Numer. Math. (2017) 136:315–342
DOI 10.1007/s00211-016-0841-y

Numerische Mathematik

On Fourier time-splitting methods for nonlinear Schrödinger equations in the semi-classical limit II. Analytic regularity

Rémi Carles[1] · **Clément Gallo**[1]

Received: 17 February 2016 / Revised: 2 August 2016 / Published online: 15 September 2016

Abstract We consider the time discretization based on Lie-Trotter splitting, for the nonlinear Schrödinger equation, in the semi-classical limit, with initial data under the form of WKB states. We show that both the exact and the numerical solutions keep a WKB structure, on a time interval independent of the Planck constant. We prove error estimates, which show that the quadratic observables can be computed with a time step independent of the Planck constant. The functional framework is based on time-dependent analytic spaces, in order to overcome a previously encountered loss of regularity phenomenon.

1 Introduction

This paper is devoted to the analysis of the numerical approximation of the solution to

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \lambda|u^\varepsilon|^{2\sigma}u^\varepsilon, \quad (t,x)\in[0,T]\times\mathbb{R}^d, \tag{1.1}$$

This work was supported by the French ANR projects SchEq (ANR-12-JS01-0005-01) and BECASIM (ANR-12-MONU-0007-04).

✉ Rémi Carles
Remi.Carles@math.cnrs.fr

Clément Gallo
Clement.Gallo@umontpellier.fr

[1] Institut Montpelliérain Alexander Grothendieck, CNRS & Univ. Montpellier, CC51, Place E. Bataillon, 34095 Montpellier, France

in the semi-classical limit $\varepsilon \to 0$. The nonlinearity is smooth and real-valued: $\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{N}$. The initial data that we consider are WKB states:

$$u^\varepsilon(0,x) = a_0(x)e^{i\phi_0(x)/\varepsilon} =: u_0^\varepsilon(x), \tag{1.2}$$

where $\phi_0 : \mathbb{R}^d \to \mathbb{R}$ is a real-valued phase, and $a_0 : \mathbb{R}^d \to \mathbb{C}$ is a possibly complex-valued amplitude. An important feature of such initial data is that in the context of the semi-classical limit for (1.1), they yield solutions which are in $L^\infty(\mathbb{R}^d)$ uniformly in ε, at least on some time interval $[0,T]$ for some $T > 0$ independent of ε. Also, note that even if $\phi_0 = 0$ (no rapid oscillation initially), for $\tau > 0$ arbitrarily small and independent of ε, $u^\varepsilon(\tau)$ takes the form of a WKB state as in (1.2) with amplitude a^ε and phase $\phi^\varepsilon \not\equiv 0$ solving (2.2)–(2.3) below (see [8]).

We consider more precisely the time discretization for (1.1) based on Fourier time splitting. We denote by X_ε^t the map $v^\varepsilon(0,\cdot) \mapsto v^\varepsilon(t,\cdot)$, where

$$i\varepsilon\partial_t v^\varepsilon + \frac{\varepsilon^2}{2}\Delta v^\varepsilon = 0. \tag{1.3}$$

We also denote by Y_ε^t the map $w^\varepsilon(0,\cdot) \mapsto w^\varepsilon(t,\cdot)$, where

$$i\varepsilon\partial_t w^\varepsilon = \lambda |w^\varepsilon|^{2\sigma} w^\varepsilon. \tag{1.4}$$

We consider the Lie-Trotter type splitting operator

$$Z_\varepsilon^t = Y_\varepsilon^t X_\varepsilon^t. \tag{1.5}$$

The Lie-Trotter operator $X_\varepsilon^t Y_\varepsilon^t$ could be handled in the same fashion. The advantage of splitting methods is that they involve sub-equations which are simpler to solve than the initial equation. In our case, (1.3) is solved explicitly by using the Fourier transform, defined by

$$\widehat{\psi}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\psi(x)dx,$$

since it becomes an ordinary differential equation

$$i\varepsilon\partial_t \widehat{v^\varepsilon} - \frac{\varepsilon^2}{2}|\xi|^2 \widehat{v^\varepsilon} = 0, \tag{1.6}$$

hence

$$\widehat{X_\varepsilon^t v}(\xi) = e^{-i\varepsilon\frac{t}{2}|\xi|^2}\widehat{v}(\xi).$$

Also, since $\lambda \in \mathbb{R}$, in (1.4) the modulus of w^ε does not depend on time, hence

$$Y_\varepsilon^t w(x) = w(x)e^{-i\lambda\frac{t}{\varepsilon}|w(x)|^{2\sigma}}. \tag{1.7}$$

In the case $\varepsilon = 1$, several results exist to prove that the Lie-Trotter time splitting is of order one, and the Strang splitting of order two [6,26]. The drawback of these proofs is that they rely on uniform Sobolev bounds for the exact solution, of the form $u \in L^\infty([0,T]; H^s(\mathbb{R}^d))$, for $s \geq 2$. However, in the framework of (1.1), these norms are not uniformly bounded as $\varepsilon \to 0$, in the sense that we rather have $\|u^\varepsilon(t)\|_{H^s} \approx \varepsilon^{-s}$, due to the oscillatory nature of u^ε.

In the case of a linear potential ($|u^\varepsilon|^{2\sigma}$ is replaced by a known function of x in (1.1)), error estimates are given in [3]; see also [14,15]. In the nonlinear case, error estimates are established in [9], but for other nonlinearities than in (1.1)–(1.2). The proof there requires either to consider a weakly nonlinear regime, that is (1.1) is replaced by

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \varepsilon\lambda|u^\varepsilon|^{2\sigma}u^\varepsilon, \quad (t,x) \in [0,T] \times \mathbb{R}^d,$$

with the same initial data (1.2), or to replace the nonlinearity in (1.1) with a smoothing nonlinearity of Poisson type. We recall in Sect. 2 why these assumptions are made in [9]. The goal of this paper is to prove error estimates which are similar to those established in [9], but for (1.1)–(1.2). Before stating our main result, we introduce a few notations. The Fourier transform is normalized as

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx.$$

A tempered distribution f is in $H^s(\mathbb{R}^d)$ if $\xi \mapsto \langle\xi\rangle^s \hat{f}(\xi)$ *belongs to* $L^2(\mathbb{R}^d)$, where

$$\langle\xi\rangle = \sqrt{1+|\xi|^2}.$$

Theorem 1.1 *Suppose that $d, \sigma \in \mathbb{N}$, $d, \sigma \geq 1$, and $\lambda \in \mathbb{R}$. Let ϕ_0, a_0 such that*

$$\int_d e^{\langle\xi\rangle^{1+\delta}}(|\hat{\phi}_0(\xi)|^2 + |\hat{a}_0(\xi)|^2)d\xi < \infty,$$

for some $\delta > 0$, and u_0^ε given by (1.2). There exist $T, \varepsilon_0, c_0 > 0$ and $(C_k)_{k\in\mathbb{N}}$ such that for all $\varepsilon \in (0, \varepsilon_0]$, the following holds:

1. *(1.1)–(1.2) has a unique solution $u^\varepsilon = S_\varepsilon^t u_0^\varepsilon \in C([0,T], H^\infty)$, where $H^\infty = \cap_{s\in\mathbb{R}} H^s$. Moreover, there exist ϕ^ε and a^ε with, for all $k \in \mathbb{N}$,*

$$\sup_{t\in[0,T]} \left(\|a^\varepsilon(t)\|_{H^k(\mathbb{R}^d)} + \|\phi^\varepsilon(t)\|_{H^k(\mathbb{R}^d)}\right) \leq C_k,$$

such that $u^\varepsilon(t,x) = a^\varepsilon(t,x)e^{i\phi^\varepsilon(t,x)/\varepsilon}$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$.

2. *For all $\Delta t \in (0, c_0]$, for all $n \in \mathbb{N}$ such that $t_n = n\Delta t \in [0,T]$, there exist ϕ_n^ε and a_n^ε with, for all $k \in \mathbb{N}$,*

$$\|a_n^\varepsilon\|_{H^k(\mathbb{R}^d)} + \|\phi_n^\varepsilon\|_{H^k(\mathbb{R}^d)} \leq C_k,$$

such that $(Z_\varepsilon^{\Delta t})^n(a_0 e^{i\phi_0/\varepsilon}) = a_n^\varepsilon e^{i\phi_n/\varepsilon}$.

3. *For all* $\Delta t \in (0, c_0]$, *for all* $n \in \mathbb{N}$ *such that* $n\Delta t \in [0, T]$, *the following error estimate holds:*

$$\left\|a_n^\varepsilon - a^\varepsilon(t_n)\right\|_{H^k} + \left\|\phi_n^\varepsilon - \phi^\varepsilon(t_n)\right\|_{H^k} \leq C_k \Delta t.$$

Example 1.2 The assumptions of Theorem 1.1 are satisfied as soon as $\hat{\phi}_0$ and $\hat{a}_0$ are compactly supported, or in the case of Gaussian functions, typically.

Note that the first two points of the theorem imply that the functions a and ϕ are not rapidly oscillatory: the oscillatory nature of both the exact and the numerical solutions is encoded in the exponential which relates the functions a and ϕ to u.

We readily infer error estimates for the wave function and for quadratic observables,

Position density: $|u^\varepsilon(t, x)|^2$.

Current density: $J^\varepsilon(t, x) = \varepsilon \operatorname{Im}\left(\overline{u}^\varepsilon(t, x)\nabla u^\varepsilon(t, x)\right)$.

Corollary 1.3 *Under the assumptions of Theorem* 1.1, *there exist* $T > 0$, $\varepsilon_0 > 0$ *and* C, c_0 *independent of* $\varepsilon \in (0, \varepsilon_0]$ *such that for all* $\Delta t \in (0, c_0]$, *for all* $n \in \mathbb{N}$ *such that* $n\Delta t \in [0, T]$, *and for all* $\varepsilon \in (0, \varepsilon_0]$,

$$\left\|(Z_\varepsilon^{\Delta t})^n u_0^\varepsilon - u^\varepsilon(t_n)\right\|_{L^2(\mathbb{R}^d)} \leq C\frac{\Delta t}{\varepsilon},$$

$$\left\|\left|(Z_\varepsilon^{\Delta t})^n u_0^\varepsilon\right|^2 - |u^\varepsilon(t_n)|^2\right\|_{L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)} \leq C\Delta t,$$

$$\left\|\operatorname{Im}\left(\varepsilon\overline{(Z_\varepsilon^{\Delta t})^n u_0^\varepsilon}\nabla(Z_\varepsilon^{\Delta t})^n u_0^\varepsilon\right) - J^\varepsilon(t_n)\right\|_{L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)} \leq C\Delta t.$$

This result is in agreement with the numerical experiments presented in [4]: to simulate the wave function u^ε, the time step must satisfy $\Delta t = o(\varepsilon)$, while to observe the quadratic observables, $\Delta t = o(1)$ can be chosen independent of ε.

It is very likely that the proof presented in this paper can be adapted to the case of Strang time-splitting schemes. The main modification to make would concern the local error estimate (presented in Sect. 6): here, we use a general abstract formula for the local error associated to Lie-Trotter schemes as established in [15]. Proving the analog of Theorem 1.1 would essentially require to adapt these computations.

This paper treats only the case of time discretization: the full numerical analysis would require to take spatial discretization as well. As pointed out in [4], the nature of the nonlinearity in (1.1), and more precisely the sign of λ (focusing or defocusing nonlinearity), alters the requirements to be made concerning the mesh size as a function of ε. We do not address this delicate issue here.

As emphasized in [9], the fact that our analysis requires the analytical solution to have the form of a WKB state forces us to consider only finite time intervals: the good news is that the length of the time interval is independent of ε, but on the other hand, the solution u^ε must be expected to keep the form of a monokinetic WKB state only for short time. This is related to the formation of singularities in compressible Euler equations, and we refer to [7,9] for more precise discussions on that topic.

However, the analytical framework presented in this paper might be useful for the long time analysis of numerical schemes, since the time dependent analytic norms that we consider were initially introduced by Ginibre and Velo to study the large time behavior of solutions to the Hartree equation (scattering theory). From this perspective, some other time discretization techniques could be tackled thanks to these tools, since several geometric numerical integrators have been proposed to overcome the lack of stability of time splitting over large time; see e.g. [5,10,16,17,21–23] and references therein.

2 Overview of the proof

We present the general strategy for the proof of Theorem 1.1 in the case of a more general nonlinearity,

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = f\left(|u^\varepsilon|^2\right)u^\varepsilon, \tag{2.1}$$

with f real-valued. For initial data of the form (1.2), it has been noticed in [9] that the numerical discretization preserves such a structure, in the sense that the numerical solution satisfies the point 2. in Theorem 1.1. Indeed, the exact solution can be represented as $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$, where a^ε and ϕ^ε are given by

$$\begin{cases} \partial_t\phi^\varepsilon + \frac{1}{2}|\nabla\phi^\varepsilon|^2 + f\left(|a^\varepsilon|^2\right) &= 0,\\ \partial_t a^\varepsilon + \nabla\phi^\varepsilon\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\phi^\varepsilon &= \frac{i\varepsilon}{2}\Delta a^\varepsilon, \end{cases} \tag{2.2}$$

with initial data

$$\phi^\varepsilon_{|t=0} = \phi_0, \quad a^\varepsilon_{|t=0} = a_0. \tag{2.3}$$

The main feature of this representation is that even though they must be expected to depend on ε, a^ε and ϕ^ε are bounded in Sobolev spaces uniformly in $\varepsilon \in (0,1]$ on some time interval $[0,T]$ for some T independent of ε.

The idea of representing the solution u^ε under this form goes back to Grenier [20]. The main features of (2.2) is that the left hand side defines a symmetrizable hyperbolic system under the assumption $f' > 0$, and the right hand side is skew adjoint (hence plays no role at the level of energy estimates). Note that in the case of (1.1), this forces $\lambda > 0$ and $\sigma = 1$ (cubic defocusing nonlinearity). For a nonlocal nonlinearity, $f(|u|^2) = K * |u|^2$, the approach of Grenier can easily be adapted if $\hat{K}$ decays at least like $|\xi|^{-2}$ for large $|\xi|$ (see e.g. [9]). The approach of Grenier has also been generalized to more general nonlinearities: see [1,13] for the defocusing case, and [28] for the focusing case. However, we do not use these results, as we now discuss.

Indeed, the splitting scheme for (2.1) amounts to some splitting technique on (2.2). Suppose that one solves the linear equation (1.3) with initial data $v^\varepsilon(0) = a_0 e^{i\phi_0/\varepsilon}$. Then the solution v^ε can be written as $v^\varepsilon(t) = a^\varepsilon(t)e^{i\phi(t)/\varepsilon}$, with a^ε and ϕ given by

$$\begin{cases} \partial_t \phi + \frac{1}{2}|\nabla\phi|^2 = 0, \quad \phi_{|t=0} = \phi_0, \\ \partial_t a^\varepsilon + \nabla\phi \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta\phi = i\frac{\varepsilon}{2}\Delta a^\varepsilon, \quad a^\varepsilon_{|t=0} = a_0. \end{cases} \tag{2.4}$$

Similarly, the solution to (1.4) with initial data $w^\varepsilon(0) = a_0 e^{i\phi_0/\varepsilon}$ can be written as $w^\varepsilon(t) = a(t)e^{i\phi(t)/\varepsilon}$, with a and ϕ given by

$$\begin{cases} \partial_t \phi + f(|a|^2) = 0, \quad \phi_{|t=0} = \phi_0, \\ \partial_t a = 0, \quad a_{|t=0} = a_0. \end{cases} \tag{2.5}$$

So computing the numerical solution amounts to solving successively (2.4) and (2.5), which turns out to be a splitting scheme on (2.2). We denote by $\mathcal{X}^t_\varepsilon : (\phi_0, a_0) \mapsto (\phi(t,\cdot), a^\varepsilon(t,\cdot))$ the flow for (2.4) and by $\mathcal{Y}^t_\varepsilon : (\phi_0, a_0) \mapsto (\phi(t,\cdot), a(t,\cdot))$ the flow for (2.5). The Lie-Trotter splitting operator we consider for (2.2) is then

$$\mathcal{Z}^t_\varepsilon = \mathcal{Y}^t_\varepsilon \mathcal{X}^t_\varepsilon \tag{2.6}$$

Now in the case of a cubic defocusing nonlinearity (which enters the framework of [20]), we face a loss of regularity issue. Indeed, the reason why (2.2) is convenient lies first in the structure of the left hand side, which enjoys symmetry properties: the splitting leading to (2.4)–(2.5) ruins this property. Suppose for instance that at time $t = 0$, $\phi_0 \in H^s(\mathbb{R}^d)$ and $a_0 \in H^k(\mathbb{R}^d)$, for large s and k. In (2.4), the first equation propagates the H^s regularity on a small time interval, provided s is large. The second equation shows that a^ε cannot be more regular than H^{s-2}, due to the last term of the left hand side. Now if we start with $\phi_0 \in H^s$ and $a_0 \in H^{s-2}$ in (2.5) (with $f(|a|^2) = |a|^2$ for a cubic defocusing nonlinearity), we see that $\phi \in H^{s-2}$ (provided $s - 2 > d/2$), and that no better regularity must be expected. So after one iteration of the operator Z^t_ε, ϕ has lost two levels of regularity. When iterating Z with a small time step Δt, this loss becomes fatal. This is why in [9], it is assumed that either f is smoothing (to regain at least two levels of regularity) or that a factor ε is present in front of f, so that (2.5) is altered to

$$\begin{cases} \partial_t \phi = 0, \quad \phi_{|t=0} = \phi_0, \\ \partial_t a = -if(|a|^2)a, \quad a_{|t=0} = a_0. \end{cases}$$

The main technical originality of this paper is based on the remark that if instead of working in Sobolev spaces, one works in *time dependent analytic spaces*, it is possible to control the loss of regularity. Such an idea goes back to [18], to solve (2.2). The fact that we consider decreasing time dependent weight to measure the analytic regularity is strongly inspired by the analysis of Ginibre and Velo in the context of long range scattering for Hartree equations [19], and is also reminiscent of the functional framework used by Chemin for the Navier–Stokes equation [12] and developed by Mouhot and Villani to prove Landau damping [27].

The main technical tools needed here are presented in Sect. 3. Thanks to these tools, we can prove that both the theoretical and the numerical solutions remain analytic in a suitable sense on some time interval $[0, T]$ with $T > 0$ independent of ε (Sects. 4 and 5).

The next key estimate is the local error estimate, presented in Sect. 6. It is based on the general formula established in [15]. As noticed in [9], we must apply this formula to the system (2.4)–(2.5) and not only to (1.3)–(1.4).

With these propagating estimates and the local error estimate, the proof of Theorem 1.1 follows from the trick known as Lady Windermere's fan (see e.g. [24]). Note however that because of the nonlinear context, where global bounds for the numerical solutions are not known a priori, the argument requires some extra care. We rely on the induction technique introduced in [25], which is sufficiently robust to be readily adapted to our case, as in [9].

3 Technical background

We recall here some of the technical tools introduced in [19]. We state the main properties established there concerning time dependent Gevrey spaces, and simplify as much as possible the framework, in view of the present context.

For $0 < \nu \leq 1$ and $\rho > 0$, we introduce the exponential weight

$$\mathrm{w}(\xi) = \exp\left(\rho \max(1, |\xi|)^{\nu}\right),$$

which is equivalent to $\exp(\rho \langle \xi \rangle^{\nu})$. Define $u_>$ and $u_<$ by:

$$\hat{u}_<(\xi) = \hat{u}(\xi)\mathbf{1}_{|\xi|\leq 1}, \quad \hat{u}_>(\xi) = \hat{u}(\xi)\mathbf{1}_{|\xi|>1}.$$

For $k, \ell \in \mathbb{R}$ and $0 \leq \ell_< < d/2$, the following families of norms are defined in [19]:

$$a \mapsto \left(\||\xi|^k \mathrm{w}(\xi)\hat{a}_>(\xi)\|_{L^2}^2 + \|\mathrm{w}(\xi)\hat{a}_<(\xi)\|_{L^2}^2\right)^{1/2},$$
$$\phi \mapsto \left(\||\xi|^{\ell+2} \mathrm{w}(\xi)\hat{\phi}_>(\xi)\|_{L^2}^2 + \||\xi|^{\ell_<}\mathrm{w}(\xi)\hat{\phi}_<(\xi)\|_{L^2}^2\right)^{1/2}.$$

The first norm is well suited to estimate amplitudes, and the second is adapted to phases. As suggested by the above notations, the indices will be different for amplitudes and phases. This can be related to the fact that in the hydrodynamical setting with $\lambda > 0$, (2.2) with $\varepsilon = 0$ is a hyperbolic system in the unknown $(\nabla\phi, a)$, and not in (ϕ, a). Indeed, eventually there will be a shift of one index between the norm in ϕ and the norm in a (see Lemma 3.3 and Proposition 4.1 below).

In the properties related to these norms which will be used in this paper, the value of $\ell_<$ is irrelevant. Therefore, we set $\ell_< = 0$, and consider only one family of norms: for $\ell \geq 0$, we set

$$\mathcal{H}_\rho^\ell = \{\psi \in L^2(\mathbb{R}^d), \quad \|\psi\|_{\mathcal{H}_\rho^\ell} < \infty\},$$
$$\text{where } \|\psi\|_{\mathcal{H}_\rho^\ell}^2 := \||\xi|^\ell \mathrm{w}(\xi)\hat{\psi}_>(\xi)\|_{L^2}^2 + \|\mathrm{w}(\xi)\hat{\psi}_<(\xi)\|_{L^2}^2 \sim \int_{\mathbb{R}^d} \langle\xi\rangle^{2\ell} e^{2\rho\langle\xi\rangle^\nu}|\hat{\psi}(\xi)|^2 d\xi.$$

Remark 3.1 The above definition is slightly different from the standard definition for Gevrey spaces, since low frequencies are smoothed out in the definition of the weight w: $\max(1, |\xi|)$ (or $\langle\xi\rangle$) in w is usually replaced with $|\xi|$.

Note that the following estimate is a straightforward consequence of this definition: for any $\alpha \in \mathbb{N}^d$, $\ell \geq 0$,

$$\|\partial^\alpha \psi\|_{\mathcal{H}^\ell_\rho} \leq \|\psi\|_{\mathcal{H}^{\ell+|\alpha|}_\rho}, \quad \forall \psi \in \mathcal{H}^{\ell+|\alpha|}_\rho. \tag{3.1}$$

Also, in view of the standard Sobolev embedding,

$$\|\psi\|_{L^\infty(\mathbb{R}^d)} \leq C\|\psi\|_{H^s(\mathbb{R}^d)},$$

valid for $s > d/2$, we have

$$\|\psi\|_{L^\infty(\mathbb{R}^d)} \leq C\|\psi\|_{\mathcal{H}^s_\rho}, \tag{3.2}$$

with the same constant C independent of $\rho \geq 0$.

The above notation may seem rather heavy: it is chosen so because the weight ρ will depend on time, as we now discuss. For a time-dependent ρ, we have:

$$\frac{d}{dt}\|\psi\|^2_{\mathcal{H}^\ell_\rho} = 2\dot{\rho}\|\psi\|^2_{\mathcal{H}^{\ell+\nu/2}_\rho} + 2\,\mathrm{Re}\,\langle\psi, \partial_t\psi\rangle_{\mathcal{H}^\ell_\rho}\,. \tag{3.3}$$

Even though ρ depends on time, we will consider below "continuous" $\mathcal{H}^\ell_\rho$ valued functions. We mean functions that belong to

$$C(I, \mathcal{H}^\ell_\rho) := \left\{\psi \in C(I, L^2) \text{ such that } \mathcal{F}^{-1}(\mathrm{w}\hat{\psi}) \in C(I, \mathcal{H}^\ell_0) = C(I, H^\ell)\right\}$$

for some interval I.

To fix the technical framework once and for all, we recall another important result from [19]. Consider the system

$$\begin{cases} \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + \lambda\,\mathrm{Re}\left(|\nabla|^{\mu-d}a\overline{a}\right) = 0, \\ \partial_t a + \nabla\phi\cdot\nabla a + \frac{1}{2}a\Delta\phi \qquad\qquad\; = 0, \end{cases} \tag{3.4}$$

for some time interval I, $0 < \mu \leq d$, and where $|\nabla|^\alpha$ denotes the Fourier multiplier defined by $\mathcal{F}(|\nabla|^\alpha g)(\xi) = |\xi|^\alpha \hat{g}(\xi)$. Lemma 3.5 from [19], which uses (3.3) as well as rather involved estimates, implies that under the assumptions

$$\ell > d/2 + 1 - \nu, \quad k \geq \nu/2, \quad \ell \geq k + 1 - \nu,$$
$$k \geq \ell + \mu - d + 1 - \nu, \quad 2k > \ell + \mu - d + 1 - \nu + d/2,$$

any solution of (3.4) on I, such that $(\phi, a) \in C(I, \mathcal{H}^{\ell+1}_\rho \times \mathcal{H}^k_\rho) \cap L^2_{\mathrm{loc}}(I, \mathcal{H}^{\ell+1+\nu/2}_\rho \times \mathcal{H}^{k+\nu/2}_\rho)$, satisfies

$$\left|\partial_t \|\phi\|^2_{\mathcal{H}^{\ell+1}_\rho} - 2\dot\rho \|\phi\|^2_{\mathcal{H}^{\ell+1+\nu/2}_\rho}\right| \leq C\left(\|\phi\|^2_{\mathcal{H}^{\ell+1+\nu/2}_\rho}\|\phi\|_{\mathcal{H}^{\ell+1}_\rho} + \|a\|_{\mathcal{H}^{k+\nu/2}_\rho}\|\phi\|_{\mathcal{H}^{\ell+1+\nu/2}_\rho}\|a\|_{\mathcal{H}^{k}_\rho}\right),$$

$$\left|\partial_t \|a\|^2_{\mathcal{H}^{k}_\rho} - 2\dot\rho \|a\|^2_{\mathcal{H}^{k+\nu/2}_\rho}\right| \leq C\left(\|a\|^2_{\mathcal{H}^{k+\nu/2}_\rho}\|\phi\|_{\mathcal{H}^{\ell+1}_\rho} + \|a\|_{\mathcal{H}^{k+\nu/2}_\rho}\|\phi\|_{\mathcal{H}^{\ell+1+\nu/2}_\rho}\|a\|_{\mathcal{H}^{k}_\rho}\right).$$

In the case of a cubic nonlinearity, we want to set $\mu = d$ in order to get a local nonlinearity. Therefore, the above algebraic conditions

$$\ell \geq k + 1 - \nu \quad \text{and} \quad k \geq \ell + 1 - \nu$$

imply $\nu \geq 1$, hence $\nu = 1$ and $k = \ell$. In view of this remark, we suppose from now on $\nu = 1$, that is, we consider analytic functions (see [19]).

Since we consider only analytic functions, we borrow from [19] the only inequalities that we will really use, which appear in [19, Lemma 3.4]:

Lemma 3.2 *Let $m \geq 0$. Then,*

1. *For $k + s > m + d/2 + 2$, and $k, s \geq m + 1$,*

$$\|\nabla\phi \cdot \nabla a\|_{\mathcal{H}^m_\rho} \leq C\|\phi\|_{\mathcal{H}^s_\rho}\|a\|_{\mathcal{H}^k_\rho}.$$

2. *For $k + s > m + 2 + d/2$, $k \geq m$ and $s \geq m + 2$,*

$$\|a\Delta\phi\|_{\mathcal{H}^m_\rho} \leq C\|\phi\|_{\mathcal{H}^s_\rho}\|a\|_{\mathcal{H}^k_\rho}.$$

3. *For $s > d/2$,*

$$\|\psi_1\psi_2\|_{\mathcal{H}^m_\rho} \leq C\left(\|\psi_1\|_{\mathcal{H}^m_\rho}\|\psi_2\|_{\mathcal{H}^s_\rho} + \|\psi_1\|_{\mathcal{H}^s_\rho}\|\psi_2\|_{\mathcal{H}^m_\rho}\right). \tag{3.5}$$

The various constants C are independent of ρ.

We infer the important lemma:

Lemma 3.3 *Set $\nu = 1$, and let $\sigma \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $\ell > d/2$, and I be some time interval. Let $(\varphi, b) \in C(I, \mathcal{H}^{\ell+1}_\rho \times \mathcal{H}^{\ell}_\rho) \cap L^2_{\rm loc}(I, \mathcal{H}^{\ell+3/2}_\rho \times \mathcal{H}^{\ell+1/2}_\rho)$. Then any solution $(\phi, a^\varepsilon) \in C(I, \mathcal{H}^{\ell+1}_\rho \times \mathcal{H}^{\ell}_\rho) \cap L^2_{\rm loc}(I, \mathcal{H}^{\ell+3/2}_\rho \times \mathcal{H}^{\ell+1/2}_\rho)$ to*

$$\begin{cases} \partial_t\phi + \frac{1}{2}\nabla\varphi\cdot\nabla\phi + \lambda|b|^{2\sigma} &= 0, \\ \partial_t a^\varepsilon + \nabla\varphi\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\varphi &= \frac{i\varepsilon}{2}\Delta a^\varepsilon, \end{cases} \tag{3.6}$$

satisfies

$$\left|\partial_t \|\phi\|^2_{\mathcal{H}^{\ell+1}_\rho} - 2\dot{\rho}\|\phi\|^2_{\mathcal{H}^{\ell+3/2}_\rho}\right| \leq C\Big(\|\phi\|^2_{\mathcal{H}^{\ell+3/2}_\rho}\|\varphi\|_{\mathcal{H}^{\ell+1}_\rho} + \|\phi\|_{\mathcal{H}^{\ell+3/2}_\rho}\|\varphi\|_{\mathcal{H}^{\ell+3/2}_\rho}\|\phi\|_{\mathcal{H}^{\ell+1}_\rho}$$
$$+\|b\|_{\mathcal{H}^{\ell+1/2}_\rho}\|\phi\|_{\mathcal{H}^{\ell+3/2}_\rho}\|b\|^{2\sigma-1}_{\mathcal{H}^{\ell}_\rho}\Big),$$
$$\left|\partial_t \|a^\varepsilon\|^2_{\mathcal{H}^{\ell}_\rho} - 2\dot{\rho}\|a^\varepsilon\|^2_{\mathcal{H}^{\ell+1/2}_\rho}\right| \leq C\Big(\|a^\varepsilon\|^2_{\mathcal{H}^{\ell+1/2}_\rho}\|\varphi\|_{\mathcal{H}^{\ell+1}_\rho}$$
$$+\|a^\varepsilon\|_{\mathcal{H}^{\ell+1/2}_\rho}\|\varphi\|_{\mathcal{H}^{\ell+3/2}_\rho}\|a^\varepsilon\|_{\mathcal{H}^{\ell}_\rho}\Big),$$

where C is independent of ε and ρ.

Proof In view of (3.3) and (3.6), we have

$$\partial_t \|\phi\|^2_{\mathcal{H}^{\ell+1}_\rho} - 2\dot{\rho}\|\phi\|^2_{\mathcal{H}^{\ell+3/2}_\rho} = -\operatorname{Re}\langle\phi, \nabla\varphi\cdot\nabla\phi\rangle_{\mathcal{H}^{\ell+1}_\rho} - 2\lambda\operatorname{Re}\left\langle\phi, |b|^{2\sigma}\right\rangle_{\mathcal{H}^{\ell+1}_\rho}.$$

Cauchy–Schwarz inequality yields

$$\left|\langle\phi, \nabla\varphi\cdot\nabla\phi\rangle_{\mathcal{H}^{\ell+1}_\rho}\right| \leq \|\phi\|_{\mathcal{H}^{\ell+3/2}_\rho}\|\nabla\varphi\cdot\nabla\phi\|_{\mathcal{H}^{\ell+1/2}_\rho}.$$

Inequality (3.5) with $m = \ell + 1/2$ and $s = \ell$ yields

$$\begin{aligned}\|\nabla\varphi\cdot\nabla\phi\|_{\mathcal{H}^{\ell+1/2}_\rho} &\leq C\left(\|\nabla\varphi\|_{\mathcal{H}^{\ell+1/2}_\rho}\|\nabla\phi\|_{\mathcal{H}^{\ell}_\rho} + \|\nabla\varphi\|_{\mathcal{H}^{\ell}_\rho}\|\nabla\phi\|_{\mathcal{H}^{\ell+1/2}_\rho}\right)\\ &\leq C\left(\|\varphi\|_{\mathcal{H}^{\ell+3/2}_\rho}\|\phi\|_{\mathcal{H}^{\ell+1}_\rho} + \|\varphi\|_{\mathcal{H}^{\ell+1}_\rho}\|\phi\|_{\mathcal{H}^{\ell+3/2}_\rho}\right),\end{aligned} \tag{3.7}$$

where we have used (3.1). The term involving b can be treated similarly. Indeed, using (3.5) on the one hand with $m = \ell + 1/2$ and $s = \ell$ and on the other hand with $m = \ell = s$, we can prove by induction on σ that

$$\||b|^{2\sigma}\|_{\mathcal{H}^{\ell+1/2}_\rho} \leq C\|b\|^{2\sigma-1}_{\mathcal{H}^{\ell}_\rho}\|b\|_{\mathcal{H}^{\ell+1/2}_\rho}, \tag{3.8}$$

hence the first inequality for Lemma 3.3.

For the second inequality,

$$\partial_t \|a^\varepsilon\|^2_{\mathcal{H}^{\ell}_\rho} - 2\dot{\rho}\|a^\varepsilon\|^2_{\mathcal{H}^{\ell+1/2}_\rho} = -2\operatorname{Re}\langle a^\varepsilon, \nabla\varphi\cdot\nabla a^\varepsilon\rangle_{\mathcal{H}^{\ell}_\rho} - \operatorname{Re}\langle a^\varepsilon, a^\varepsilon\Delta\varphi\rangle_{\mathcal{H}^{\ell}_\rho}$$
$$+\varepsilon\operatorname{Re}\langle a^\varepsilon, i\Delta a^\varepsilon\rangle_{\mathcal{H}^{\ell}_\rho}.$$

Remark that

$$\operatorname{Re}\langle a^\varepsilon, i\Delta a^\varepsilon\rangle_{\mathcal{H}^{\ell}_\rho} = 0,$$

so the Laplacian term is not present in energy estimates, which are therefore independent of ε. Like before, Cauchy–Schwarz inequality yields

$$|\left\langle a^\varepsilon, \nabla\varphi \cdot \nabla a^\varepsilon\right\rangle_{\mathcal{H}^\ell_\rho}| \le \|a^\varepsilon\|_{\mathcal{H}^{\ell+1/2}_\rho} \|\nabla\varphi \cdot \nabla a^\varepsilon\|_{\mathcal{H}^{\ell-1/2}_\rho}.$$

The last term is estimated thanks to the first point in Lemma 3.2, with

$$m = \ell - \frac{1}{2}, \quad k = \ell + \frac{1}{2}, \quad s = \ell + 1.$$

Similarly,

$$|\left\langle a^\varepsilon, a^\varepsilon \Delta\varphi\right\rangle_{\mathcal{H}^\ell_\rho}| \le \|a^\varepsilon\|_{\mathcal{H}^{\ell+1/2}_\rho} \|a^\varepsilon \Delta\varphi\|_{\mathcal{H}^{\ell-1/2}_\rho},$$

and the last term is estimated thanks to the second point in Lemma 3.2, with

$$m = \ell - \frac{1}{2}, \quad k = \ell, \quad s = \ell + \frac{3}{2}.$$

The lemma follows easily. □

4 A fundamental estimate

In the framework of Theorem 1.1, the initial datum $u^\varepsilon_{|t=0} = a_0 e^{i\phi_0/\varepsilon}$ belongs to H^∞, so the existence of $T^\varepsilon > 0$ (depending a priori on ε), and of a unique solution $u^\varepsilon \in C([0, T^\varepsilon], H^\infty)$ to (1.1)–(1.2), stems from standard theory (see e.g. [11]). The fact that the existence time may be chosen independent of ε, along with the rest of the first point of Theorem 1.1, stems from Proposition 4.1 below.

For a decreasing function ρ, we introduce the norm defined by

$$|||\psi|||^2_{\ell,t} = \max\left(\sup_{0\le s\le t} \|\psi(s)\|^2_{\mathcal{H}^\ell_{\rho(s)}}, 2\int_0^t |\dot\rho(s)| \|\psi(s)\|^2_{\mathcal{H}^{\ell+1/2}_{\rho(s)}} ds\right). \tag{4.1}$$

Proposition 4.1 *Let $\lambda \in \mathbb{R}$, $\ell > d/2 + 1$, $M_0 > 0$ and $(\phi_0, a_0) \in \mathcal{H}^{\ell+1}_{M_0} \times \mathcal{H}^\ell_{M_0}$.*

1. *There exists $M \gg 1$ such that if $T < M_0/M$ and $\rho(t) = M_0 - Mt$, (2.2)–(2.3) has a unique solution*

$$(\phi^\varepsilon, a^\varepsilon) \in C([0, T], \mathcal{H}^{\ell+1}_\rho \times \mathcal{H}^\ell_\rho) \cap L^2([0, T], \mathcal{H}^{\ell+3/2}_\rho \times \mathcal{H}^{\ell+1/2}_\rho),$$

with

$$|||\phi^\varepsilon|||^2_{\ell+1,T} \le 2\|\phi_0\|^2_{\mathcal{H}^{\ell+1}_{M_0}} + \|a_0\|^{4\sigma}_{\mathcal{H}^\ell_{M_0}}, \quad |||a^\varepsilon|||^2_{\ell,T} \le 2\|a_0\|^2_{\mathcal{H}^\ell_{M_0}}. \tag{4.2}$$

2. *If* $R > 0$ *and* $(\phi_0, a_0), (\varphi_0, b_0) \in \mathcal{H}^{\ell+1}_{M_0} \times \mathcal{H}^{\ell}_{M_0}$, *with*

$$\|\phi_0\|_{\mathcal{H}^{\ell+1}_{M_0}} + \|a_0\|_{\mathcal{H}^{\ell}_{M_0}} \leq R, \quad \|\varphi_0\|_{\mathcal{H}^{\ell+1}_{M_0}} + \|b_0\|_{\mathcal{H}^{\ell}_{M_0}} \leq R,$$

there exists $K = K(R)$ *such that if* M *is chosen sufficiently large such that according to the first part of the proposition,* (2.2)–(2.3) *has solutions* $(\phi^\varepsilon, a^\varepsilon)$ *and* $(\varphi^\varepsilon, b^\varepsilon)$ *on* $[0, T]$ *corresponding respectively to the initial data* (ϕ_0, a_0) *and* (φ_0, b_0) *(with the same choice of* ρ *and the same assumption* $T < M_0/M$*), then*

$$|||\phi^\varepsilon - \varphi^\varepsilon|||_{\ell+1,T} + |||a^\varepsilon - b^\varepsilon|||_{\ell,T} \leq K \left(\|\phi_0 - \varphi_0\|_{\mathcal{H}^{\ell+1}_{M_0}} + \|a_0 - b_0\|_{\mathcal{H}^{\ell}_{M_0}} \right).$$

Remark 4.2 The proof yields a rather implicit dependence of M upon M_0 and (ϕ_0, a_0). As a consequence, it is not clear how to choose the best possible T, even for initial data whose Fourier transform is compactly supported. For our present concern, the important information is that we get some positive T independent of ε.

Proof To construct the solution, we use the standard scheme from hyperbolic symmetric systems (see e.g. [2]), that is, we consider the iterative scheme defined by

$$\begin{cases} \partial_t \phi^\varepsilon_{j+1} + \frac{1}{2}\nabla\phi^\varepsilon_j \cdot \nabla\phi^\varepsilon_{j+1} + f(|a^\varepsilon_j|^2) = 0, & \phi^\varepsilon_{j+1|t=0} = \phi_0, \\ \partial_t a^\varepsilon_{j+1} + \nabla\phi^\varepsilon_j \cdot \nabla a^\varepsilon_{j+1} + \frac{1}{2} a^\varepsilon_{j+1}\Delta\phi^\varepsilon_j = \frac{i\varepsilon}{2}\Delta a^\varepsilon_{j+1}, & a^\varepsilon_{j+1|t=0} = a_0, \end{cases} \tag{4.3}$$

with $f(|a|^2) = \lambda|a|^{2\sigma}$, initialized with $(\phi^\varepsilon_0, a^\varepsilon_0)(t) = (\phi_0, a_0)$. For functions at the level of regularity of the norm (4.1) with $\ell > d/2$, the above scheme is well defined: if $|||\phi^\varepsilon_j|||_{\ell+1,T} + |||a^\varepsilon_j|||_{\ell,T}$ is finite, then ϕ^ε_{j+1} and a^ε_{j+1} are well-defined. Indeed, in the first equation, ϕ^ε_{j+1} solves a linear transport equation with smooth coefficients, and the second equation is equivalent to the linear Schrödinger equation

$$i\varepsilon\partial_t v^\varepsilon_{j+1} + \frac{\varepsilon^2}{2}\Delta v^\varepsilon_{j+1} = -\left(\partial_t\phi^\varepsilon_j + \frac{1}{2}|\nabla\phi^\varepsilon_j|^2\right) v^\varepsilon_{j+1}, \quad v^\varepsilon_{j+1|t=0} = a_0 e^{i\phi^\varepsilon_j(0)/\varepsilon},$$

through the relation $v^\varepsilon_{j+1} = a^\varepsilon_{j+1} e^{i\phi^\varepsilon_j/\varepsilon}$. This is a linear Schrödinger equation with a smooth and bounded external time-dependent potential, for which the existence of an L^2-solution is granted.

The proof of the first part of the proposition goes in two steps: first, we prove that the sequence $(|||\phi^\varepsilon_j|||_{\ell+1,T} + |||a^\varepsilon_j|||_{\ell,T})_{j\geq 0}$ is bounded for some $T > 0$ sufficiently small, but independent of ε. Then we show that up to decreasing T, the series

$$\sum_{j\geq 0} \left(|||\phi^\varepsilon_{j+1} - \phi^\varepsilon_j|||_{\ell+1,T} + |||a^\varepsilon_{j+1} - a^\varepsilon_j|||_{\ell,T} \right)$$

is converging. Note that unlike in the case of hyperbolic symmetric systems in Sobolev spaces, the regularity is the same at the two steps of the proof (in Sobolev spaces, the standard proof involves first a bound in the large norm, then convergence in the small norm). This can be seen as an illustration of the fact that these time-dependent norms, through the decay of the weight ρ, "neutralize" some nonlinear effects.

First step: the sequence is bounded By integration, Lemma 3.3 yields, for a decreasing $\rho(t)$ and $T > 0$ to be chosen later,

$$\begin{aligned}
|||\phi^\varepsilon_{j+1}|||^2_{\ell+1,T} \le \|\phi_0\|^2_{\mathcal{H}^{\ell+1}_{\rho(0)}} &+ C\int_0^T \frac{1}{|\dot\rho(t)|}|\dot\rho(t)|\|\phi^\varepsilon_{j+1}(t)\|^2_{\mathcal{H}^{\ell+3/2}_{\rho(t)}}\|\phi^\varepsilon_j(t)\|_{\mathcal{H}^{\ell+1}_{\rho(t)}}dt\\
&+ C\int_0^T \frac{1}{|\dot\rho(t)|}|\dot\rho(t)|\|\phi^\varepsilon_{j+1}(t)\|_{\mathcal{H}^{\ell+3/2}_{\rho(t)}}\|\phi^\varepsilon_j(t)\|_{\mathcal{H}^{\ell+3/2}_{\rho(t)}}\|\phi^\varepsilon_{j+1}(t)\|_{\mathcal{H}^{\ell+1}_{\rho(t)}}dt\\
&+ C\int_0^T \frac{1}{|\dot\rho(t)|}|\dot\rho(t)|\|a^\varepsilon_j(t)\|_{\mathcal{H}^{\ell+1/2}_{\rho(t)}}\|\phi^\varepsilon_{j+1}(t)\|_{\mathcal{H}^{\ell+3/2}_{\rho(t)}}\|a^\varepsilon_j(t)\|^{2\sigma-1}_{\mathcal{H}^{\ell}_{\rho(t)}}dt,\\
|||a^\varepsilon_{j+1}|||^2_{\ell,T} \le \|a_0\|^2_{\mathcal{H}^{\ell}_{\rho(0)}} &+ C\int_0^T \frac{1}{|\dot\rho(t)|}|\dot\rho(t)|\|a^\varepsilon_{j+1}(t)\|^2_{\mathcal{H}^{\ell+1/2}_{\rho(t)}}\|\phi^\varepsilon_j(t)\|_{\mathcal{H}^{\ell+1}_{\rho(t)}}dt\\
&+ C\int_0^T \frac{1}{|\dot\rho(t)|}|\dot\rho(t)|\|a^\varepsilon_{j+1}(t)\|_{\mathcal{H}^{\ell+1/2}_{\rho(t)}}\|\phi^\varepsilon_j(t)\|_{\mathcal{H}^{\ell+3/2}_{\rho(t)}}\|a^\varepsilon_{j+1}(t)\|_{\mathcal{H}^{\ell}_{\rho(t)}}dt.
\end{aligned}$$

Hölder and Cauchy–Schwarz inequalities yield

$$\begin{aligned}
|||\phi^\varepsilon_{j+1}|||^2_{\ell+1,T} \le \|\phi_0\|^2_{\mathcal{H}^{\ell+1}_{\rho(0)}} &+ C\left(\sup_{0\le t\le T}\frac{1}{|\dot\rho(t)|}\right)|||\phi^\varepsilon_{j+1}|||^2_{\ell+1,T}|||\phi^\varepsilon_j|||_{\ell+1,T}\\
&+ C\left(\sup_{0\le t\le T}\frac{1}{|\dot\rho(t)|}\right)|||\phi^\varepsilon_{j+1}|||_{\ell+1,T}|||a^\varepsilon_j|||^{2\sigma}_{\ell,T},\\
|||a^\varepsilon_{j+1}|||^2_{\ell,T} \le \|a_0\|^2_{\mathcal{H}^{\ell}_{\rho(0)}} &+ C\left(\sup_{0\le t\le T}\frac{1}{|\dot\rho(t)|}\right)|||a^\varepsilon_{j+1}|||^2_{\ell,T}|||\phi^\varepsilon_j|||_{\ell+1,T}.
\end{aligned}$$

Recall that $M_0 > 0$ is given. Take $\phi_0 \in \mathcal{H}^{\ell+1}_{M_0}$, $a_0 \in \mathcal{H}^{\ell}_{M_0}$ and set $\rho(t) = M_0 - Mt$. Under the condition

$$\frac{C}{M}|||\phi^\varepsilon_j|||_{\ell+1,T} \le \frac{1}{4}, \tag{4.4}$$

the previous inequalities imply

$$\begin{aligned}
\frac{1}{2}|||\phi^\varepsilon_{j+1}|||^2_{\ell+1,T} &\le \|\phi_0\|^2_{\mathcal{H}^{\ell+1}_{M_0}} + \frac{C^2}{M^2}|||a^\varepsilon_j|||^{4\sigma}_{\ell,T},\\
\frac{3}{4}|||a^\varepsilon_{j+1}|||^2_{\ell,T} &\le \|a_0\|^2_{\mathcal{H}^{\ell}_{M_0}}.
\end{aligned}$$

Let us now choose $M = |\dot\rho(t)|$ sufficiently large such that (4.4) holds for $j = 0$ and such that

$$2\|\phi_0\|^2_{\mathcal{H}^{\ell+1}_{M_0}} + \frac{2C^2}{M^2}\left(\frac{4}{3}|||a_0^\varepsilon|||^2_{\ell,T}\right)^{2\sigma} \leq \frac{M^2}{16C^2},$$
$$\frac{2C^2}{M^2}\left(\frac{4}{3}\right)^{2\sigma} \leq 1.$$

Note that in view of (4.1), for all $T < M_0/M$ (so that ρ remains positive on $[0, T]$),

$$\begin{aligned}|||a_0^\varepsilon|||^2_{\ell,T} &= \max\left(\|a_0\|^2_{\mathcal{H}^\ell_{M_0}}, \int \langle\xi\rangle^{2\ell} e^{2M_0\langle\xi\rangle}|\hat{a}_0(\xi)|^2 \int_0^T 2M\,\langle\xi\rangle^{-2Mt\langle\xi\rangle}\, dt d\xi\right)\\ &= \|a_0\|^2_{\mathcal{H}^\ell_{M_0}},\end{aligned}$$

and similarly

$$|||\phi_0^\varepsilon|||^2_{\ell+1,T} = \|\phi_0\|^2_{\mathcal{H}^{\ell+1}_{M_0}},$$

so that our constraint on M only depends on $\|\phi_0\|_{\mathcal{H}^{\ell+1}_{M_0}}$ and $\|a_0\|_{\mathcal{H}^\ell_{M_0}}$. Then, for $T < M_0/M$, the above inequalities yield, by induction, for all $j \geq 1$,

$$\begin{aligned}|||\phi_j^\varepsilon|||^2_{\ell+1,T} &\leq 2\|\phi_0\|^2_{\mathcal{H}^{\ell+1}_{M_0}} + \frac{2C^2}{M^2}\left(\frac{4}{3}\|a_0\|^2_{\mathcal{H}^\ell_{M_0}}\right)^{2\sigma} \leq 2\|\phi_0\|^2_{\mathcal{H}^{\ell+1}_{M_0}} + \|a_0\|^{4\sigma}_{\mathcal{H}^\ell_{M_0}},\\ |||a_j^\varepsilon|||^2_{\ell,T} &\leq \frac{4}{3}\|a_0\|^2_{\mathcal{H}^\ell_{M_0}}.\end{aligned}$$

Second step: the sequence converges For $j \geq 1$, consider the difference of two successive iterates: in the case of the phase, we have

$$\partial_t(\phi^\varepsilon_{j+1} - \phi^\varepsilon_j) + \frac{1}{2}\left(\nabla\phi^\varepsilon_j \cdot \nabla\phi^\varepsilon_{j+1} - \nabla\phi^\varepsilon_{j-1}\cdot\nabla\phi^\varepsilon_j\right) + f(|a^\varepsilon_j|^2) - f(|a^\varepsilon_{j-1}|^2) = 0,$$

along with zero initial data. Inserting the term $|\nabla\phi^\varepsilon_j|^2$, and denoting by $\delta\phi^\varepsilon_{j+1} = \phi^\varepsilon_{j+1} - \phi^\varepsilon_j$, we can rewrite the above equation as

$$\partial_t\delta\phi^\varepsilon_{j+1} + \frac{1}{2}\left(\nabla\phi^\varepsilon_j\cdot\nabla\delta\phi^\varepsilon_{j+1} + \nabla\delta\phi^\varepsilon_j\cdot\nabla\phi^\varepsilon_j\right) + f(|a^\varepsilon_j|^2) - f(|a^\varepsilon_{j-1}|^2) = 0.$$

(3.3) yields, along with Cauchy–Schwarz inequality as in the first step of the proof of Proposition 4.1:

$$\begin{aligned}
|||\delta\phi^{\varepsilon}_{j+1}|||^2_{\ell+1,T} \leq &\int_0^T \|\delta\phi^{\varepsilon}_{j+1}(t)\|_{\mathcal{H}^{\ell+3/2}_{\rho(t)}} \|\nabla\phi^{\varepsilon}_j \cdot \nabla\delta\phi^{\varepsilon}_{j+1}\|_{\mathcal{H}^{\ell+1/2}_{\rho(t)}} dt \\
&+ \int_0^T \|\delta\phi^{\varepsilon}_{j+1}(t)\|_{\mathcal{H}^{\ell+3/2}_{\rho(t)}} \|\nabla\delta\phi^{\varepsilon}_j \cdot \nabla\phi^{\varepsilon}_j\|_{\mathcal{H}^{\ell+1/2}_{\rho(t)}} dt \\
&+ 2\int_0^T \|\delta\phi^{\varepsilon}_{j+1}(t)\|_{\mathcal{H}^{\ell+3/2}_{\rho(t)}} \|f(|a^{\varepsilon}_j|^2) - f(|a^{\varepsilon}_{j-1}|^2)\|_{\mathcal{H}^{\ell+1/2}_{\rho(t)}} dt.
\end{aligned}$$

The first two terms are estimated thanks to the last point in Lemma 3.2, as in (3.7).

Since $f(|z|^2)$ is a polynomial in $(z, \bar{z})$, the last point of Lemma 3.2 yields

$$\begin{aligned}
&\|f(|a^{\varepsilon}_j|^2) - f(|a^{\varepsilon}_{j-1}|^2)\|_{\mathcal{H}^{\ell+1/2}_{\rho}} \leq C\left(\|a^{\varepsilon}_j\|^{2\sigma-2}_{\mathcal{H}^{\ell}_{\rho}} + \|a^{\varepsilon}_{j-1}\|^{2\sigma-2}_{\mathcal{H}^{\ell}_{\rho}}\right) \\
&\times\left(\left(\|a^{\varepsilon}_j\|_{\mathcal{H}^{\ell}_{\rho}} + \|a^{\varepsilon}_{j-1}\|_{\mathcal{H}^{\ell}_{\rho}}\right)\|\delta a^{\varepsilon}_j\|_{\mathcal{H}^{\ell+1/2}_{\rho}}\right. \\
&\left.+\left(\|a^{\varepsilon}_j\|_{\mathcal{H}^{\ell+1/2}_{\rho}} + \|a^{\varepsilon}_{j-1}\|_{\mathcal{H}^{\ell+1/2}_{\rho}}\right)\|\delta a^{\varepsilon}_j\|_{\mathcal{H}^{\ell}_{\rho}}\right),
\end{aligned}$$

where we have also denoted $\delta a^{\varepsilon}_j = a^{\varepsilon}_j - a^{\varepsilon}_{j-1}$. We conclude:

$$|||\delta\phi^{\varepsilon}_{j+1}|||^2_{\ell+1,T} \leq \frac{K}{M}\left(|||\delta\phi^{\varepsilon}_{j+1}|||^2_{\ell+1,T} + |||\delta\phi^{\varepsilon}_j|||^2_{\ell+1,T} + |||\delta a^{\varepsilon}_j|||^2_{\ell,T}\right),$$

where K stems from the first step. For M sufficiently large,

$$|||\delta\phi^{\varepsilon}_{j+1}|||^2_{\ell+1,T} \leq \frac{2K}{M}\left(|||\delta\phi^{\varepsilon}_j|||^2_{\ell+1,T} + |||\delta a^{\varepsilon}_j|||^2_{\ell,t}\right).$$

Similarly, $\delta a^{\varepsilon}_{j+1}$ solves

$$\partial_t\delta a^{\varepsilon}_{j+1} + \nabla\phi^{\varepsilon}_j \cdot \nabla\delta a^{\varepsilon}_{j+1} + \nabla\delta\phi^{\varepsilon}_j \cdot \nabla a^{\varepsilon}_j + \frac{1}{2}\delta a^{\varepsilon}_{j+1}\Delta\phi^{\varepsilon}_j + \frac{1}{2}a^{\varepsilon}_j\Delta\delta\phi^{\varepsilon}_j = i\frac{\varepsilon}{2}\Delta\delta a^{\varepsilon}_{j+1}.$$

The last term is skew-symmetric, and thus does not appear in energy estimates. Resuming the same estimates as in the proof of Lemma 3.3, we come up with:

$$|||\delta a^{\varepsilon}_{j+1}|||^2_{\ell,T} \leq \frac{K}{M}\left(|||\delta a^{\varepsilon}_{j+1}|||^2_{\ell,T} + |||\delta\phi^{\varepsilon}_j|||^2_{\ell+1,T} + |||\delta a^{\varepsilon}_j|||^2_{\ell,T}\right),$$

hence

$$|||\delta a^{\varepsilon}_{j+1}|||^2_{\ell,T} \leq \frac{2K}{M}\left(|||\delta\phi^{\varepsilon}_j|||^2_{\ell+1,T} + |||\delta a^{\varepsilon}_j|||^2_{\ell,T}\right),$$

up to increasing M (hence decreasing T). For M possibly even larger, we infer that the series

$$\sum_{j\geq 0}\left(|||\phi^\varepsilon_{j+1}-\phi^\varepsilon_j|||_{\ell+1,T}+|||a^\varepsilon_{j+1}-a^\varepsilon_j|||_{\ell,T}\right)$$

converges geometrically. Uniqueness is a direct consequence of the estimates used in this second step. (4.2) is obtained by letting j go to infinity in the estimates at the end of the first step.

The Lipschitzean property of the flow follows from calculations similar to those of the second step of the proof. □

5 Bounds on the numerical solution

Proposition 5.1 *Let $\lambda \in \mathbb{R}$, $\sigma \in \mathbb{N}$, and let $(\phi^\varepsilon, a^\varepsilon)$ be the solution of either of the systems (2.2), (2.4), or (2.5), with the notation $f(|z|^2)=\lambda|z|^{2\sigma}$. Let $s>d/2+1$, $\mu>0$, and $\ell\geq s$. Suppose that $(\phi^\varepsilon, a^\varepsilon)$ satisfies*

$$(\phi^\varepsilon, a^\varepsilon)\in C([0,T],\mathcal{H}^{s+1}_\rho\times\mathcal{H}^s_\rho),$$

where $\rho(t)=M_0-Mt$ and $0<T<M_0/M$, with

$$\sup_{t\in[0,T]}\|\phi^\varepsilon(t)\|_{\mathcal{H}^{s+1}_{\rho(t)}}+\sup_{t\in[0,T]}\|a^\varepsilon(t)\|_{\mathcal{H}^s_{\rho(t)}}\leq\mu.$$

Then, up to increasing M (and decreasing T),

$$\|\phi^\varepsilon(t)\|_{\mathcal{H}^{\ell+1}_{\rho(t)}}+\|a^\varepsilon(t)\|_{\mathcal{H}^{\ell}_{\rho(t)}}\leq\|\phi^\varepsilon(0)\|_{\mathcal{H}^{\ell+1}_{M_0}}+\|a^\varepsilon(0)\|_{\mathcal{H}^{\ell}_{M_0}},\quad\forall t\in[0,T].$$

Note that the assumption carries over a regularity at level $s>d/2+1$, while the conclusion addresses the regularity at level $\ell\geq s$: the above proposition may be viewed as a tame estimate result.

Proof First, remark that $|||\phi^\varepsilon(T)|||_{s+1}+|||a^\varepsilon(T)|||_s$ is a non-increasing function of M, provided that the constraint $T<M_0/M$ remains fulfilled.

Second, note that it suffices to establish the result in the case of (2.2), since the other systems contain fewer terms, and we will estimate each term present in (2.2).

The idea of the result is then to view (3.3) as a parabolic estimate, with diffusive coefficient $-\dot\rho=M$. Indeed, like in the proof of Lemma 3.3, we have

$$\partial_t\|\phi^\varepsilon\|^2_{\mathcal{H}^{\ell+1}_\rho}+2M\|\phi^\varepsilon\|^2_{\mathcal{H}^{\ell+3/2}_\rho}\leq C\|\phi^\varepsilon\|_{\mathcal{H}^{\ell+3/2}_\rho}\left(\|\nabla\phi^\varepsilon\cdot\nabla\phi^\varepsilon\|_{\mathcal{H}^{\ell+1/2}_\rho}+\||a^\varepsilon|^{2\sigma}\|_{\mathcal{H}^{\ell+1/2}_\rho}\right),$$
$$\partial_t\|a^\varepsilon\|^2_{\mathcal{H}^{\ell}_\rho}+2M\|a^\varepsilon\|^2_{\mathcal{H}^{\ell+1/2}_\rho}\leq C\|a^\varepsilon\|_{\mathcal{H}^{\ell+1/2}_\rho}\left(\|\nabla\phi^\varepsilon\cdot\nabla a^\varepsilon\|_{\mathcal{H}^{\ell-1/2}_\rho}+\|a^\varepsilon\Delta\phi^\varepsilon\|_{\mathcal{H}^{\ell-1/2}_\rho}\right).$$

We then invoke Lemma 3.2 once more. Since the first two points in Lemma 3.2 involve the constraint $k, s \geq m$, we can rely only on (3.5). We have

$$\|\nabla \phi^\varepsilon \cdot \nabla \phi^\varepsilon\|_{\mathcal{H}_\rho^{\ell+1/2}} \leq C\|\nabla \phi^\varepsilon\|_{\mathcal{H}_\rho^{\ell+1/2}}\|\nabla \phi^\varepsilon\|_{\mathcal{H}_\rho^{s-1}} \leq C\|\phi^\varepsilon\|_{\mathcal{H}_\rho^{\ell+3/2}}\|\phi^\varepsilon\|_{\mathcal{H}_\rho^{s}},$$

since $s > d/2 + 1$. We have already used the estimate

$$\||a^\varepsilon|^{2\sigma}\|_{\mathcal{H}_\rho^{\ell+1/2}} \leq C\|a^\varepsilon\|_{\mathcal{H}_\rho^s}^{2\sigma-1}\|a^\varepsilon\|_{\mathcal{H}_\rho^{\ell+1/2}},$$

so that Young inequality yields

$$\partial_t\|\phi^\varepsilon\|^2_{\mathcal{H}_\rho^{\ell+1}} + 2M\|\phi^\varepsilon\|^2_{\mathcal{H}_\rho^{\ell+3/2}} \leq C\left(\mu + \mu^{2\sigma-1}\right)\left(\|\phi^\varepsilon\|^2_{\mathcal{H}_\rho^{\ell+3/2}} + \|a^\varepsilon\|^2_{\mathcal{H}_\rho^{\ell+1/2}}\right).$$

Again, (3.5) yields

$$\begin{aligned}\|\nabla \phi^\varepsilon \cdot \nabla a^\varepsilon\|_{\mathcal{H}_\rho^{\ell-1/2}} &\leq C\left(\|\nabla \phi^\varepsilon\|_{\mathcal{H}_\rho^{\ell-1/2}}\|\nabla a^\varepsilon\|_{\mathcal{H}_\rho^{s-1}} + \|\nabla \phi^\varepsilon\|_{\mathcal{H}_\rho^{s-1}}\|\nabla a^\varepsilon\|_{\mathcal{H}_\rho^{\ell-1/2}}\right)\\ &\leq C\left(\|\phi^\varepsilon\|_{\mathcal{H}_\rho^{\ell+1/2}}\|a^\varepsilon\|_{\mathcal{H}_\rho^{s}} + \|\phi^\varepsilon\|_{\mathcal{H}_\rho^{s}}\|a^\varepsilon\|_{\mathcal{H}_\rho^{\ell+1/2}}\right),\end{aligned}$$

and

$$\begin{aligned}\|a^\varepsilon \Delta \phi^\varepsilon\|_{\mathcal{H}_\rho^{\ell-1/2}} &\leq C\left(\|\Delta \phi^\varepsilon\|_{\mathcal{H}_\rho^{\ell-1/2}}\|a^\varepsilon\|_{\mathcal{H}_\rho^{s-1}} + \|\Delta \phi^\varepsilon\|_{\mathcal{H}_\rho^{s-1}}\|a^\varepsilon\|_{\mathcal{H}_\rho^{\ell-1/2}}\right)\\ &\leq C\left(\|\phi^\varepsilon\|_{\mathcal{H}_\rho^{\ell+3/2}}\|a^\varepsilon\|_{\mathcal{H}_\rho^{s}} + \|\phi^\varepsilon\|_{\mathcal{H}_\rho^{s+1}}\|a^\varepsilon\|_{\mathcal{H}_\rho^{\ell+1/2}}\right).\end{aligned}$$

We end up with

$$\begin{aligned}\partial_t\left(\|\phi^\varepsilon\|^2_{\mathcal{H}_\rho^{\ell+1}} + \|a^\varepsilon\|^2_{\mathcal{H}_\rho^{\ell}}\right) &+ 2M\left(\|\phi^\varepsilon\|^2_{\mathcal{H}_\rho^{\ell+3/2}} + \|a^\varepsilon\|^2_{\mathcal{H}_\rho^{\ell+1/2}}\right)\\ &\leq C\left(\mu + \mu^{2\sigma-1}\right)\left(\|\phi^\varepsilon\|^2_{\mathcal{H}_\rho^{\ell+3/2}} + \|a^\varepsilon\|^2_{\mathcal{H}_\rho^{\ell+1/2}}\right).\end{aligned}$$

Choosing $2M \geq C\left(\mu + \mu^{2\sigma-1}\right)$ thus yields the result. □

We readily infer:

Corollary 5.2 *Let $\ell \geq s > d/2+1$, and $\tau > 0$. Suppose that the numerical solution*

$$\mathcal{Z}_\varepsilon^t\begin{pmatrix}\phi_0^\varepsilon\\ a_0^\varepsilon\end{pmatrix} = \begin{pmatrix}\phi_t^\varepsilon\\ a_t^\varepsilon\end{pmatrix}$$

satisfies

$$\sup_{t\in[0,\tau]}\|\phi_t^\varepsilon\|_{\mathcal{H}_{\rho(t)}^{s+1}} + \sup_{t\in[0,\tau]}\|a_t^\varepsilon\|_{\mathcal{H}_{\rho(t)}^{s}} \leq \mu,$$

where $\rho(t) = M_0 - Mt$*. Then, up to increasing* M *(and possibly decreasing* τ*),*

$$\|\phi_t^\varepsilon\|_{\mathcal{H}^{\ell+1}_{\rho(t)}} + \|a_t^\varepsilon\|_{\mathcal{H}^{\ell}_{\rho(t)}} \le \|\phi_0^\varepsilon\|_{\mathcal{H}^{\ell+1}_{M_0}} + \|a_0^\varepsilon\|_{\mathcal{H}^{\ell}_{M_0}}, \quad \forall t \in [0,\tau].$$

Proof Proposition 5.1 yields possibly three values for M, according to which flow is considered. Note that this value depends only on μ, and recall that if $\tilde{M} \ge M$, $\tilde{\rho}(t) = M_0 - \tilde{M}t$, we have

$$\|f\|_{\mathcal{H}^s_{\tilde{\rho}(t)}} \le \|f\|_{\mathcal{H}^s_{\rho(t)}}, \quad 0 \le t < M_0/\tilde{M} = \min\left(M_0/M, M_0/\tilde{M}\right).$$

Therefore, considering the maximum of the two values of M given by Proposition 5.1 (for the flows associated to (2.4) and (2.5)) yields the corollary. □

6 Local error estimate

We resume the computations from [9], based on the general formula established in [15]. For a possibly nonlinear operator A, we denote by $\mathcal{E}_A$ the associated flow:

$$\partial_t \mathcal{E}_A(t,v) = A\left(\mathcal{E}_A(t,v)\right); \quad \mathcal{E}_A(0,v) = v.$$

Theorem 6.1 (Theorem 1 from [15]) *Suppose that* $F(u) = A(u) + B(u)$*, and denote by*

$$\mathcal{S}^t(u) = \mathcal{E}_F(t,u) \text{ and } \mathcal{Z}^t(u) = \mathcal{E}_B(t, \mathcal{E}_A(t,u))$$

the exact flow and the Lie-Trotter flow, respectively. Let $\mathcal{L}(t,u) = \mathcal{Z}^t(u) - \mathcal{S}^t(u)$*. We have the exact formula*

$$\begin{aligned}\mathcal{L}(t,u) = \int_0^t \int_0^{\tau_1} &\partial_2 \mathcal{E}_F\left(t-\tau_1, \mathcal{Z}^{\tau_1}(u)\right) \partial_2 \mathcal{E}_B\left(\tau_1 - \tau_2, \mathcal{E}_A(\tau_1,u)\right) \\ &\times [B,A]\left(\mathcal{E}_B\left(\tau_2, \mathcal{E}_A(\tau_1,u)\right)\right) d\tau_2 d\tau_1,\end{aligned}$$

where $[B,A](v) = B'(v)A(v) - A'(v)B(v)$*.*

In the case of the Lie-Trotter splitting (1.5) for Eq. (1.1), we would have

$$A = i\frac{\varepsilon}{2}\Delta; \quad B(v) = -\frac{i}{\varepsilon} f(|v|^2)v, \quad f(|v|^2) = \lambda |v|^{2\sigma}; \quad F(v) = A(v) + B(v),$$

where we have omitted the dependence upon ε in the notations for the sake of brevity.

However, as pointed out in [9], using the above result directly in terms of the wave function u^ε does not seem convenient. In the context of WKB regime, we rather consider the operators A and B defined by

$$A\begin{pmatrix}\phi \\ a\end{pmatrix} = \begin{pmatrix} -\frac{1}{2}|\nabla\phi|^2 \\ -\nabla\phi\cdot\nabla a - \frac{1}{2}a\Delta\phi + i\frac{\varepsilon}{2}\Delta a\end{pmatrix}, \quad B\begin{pmatrix}\phi \\ a\end{pmatrix} = \begin{pmatrix} -f(|a|^2) \\ 0\end{pmatrix}. \quad (6.1)$$

We note that with this approach, neither A nor B is a linear operator.

Lemma 6.2 *Let A and B defined by (6.1). Their commutator is given by*

$$[A, B]\begin{pmatrix}\phi\\ a\end{pmatrix} = \begin{pmatrix}\nabla\phi\cdot\nabla f\left(|a|^2\right) - \operatorname{div}\left(|a|^2\nabla\phi + \varepsilon\operatorname{Im}(\overline{a}\nabla a)\right) f'(|a|^2)\\ \nabla a\cdot\nabla f\left(|a|^2\right) + \frac{1}{2}a\Delta f\left(|a|^2\right)\end{pmatrix}.$$

As a consequence, if $\ell > d/2 + 3$, $\rho > 0$, $\|\phi\|_{\mathcal{H}^{\ell+1}_\rho} \le \mu$, $\|a\|_{\mathcal{H}^\ell_\rho} \le \mu$, then there exists $C = C(\mu)$ independent of $\varepsilon \in [0, 1]$ such that

$$\begin{pmatrix}\varphi\\ b\end{pmatrix} = [A, B]\begin{pmatrix}\phi\\ a\end{pmatrix} \quad \textit{satisfies} \begin{cases}\|\varphi\|_{\mathcal{H}^{\ell-2}_\rho} \le C\left(\|\phi\|_{\mathcal{H}^{\ell+1}_\rho} + \|a\|_{\mathcal{H}^\ell_\rho}\right),\\ \|b\|_{\mathcal{H}^{\ell-3}_\rho} \le C\|a\|_{\mathcal{H}^\ell_\rho}.\end{cases}$$

Proof Like in [9], we have

$$A'\begin{pmatrix}\phi\\ a\end{pmatrix}\begin{pmatrix}\varphi\\ b\end{pmatrix} = \begin{pmatrix}-\nabla\phi\cdot\nabla\varphi\\ -\nabla\phi\cdot\nabla b - \nabla\varphi\cdot\nabla a - \frac{1}{2}b\Delta\phi - \frac{1}{2}a\Delta\varphi + i\frac{\varepsilon}{2}\Delta b\end{pmatrix},$$

whereas unlike in [9], we consider a function f which is not necessarily linear, so that the linearized operator of B is given by

$$B'\begin{pmatrix}\phi\\ a\end{pmatrix}\begin{pmatrix}\varphi\\ b\end{pmatrix} = \begin{pmatrix}-2\operatorname{Re}(\overline{a}b) f'(|a|^2)\\ 0\end{pmatrix}$$

and thus

$$B'\begin{pmatrix}\phi\\ a\end{pmatrix}\left(A\begin{pmatrix}\phi\\ a\end{pmatrix}\right) = \begin{pmatrix}\left(2\operatorname{Re}(\overline{a}\nabla a\cdot\nabla\phi) + |a|^2\Delta\phi + \varepsilon\operatorname{Im}(\overline{a}\Delta a)\right) f'(|a|^2)\\ 0\end{pmatrix}.$$

The explicit formula for $[A, B]$ follows as in [9]. The estimates then follow directly from (3.5) and (3.1). □

We have the explicit formula

$$\mathcal{Y}^t_\varepsilon\begin{pmatrix}\phi\\ a\end{pmatrix} = \mathcal{E}_B\left(t, \begin{pmatrix}\phi\\ a\end{pmatrix}\right) = \begin{pmatrix}\phi - tf(|a|^2)\\ a\end{pmatrix}, \tag{6.2}$$

and we readily infer

$$\partial_2\mathcal{E}_B\left(t, \begin{pmatrix}\phi\\ a\end{pmatrix}\right)\begin{pmatrix}\varphi\\ b\end{pmatrix} = \begin{pmatrix}\varphi - 2\sigma\lambda t|a|^{2\sigma-2}\operatorname{Re}(\overline{a}b)\\ b\end{pmatrix}. \tag{6.3}$$

Finally, we compute that

$$\begin{pmatrix}\varphi(t)\\ b(t)\end{pmatrix} = \partial_2\mathcal{E}_F\left(t, \begin{pmatrix}\phi_0\\ a_0\end{pmatrix}\right)\begin{pmatrix}\varphi_0\\ b_0\end{pmatrix}$$

solves the system

$$\begin{cases} \partial_t \varphi + \nabla\phi \cdot \nabla\varphi + 2\sigma\lambda |a|^{2\sigma-2}\operatorname{Re}(\overline{a}b) = 0; \quad \varphi_{|t=0} = \varphi_0, \\ \partial_t b + \nabla\phi \cdot \nabla b + \nabla\varphi \cdot \nabla a + \frac{1}{2}\left(b\Delta\phi + a\Delta\varphi\right) = i\frac{\varepsilon}{2}\Delta b; \quad b_{|t=0} = b_0, \end{cases} \tag{6.4}$$

where $\begin{pmatrix}\phi(t)\\ a(t)\end{pmatrix} = \mathcal{E}_F\left(t, \begin{pmatrix}\phi_0\\ a_0\end{pmatrix}\right)$.

Lemma 6.3 *Let $\ell > d/2+1$, $s \geq \ell$ and $(\varphi_0, b_0) \in \mathcal{H}^{\ell+1}_{M_0} \times \mathcal{H}^{\ell}_{M_0}$. Assume that $(\phi, a) \in C([0,T], \mathcal{H}^{s+1}_\rho \times \mathcal{H}^s_\rho) \cap L^2([0,T], \mathcal{H}^{s+3/2}_\rho \times \mathcal{H}^{s+1/2}_\rho)$. Then for M sufficiently large and $T < M_0/M$, the solution to (6.4) satisfies*

$$|||\varphi|||^2_{\ell+1,T} + |||b|||^2_{\ell,T} \leq 4\|\varphi_0\|^2_{\mathcal{H}^{\ell+1}_{M_0}} + 4\|b_0\|^2_{\mathcal{H}^{\ell}_{M_0}}.$$

Proof The proof is quite similar to the one of Lemma 3.3 and Proposition 4.1. We take the $\mathcal{H}^{\ell+1}_\rho$ scalar product of the first equation in (6.4) with φ, and the $\mathcal{H}^\ell_\rho$ scalar product of the second one with b. We get

$$\begin{aligned} \partial_t \|\varphi\|^2_{\mathcal{H}^{\ell+1}_\rho} + 2M\|\varphi\|^2_{\mathcal{H}^{\ell+3/2}_\rho} &\leq C\|\varphi\|_{\mathcal{H}^{\ell+3/2}_\rho}\Big(\|\nabla\phi\cdot\nabla\varphi\|_{\mathcal{H}^{\ell+1/2}_\rho} \\ &\quad + \||a|^{2\sigma-2}\operatorname{Re}(\overline{a}b)\|_{\mathcal{H}^{\ell+1/2}_\rho}\Big), \\ \partial_t \|b\|^2_{\mathcal{H}^{\ell}_\rho} + 2M\|b\|^2_{\mathcal{H}^{\ell+1/2}_\rho} &\leq C\|b\|_{\mathcal{H}^{\ell+1/2}_\rho}\Big(\|\nabla\phi\cdot\nabla b\|_{\mathcal{H}^{\ell-1/2}_\rho} + \|\nabla\varphi\cdot\nabla a\|_{\mathcal{H}^{\ell-1/2}_\rho} \\ &\quad + \|b\Delta\phi\|_{\mathcal{H}^{\ell-1/2}_\rho} + \|a\Delta\varphi\|_{\mathcal{H}^{\ell-1/2}_\rho}\Big). \end{aligned}$$

Then, the use of (3.5) with $m > d/2$ and integration in time yield, with estimates similar to those presented in the proof of Proposition 4.1,

$$\begin{aligned} &\|\varphi(t)\|^2_{\mathcal{H}^{\ell+1}_{\rho(t)}} + 2M\int_0^t \|\varphi(\tau)\|^2_{\mathcal{H}^{\ell+3/2}_{\rho(\tau)}}d\tau \leq \|\varphi_0\|^2_{\mathcal{H}^{\ell+1}_{M_0}} \\ &\quad + C\int_0^t \|\varphi(\tau)\|_{\mathcal{H}^{\ell+3/2}_{\rho(\tau)}}\|\phi(\tau)\|_{\mathcal{H}^{\ell+3/2}_{\rho(\tau)}}\|\varphi(\tau)\|_{\mathcal{H}^{m+1}_{\rho(\tau)}}d\tau \\ &\quad + C\int_0^t \|\varphi(\tau)\|^2_{\mathcal{H}^{\ell+3/2}_{\rho(\tau)}}\|\phi(\tau)\|_{\mathcal{H}^{m+1}_{\rho(\tau)}}d\tau \\ &\quad + C\int_0^t \|\varphi(\tau)\|_{\mathcal{H}^{\ell+3/2}_{\rho(\tau)}}\|a(\tau)\|_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}}\|a(\tau)\|^{2\sigma-2}_{\mathcal{H}^{m}_{\rho(\tau)}}\|b(\tau)\|_{\mathcal{H}^{m}_{\rho(\tau)}}d\tau \\ &\quad + C\int_0^t \|\varphi(\tau)\|_{\mathcal{H}^{\ell+3/2}_{\rho(\tau)}}\|b(\tau)\|_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}}\|a(\tau)\|^{2\sigma-1}_{\mathcal{H}^{m}_{\rho(\tau)}}d\tau, \end{aligned}$$

$$
\begin{aligned}
&\|b(t)\|^2_{\mathcal{H}^\ell_{\rho(t)}} + 2M\int_0^t \|b(\tau)\|^2_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}} d\tau \le \|b_0\|^2_{\mathcal{H}^\ell_{M_0}} \\
&\quad + C\int_0^t \|b(\tau)\|_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}} \|\phi(\tau)\|_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}} \|b(\tau)\|_{\mathcal{H}^{m+1}_{\rho(\tau)}} d\tau \\
&\quad + C\int_0^t \|b(\tau)\|^2_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}} \|\phi(\tau)\|_{\mathcal{H}^{m+1}_{\rho(\tau)}} d\tau \\
&\quad + C\int_0^t \|b(\tau)\|_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}} \|\varphi(\tau)\|_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}} \|a(\tau)\|_{\mathcal{H}^{m+1}_{\rho(\tau)}} d\tau \\
&\quad + C\int_0^t \|b(\tau)\|_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}} \|a(\tau)\|_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}} \|\varphi(\tau)\|_{\mathcal{H}^{m+1}_{\rho(\tau)}} d\tau \\
&\quad + C\int_0^t \|b(\tau)\|_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}} \|b(\tau)\|_{\mathcal{H}^{\ell-1/2}_{\rho(\tau)}} \|\phi(\tau)\|_{\mathcal{H}^{m+2}_{\rho(\tau)}} d\tau \\
&\quad + C\int_0^t \|b(\tau)\|_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}} \|\phi(\tau)\|_{\mathcal{H}^{\ell+3/2}_{\rho(\tau)}} \|b(\tau)\|_{\mathcal{H}^{m}_{\rho(\tau)}} d\tau \\
&\quad + C\int_0^t \|b(\tau)\|_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}} \|a(\tau)\|_{\mathcal{H}^{\ell-1/2}_{\rho(\tau)}} \|\varphi(\tau)\|_{\mathcal{H}^{m+2}_{\rho(\tau)}} d\tau \\
&\quad + C\int_0^t \|b(\tau)\|_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}} \|\varphi(\tau)\|_{\mathcal{H}^{\ell+3/2}_{\rho(\tau)}} \|a(\tau)\|_{\mathcal{H}^{m}_{\rho(\tau)}} d\tau
\end{aligned}
$$

We choose $m = \ell > d/2 + 1$ in the estimate for φ and $m = \ell - 1 > d/2$ in the estimate for b. Denoting

$$
\mu = |||\phi|||_{\ell+1,T} + |||a|||_{\ell,T},
$$

and

$$
\|\psi\|^2_{L^2_t \mathcal{H}^k_\rho} = \int_0^t \|\psi(\tau)\|^2_{\mathcal{H}^k_{\rho(\tau)}} d\tau,
$$

since the $\mathcal{H}^k_\rho$ norms are increasing with k, Cauchy–Schwarz in time yields

$$
\begin{aligned}
&\|\varphi(t)\|^2_{\mathcal{H}^{\ell+1}_{\rho(t)}} + 2M\|\varphi\|^2_{L^2_t \mathcal{H}^{\ell+3/2}_\rho} \le \|\varphi_0\|^2_{\mathcal{H}^{\ell+1}_{M_0}} + C\|\varphi\|_{L^2_t \mathcal{H}^{\ell+3/2}_\rho} \\
&\quad \times \left(\frac{\mu}{\sqrt{M}} \sup_{0\le\tau\le t} \|\varphi(\tau)\|_{\mathcal{H}^{\ell+1}_{\rho(\tau)}} + \mu\|\varphi\|_{L^2_t \mathcal{H}^{\ell+3/2}_\rho} + \frac{\mu^{2\sigma-1}}{\sqrt{M}} \sup_{0\le\tau\le t} \|b(\tau)\|_{\mathcal{H}^\ell_{\rho(\tau)}} + \mu^{2\sigma-1}\|b\|_{L^2_t \mathcal{H}^{\ell+1/2}_\rho} \right),
\end{aligned}
$$

$$
\begin{aligned}
&\|b(t)\|^2_{\mathcal{H}^\ell_{\rho(t)}} + 2M\|b\|^2_{L^2_t \mathcal{H}^{\ell+1/2}_\rho} \le \|b_0\|^2_{\mathcal{H}^\ell_{M_0}} + C\mu\|b\|_{L^2_t \mathcal{H}^{\ell+1/2}_\rho} \\
&\quad \times \left(\frac{1}{\sqrt{M}} \sup_{0\le\tau\le t} \|b(\tau)\|_{\mathcal{H}^\ell_{\rho(\tau)}} + \|b\|_{L^2_t \mathcal{H}^{\ell+1/2}_\rho} + \|\varphi\|_{L^2_t \mathcal{H}^{\ell+3/2}_\rho} \right. \\
&\quad \left. + \frac{1}{\sqrt{M}} \sup_{0\le\tau\le t} \|\varphi(\tau)\|_{\mathcal{H}^{\ell+1}_{\rho(\tau)}} \right).
\end{aligned}
$$

Adding the last two inequalities, we deduce, for $\mu \leq C$,

$$\begin{aligned}
&\|\varphi(t)\|^2_{\mathcal{H}^{\ell+1}_{\rho(t)}} + \|b(t)\|^2_{\mathcal{H}^{\ell}_{\rho(t)}} + 2M\|\varphi\|^2_{L^2_t\mathcal{H}^{\ell+3/2}_\rho} + 2M\|b\|^2_{L^2_t\mathcal{H}^{\ell+1/2}_\rho} \\
&\leq \|\varphi_0\|^2_{\mathcal{H}^{\ell+1}_{M_0}} + \|b_0\|^2_{\mathcal{H}^{\ell}_{M_0}} \\
&+ C\frac{\mu}{M}\left(\sup_{0\leq\tau\leq t}\|\varphi(\tau)\|^2_{\mathcal{H}^{\ell+1}_{\rho(\tau)}} + \sup_{0\leq\tau\leq t}\|b(\tau)\|^2_{\mathcal{H}^{\ell}_{\rho(\tau)}}\right. \\
&\left. + M\|\varphi\|^2_{L^2_t\mathcal{H}^{\ell+3/2}_\rho} + M\|b\|^2_{L^2_t\mathcal{H}^{\ell+1/2}_\rho}\right),
\end{aligned}$$

from which the inequality of the lemma easily follows provided M is sufficiently large. □

We infer the WKB local error estimate:

Theorem 6.4 (Local error estimate for WKB states) *Let $\ell > d/2 + 5$, $M_0 > 0$, $M \gg 1$, $\rho(t) = M_0 - Mt$ and $\mu > 0$. Let $(\phi_0, a_0) \in \mathcal{H}^{\ell+1}_{M_0} \times \mathcal{H}^{\ell}_{M_0}$ such that*

$$\|\phi_0\|_{\mathcal{H}^{\ell+1}_{M_0}} \leq \mu, \quad \|a_0\|_{\mathcal{H}^{\ell}_{M_0}} \leq \mu.$$

There exist $C, c_0 > 0$ (depending on μ) independent of $\varepsilon \in (0, 1]$ such that

$$\mathcal{L}\left(t, \begin{pmatrix}\phi_0\\ a_0\end{pmatrix}\right) := \mathcal{Z}^t_\varepsilon\begin{pmatrix}\phi_0\\ a_0\end{pmatrix} - \mathcal{S}^t_\varepsilon\begin{pmatrix}\phi_0\\ a_0\end{pmatrix} = \begin{pmatrix}\Psi^\varepsilon(t)\\ A^\varepsilon(t)\end{pmatrix},$$

satisfies

$$\|\Psi^\varepsilon(t)\|_{\mathcal{H}^{\ell-3}_{\rho(t)}} + \|A^\varepsilon(t)\|_{\mathcal{H}^{\ell-4}_{\rho(t)}} \leq Ct^2, \quad 0 \leq t \leq c_0.$$

The above result obviously involves a loss of regularity, between the initial assumptions and the conclusion. It is important to note that the local error estimate is used only once in the final Lady Windermere's fan argument presented in the next section, so this loss is not a serious problem.

Proof Let $t \in [0, c_0]$, and fix τ_1, τ_2 such that $0 \leq \tau_2 \leq \tau_1 \leq t$. Introduce the following intermediary notations:

$$\begin{aligned}
&\begin{pmatrix}\phi_1\\ a_1^\varepsilon\end{pmatrix} = \mathcal{E}_A\left(\tau_1, \begin{pmatrix}\phi_0\\ a_0\end{pmatrix}\right), \\
&\begin{pmatrix}\phi_2^\varepsilon\\ a_2^\varepsilon\end{pmatrix} = \mathcal{E}_B\left(\tau_2, \begin{pmatrix}\phi_1\\ a_1^\varepsilon\end{pmatrix}\right), \qquad \begin{pmatrix}\tilde\phi_2^\varepsilon\\ \tilde a_2^\varepsilon\end{pmatrix} = \mathcal{E}_B\left(\tau_1, \begin{pmatrix}\phi_1\\ a_1^\varepsilon\end{pmatrix}\right) \\
&\begin{pmatrix}\phi_3^\varepsilon\\ a_3^\varepsilon\end{pmatrix} = [B, A]\begin{pmatrix}\phi_2^\varepsilon\\ a_2^\varepsilon\end{pmatrix}, \qquad \begin{pmatrix}\phi_4^\varepsilon\\ a_4^\varepsilon\end{pmatrix} = \partial_2\mathcal{E}_B\left(\tau_1 - \tau_2, \begin{pmatrix}\phi_1\\ a_1^\varepsilon\end{pmatrix}\right)\begin{pmatrix}\phi_3^\varepsilon\\ a_3^\varepsilon\end{pmatrix}.
\end{aligned}$$

Then in view of Theorem 6.1, we have

$$\begin{pmatrix}\Psi^\varepsilon(t)\\ A^\varepsilon(t)\end{pmatrix} = \int_0^t\int_0^{\tau_1} \partial_2\mathcal{E}_F\left(t-\tau_1, \begin{pmatrix}\tilde{\phi}_2^\varepsilon\\ \tilde{a}_2^\varepsilon\end{pmatrix}\right)\begin{pmatrix}\phi_4^\varepsilon\\ a_4^\varepsilon\end{pmatrix} d\tau_2 d\tau_1.$$

Since $\ell > d/2+1$, Proposition 4.1 for $\lambda = 0$ ensures that $(\phi_1, a_1^\varepsilon) \in \mathcal{H}^{\ell+1}_{\rho(\tau_1)} \times \mathcal{H}^{\ell}_{\rho(\tau_1)}$ is well defined provided $\tau_1 \leq c_0 < M_0/M$, with (according to (4.2) where we can remove the $\|a_0\|^{4\sigma}_{\mathcal{H}^\ell_{M_0}}$ term because $\lambda = 0$)

$$\|\phi_1\|_{\mathcal{H}^{\ell+1}_{\rho(\tau_1)}} \leq 2\mu, \quad \|a_1^\varepsilon\|_{\mathcal{H}^{\ell}_{\rho(\tau_1)}} \leq 2\mu.$$

(6.2) writes $(\phi_2^\varepsilon, a_2^\varepsilon) = (\phi_1^\varepsilon - \lambda\tau_2|a_1^\varepsilon|^{2\sigma}, a_1^\varepsilon)$ and thus (3.5) yields (in the calculations below, the constant C may depend on μ and may change from line to line)

$$\|\phi_2^\varepsilon\|_{\mathcal{H}^{\ell}_{\rho(\tau_1)}} \leq 2\mu + C\mu^{2\sigma} \leq C\mu, \quad \|a_2^\varepsilon\|_{\mathcal{H}^{\ell}_{\rho(\tau_1)}} \leq 2\mu,$$

because $\ell > d/2$. Similarly,

$$\|\tilde{\phi}_2^\varepsilon\|_{\mathcal{H}^{\ell}_{\rho(\tau_1)}} \leq C\mu, \quad \|\tilde{a}_2^\varepsilon\|_{\mathcal{H}^{\ell}_{\rho(\tau_1)}} \leq 2\mu. \tag{6.5}$$

Next, since $\ell - 1 > d/2+3$, Lemma 6.2 implies

$$\|\phi_3^\varepsilon\|_{\mathcal{H}^{\ell-3}_{\rho(\tau_1)}} \leq C\mu, \quad \|a_3^\varepsilon\|_{\mathcal{H}^{\ell-4}_{\rho(\tau_1)}} \leq C\mu.$$

In view of (6.3), we have

$$\phi_4^\varepsilon = \phi_3^\varepsilon - 2\sigma\lambda(\tau_1-\tau_2)|a_1^\varepsilon|^{2\sigma-2}\,\mathrm{Re}\left(\overline{a}_1^\varepsilon a_3^\varepsilon\right), \quad a_4^\varepsilon = a_3^\varepsilon,$$

and therefore

$$\|\phi_4^\varepsilon\|_{\mathcal{H}^{\ell-3}_{\rho(\tau_1)}} \leq C\mu, \quad \|a_4^\varepsilon\|_{\mathcal{H}^{\ell-4}_{\rho(\tau_1)}} \leq C\mu, \tag{6.6}$$

since $\ell - 3 > d/2$ and thanks to (3.5).

Finally, we prove that if $\ell > d/2+5$, the $\mathcal{H}^{\ell-3}_{\rho(t)} \times \mathcal{H}^{\ell-4}_{\rho(t)}$ norm of

$$\begin{pmatrix}\phi_5^\varepsilon\\ a_5^\varepsilon\end{pmatrix} = \partial_2\mathcal{E}_F\left(t-\tau_1, \begin{pmatrix}\tilde{\phi}_2^\varepsilon\\ \tilde{a}_2^\varepsilon\end{pmatrix}\right)\begin{pmatrix}\phi_4^\varepsilon\\ a_4^\varepsilon\end{pmatrix}$$

is uniformly bounded in t, τ_1, τ_2 as long as $0 \leq \tau_2 \leq \tau_1 \leq t \leq T < M_0/M$. For this purpose, first note that since $\ell-1 > d/2+1$, it follows from (6.5) and Proposition 4.1 that we can choose $M = M(\mu)$ sufficiently large such that if $0 < T - \tau_1 < \rho(\tau_1)/M$,

$$\begin{pmatrix}\phi\\ a\end{pmatrix}(\tau)=\mathcal{E}_F\left(\tau-\tau_1,\begin{pmatrix}\tilde{\phi}_2^\varepsilon\\ \tilde{a}_2^\varepsilon\end{pmatrix}\right)\text{ is such that}$$

$$\begin{pmatrix}\phi\\ a\end{pmatrix}\in C\left([\tau_1,T],\mathcal{H}_\rho^\ell\times\mathcal{H}_\rho^{\ell-1}\right)\cap L^2\left([\tau_1,T],\mathcal{H}_\rho^{\ell+1/2}\times\mathcal{H}_\rho^{\ell-1/2}\right),$$

with

$$\max\Bigg(\sup_{\tau_1\le\tau\le T}\|\phi(\tau)\|^2_{\mathcal{H}^\ell_{\rho(\tau)}},\ \sup_{\tau_1\le\tau\le T}\|a(\tau)\|^2_{\mathcal{H}^{\ell-1}_{\rho(\tau)}},$$
$$2M\int_{\tau_1}^T\|\phi(\tau)\|^2_{\mathcal{H}^{\ell+1/2}_{\rho(\tau)}}d\tau,\ 2M\int_{\tau_1}^T\|a(\tau)\|^2_{\mathcal{H}^{\ell-1/2}_{\rho(\tau)}}d\tau\Bigg)\le C(\mu+\mu^{2\sigma}).$$

(Note that $\rho(\tau)=\rho(\tau_1)-M(\tau-\tau_1)$). Then, thanks to (6.6) and Lemma 6.3, since $\ell-4>d/2+1$ and $s=\ell-1\ge\ell-4$, choosing possibly $M=M(\mu)$ even larger,

$$\max\left(\|\phi_5^\varepsilon\|_{\mathcal{H}^{\ell-3}_{\rho(t)}},\|a_5^\varepsilon\|_{\mathcal{H}^{\ell-4}_{\rho(t)}}\right)\le C\mu.$$

The theorem follows. □

Back to the wave functions, we obtain an estimate similar to the one presented in [15, Section 4.2.2]:

Corollary 6.5 *Under the assumptions of Theorem 6.4, denoting*

$$\begin{pmatrix}\phi_t^\varepsilon\\ a_t^\varepsilon\end{pmatrix}=\mathcal{Z}_\varepsilon^t\begin{pmatrix}\phi_0\\ a_0\end{pmatrix},\quad\begin{pmatrix}\phi^\varepsilon(t)\\ a^\varepsilon(t)\end{pmatrix}=\mathcal{S}_\varepsilon^t\begin{pmatrix}\phi_0\\ a_0\end{pmatrix},$$

there exist $C,c_0>0$ (depending on μ) independent of $\varepsilon\in(0,1]$ such that

$$\left\|a_t^\varepsilon e^{i\phi_t^\varepsilon/\varepsilon}-a^\varepsilon(t)e^{i\phi^\varepsilon(t)/\varepsilon}\right\|_{L^2}\le C\frac{t^2}{\varepsilon},\quad 0\le t\le c_0.$$

Proof With the same notations as in Theorem 6.4, the Sobolev embedding of $\mathcal{H}^{\ell-4}_{\rho(t)}$ into L^∞ ($\ell>d/2+5$) ensures

$$\begin{aligned}\left\|a_t^\varepsilon e^{i\phi_t^\varepsilon/\varepsilon}-a^\varepsilon(t)e^{i\phi^\varepsilon(t)/\varepsilon}\right\|_{L^2}&\le\left\|a_t^\varepsilon-a^\varepsilon(t)\right\|_{L^2}+\left\|a^\varepsilon(t)\left(e^{i\phi_t^\varepsilon/\varepsilon}-e^{i\phi^\varepsilon(t)/\varepsilon}\right)\right\|_{L^2}\\&\le\left\|A^\varepsilon(t)\right\|_{L^2}+\frac{1}{\varepsilon}\left\|a^\varepsilon(t)\right\|_{L^\infty}\left\|\Psi^\varepsilon(t)\right\|_{L^2}\le\frac{Ct^2}{\varepsilon}.\end{aligned}$$

□

This result will not be used in the sequel, but shows how a $1/\varepsilon$ factor appears when going back to the wave function, in agreement with the observations in [4]. The above computation also shows how to infer the first point in Corollary 1.3 from Theorem 1.1.

7 Lady Windermere's fan

Let $M_0 > 0$, $\ell > d/2 + 5$, and $v_0 = (\phi_0, a_0) \in \mathcal{H}^{\ell+1}_{M_0} \times \mathcal{H}^{\ell}_{M_0}$. For the sake of conciseness, we use the following notations: for $t > 0$, $n \in \mathbb{N}$ and $\Delta t > 0$,

$$v^\varepsilon(t) = (\phi^\varepsilon(t), a^\varepsilon(t)) = \mathcal{S}^t_\varepsilon v_0, \quad v^\varepsilon_n = (\phi^\varepsilon_n, a^\varepsilon_n) = \left(\mathcal{Z}^{\Delta t}_\varepsilon\right)^n v_0.$$

For $\rho > 0$ and $v = (\phi, a) \in \mathcal{H}^{\ell+1}_\rho \times \mathcal{H}^\ell_\rho$, we also denote

$$\|v\|_{\rho,\ell} = \|\phi\|_{\mathcal{H}^{\ell+1}_\rho} + \|a\|_{\mathcal{H}^\ell_\rho}.$$

According to Proposition 4.1, if $M > 0$ is sufficiently large, $T < M_0/M$ and $\rho(t) = M_0 - Mt$, (2.2)–(2.3) has a unique solution $v^\varepsilon \in C([0, T], \mathcal{H}^{\ell+1}_\rho \times \mathcal{H}^\ell_\rho)$, with

$$\sup_{0\le t\le T} \|v^\varepsilon(t)\|_{\rho(t),\ell} \le R,$$

where $R = 2\|v_0\|_{M_0,\ell}$.

We recall the notation $t_n = n\Delta t$, and we set $\rho_n = \rho(t_n)$. We now prove by induction on n that there exists $c_0 > 0$ such that if $\Delta t \in (0, c_0]$, for every $n \ge 0$ such that $n\Delta t \le T$, we have

$$\|v^\varepsilon_n\|_{\rho_n,\ell-4} \le R + \delta, \tag{7.1}$$

$$\|v^\varepsilon_n - v^\varepsilon(t_n)\|_{\rho_n,\ell-4} \le \gamma\Delta t, \tag{7.2}$$

$$\|v^\varepsilon_n\|_{\rho_n,\ell} \le R/2, \tag{7.3}$$

for some $\delta, \gamma > 0$ that will be given later. $(7.1)_n$–$(7.2)_n$–$(7.3)_n$ obviously hold for $n = 0$. Let $n > 0$ such that $n\Delta t \le T$ and assume that $(7.1)_j$–$(7.2)_j$–$(7.3)_j$ hold for all $j \in \{0, \dots, n-1\}$. Then, for all $j \in \{0, \dots, n-2\}$, $(7.1)_{j+1}$ yields

$$\left\|\mathcal{Z}^{\Delta t}_\varepsilon v^\varepsilon_j\right\|_{\rho_{j+1},\ell-4} = \left\|v^\varepsilon_{j+1}\right\|_{\rho_{j+1},\ell-4} \le R + \delta. \tag{7.4}$$

On the other hand, for $j \in \{0, \dots, n-2\}$, we also have

$$\begin{aligned}\left\|\mathcal{S}^{\Delta t}_\varepsilon v^\varepsilon_j\right\|_{\rho_{j+1},\ell-4} &\le \left\|\mathcal{S}^{\Delta t}_\varepsilon v^\varepsilon_j - \mathcal{S}^{\Delta t}_\varepsilon v^\varepsilon(t_j)\right\|_{\rho_{j+1},\ell-4} + \left\|v^\varepsilon(t_{j+1})\right\|_{\rho_{j+1},\ell-4} \\ &\le \left\|\mathcal{S}^{\Delta t}_\varepsilon v^\varepsilon_j - \mathcal{S}^{\Delta t}_\varepsilon v^\varepsilon(t_j)\right\|_{\rho_{j+1},\ell-4} + R.\end{aligned}$$

From $(7.3)_j$, $\|v^\varepsilon_j\|_{\rho_j,\ell-4} \le R/2$, whereas $\|v^\varepsilon(t_j)\|_{\rho_j,\ell-4} \le R$ by choice of R. Thus, since $\ell - 4 > d/2 + 1$, Proposition 4.1 and $(7.2)_j$ imply (up to increasing M)

$$\left\|\mathcal{S}^{\Delta t}_\varepsilon v^\varepsilon_j - \mathcal{S}^{\Delta t}_\varepsilon v^\varepsilon(t_j)\right\|_{\rho_{j+1},\ell-4} \le K(R)\gamma\Delta t.$$

Therefore, if $c_0 > 0$ is chosen sufficiently small such that $K(R)\gamma c_0 \le \delta$, we have

$$\left\| \mathcal{S}_\varepsilon^{\Delta t} v_j^\varepsilon \right\|_{\rho_{j+1},\ell-4} \le R + \delta, \tag{7.5}$$

and (7.4), (7.5) and Proposition 4.1 ensure that for all $j \in \{0, \dots, n-2\}$,

$$\begin{aligned} &\left\| \left(\mathcal{S}_\varepsilon^{\Delta t}\right)^{n-1-j} \mathcal{Z}_\varepsilon^{\Delta t} v_j^\varepsilon - \left(\mathcal{S}_\varepsilon^{\Delta t}\right)^{n-1-j} \mathcal{S}_\varepsilon^{\Delta t} v_j^\varepsilon \right\|_{\rho_n,\ell-4} \\ &\quad \le K(R+\delta) \left\| \mathcal{Z}_\varepsilon^{\Delta t} v_j^\varepsilon - \mathcal{S}_\varepsilon^{\Delta t} v_j^\varepsilon \right\|_{\rho_{j+1},\ell-4}. \end{aligned}$$

Moreover, the last estimate also holds for $j = n-1$ if K is replaced by 1. According to $(7.3)_j$ and Theorem 6.4, we deduce that for all $j \in \{0, \dots, n-1\}$,

$$\begin{aligned} &\left\| \left(\mathcal{S}_\varepsilon^{\Delta t}\right)^{n-1-j} \mathcal{Z}_\varepsilon^{\Delta t} v_j^\varepsilon - \left(\mathcal{S}_\varepsilon^{\Delta t}\right)^{n-1-j} \mathcal{S}_\varepsilon^{\Delta t} v_j^\varepsilon \right\|_{\rho_n,\ell-4} \\ &\quad \le \max(1, K(R+\delta)) C(R/2) \Delta t^2. \end{aligned} \tag{7.6}$$

Piling up the last inequality for $j \in \{0, \dots, n-1\}$, we conclude

$$\begin{aligned} \|v_n^\varepsilon - v^\varepsilon(t_n)\|_{\rho_n,\ell-4} &\le \sum_{j=0}^{n-1} \left\| \left(\mathcal{S}_\varepsilon^{\Delta t}\right)^{n-1-j} \mathcal{Z}_\varepsilon^{\Delta t} v_j^\varepsilon - \left(\mathcal{S}_\varepsilon^{\Delta t}\right)^{n-1-j} \mathcal{S}_\varepsilon^{\Delta t} v_j^\varepsilon \right\|_{\rho_n,\ell-4} \\ &\le n \max(1, K(R+\delta)) C(R/2) \Delta t^2 \\ &\le \max(1, K(R+\delta)) C(R/2) T \Delta t, \end{aligned}$$

which proves 7.2_n with $\gamma = \max(1, K(R+\delta))C(R/2)T$. Then, $(7.2)_n$ yields

$$\|v_n^\varepsilon\|_{\rho_n,\ell-4} \le \|v_n^\varepsilon - v^\varepsilon(t_n)\|_{\rho_n,\ell-4} + \|v^\varepsilon(t_n)\|_{\rho_n,\ell-4} \le \gamma \Delta t + R. \tag{7.7}$$

Note that it does not prove $(7.1)_n$ yet, because the choice of $\delta = \gamma c_0$ may be incompatible with the previous constraint $K(R)\gamma c_0 \le \delta$. However, $(7.3)_n$ follows from (7.7) and Corollary 5.2, once we have noticed that the proof of (7.7) also works if $v_n^\varepsilon = \mathcal{Z}_\varepsilon^{\Delta t} v_{n-1}^\varepsilon$ is replaced by $\mathcal{Z}_\varepsilon^t v_{n-1}^\varepsilon$ (and t_n by $t_{n-1} + t$), for any $0 \le t \le \Delta t$, so that

$$\begin{aligned} &\mathcal{Z}_\varepsilon^t \left(\mathcal{Z}_\varepsilon^{\Delta t}\right)^{n-1} v_0 - \mathcal{S}_\varepsilon^{t+(n-1)\Delta t} v_0 \\ &\quad = \mathcal{Z}_\varepsilon^t v_{n-1}^\varepsilon - \mathcal{S}_\varepsilon^t v_{n-1}^\varepsilon + \sum_{j=0}^{n-2} \left[\mathcal{S}_\varepsilon^t \left(\mathcal{S}_\varepsilon^{\Delta t}\right)^{n-2-j} \mathcal{Z}_\varepsilon^{\Delta t} v_j^\varepsilon - \mathcal{S}_\varepsilon^t \left(\mathcal{S}_\varepsilon^{\Delta t}\right)^{n-2-j} \mathcal{S}_\varepsilon^{\Delta t} v_j^\varepsilon \right]. \end{aligned}$$

Then, $(7.1)_n$ follows from $(7.3)_n$, and any positive value for δ is admissible.

References

1. Alazard, T., Carles, R.: Supercritical geometric optics for nonlinear Schrödinger equations. Arch. Ration. Mech. Anal. **194**, 315–347 (2009)
2. Alinhac, S., Gérard, P.: Pseudo-differential operators and the Nash-Moser theorem, vol. 82 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI. Translated from the 1991 French original by Stephen S. Wilson (2007)
3. Bao, W., Jin, S., Markowich, P.A.: On time-splitting spectral approximations for the Schrödinger equation in the semiclassical regime. J. Comput. Phys. **175**, 487–524 (2002)
4. Bao, W., Jin, S., Markowich, P.A.: Numerical study of time-splitting spectral discretizations of nonlinear Schrödinger equations in the semiclassical regimes. SIAM J. Sci. Comput. **25**, 27–64 (2003)
5. Benettin, G., Giorgilli, A.: On the Hamiltonian interpolation of near-to-the-identity symplectic mappings with application to symplectic integration algorithms. J. Stat. Phys. **74**, 1117–1143 (1994)
6. Besse, C., Bidégaray, B., Descombes, S.: Order estimates in time of splitting methods for the nonlinear Schrödinger equation. SIAM J. Numer. Anal. **40**, 26–40 (2002)
7. Besse, C., Carles, R., Méhats, F.: An asymptotic preserving scheme based on a new formulation for NLS in the semiclassical limit. Multiscale Model. Simul. **11**, 1228–1260 (2013)
8. Carles, R.: Semi-classical analysis for nonlinear Schrödinger equations. World Scientific Publishing Co Pte. Ltd., Hackensack (2008)
9. Carles, R.: On Fourier time-splitting methods for nonlinear Schrödinger equations in the semi-classical limit. SIAM J. Numer. Anal. **51**, 3232–3258 (2013)
10. Castella, F., Chartier, P., Méhats, F., Murua, A.: Stroboscopic averaging for the nonlinear Schrödinger equation. Found. Comput. Math. **15**, 519–559 (2015)
11. Cazenave, T., Haraux, A.: An introduction to semilinear evolution equations, vol. 13 of Oxford Lecture Series in Mathematics and its Applications, The Clarendon Press Oxford University Press, New York, 1998. Translated from the 1990 French original by Yvan Martel and revised by the authors (1998)
12. Chemin, J.-Y.: Le système de Navier-Stokes incompressible soixante dix ans après Jean Leray. In: Actes des Journées Mathématiques à la Mémoire de Jean Leray, vol. 9 of Sémin. Congr., Soc. Math. pp. 99–123. France, Paris (2004)
13. Chiron, D., Rousset, F.: Geometric optics and boundary layers for nonlinear Schrödinger equations. Commun. Math. Phys. **288**, 503–546 (2009)
14. Descombes, S., Thalhammer, M.: An exact local error representation of exponential operator splitting methods for evolutionary problems and applications to linear Schrödinger equations in the semi-classical regime. BIT **50**, 729–749 (2010)
15. Descombes, S., Thalhammer, M.: The Lie-Trotter splitting for nonlinear evolutionary problems with critical parameters. A compact local error representation and application to nonlinear Schrödinger equations in the semi-classical regime. IMA J. Numer. Anal. **33**, 722–745 (2013)
16. Faou, E.: Geometric numerical integration and Schrödinger equations. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich (2012)
17. Faou, E., Grébert, B.: Hamiltonian interpolation of splitting approximations for nonlinear PDEs. Found. Comput. Math. **11**, 381–415 (2011)
18. Gérard, P.: Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire, In: Séminaire sur les Équations aux Dérivées Partielles, 1992–1993, École; Polytech., Palaiseau, pp. Exp. No. XIII, 13 (1993). http://www.numdam.org
19. Ginibre, J., Velo, G.: Long range scattering and modified wave operators for some Hartree type equations. III. Gevrey spaces and low dimensions. J. Differ. Equ. **175**, 415–501 (2001)
20. Grenier, E.: Semiclassical limit of the nonlinear Schrödinger equation in small time. Proc. Am. Math. Soc. **126**, 523–530 (1998)
21. Hairer, E., Lubich, C.: Oscillations over long times in numerical Hamiltonian systems. In: Highly oscillatory problems, vol. 366 of London Math. Soc. Lecture Note Ser., pp. 1–24. Cambridge Univ. Press, Cambridge (2009)
22. Hairer, E., Lubich, C.: Modulated Fourier expansions for continuous and discrete oscillatory systems. In: Foundations of computational mathematics, Budapest, 2011, vol. 403 of London Math. Soc. Lecture Note Ser. pp. 113–128. Cambridge Univ. Press, Cambridge (2013)
23. Hairer, E., Lubich, C., Wanner, G.: Geometric numerical integration, vol. 31 of Springer Series in Computational Mathematics, Springer, Heidelberg, 2010. Structure-preserving algorithms for ordinary differential equations. Reprint of the second (2006) edition (2010)

24. Hairer, E., Nørsett, S.P., Wanner, G.: Solving ordinary differential equations. I, vol. 8 of Springer Series in Computational Mathematics, 2nd edn. Springer, Berlin (1993) **(Nonstiff problems)**
25. Holden, H., Lubich, C., Risebro, N.H.: Operator splitting for partial differential equations with Burgers nonlinearity. Math. Comp. **82**, 173–185 (2013)
26. Lubich, C.: On splitting methods for Schrödinger-Poisson and cubic nonlinear Schrödinger equations. Math. Comp. **77**, 2141–2153 (2008)
27. Mouhot, C., Villani, C.: On Landau damping. Acta Math. **207**, 29–201 (2011)
28. Thomann, L.: Instabilities for supercritical Schrödinger equations in analytic manifolds. J. Differ. Equ. **245**, 249–280 (2008)