left\| \gamma^{i\frac{1}{2}} \left( \boldsymbol{\sigma}_t^i(\mathbf{u}^i) + \frac{1}{\gamma^i} [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i} \right) \right\|_{0,\Gamma_C^i}^2 \\
 & \quad + \frac{|1-\theta|}{2\beta_3} \left\| \gamma^{i\frac{1}{2}} \left( \boldsymbol{\sigma}_t^i(\mathbf{u}^i) \right. \right. \\
 & \quad \left. \left. + \frac{1}{\gamma^i} [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i} \right) \right\|_{0,\Gamma_C^i}^2 \\
 & \quad + \frac{|1-\theta|\beta_3}{2} \left\| \gamma^{i\frac{1}{2}} \boldsymbol{\sigma}_t^i(\mathbf{v}_h^i - \mathbf{u}_h^i) \right\|_{0,\Gamma_C^i}^2. \tag{22}
 \end{aligned}$$

In a similar way, we can upper bound the third integral term of (20). Noting that:

$$\begin{aligned}
 & \left\| \gamma^{i-\frac{1}{2}} [\mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h - \mathbf{u})]_{\gamma^i s^i} \right\|_{0,\Gamma_C^i}^2 + \left\| \gamma^{i-\frac{1}{2}} [P_{n,\gamma^i}^i(\mathbf{v}_h - \mathbf{u})]_+ \right\|_{0,\Gamma_C^i}^2 \\
 & \leq 2 \left\| \gamma^{i-\frac{1}{2}} (\llbracket \mathbf{u} - \mathbf{v}_h \rrbracket_n^i + \llbracket \mathbf{u} - \mathbf{v}_h \rrbracket_t^i) \right\|_{0,\Gamma_C^i}^2 + 2 \left\| \gamma^{i\frac{1}{2}} \boldsymbol{\sigma}^i(\mathbf{u}^i - \mathbf{v}_h^i) \mathbf{n}^i \right\|_{0,\Gamma_C^i}^2 \\
 & \leq 2 \sum_{i=1}^2 \left( \left\| \gamma^{i-\frac{1}{2}} (\mathbf{u}^i - \mathbf{v}_h^i) \right\|_{0,\Gamma_C^i}^2 \right) + 2 \left\| \gamma^{i\frac{1}{2}} \boldsymbol{\sigma}^i(\mathbf{u}^i - \mathbf{v}_h^i) \mathbf{n}^i \right\|_{0,\Gamma_C^i}^2, \tag{23}
 \end{aligned}$$

and using estimates (21) and (22) in (20), we obtain:

$$\begin{aligned}
 & \frac{\alpha}{2} \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{u}_h^i\|_{1,\Omega^i}^2 \leq \frac{C^2}{2\alpha} \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{v}_h^i\|_{1,\Omega^i}^2 \\
 & \quad + \frac{1}{2} \sum_{i=1}^2 \left( \left( \frac{\beta_1 \theta^2}{2} + \beta_2 \right) \left\| \gamma^{i\frac{1}{2}} \boldsymbol{\sigma}^i(\mathbf{u}^i - \mathbf{v}_h^i) \mathbf{n}^i \right\|_{0,\Gamma_C^i}^2 + 2\beta_2 \left\| \gamma^{i-\frac{1}{2}} (\mathbf{u}^i - \mathbf{v}_h^i) \right\|_{0,\Gamma_C^i}^2 \right. \\
 & \quad \left. + \left( -1 + \frac{1}{2\beta_2} + \frac{|1-\theta|}{2\beta_3} \right) \right. \\
 & \quad \left. \left( \left\| \gamma^{i\frac{1}{2}} \left( \boldsymbol{\sigma}_t^i(\mathbf{u}^i) + \frac{1}{\gamma^i} [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i} \right) \right\|_{0,\Gamma_C^i}^2 + \left\| \gamma^{i\frac{1}{2}} \left( \boldsymbol{\sigma}_n^i(\mathbf{u}^i) + \frac{1}{\gamma^i} [P_{n,\gamma^i}^i(\mathbf{u}_h)]_+ \right) \right\|_{0,\Gamma_C^i}^2 \right) \right. \\
 & \quad \left. + \left( \frac{1}{2\beta_1} + \theta + \frac{|1-\theta|\beta_3}{2} \right) \left\| \gamma^{i\frac{1}{2}} \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{u}_h^i) \mathbf{n}^i \right\|_{0,\Gamma_C^i}^2 \right). \tag{24}
 \end{aligned}$$

We use now the estimate (18) to get:

$$\|\gamma^{i\frac{1}{2}}\sigma^i(\mathbf{v}_h^i - \mathbf{u}_h^i)\mathbf{n}^i\|_{0,\Gamma_C^i}^2 \leq C\gamma_0^{\frac{1}{2}}\|\mathbf{v}_h^i - \mathbf{u}_h^i\|_{1,\Omega^i}^2 \leq C\gamma_0^{\frac{1}{2}}(\|\mathbf{v}_h^i - \mathbf{u}^i\|_{1,\Omega^i}^2 + \|\mathbf{u}_h^i - \mathbf{u}^i\|_{1,\Omega^i}^2) \tag{25}$$

For a fixed  $\theta \in \mathbb{R}$  we choose  $\beta_2$  and  $\beta_3$  large enough that:

$$-1 + \frac{1}{2\beta_2} + \frac{|1 - \theta|}{2\beta_3} < -\frac{1}{2}$$

Choosing  $\gamma_0$  small enough in (25) and putting the estimate in (24), we establish the first statement of the theorem.

We consider now the case  $\theta = -1$  in which (24) becomes:

$$\begin{aligned} \frac{\alpha}{2} \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{u}_h^i\|_{1,\Omega^i}^2 &\leq \frac{C^2}{2\alpha} \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{v}_h^i\|_{1,\Omega^i}^2 \\ &+ \frac{1}{2} \sum_{i=1}^2 \left( \left( \frac{\beta_1}{2} + \beta_2 \right) \|\gamma^{i\frac{1}{2}}\sigma^i(\mathbf{u}^i - \mathbf{v}_h^i)\mathbf{n}^i\|_{0,\Gamma_C^i}^2 + 2\beta_2 \|\gamma^{i-\frac{1}{2}}(\mathbf{u}^i - \mathbf{v}_h^i)\|_{0,\Gamma_C^i}^2 \right. \\ &+ \left. \left( -1 + \frac{1}{2\beta_2} + \frac{1}{\beta_3} \right) \right. \\ &\left. \left( \|\gamma^{i\frac{1}{2}}(\sigma_n^i(\mathbf{u}^i) + \frac{1}{\gamma^i}[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i})\|_{0,\Gamma_C^i}^2 + \|\gamma^{i\frac{1}{2}}(\sigma_n^i(\mathbf{u}^i) + \frac{1}{\gamma^i}[P_{n,\gamma^i}^i(\mathbf{u}_h)]_+)\|_{0,\Gamma_C^i}^2 \right) \right. \\ &\left. + \left( \frac{1}{2\beta_1} - 1 + \beta_3 \right) \left( \|\gamma^{i\frac{1}{2}}\sigma^i(\mathbf{v}_h^i - \mathbf{u}_h^i)\mathbf{n}^i\|_{0,\Gamma_C^i}^2 \right) \right). \end{aligned}$$

Let be given  $\eta > 0$ . Set  $\beta_1 = \frac{1}{2\eta}$ ,  $\beta_2 = 1 + \frac{1}{\eta}$ ,  $\beta_3 = 1 + \eta$ . And so we arrive at:

$$\begin{aligned} \frac{\alpha}{2} \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{u}_h^i\|_{1,\Omega^i}^2 &\leq \frac{C^2}{2\alpha} \sum_{i=1}^2 \|\mathbf{u}^i - \mathbf{v}_h^i\|_{1,\Omega^i}^2 \\ &+ \frac{1}{2} \sum_{i=1}^2 \left( \left( \frac{5}{4\eta} + 1 \right) \|\gamma^{i\frac{1}{2}}\sigma^i(\mathbf{u}^i - \mathbf{v}_h^i)\mathbf{n}^i\|_{0,\Gamma_C^i}^2 + 2\frac{1+\eta}{\eta} \|\gamma^{i-\frac{1}{2}}(\mathbf{u}^i - \mathbf{v}_h^i)\|_{0,\Gamma_C^i}^2 \right. \\ &- \frac{\eta}{2(1+\eta)} \left( \|\gamma^{i\frac{1}{2}}(\sigma_n^i(\mathbf{u}^i) + \frac{1}{\gamma^i}[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i})\|_{0,\Gamma_C^i}^2 \right. \\ &+ \left. \|\gamma^{i\frac{1}{2}}(\sigma_n^i(\mathbf{u}^i) + \frac{1}{\gamma^i}[P_{n,\gamma^i}^i(\mathbf{u}_h)]_+)\|_{0,\Gamma_C^i}^2 \right) \\ &\left. + 2\eta \|\gamma^{i\frac{1}{2}}\sigma^i(\mathbf{v}_h^i - \mathbf{u}_h^i)\mathbf{n}^i\|_{0,\Gamma_C^i}^2 \right) \end{aligned}$$

Set  $\eta = \frac{\alpha}{16C^2\gamma_0}$ , where  $C$  is the constant in (25) to conclude the proof of the theorem. □

**Theorem 2.6** *Suppose that  $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2)$  is a solution to problem (1–5) and belongs to  $(H^{\frac{3}{2}+\nu}(\Omega^1))^d \times (H^{\frac{3}{2}+\nu}(\Omega^2))^d$  with  $0 < \nu \leq \frac{1}{2}$  if  $k = 1$  and  $0 < \nu \leq 1$  if  $k = 2$  ( $k$  is the degree of the finite element method). If  $\theta = -1$  or  $\gamma_0$  is sufficiently small, the solution  $\mathbf{u}_h$  to the problem (15) satisfies the following estimate:*

$$\sum_{i=1}^2 \left( \|\mathbf{u}^i - \mathbf{u}_h^i\|_{1,\Omega^i}^2 + \frac{1}{2} \|\gamma^{i\frac{1}{2}}(\sigma_n^i(\mathbf{u}^i) + \frac{1}{\gamma^i}[P_{n,\gamma^i}^i(\mathbf{u}_h)]_+)\|_{0,\Gamma_C^i}^2 + \frac{1}{2} \|\gamma^{i\frac{1}{2}}(\sigma_t^i(\mathbf{u}^i) + \frac{1}{\gamma^i}[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i})\|_{0,\Gamma_C^i}^2 \right) \leq Ch^{1+2\nu} \sum_{i=1}^2 \|\mathbf{u}^i\|_{\frac{3}{2}+\nu,\Omega^i}^2 \tag{26}$$

where  $C$  is a constant independent of  $\mathbf{u}$  and  $h$ .

*Proof* To establish (26) we need to bound the right terms in estimate (19). We choose  $\mathbf{v}_h^i = \mathcal{I}_h^i \mathbf{u}^i$  where  $\mathcal{I}_h^i$  stands for the Lagrange interpolation operator mapping onto  $\mathbf{V}_h^i$ . The estimation of the Lagrange interpolation error in the  $H^1$ -norm on a domain is classical (see, e.g. [3,9] and [10])

$$\|\mathbf{u}^i - \mathcal{I}_h^i \mathbf{u}^i\|_{1,\Omega^i} \leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}^i\|_{\frac{3}{2}+\nu,\Omega^i} \tag{27}$$

for  $-\frac{1}{2} < \nu \leq k - \frac{1}{2}$ .

Let  $E$  in  $\Gamma_C^i$  be an edge of triangle  $K \in T_h^i$ , we have:

$$\|\gamma^{i-\frac{1}{2}}(\mathbf{u}^i - \mathcal{I}_h^i \mathbf{u}^i)\|_{0,E} \leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}^i\|_{1+\nu,E}$$

A summation on all the edges  $E$ , with the trace theorem yields:

$$\|\gamma^{i-\frac{1}{2}}(\mathbf{u}^i - \mathcal{I}_h^i \mathbf{u}^i)\|_{0,\Gamma_C^i} \leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}^i\|_{1+\nu,\Gamma_C^i} \leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}^i\|_{\frac{3}{2}+\nu,\Omega^i} \tag{28}$$

From Appendix A of [8] (see also [12]), we get the following estimate:

$$\|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{u}^i - \mathcal{I}_h^i \mathbf{u}^i) \mathbf{n}^i\|_{0,\Gamma_C^i} \leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}^i\|_{\frac{3}{2}+\nu,\Omega^i} \tag{29}$$

By inserting (27), (28) and (29) onto (19) we get (26). □

### 3 Numerical experiments

In this section, we test the Nitsche unbiased method (15) for two/three-dimensional contact between two elastic bodies  $\Omega^1$  and  $\Omega^2$ . The first body is a disk/sphere and the second is a rectangle/rectangular cuboid. This situation is not strictly a Hertz type contact problem because  $\Omega^2$  is bounded.

The tests are performed with  $P_1$  and  $P_2$  Lagrange finite elements. The finite element library Getfem++ is used. The discrete contact problem is solved by using a generalized

Newton method. Further details on generalized Newton's method applied to contact problems can be found for instance in [25] and the references therein. The accuracy of the method is discussed for the different cases with respect to the finite element used, the mesh size, and the value of the parameters  $\theta$  and  $\gamma_0$ . We perform experiments with a frictionless contact to compare the results of the formulation with other ones using Nitsche's method (given mainly in [8, 11]). Moreover, we present the convergence curves for frictional contact in Figs. 11 and 12.

The numerical tests in two dimensions (resp. three dimensions) are performed on a domain  $\Omega = ] - 0.5, 0.5[$  (resp.  $\Omega = ] - 0.5, 0.5[$ ) containing the two bodies  $\Omega^1$  and  $\Omega^2$ . The first body is a disk of radius 0.25 and center (0,0) (resp. a sphere of radius 0.25 and center (0,0,0)), and the second is a rectangle  $] - 0.5, 0.5[ \times ] - 0.5, -0.25[$  (resp.  $\Omega^2 = ] - 0.5, 0.5[ \times ] - 0.5, 0.25[$ ). The contact surface  $\Gamma_C^1$  is the lower semicircle and  $\Gamma_C^2$  is the top surface of  $\Omega^2$  (i.e.  $\Gamma_C^1 = \{\mathbf{x} \in \partial\Omega^1; x_2 \leq 0\}$  and  $\Gamma_C^2 = \{\mathbf{x} \in \partial\Omega^2; x_2 = -0.25\}$  (resp.  $\Gamma_C^1 = \{\mathbf{x} \in \partial\Omega^1; x_3 \leq 0\}$  and  $\Gamma_C^2 = \{\mathbf{x} \in \partial\Omega^2; x_3 = -0.25\}$ )). A Dirichlet condition is prescribed at the bottom of the rectangle (resp. cuboid). Since no Dirichlet condition is applied on  $\Omega^1$  the problem is only semi-coercive. To overcome the non-definiteness coming from the free rigid motions, the horizontal displacement is prescribed to be zero on the two points of coordinates (0,0) and (0,0.1) (resp. (0,0,0) and (0,0,0.1)) which blocks the horizontal translation and the rigid rotation. The projector  $\Pi^1$  is defined from  $\Gamma_C^1$  to  $\Gamma_C^2$  in the vertical direction. All remaining parts of the boundaries are considered traction free. For simplicity, we consider a dimensionless configuration with Lamé coefficients  $\lambda = 1$  and  $\mu = 1$  and a volume density of vertical force  $f_v = -0.25$ .

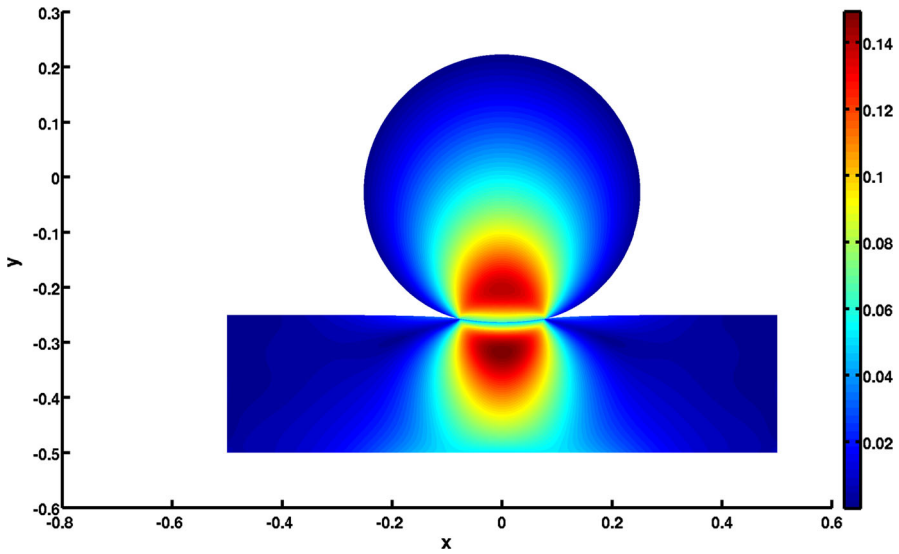
The expression of the exact solution being unknown, the convergence is studied with respect to a reference solution computed with a  $P_2$  element on a very fine mesh for  $\theta = -1$  (see Figs. 2 and 3).

To show the quality of the approximation method we plot in Fig. 4 the contact stress profile on the second boundary and we compare it to Hertz's solution. The diagrams in Fig. 4 correspond to the pressure profiles for the reference fine mesh with quadratic elements. The vertical green arrows correspond to values of the contact pressure field at quadrature points. The blue solid line represents the analytically calculated Hertz's pressure profile. The left diagram correspond to the bidimensional case and the right one is the obtained pressure at quadrature points of the elements crossing the plan  $y = 0$  in the three dimensional case.

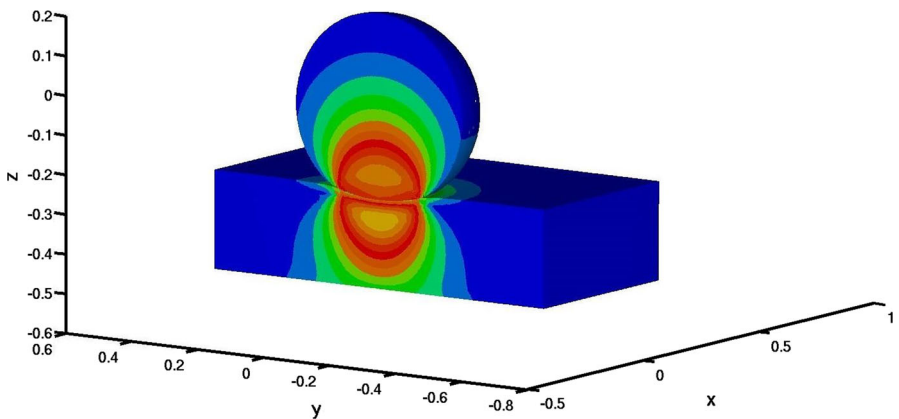
### 3.1 Convergence in the two dimensional frictionless case

We perform a numerical convergence study on the three methods  $\theta = 1$ ,  $\theta = 0$  and  $\theta = -1$  for a fixed parameter  $\gamma_0 = \frac{1}{100}$  (chosen small in order to obtain convergence for the three cases) and friction coefficients  $s^1 = s^2 = 0$ . In each case we plot the relative error in percentage in the  $H^1$ -norm of the displacement in the two bodies and the error of the  $L^2$  norm of the Nitsche's contact condition on  $\Gamma_C^1$  and  $\Gamma_C^2$ . The error of the Nitsche's contact condition is equal to:





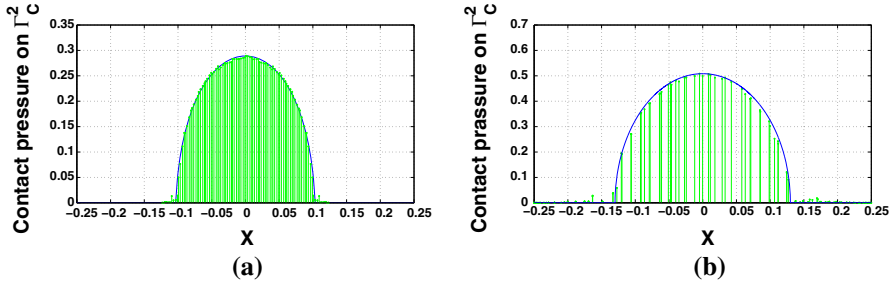
**Fig. 2** 2D Numerical reference solution with contour plot of Von Mises stress.  $h = 1/400$ ,  $\gamma_0 = 1/100$  and  $P_2$  elements



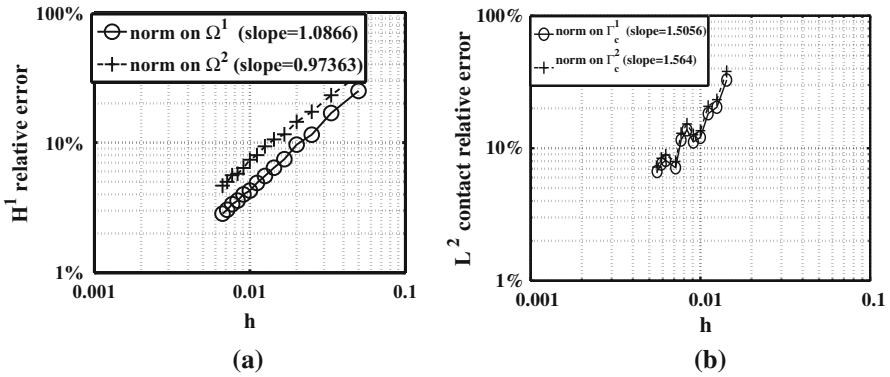
**Fig. 3** Cross-section of 3D numerical reference solution with contour plot of Von Mises stress.  $h = 1/50$ ,  $\gamma_0 = 1/100$  and  $P_2$  elements

$$\frac{\|\gamma^{i\frac{1}{2}}(\sigma_n^i(\mathbf{u}_{ref}^{hi}) + \frac{1}{\gamma^i}[P_{n,\gamma}^i(\mathbf{u}_h)]_+)\|_{0,\Gamma_C^i}}{\|\gamma^{\frac{1}{2}}\sigma_n^i(\mathbf{u}_{ref}^{hi})\|_{0,\Gamma_C^i}}, \text{ where } \mathbf{u}_{ref}^{hi} \text{ is the reference solution on } \Omega^i.$$

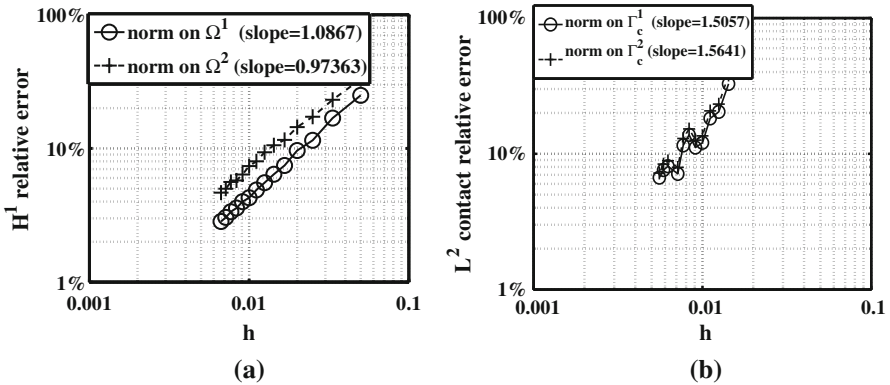
On Figs. 5, 6 and 7 the curves of relative error in percentage for Lagrange  $P_1$  finite elements are plotted. The convergence rate in a  $H^1$ -norm is about 1 for the three values of  $\theta$  which is in this case optimal, according to Theorem 2.6. On Figs. 8, 9 and 10 the same experiments are reported for Lagrange  $P_2$  finite elements. The convergence rate for the three cases is about 1.5 which corresponds to optimality as well.



**Fig. 4** Contact pressure profile for the 2D and 3D cases (Hertz solution in blue solid line and computed solution in vertical green lines). **a** 2D, **b** 3D (for  $y = 0$ ) (color figure online)



**Fig. 5** Convergence curves in 2D for the method  $\theta = 1$ , with  $\gamma_0 = 1/100$  and  $P_1$  finite elements for the relative  $H^1$ -norm of the error **(a)** and the relative  $L^2(\Gamma_C)$ -norm of the error **(b)**



**Fig. 6** Convergence curves in 2D for the method  $\theta = 0$ , with  $\gamma_0 = 1/100$  and  $P_1$  finite elements for the relative  $H^1$ -norm of the error **(a)** and the relative  $L^2(\Gamma_C)$ -norm of the error **(b)**

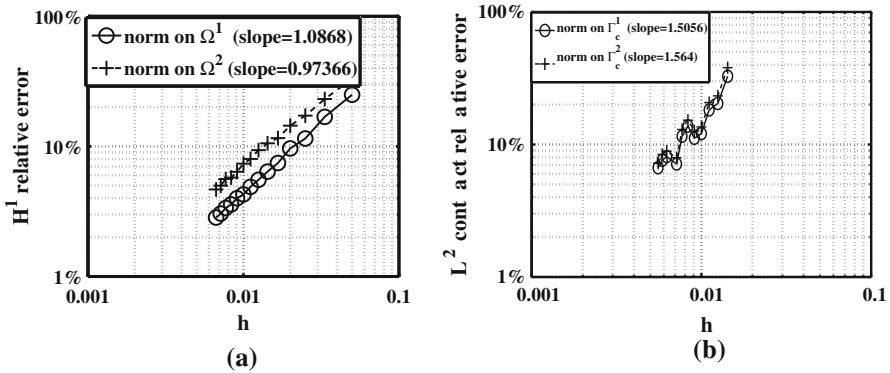


Fig. 7 Convergence curves in 2D for the method  $\theta = -1$ , with  $\gamma_0 = 1/100$  and  $P_1$  finite elements for the relative  $H^1$ -norm of the error (a) and the relative  $L^2(\Gamma_C)$ -norm of the error (b)

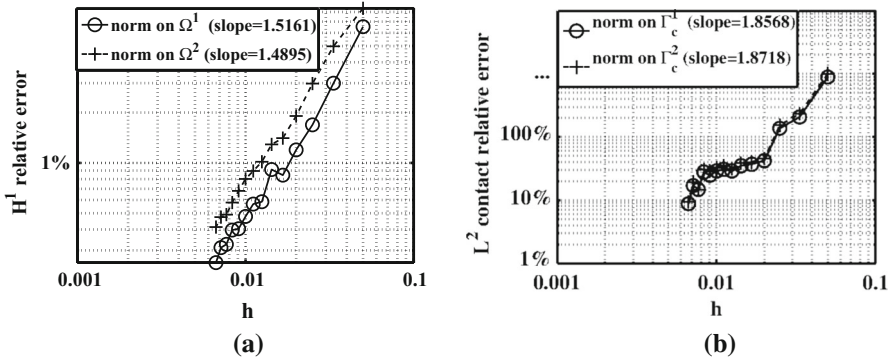


Fig. 8 Convergence curves in 2D for the method  $\theta = 1$ , with  $\gamma_0 = 1/100$  and  $P_2$  finite elements for the relative  $H^1$ -norm of the error (a) and the relative  $L^2(\Gamma_C)$ -norm of the error (b)

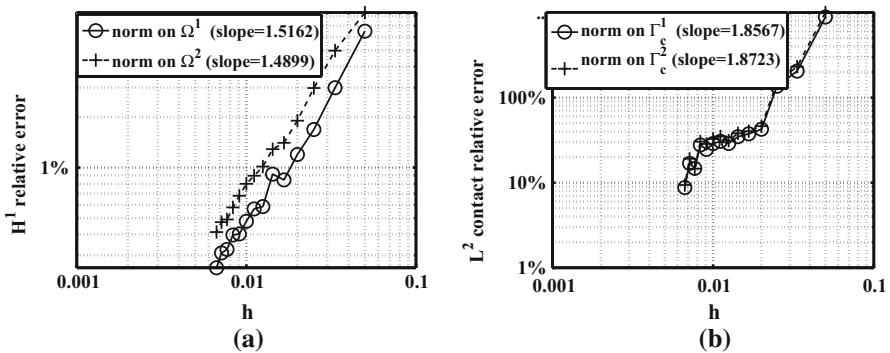
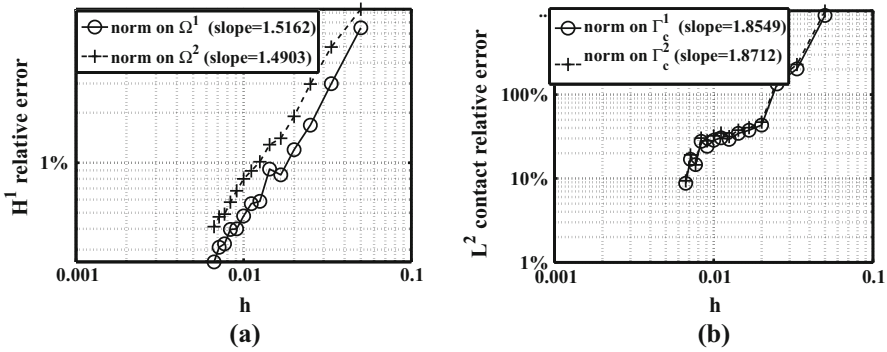
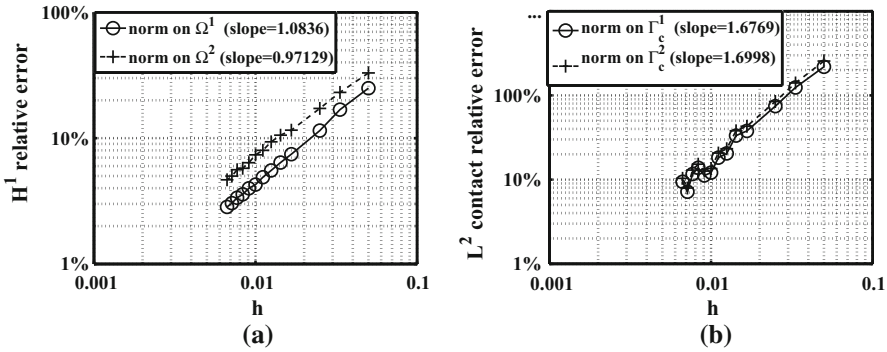


Fig. 9 Convergence curves in 2D for the method  $\theta = 0$ , with  $\gamma_0 = 1/100$  and  $P_2$  finite elements for the relative  $H^1$ -norm of the error (a) and the relative  $L^2(\Gamma_C)$ -norm of the error (b)



**Fig. 10** Convergence curves in 2D for the method  $\theta = -1$ , with  $\gamma_0 = 1/100$  and  $P_2$  finite elements for the relative  $H^1$ -norm of the error (a) and the relative  $L^2(\Gamma_C)$ -norm of the error (b)



**Fig. 11** Convergence curves in 2D frictional case for the method  $\theta = -1$ , with  $\gamma_0 = 1/100$  with  $P_1$  finite elements for the relative  $H^1$ -norm of the error (a) for the  $L^2(\Gamma_C)$ -norm of the error (b)

### 3.2 Convergence in 2D frictional contact case

We establish, as well, the convergence curves for a frictional contact (Tresca friction) with a friction coefficient  $s^1 = 0.1$  with the method  $\theta = -1$ , for a Nitsche’s parameter  $\gamma_0 = \frac{1}{100}$ . The frictional contact curves are presented for  $P_1$  and  $P_2$  Lagrange elements in Figs. 11 and 12. Similar curves are obtained with other values of  $\theta$ . We mention here that this numerical validation is the first one for Nitsche’s method with frictional contact since in [5] no numerical study was performed. This validation confirms optimal convergence with a convergence rate close to the frictionless case.

### 3.3 Convergence in the three dimensional case

The three-dimensional tests are similar to the two-dimensional ones. The error curves with  $\theta = -1$  and  $P_1$  Lagrange elements are presented in Fig. 13. Very similar conclusions can be drawn compared with the two-dimensional case.

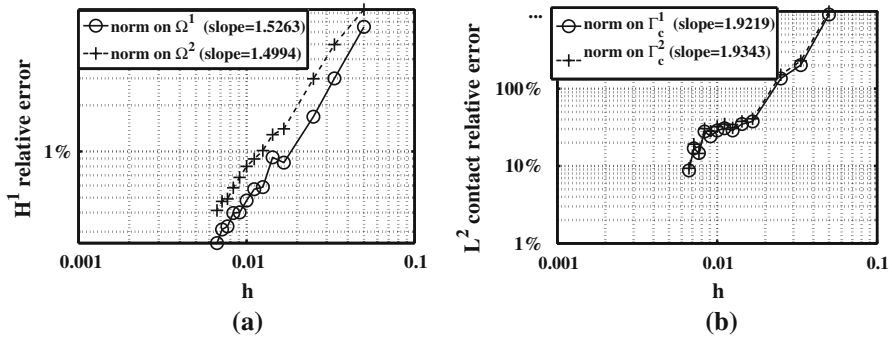


Fig. 12 Convergence curves in 2D frictional case for the method  $\theta = -1$ , with  $\gamma_0 = 1/100$  with  $P_2$  finite elements for the relative  $H^1$ -norm of the error (a) for the  $L^2(\Gamma_C)$ -norm of the error (b)

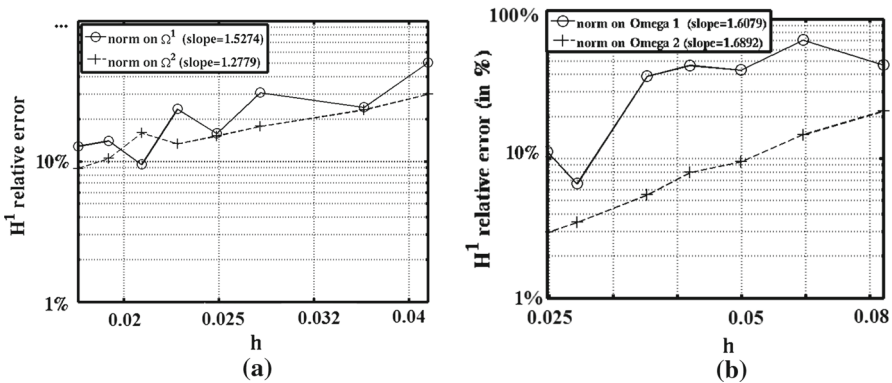


Fig. 13 Convergence curves in 3D for the method  $\theta = -1$ , with  $\gamma_0 = 1/100$  for the relative  $H^1$ -norm of the error with  $P_1$  finite elements (a) and  $P_2$  finite elements (b)

As expected the optimal convergence is obtained in  $H^1$  and  $L^2(\Gamma_C)$ -norm for all methods in good accordance with Theorem 2.6.

### 3.4 Comparison with other methods

To better compare the proposed method with other ones we present in the following the convergence curves of our test case with the convergence curves of the biased Nitsche's formulation and the augmented Lagrangian method [8, 16], see Figs. 14 and 15.

The curves are exactly the same for  $P_1$  elements and very similar for  $P_2$  ones and the convergence rate of the unbiased Nitsche's method is equal to other formulations' rate. We note that, for different values of  $\theta$  the convergence is obtained for Nitsche's method (biased and unbiased) and the augmented Lagrangian method generally with a close number of iterations of the Newton algorithm.

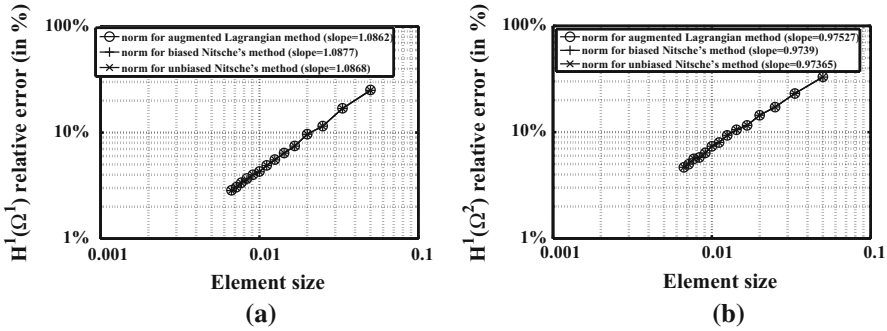


Fig. 14 Comparison of convergence curves in 2D frictionless case for the method  $\theta = -1$ , with  $\gamma_0 = 1/100$  and  $P_1$  finite elements for the relative  $H^1$ -norm of the error on  $\Omega^1$  (a) and on  $\Omega^2$  (b) for different formulations of contact

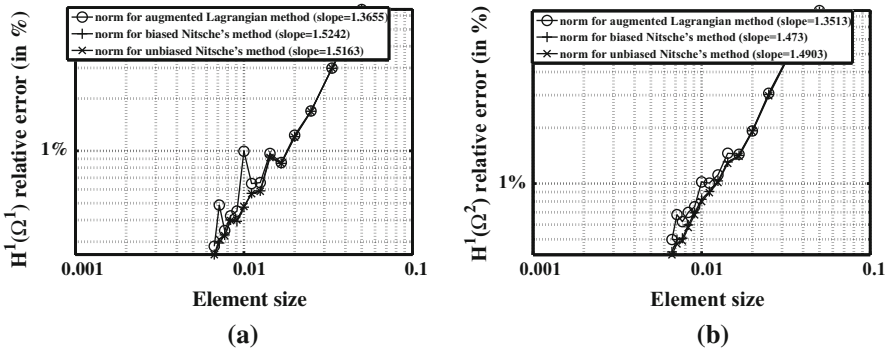


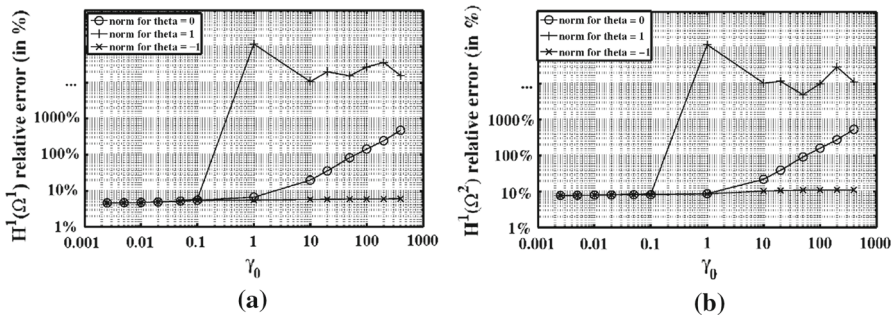
Fig. 15 Comparison of convergence curves in 2D frictionless case for the method  $\theta = -1$ , with  $\gamma_0 = 1/100$  and  $P_2$  finite elements for the relative  $H^1$ -norm of the error on  $\Omega^1$  (a) and on  $\Omega^2$  (b) for different formulations of contact

### 3.5 Influence of the Nitsche parameter

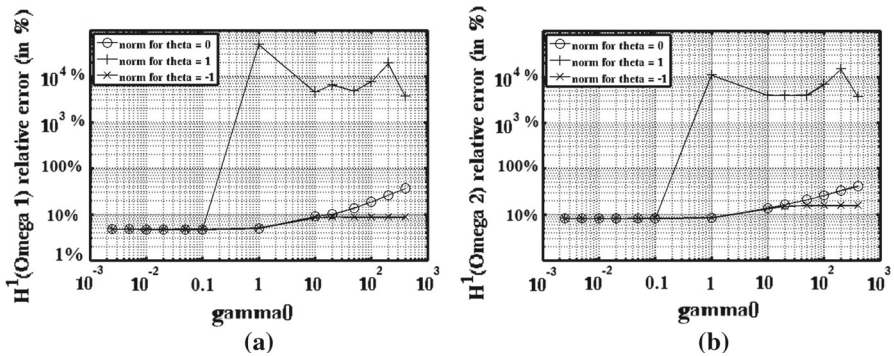
The influence of  $\gamma_0$  on the  $H^1$ -norm of the error for  $P_2$  elements is plotted in Fig. 16 in the frictionless case and on Fig. 17 with a friction coefficient  $s^1 = 0.1$ . It is remarkable that the error curves for the smallest value of  $\gamma_0$  are rather the same for the three values of  $\theta$ .

The variant  $\theta = 1$  is the most influenced by the value of  $\gamma_0$ . It converges only for  $\gamma_0$  very small ( $\leq 10^{-1}$ ). The method for  $\theta = 0$  gives a much large window of choice of  $\gamma_0$  though it has to remain small to keep a good solution. In agreement with the theoretical result of Theorem 2.6, the influence of  $\gamma_0$  on the method  $\theta = -1$  is limited.

So the choice of  $\gamma_0$  depends on the considered version. We can always choose  $\theta = -1$  to insure the stability and convergence independently on  $\gamma_0$ . In this case we loose symmetry and we need to introduce  $\sigma(v_h)$  into the weak formulation. The version  $\theta = 1$  allows to keep symmetry, however it requires that  $\gamma_0$  be rather small. The version  $\theta = 0$  can be seen as good compromise since it is the simplest and it remains stable and converges optimally even for moderate values of  $\gamma_0$ . A strategy to guarantee



**Fig. 16** Influence of  $\gamma_0$  on the  $H^1$ -norm error for different values of  $\theta$  in the 2D frictionless case and with  $P_2$  finite elements on  $\Omega^1$  (a) and on  $\Omega^2$  (b)



**Fig. 17** Influence of  $\gamma_0$  on the  $H^1$ -norm error for different values of  $\theta$  in the 2D frictional case and with  $P_2$  finite elements on  $\Omega^1$  (a) and on  $\Omega^2$  (b)

the coercivity of the problem and then an optimal convergence is of course to consider a sufficiently small  $\gamma_0$ . However, the price to pay is an ill-conditioned discrete problem. The study presented in [25] shows that Newton’s method has important difficulties to converge when  $\gamma_0$  is very small because the nonlinear discrete system (15) becomes very stiff in this case.

**Conclusion**

A theoretical and numerical study of Nitsche’s method was carried out for the Signorini problem in [6, 8]. These analysis prove the performance of this type of formulation for contact between an elastic body and a rigid support. In this work we adapt Nitsche’s method to the two elastic bodies contact problem and we proposed an unbiased method that could be directly applicable to multi-body contact and self-contact. The method was analysed and we proved its consistency, well-posedness and optimal convergence. For the numerical study, the accuracy of the method was discussed for the Hertz problem with different types of finite elements, for variations of the mesh size and the value of the parameters  $\theta$  and  $\gamma_0$ . Frictionless and frictional situations have been considered, as well as two- and three-dimensional cases. The theoretical results are, generally, con-

firmed by numerical tests, especially the optimal convergence and the influence of the parameter  $\gamma_0$ . Since the analysis in the small strain case are promising, forthcoming studies will be for the non-linear materials in the large deformation framework. In this case, our goal is to provide a construction of the method similar to the linear case and the corresponding numerical study.

As well, other solvers than semi-smooth Newton could be considered for improved computational efficiency. For instance highly efficient multigrid methods have been designed for mortar-type discretizations of contact problems in [29]. The adaptation of multigrid techniques to Nitsche’s discretization of contact is still an open issue and could be considered as a perspective (see [12] for multigrid with Nitsche for interface problems).

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## Appendix

### A Strong–weak formulation equivalence

Let  $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2)$  be a sufficiently regular solution to the problem (13). Using the definitions of  $A_\theta$ ,  $P_{\gamma^i}^i(\mathbf{u})$  and  $P_{\theta\gamma^i}^i(\mathbf{v})$ , we obtain:

$$\begin{aligned} \mathbf{a}(\mathbf{u}, \mathbf{v}) &= \frac{1}{2} \int_{\Gamma_c^1} \theta \gamma^1 \sigma_n^1(\mathbf{u}^1) \sigma_n^1(\mathbf{v}^1) d\Gamma - \frac{1}{2} \int_{\Gamma_c^2} \theta \gamma^2 \sigma_n^2(\mathbf{u}^2) \sigma_n^2(\mathbf{v}^2) d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_c^1} \theta \gamma^1 \boldsymbol{\sigma}_t^1(\mathbf{u}^1) \cdot \boldsymbol{\sigma}_t^1(\mathbf{v}^1) d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_c^2} \theta \gamma^2 \boldsymbol{\sigma}_t^2(\mathbf{u}^2) \cdot \boldsymbol{\sigma}_t^2(\mathbf{v}^2) d\Gamma + \frac{1}{2} \int_{\Gamma_c^1} \frac{1}{\gamma^1} [ [u ]_n^1 - g_n^1 - \gamma^1 \sigma_n^1(\mathbf{u}^1) ]_+ \\ &\quad (v_n^1 + v_n^2 \circ \Pi^1 - \theta \gamma^1 \sigma_n^1(\mathbf{v}^1)) d\Gamma \\ &\quad + \frac{1}{2} \int_{\Gamma_c^2} \frac{1}{\gamma^2} [ [u ]_n^2 - g_n^2 - \gamma^2 \sigma_n^2(\mathbf{u}^2) ]_+ (v_n^2 + v_n^1 \circ \Pi^2 - \theta \gamma^2 \sigma_n^2(\mathbf{v}^2)) d\Gamma \\ &\quad + \frac{1}{2} \int_{\Gamma_c^1} \frac{1}{\gamma^1} [ [u ]_t^1 - \gamma^1 \boldsymbol{\sigma}_t^1(\mathbf{u}^1) ]_{\gamma^1 s^1} \cdot (\mathbf{v}_t^1 - \mathbf{v}_t^2 \circ \Pi^1 - \theta \gamma^1 \boldsymbol{\sigma}_t^1(\mathbf{v}^1)) d\Gamma \\ &\quad + \frac{1}{2} \int_{\Gamma_c^2} \frac{1}{\gamma^2} [ [u ]_t^2 - \gamma^2 \boldsymbol{\sigma}_t^2(\mathbf{u}^2) ]_{\gamma^2 s^2} \cdot (\mathbf{v}_t^2 - \mathbf{v}_t^1 \circ \Pi^2 - \theta \gamma^2 \boldsymbol{\sigma}_t^2(\mathbf{v}^2)) d\Gamma = L(\mathbf{v}). \end{aligned}$$



Using Green’s formula we can write

$$\begin{aligned} \mathbf{a}(\mathbf{u}, \mathbf{v}) = & - \int_{\Omega^1} \mathbf{div} \boldsymbol{\sigma}^1(\mathbf{u}^1) \cdot \mathbf{v}^1 d\Omega - \int_{\Omega^2} \mathbf{div} \boldsymbol{\sigma}^2(\mathbf{u}^2) \cdot \mathbf{v}^2 d\Omega \\ & + \int_{\partial\Omega^1} \boldsymbol{\sigma}^1(\mathbf{u}^1) \mathbf{n}^1 \cdot \mathbf{v}^1 d\Gamma + \int_{\partial\Omega^2} \boldsymbol{\sigma}^2(\mathbf{u}^2) \mathbf{n}^2 \cdot \mathbf{v}^2 d\Gamma. \end{aligned}$$

If we take  $\mathbf{v} = (\mathbf{v}^1, \mathbf{0})$  with  $\mathbf{v}^1 = \mathbf{0}$  on  $\partial\Omega^1$ , we obtain:

$$\int_{\Omega^1} \mathbf{div} \boldsymbol{\sigma}^1(\mathbf{u}^1) \cdot \mathbf{v}^1 d\Omega = \int_{\Omega^1} \mathbf{f}^1 \cdot \mathbf{v}^1 d\Omega \forall \mathbf{v}^1,$$

which yields (1a) for  $i = 1$ . In the same way we establish (1a) for  $i = 2$ .

To establish (2), (3), (4) and (5), we consider a displacement field  $\mathbf{v}$  that vanishes on the boundary except on the contact surfaces where  $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$ . Then (13) and (1a) gives

$$\begin{aligned} & \int_{\Gamma_c^1} \sigma_n^1(\mathbf{u}^1) v_n^1 d\Gamma + \int_{\Gamma_c^1} \boldsymbol{\sigma}_t^1(\mathbf{u}^1) \cdot \mathbf{v}_t^1 d\Gamma \\ & + \int_{\Gamma_c^2} \sigma_n^2(\mathbf{u}^2) v_n^2 d\Gamma + \int_{\Gamma_c^2} \boldsymbol{\sigma}_t^2(\mathbf{u}^2) \cdot \mathbf{v}_t^2 d\Gamma \\ & - \frac{1}{2} \int_{\Gamma_c^1} \theta \gamma^1 \sigma_n^1(\mathbf{u}^1) \sigma_n^1(\mathbf{v}^1) d\Gamma - \frac{1}{2} \int_{\Gamma_c^2} \theta \gamma^2 \sigma_n^2(\mathbf{u}^2) \sigma_n^2(\mathbf{v}^2) d\Gamma \\ & - \frac{1}{2} \int_{\Gamma_c^1} \theta \gamma^1 \boldsymbol{\sigma}_t^1(\mathbf{u}^1) \cdot \boldsymbol{\sigma}_t^1(\mathbf{v}^1) d\Gamma \\ & - \frac{1}{2} \int_{\Gamma_c^2} \theta \gamma^2 \boldsymbol{\sigma}_t^2(\mathbf{u}^2) \cdot \boldsymbol{\sigma}_t^2(\mathbf{v}^2) d\Gamma + \frac{1}{2} \int_{\Gamma_c^1} \frac{1}{\gamma^1} [ \llbracket u \rrbracket_n^1 - g_n^1 \\ & - \gamma^1 \sigma_n^1(\mathbf{u}^1) ]_+ (v_n^1 + v_n^2 \circ \Pi^1 - \theta \gamma^1 \sigma_n^1(\mathbf{v}^1)) d\Gamma \\ & + \frac{1}{2} \int_{\Gamma_c^2} \frac{1}{\gamma^2} [ \llbracket u \rrbracket_n^2 - g_n^2 \\ & - \gamma^2 \sigma_n^2(\mathbf{u}^2) ]_+ \\ & (v_n^2 + v_n^1 \circ \Pi^2 - \theta \gamma^2 \sigma_n^2(\mathbf{v}^2)) d\Gamma \\ & + \frac{1}{2} \int_{\Gamma_c^1} \frac{1}{\gamma^1} [ \llbracket \mathbf{u} \rrbracket_t^1 - \gamma^1 \boldsymbol{\sigma}_t^1(\mathbf{u}^1) ]_{\gamma^1 s^1} \cdot (\mathbf{v}_t^1 - \mathbf{v}_t^2 \circ \Pi^1 - \theta \gamma^1 \boldsymbol{\sigma}_t^1(\mathbf{v}^1)) d\Gamma \\ & + \frac{1}{2} \int_{\Gamma_c^2} \frac{1}{\gamma^2} [ \llbracket \mathbf{u} \rrbracket_t^2 - \gamma^2 \boldsymbol{\sigma}_t^2(\mathbf{u}^2) ]_{\gamma^2 s^2} \cdot (\mathbf{v}_t^2 - \mathbf{v}_t^1 \circ \Pi^2 - \theta \gamma^2 \boldsymbol{\sigma}_t^2(\mathbf{v}^2)) d\Gamma = 0. \end{aligned} \tag{30}$$

We need to discuss two cases:  $\theta \neq 0$  and  $\theta = 0$ .

**Case 1**  $\theta \neq 0$ : In (30), let us consider  $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$  such that:

$$\begin{cases} \mathbf{v}^1 = \mathbf{0} \text{ and } \sigma_t^1(\mathbf{v}^1) = \mathbf{0}, \sigma_n^1(\mathbf{v}^1) \neq 0 & \text{on } \Gamma_C^1 \text{ and} \\ \mathbf{v}^2 = \mathbf{0} \text{ and } \sigma^2(\mathbf{v}^2)\mathbf{n}^2 = \mathbf{0} & \text{on } \Gamma_C^2, \end{cases} \tag{31}$$

so,

$$\frac{\theta}{2} \int_{\Gamma_C^1} \left( [\llbracket u \rrbracket_n^1 - g_n^1 - \gamma^1 \sigma_n^1(\mathbf{u}^1)]_+ + \gamma^1 \sigma_n^1(\mathbf{u}^1) \right) \sigma_n^1(\mathbf{v}^1) d\Gamma = 0 \quad \forall \mathbf{v} \text{ satisfying (31)}.$$

Then:

$$\sigma_n^1(\mathbf{u}^1) = -\frac{1}{\gamma^1} [\llbracket u \rrbracket_n^1 - g_n^1 - \gamma^1 \sigma_n^1(\mathbf{u}^1)]_+,$$

which implies (2). Arguing in the same way we obtain (3) and the friction conditions (4).

*Remark A.1* We can show that  $\mathbf{v}$  satisfying (31) can be built by considering  $\mathbf{s}(\mathbf{x})$  the curvilinear coordinate system on the boundary  $\Gamma_C$  and  $d(\mathbf{x})$  the signed distance to  $\Gamma_C$ . Then, for  $\mathbf{g}$  a given vector field of  $\mathbb{R}^d$ ,  $\mathbf{u}(\mathbf{x}) = B^{-1}(\mathbf{s}(\mathbf{x}))\mathbf{g}(\mathbf{s}(\mathbf{x}))d(\mathbf{x})$  satisfies  $\mathbf{u}(\mathbf{x}) = \mathbf{0}$  and  $\sigma(\mathbf{u})\mathbf{n} = \mathbf{g}$  on  $\Gamma_C$ , with  $B_{il} = A_{ijkl}n_k n_j$ ,  $A$  being the elasticity tensor.

To obtain the second Newton law, we use Nitsche’s writing of (2) and (3) in (30) with:  $\mathbf{v}_t = \mathbf{0}$  and  $\sigma_t = \mathbf{0}$  and  $v_n^2 = -v_n^1 \circ \Pi^2$ :

$$\int_{\Gamma_C^1} \sigma_n^1(\mathbf{u}^1)v_n^1 d\Gamma - \int_{\Gamma_C^2} \sigma_n^2(\mathbf{u}^2)v_n^1 \circ \Pi^2 d\Gamma = 0 \quad \forall v_n^1.$$

Then:

$$\int_{\Gamma_C^1} [\sigma_n^1(\mathbf{u}^1) - J^1 \sigma_n^2(\mathbf{u}^2 \circ \Pi^1)] v_n^1 d\Gamma = 0 \quad \forall v_n^1.$$

For  $v_n^1 = v_n^2 = 0$  and  $\mathbf{v}_t^2 = \mathbf{v}_t^1 \circ \Pi^2$  and using (4) in (30), we have similiary

$$\int_{\Gamma_C^1} [\sigma_t^1(\mathbf{u}^1) + J^1 \sigma_t^2(\mathbf{u}^2 \circ \Pi^1)] \cdot \mathbf{v}_t^1 d\Gamma = 0 \quad \forall \mathbf{v}_t^1,$$

and we have (5).

**Case 2**  $\theta = 0$ : Let us take  $\mathbf{v}_t^1 = \mathbf{v}_t^2 = \mathbf{0}$  and  $v_n^2 = -v_n^1 \circ \Pi^2$ ,  $v_n^1 = -v_n^2 \circ \Pi^1$ , then (30) reads:

$$\int_{\Gamma_C^1} [\sigma_n^1(\mathbf{u}^1) - J^1 \sigma_n^2(\mathbf{u}^2 \circ \Pi^1)] v_n^1 d\Gamma = 0 \quad \forall v_n^1.$$

Let us take, now  $v_n^1 = v_n^2 = 0$  and  $\mathbf{v}_t^2 = \mathbf{v}_t^1 \circ \Pi^2$ ,  $\mathbf{v}_t^1 = \mathbf{v}_t^2 \circ \Pi^1$ , then (30) reads:

$$\int_{\Gamma_C^1} [\sigma_t^1(\mathbf{u}^1) + J^1 \sigma_t^2(\mathbf{u}^2 \circ \Pi^1)] \cdot \mathbf{v}_t^1 d\Gamma = 0 \quad \forall \mathbf{v}_t^1,$$

and we have (5).

Let  $\mathbf{v}^2 = 0$  on  $\Gamma_C^2$ . Taking  $\mathbf{v}_t^1 = 0$ , we get:

$$\int_{\Gamma_C^1} \left[ \sigma_n^1(\mathbf{u}^1) + \frac{1}{2\gamma^1} [[u]_n^1 - g_n^1 - \gamma^1 \sigma_n^1(\mathbf{u}^1)]_+ + J^1 \frac{1}{2\gamma^2} [[u]_n^2 \circ \Pi^1 - g_n^2 \circ \Pi^1 - \gamma^2 \sigma_n^2(\mathbf{u}^2 \circ \Pi^1)]_+ \right] v_n^1 d\Gamma = 0 \forall v_n^1.$$

Then:

$$\sigma_n^1(\mathbf{u}^1) = -\frac{1}{2} \left[ \frac{1}{\gamma^1} [[u]_n^1 - g_n^1 - \gamma^1 \sigma_n^1(\mathbf{u}^1)]_+ + \frac{J^1}{\gamma^2} [[u]_n^1 - g_n^1 - \gamma^2 \sigma_n^2(\mathbf{u}^2 \circ \Pi^1)]_+ \right].$$

Since  $J^1 > 0$ ,  $\sigma_n^1(\mathbf{u}^1) \leq 0$  and so we obtain (2b). The second Newton law (5) yields:

$$\sigma_n^1(\mathbf{u}^1) = -\frac{1}{2} \left[ \frac{1}{\gamma^1} ([u]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) \right]_+ + \left[ \frac{J^1}{\gamma^2} ([u]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) \right]_+. \quad (32)$$

We discuss both cases:

If  $\sigma_n^1(\mathbf{u}^1) = 0$ :

$$\frac{1}{2} \left( \frac{1}{\gamma^1} + \frac{J^1}{\gamma^2} \right) [[u]_n^1 - g_n^1]_+ = 0 \text{ then } [[u]_n^1 \leq g_n^1.$$

If  $\sigma_n^1(\mathbf{u}^1) < 0$ :

$$\frac{1}{\gamma^1} ([u]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) > 0 \quad \text{or} \quad \frac{J^1}{\gamma^2} ([u]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) > 0 \quad \text{or both.}$$

1. If we suppose first that:  $\frac{1}{\gamma^1} ([u]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) > 0$  and  $\frac{J^1}{\gamma^2} ([u]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) > 0$ , the Eq. (32) holds:

$$\sigma_n^1(\mathbf{u}^1) = -\frac{1}{2} \left[ \left( \frac{1}{\gamma^1} + \frac{J^1}{\gamma^2} \right) ([u]_n^1 - g_n^1) - 2\sigma_n^1(\mathbf{u}^1) \right] \quad \text{then} \quad [[u]_n^1 = g_n^1.$$

2. If now there only holds  $\frac{1}{\gamma^1}(\llbracket u \rrbracket_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) > 0$  and  $\frac{J^1}{\gamma^2}(\llbracket u \rrbracket_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) = 0$ , we can write (32):

$$\begin{aligned} \sigma_n^1(\mathbf{u}^1) &= -\frac{1}{2\gamma^1}(\llbracket u \rrbracket_n^1 - g_n^1) + \frac{1}{2}\sigma_n^1(\mathbf{u}^1). \\ \text{So } \sigma_n^1(\mathbf{u}^1) &= -\frac{1}{\gamma}(\llbracket u \rrbracket_n^1 - g_n^1). \end{aligned}$$

Then, since  $\sigma_n(\mathbf{u}^1) < 0: \llbracket u \rrbracket_n^1 > g_n^1$ . But in this case,

$$\frac{J^1}{\gamma^2}(\llbracket u \rrbracket_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) > 0,$$

and this contradicts the assumption  $\frac{J^1}{\gamma^2}(\llbracket u \rrbracket_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) = 0$ . So, this case is absurd. In a similar way we get contradiction for the case  $\frac{J^1}{\gamma^2}(\llbracket u \rrbracket_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) < 0$ .

To conclude, we establish that: if  $\sigma_n^1(\mathbf{u}^1) = 0, \llbracket u \rrbracket_n^1 \leq g_n^1$  and if  $\sigma_n^1(\mathbf{u}^1) < 0, \llbracket u \rrbracket_n^1 = g_n^1$ ; and this is equivalent to (2a) and (2c).

We suppose, now, that  $v_n^1 = 0$  and  $\mathbf{v}^2 = \mathbf{0}$ . We get:

$$\begin{aligned} \int_{\Gamma_C^1} \left[ \sigma_t^1(\mathbf{u}^1) + \frac{1}{2\gamma^1}[\llbracket \mathbf{u} \rrbracket_t^1 - \gamma^1 \sigma_t^1(\mathbf{u}^1)]_{\gamma^1 s^1} - \frac{J^1}{2\gamma^2}[\llbracket \mathbf{u} \rrbracket_t^2 \circ \Pi^1 \right. \\ \left. - \gamma^2 \sigma_t^2(\mathbf{u}^2 \circ \Pi^1)]_{\gamma^2 s^2} \right] \cdot \mathbf{v}_t^1 d\Gamma = 0 \forall \mathbf{v}_t^1. \end{aligned}$$

Then, using the property:  $\forall \gamma > 0, [x]_{\gamma s} = \gamma \left[ \frac{x}{\gamma} \right]_s$ , it yields:

$$\sigma_t^1(\mathbf{u}^1) + \frac{1}{2} \left[ \frac{\llbracket \mathbf{u} \rrbracket_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1) \right]_{s^1} - \frac{J^1}{2} \left[ \frac{\llbracket \mathbf{u} \rrbracket_t^2 \circ \Pi^1}{\gamma^2} - \sigma_t^2(\mathbf{u}^2 \circ \Pi^1) \right]_{s^2} = 0.$$

We use the Newton law (5) and the condition (6) to obtain:

$$\sigma_t^1(\mathbf{u}^1) + \frac{1}{2} \left[ \frac{\llbracket \mathbf{u} \rrbracket_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1) \right]_{s^1} + \frac{1}{2} \left[ J^1 \frac{\llbracket \mathbf{u} \rrbracket_t^1}{\gamma^2} - \sigma_t^1(\mathbf{u}^1) \right]_{s^1} = 0. \tag{33}$$

1. If  $\| \frac{\llbracket \mathbf{u} \rrbracket_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1) \| < s^1$  and  $\| J^1 \frac{\llbracket \mathbf{u} \rrbracket_t^1}{\gamma^2} - \sigma_t^1(\mathbf{u}^1) \| < s^1$ :

$\frac{\llbracket \mathbf{u} \rrbracket_t^1}{2} \left( \frac{1}{\gamma^1} + \frac{J^1}{\gamma^2} \right) = 0$ ; so  $\llbracket \mathbf{u} \rrbracket_t^1 = 0$ . In this case we obtain:  $\sigma_t^1(\mathbf{u}^1) = \left[ \sigma_t^1(\mathbf{u}^1) \right]_{s^1}$ , and so:  $\| \sigma_t^1(\mathbf{u}^1) \| < s^1$ .

2. If  $\| \frac{[\mathbf{u}]_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1) \| \geq s^1$  and  $\| J^1 \frac{[\mathbf{u}]_t^1}{\gamma^2} - \sigma_t^1(\mathbf{u}^1) \| \geq s^1$ :

$$\sigma_t^1(\mathbf{u}^1) + \frac{s^1}{2} \frac{\frac{[\mathbf{u}]_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1)}{\| \frac{[\mathbf{u}]_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1) \|} + \frac{s^1}{2} \frac{J^1 \frac{[\mathbf{u}]_t^1}{\gamma^2} - \sigma_t^1(\mathbf{u}^1)}{\| J^1 \frac{[\mathbf{u}]_t^1}{\gamma^2} - \sigma_t^1(\mathbf{u}^1) \|} = 0. \tag{34}$$

The Eq. (34) shows that  $\sigma_t^1(\mathbf{u}^1)$  and  $[\mathbf{u}]_t^1$  are collinear.

$$\text{So: } \left\{ \begin{array}{l} \frac{\frac{[\mathbf{u}]_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1)}{\| \frac{[\mathbf{u}]_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1) \|} = \frac{J^1 \frac{[\mathbf{u}]_t^1}{\gamma^2} - \sigma_t^1(\mathbf{u}^1)}{\| J^1 \frac{[\mathbf{u}]_t^1}{\gamma^2} - \sigma_t^1(\mathbf{u}^1) \|}, \\ \text{or} \\ \frac{\frac{[\mathbf{u}]_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1)}{\| \frac{[\mathbf{u}]_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1) \|} = - \frac{J^1 \frac{[\mathbf{u}]_t^1}{\gamma^2} - \sigma_t^1(\mathbf{u}^1)}{\| J^1 \frac{[\mathbf{u}]_t^1}{\gamma^2} - \sigma_t^1(\mathbf{u}^1) \|} \quad (*), \end{array} \right.$$

and we obtain, from (34):

$$\left\{ \begin{array}{l} \sigma_t^1(\mathbf{u}^1) = -s^1 \frac{\frac{[\mathbf{u}]_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1)}{\| \frac{[\mathbf{u}]_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1) \|} = -\frac{1}{\gamma^1} [ [\mathbf{u}]_t^1 - \gamma^1 \sigma_t^1(\mathbf{u}^1) ]_{\gamma^1 s^1}, \\ \text{and this is equivalent to (4).} \\ \text{or} \\ \sigma_t^1(\mathbf{u}^1) = 0 \text{ which is impossible in } (*). \end{array} \right.$$

3. If now  $\| \frac{[\mathbf{u}]_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1) \| < s^1$  and  $\| J^1 \frac{[\mathbf{u}]_t^1}{\gamma^2} - \sigma_t^1(\mathbf{u}^1) \| \geq s^1$ :

$$\sigma_t^1(\mathbf{u}^1) + \frac{1}{2} \left( \frac{[\mathbf{u}]_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1) \right) + \frac{s^1}{2} \frac{J^1 \frac{[\mathbf{u}]_t^1}{\gamma^2} - \sigma_t^1(\mathbf{u}^1)}{\| J^1 \frac{[\mathbf{u}]_t^1}{\gamma^2} - \sigma_t^1(\mathbf{u}^1) \|} = 0.$$

Projecting on  $\frac{\sigma_t^1(\mathbf{u}^1)}{\| \sigma_t^1(\mathbf{u}^1) \|}$  and setting  $a = \| \sigma_t^1(\mathbf{u}^1) \|$ ;  $b = \frac{\sigma_t^1(\mathbf{u}^1) \cdot [\mathbf{u}]_t^1}{\gamma^1 \| \sigma_t^1(\mathbf{u}^1) \|}$ ,

$$\text{we get: } \left\{ \begin{array}{l} |b - a| < s^1 \text{ and } |b J^1 \frac{\gamma^1}{\gamma^2} - a| \geq s^1 \\ \text{and} \\ (b + a) + \epsilon s^1 = 0; \text{ where } \epsilon = \text{sign}(b J^1 \frac{\gamma^1}{\gamma^2} - a) = \pm 1. \end{array} \right.$$

Let  $\epsilon = +1$ ; so,  $a = -b - s^1$  and we obtain:

$$\left\{ \begin{array}{l} b - a = 2b + s^1 \text{ and } |b - a| < s^1 \\ \text{and} \\ b J^1 \frac{\gamma^1}{\gamma^2} - a = (J^1 \frac{\gamma^1}{\gamma^2} + 1)b + s^1 \text{ and } b J^1 \frac{\gamma^1}{\gamma^2} - a \geq s^1. \end{array} \right.$$

$$\text{So: } \left\{ \begin{array}{l} -s^1 < b < 0 \\ \text{and} \\ (J^1 \frac{\gamma^1}{\gamma^2} + 1)b \geq 0 \end{array} \right. \text{ which is absurd.}$$

Let  $\epsilon = -1$ ; so  $a = -b + s^1$  and we obtain:

$$\left\{ \begin{array}{l} b - a = 2b - s^1 \text{ and } |b - a| < s^1 \\ \text{and} \\ bJ^1 \frac{\gamma^1}{\gamma^2} - a = (J^1 \frac{\gamma^1}{\gamma^2} + 1)b - s^1 \text{ and } bJ^1 \frac{\gamma^1}{\gamma^2} - a \leq -s^1, \end{array} \right.$$

$$\text{so: } \left\{ \begin{array}{l} 0 < b < s^1 \\ \text{and} \\ (J^1 \frac{\gamma^1}{\gamma^2} + 1)b \leq 0 \end{array} \right. , \text{ which is absurd.}$$

4. If  $\| \frac{[\mathbf{u}]_t^1}{\gamma^1} - \sigma_t^1(\mathbf{u}^1) \| \geq s^1$  and  $\| J^1 \frac{[\mathbf{u}]_t^1}{\gamma^2} - \sigma_t^1(\mathbf{u}^1) \| < s^1$ :

$$\text{We argue in the same way putting } a = \| \sigma_t^1(\mathbf{u}^1) \|; b = J^1 \frac{\sigma_t^1(\mathbf{u}^1) \cdot [\mathbf{u}]_t^1}{\gamma^2 \| \sigma_t^1(\mathbf{u}^1) \|}.$$

Thus, we establish the friction condition (4) for  $i = 1$ . In the same way, when supposing  $\mathbf{v}^1 = 0$ , we get (2a)–(2b)–(2c) and (4) for  $i = 2$ .

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