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Ultraconvergence of high order FEMs for elliptic problems with variable coefficients

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Abstract In this paper, we investigate local ultraconvergence properties of the high-order finite element method (FEM) for second order elliptic problems with variable coefficients. Under suitable regularity and mesh conditions, we show that at an interior vertex, which is away from the boundary with a fixed distance, the gradient of the post-precessed kth ($k \geq 2$) order finite element solution converges to the gradient of the exact solution with order $\mathcal{O}(h^{k+2}(\ln h)^3)$. The proof of this ultraconvergence property depends on a new interpolating operator, some new estimates for the discrete Green's function, a symmetry theory derived in [26], and the Richardson extrapolation technique in [20]. Numerical experiments are performed to demonstrate our theoretical findings.

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1 Introduction

In this paper, we consider the ultraconvergence of FE approximation for the following elliptic problem

$$\begin{cases} \mathcal{L}u(y) \equiv -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial u}{\partial y_j}\right) = f(y) & \text{in} \quad \Omega, \\ u(y) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \Re^n$ ($n = 2, 3$) is a bounded polygon ($n = 2$) or polyhedron ($n = 3$), and $A = (a_{ij}) \in (C^\infty(\Omega))^{n^2}$ is a uniformly positive definite matrix in the sense that there exists $\delta > 0$ satisfying

$$a_{ij}\xi_i\xi_j \geq \delta\xi_i\xi_i, \xi \in \Re^n.$$

Note that throughout the paper, the Einstein convention is used: the summation will be taken over all repeated indices.

The study of optimal convergence and superconverence/ultraconvergence properties has been an area of active research, see [1,2,4–9,13–15] and [17–37] for an uncompleted list of references). For instances, Bank and Xu (see [5]) proved that, for linear finite element, the recovered gradient $Q_h\nabla u^h$ converges with order $O(h^{1+\min(1,\sigma)}|\log h|^{\frac{1}{2}})$, where Q_h is a global L^2 projection, the underlying mesh $\mathcal{T}_h$ is quasi-uniform and satisfies the so-called (α, σ)- parallelogram property. Huang and Xu (see [15]) obtained that, for second-degree finite element, the recovered gradient $Q_h\nabla u^h$ converges with order $O(h^{2+\min(1,\sigma)}|\log h|^{\frac{1}{2}})$.

It is known that when the underlying mesh has a certain *local symmetry property*, the corresponding finite element solution has some *natrual* superconvergence properties. For instances, Schatz, Sloan and Wahlbin discovered in [22,23,27] that at a *local symmetric vertex* x_0, when k is even, the finite element solution converges to the exact solution with order $\mathcal{O}(h^{k+2-\varepsilon})$, where $\varepsilon > 0$ can be arbitrary small; when k is odd, the discrete gradient of the finite element solution converges to the gradient of the exact solution with order $\mathcal{O}(h^{k+1-\varepsilon})$, where a vertex x_0 is called *local symmetric* if there exists some radius $d > 0$ such that the underlying mesh is symmetric in the neighborhood $B(x_0, d) = \{y\colon |x_0 - y| \leq d\}$. The natural superconvergence or even ultraconvergence of the finite element solution has been also investigated by other techniques such as the so-called *weak estimate*, see [6,14,18,19,36] for an uncompleted list of references.

To obtain better superconvergence/ultraconvergence result, one natural idea is to post-process the finite element solution on some local symmetric mesh, see e.g., [6,7,13,18–20] and references therein. Along this direction, Lin found in [19] that the gradient of a special interpolation of some odd order finite element solution on a uniform rectangular mesh converges with order $\mathcal{O}(h^{k+2}|\ln h|)$ at an interior vertex. Zhang et al. discovered in [30,33] that the recovered gradient (see e.g. SPR by [34,37] and PPR by [30]) of some even order finite element solution on some uniform rectangular meshes converges with order $\mathcal{O}(h^{k+2}|\ln h|)$ as well. M. Asadzadeh, A. Schatz, and W. Wendland showed that the discrete gradient of some extrapolated finite element solution superconverges with order $\mathcal{O}(h^{k+1}|\ln h|)$ under suitable local symmetric mesh.

All aforementioned ultraconvergence results are valid only for elliptic equations with constant coefficients. Recall that the classical superconvergence analysis for variable-coefficient problems is done by estimating the difference between the variable coefficient bilinear form and its corresponding (piecewise) constant coefficient bilinear form. Since this difference is only a one-order-higher term, we can not obtain ultraconvergence results of the corresponding finite element solution for variable-coefficient problems. In other words, we can not use this approach to prove the ultraconvergence of the post-processed FE solution for variable-coefficient problems.

In this paper, we propose a novel local interpolation operator to post-process finite element solutions for variable-coefficient problems. Unlike the classical interpolation operator defined in [19,20], our local operator interpolates the value of the original finite element solution at all vertices of the underlying mesh in a patch instead of interpolating all nodes in a relatively smaller-sized patch.

To prove the ultraconvergence property of our post-processed FE solution, we first investigate properties of the so-called *discrete Green's function* in the whole domain $\Re^n$. We found the difference between two errors, one is the FE projection error of the Green's function with variable coefficients, another is the FE projection error of the Green's function with constant coefficients by fixing one point value of the corresponding variable coefficients, is of order almost $O(h^2)$. Combining with some further nice properties for the FE projection error of the Green's function (see Theorem 2.1 for the details), we show that for even k and translation invariant mesh, between two interior vertices y_1 and y_2 satisfying $|y_1 - y_2| \lesssim h$, there holds

$$|(u-u^h)(y_1) - (u-u^h)(y_2)| \lesssim h^{k+3}|\ln h|^3 \|u\|_{W^{k+3,\infty}(\Re^n)}, \tag{1.2}$$

where u and u^h are the exact and finite element solutions, respectively, of the following problem

$$\mathcal{L}u(y) = f(y) \quad \text{in} \quad \Re^n, \quad u \quad \text{has a compact support.}$$

The inequality (1.2) plays a critical role in the proof of our main result

$$|\nabla(u - \Pi_{2kh}^{2k}u^h)(y_0)| \lesssim h^{k+2}|\ln h|^3 \|u\|_{W^{k+3,\infty}(\Re^n)}, \tag{1.3}$$

where k is even and Π_{2kh}^{2k} is our local operator which interpolates a continuous function to a polynomial of order $2k$ in a mesh-patch of size $2kh$.

Note that the above estimate is only valid for the case that the degree k is even. To obtain the same ultra-convergence property for the case that k is odd, we need to do some special treatment. Towards this end, we first extrapolate the finite element solution to obtain

$$|(u-Pu^h)(y_1) - (u-Pu^h)(y_2)| \lesssim h^{k+3}|\ln h|^3 \|u\|_{W^{k+3,\infty}(\Re^n)}, \tag{1.4}$$

where P is the extrapolation operator defined in Section 4. Subsequently, we obtain

$$|\nabla(u - \Pi_{2kh}^{2k}(Pu^h))(y_0)| \lesssim h^{k+2}|\ln h|^3 \|u\|_{W^{k+3,\infty}(\Re^n)}. \tag{1.5}$$

Based on (1.3) and (1.5) for $\Re^n$, we establish similar estimates on the bounded region Ω for the problem (1.1) with help of interior analysis (see [26] et al.) and negative norm estimate (see [21]).

Other than variable coefficients, we also would like to emphasize that our results are valid for any locally symmetric mesh, particularly for simplical meshes. Comparing with the best known gradient superconvergence result for variable coefficients by Schatz-Sloan-Wahlbin [23,26]

$$\left|\frac{\partial u(x_0)}{\partial x_i} - \frac{\partial \hat{u}^h(x_0,\beta)}{\partial x_i}\right| \lesssim h^{k+\delta'} \ln\left(\frac{1}{h}\right), \quad \text{if} \quad k \quad \text{is odd} \tag{1.6}$$

and by Asadzadeh et al. [2]

$$\left|\frac{\partial u(x_0)}{\partial x_i} - \frac{\partial \hat{v}^h(x_0,\beta)}{\partial x_i}\right| \lesssim h^{k+\delta'} \ln\left(\frac{1}{h}\right), \quad \text{if} \quad k \geq 2 \tag{1.7}$$

where $0 < \delta' < 1$, our results raise the superocnvergence order to $h^{k+2}|\ln h|^3$, an ultraconvergence result.

The rest of this paper is outlined as follows. In Sect. 2, we discuss the discrete Green's functions in the whole domain $\Re^n$ over a uniform conforming partitions. In Sects. 3 and 4, we investigate the ultraconvergence of the finite element solution for the problems in the whole domain $\Re^n$ over a uniform conforming partitions, where Sect. 3 is for the even order and Sect. 4 is for the odd order finite element solution. In Sect. 5, we apply our theory to problem (1.1). The numerical experiments supporting our theory are presented in Sect. 6.

2 Discrete Green's functions in $\Re^n$

This section is dedicated to a discussion of the finite element approximation properties of the Green's function. For any positive definite coefficient matrix $B = (b_{ij})_{n\times n}$, we define the associated bilinear form

$$a_B(\psi,\phi) = \int_{\Re^n} b_{ij}(y)\frac{\partial\psi(y)}{\partial y_i}\frac{\partial\phi(y)}{\partial y_j}dy \quad \psi,\phi \in H_0^1(\Re^n),$$

where $H_0^1(\Re^n) = \{v \in H^1(\Re^n)|v \text{ has a compact support}\}$. In particular, we denote $a(\cdot,\cdot) = a_A(\cdot,\cdot)$ for simplicity. Let the Green function G_z^B be defined by

$$a_B(G_z^B, w) = w(z), \quad \forall w \in W^{1,q}(\Re^n) \cap H_0^1(\Re^n), \forall q > n. \tag{2.1}$$

For a given point $z \in \Re^n$, we define the shift of A by $A_z = (a_{ij}(y+z))$ and the constant matrix $\overline{A}_z = (a_{ij}(z))$. For simplicity, we denote

$$G_z = G_z^A \quad \overline{G}_z = G_z^{\overline{A}_z}.$$

Let $\mathcal{T}_h$ be a uniform conforming partitions of $\Re^n$ and $\mathcal{N}_h$ be the set of all vertices of $\mathcal{T}_h$. We assume that $\mathcal{T}_h$ is symmetric in the sense that each vertex $y \in \mathcal{N}_h$ is a *symmetric center* of the mesh $\mathcal{T}_h$. That is to say, for all $z \in \mathcal{N}_h$, $2z - y \in \mathcal{N}_h$. Let

$$S_h = \{v_h \in C(\Re^n) : (v_h)|_e \in P_k, \forall e \in \mathcal{T}_h\}$$

be the associated finite element space of degree k and let $S_h^0 = S_h \cap H_0^1(\Re^n)$, we introduce the finite element projector $R_h^B : H_0^1(\Re^n) \to S_h^0$ for all $\psi \in S_h^0(\Re^n)$ by

$$a_B(w - R_h^B w, \psi) = 0. \tag{2.2}$$

In particular, we denote $R_h = R_h^A$, $\tilde{R}_h^z = R_h^{A_z}$, $\overline{R}_h^z = R_h^{\overline{A}_z}$.
The following approximation property

$$\|G_z^B - R_h^B G_z^B\|_{W^{1,\infty}(\Re^n \setminus B(z,d))} \lesssim h^k d^{1-k-n} |\ln h|, \forall d > 0, \tag{2.3}$$

is shown in [24]. Usually, the estimate (2.3) can be used to derive the vertex-wise convergence or superconvergence of $|(u - R_h u)(x_0)|$, $x_0 \in \mathcal{N}_h$, where $u \in H_0^1(\Re^n)$ is the solution of

$$a(u, v) = (f, v), \forall v \in H_0^1(\Re^n) \tag{2.4}$$

and $R_h u = u^h \in S_h^0$ is the finite element solution satisfying

$$a(u^h, v_h) = (f, v_h), \forall v_h \in S_h^0. \tag{2.5}$$

To derive the ultraconvergence of $R_h u$, we need to discuss further approximation properties of the discrete Green's functions $R_h G_z$. Precisely, in this section, we shall estimate the following three quantities defined for all $y, z \in \Re^n$ by

$$\alpha(y, z) = G_z(y) - R_h G_z(y) - [\overline{G}_z(y) - \overline{R}_h^z \overline{G}_z(y)], \tag{2.6}$$

$$\alpha_1(y, z) = G_0(y) - R_h G_0(y) - [G_z(y + z) - (R_h G_z)(y + z)], \tag{2.7}$$

$$\alpha_2(y, z) = \overline{G}_0(y) - \overline{R}_h^0 \overline{G}_0(y) - [\overline{G}_z(y + z) - \overline{R}_h^z G_z(y + z)] \tag{2.8}$$

which depend on the smoothness of variable coefficient $A = (a_{ij})$.
We have the following Theorem.

Theorem 2.1 *Let $a_{ij} \in C^\infty(\Re^n)$, $1 \le i, j \le n$. Then*

$$\int_{\Re^n} |\nabla_y \alpha(y, z)| dy \lesssim h^2 |\ln h|^3. \tag{2.9}$$

Moreover if $|z| \lesssim h$, then we have

$$\int_{\Re^n} |y||\nabla_y \alpha_1(y, z)| dy \lesssim h^3 |\ln h|^3, \tag{2.10}$$

$$\int_{\Re^n} |y||\nabla_y \alpha_2(y, z)| dy \lesssim h^3 |\ln h|^3, \tag{2.11}$$

and

$$\int_{\Re^n} |\nabla(\alpha(y,0)-\alpha(y+z,z))|dy \lesssim h^3|\ln h|^3. \tag{2.12}$$

Before proceeding the proof of Theorem 2.1, we first introduce some lemmas.

Lemma 2.2 *Let $a_{ij} \in C^\infty(\Re^n)$, $1 \le i, j \le n$ and $E(y) = G_z(y) - \overline{G}_z(y)$, $y \in \Re^n$. Then*

$$\|E\|_{W^{l,\infty}(\Re^n \setminus B(z,d))} \lesssim d^{3-n-l}|\ln d|, \quad \forall d > 0. \tag{2.13}$$

Consequently,

$$\|E - R_h E\|_{W^{1,1}(\Re^n)} \lesssim h^2|\ln h|^2. \tag{2.14}$$

Proof We first show (2.13) for the case $l = 1$. We denote $\chi_t(s) = \frac{\partial E(s)}{\partial s_t}$, $t = 1, \ldots, n$. One observes that

$$\frac{\partial}{\partial s_i}\left(a_{ij}(s)\frac{\partial \chi_t(s)}{\partial s_j}\right) = \frac{\partial^2}{\partial s_i \partial s_t}\left(a_{ij}(s)\frac{\partial E(s)}{\partial s_j}\right) - \frac{\partial}{\partial s_i}\left(\frac{\partial a_{ij}(s)}{\partial s_t}\frac{\partial E(s)}{\partial s_j}\right). \tag{2.15}$$

The fact that $w(z) = a(G_z, w) = a_{\overline{A}_z}(\overline{G}_z, w)$, $\forall w \in W^{1,q}(\Re^n) \cap H_0^1(\Re^n)$, $\forall q > n$ yields that

$$\frac{\partial}{\partial s_i}\left(a_{ij}(s)\frac{\partial G_z(s)}{\partial s_j}\right) = \frac{\partial}{\partial s_i}\left(a_{ij}(z)\frac{\partial \overline{G}_z(s)}{\partial s_j}\right)$$

and thus

$$\begin{aligned}\frac{\partial^2}{\partial s_i \partial s_t}\left(a_{ij}(s)\frac{\partial E(s)}{\partial s_j}\right) &= \frac{\partial^2}{\partial s_i \partial s_t}\left[(a_{ij}(z) - a_{ij}(s))\frac{\partial \overline{G}_z(s)}{\partial s_j}\right] \\ &= \frac{\partial}{\partial s_i}\left[(a_{ij}(z) - a_{ij}(s))\frac{\partial^2 \overline{G}_z(s)}{\partial s_j \partial s_t}\right] - \frac{\partial}{\partial s_i}\left[\frac{\partial a_{ij}(s)}{\partial s_t}\frac{\partial \overline{G}_z(s)}{\partial s_j}\right].\end{aligned} \tag{2.16}$$

Combining (2.15) and (2.16), we have

$$\begin{aligned}\mathcal{L}\chi_t(s) &= \frac{\partial}{\partial s_i}\left[(a_{ij}(z) - a_{ij}(s))\frac{\partial^2 \overline{G}_z(s)}{\partial s_j \partial s_t}\right] \\ &\quad - \frac{\partial}{\partial s_i}\left[\frac{\partial a_{ij}(s)}{\partial s_t}\frac{\partial \overline{G}_z(s)}{\partial s_j}\right] - \frac{\partial}{\partial s_i}\left(\frac{\partial a_{ij}(s)}{\partial s_t}\frac{\partial E(s)}{\partial s_j}\right).\end{aligned} \tag{2.17}$$

Let $y \in \Re^n \backslash B(z, d)$. Then

$$\chi_t(y) = \int_{\Re^n} a_{ij}(s) \frac{\partial \chi_k(s)}{\partial s_j} \frac{\partial G_y(s)}{\partial s_i} ds = \int_{\Re^n} \left[(a_{ij}(z) - a_{ij}(s)) \frac{\partial^2 \overline{G}_z(s)}{\partial s_j \partial s_t} \right] \frac{\partial G_y(s)}{\partial s_i} ds$$
$$- \int_{\Re^n} \frac{\partial a_{ij}(s)}{\partial s_t} \frac{\partial \overline{G}_z(s)}{\partial s_j} \frac{\partial G_y(s)}{\partial s_i} ds - \int_{\Re^n} \frac{\partial a_{ij}(s)}{\partial s_t} \frac{\partial E(s)}{\partial s_j} \frac{\partial G_y(s)}{\partial s_i} ds$$
$$= J_1(y) + J_2(y) + J_3(y). \tag{2.18}$$

We first estimate $J_1(y)$. For all $d > 0$, let $d_0 = \max\{2d, 1\}$ and $\Omega_0 = B(z, d_0) \cup B(y, d_0)$. Then

$$J_1(y) = \left(\int_{B(z,\frac{d}{2})} + \int_{B(y,\frac{d}{2})} + \int_{\Omega_0 \setminus (B(y,\frac{d}{2}) \cup B(z,\frac{d}{2}))} + \int_{\Re^n \backslash \Omega_0} \right)$$
$$\times \left[(a_{ij}(z) - a_{ij}(s)) \frac{\partial^2 \overline{G}_z(s)}{\partial s_j \partial s_t} \right] \frac{\partial G_y(s)}{\partial s_i} ds$$
$$= J_{1,1}(y) + J_{1,2}(y) + J_{1,3}(y) + J_{1,4}(y). \tag{2.19}$$

Moreover, by [11] and [12],

$$|\nabla^2 \overline{G}_z(s)| \lesssim |s - z|^{-n}, \quad |\nabla G_y(s)| \lesssim |y - s|^{1-n}$$

and the fact that $a_{ij} \in C^\infty$ yields that

$$|a_{ij}(z) - a_{ij}(s)| \lesssim |s - z|.$$

When $y \in \Re^n \setminus B(z, d)$, $s \in B(z, \frac{d}{2})$, we have $|y - s| \geq \frac{d}{2}$ and thus

$$|J_{1,1}(y)| \lesssim \int_{B(z,d/2)} |s - z|^{1-n} |y - s|^{1-n} ds \lesssim d^{1-n} \int_{B(z,d/2)} |s - z|^{1-n} ds \lesssim d^{2-n}.$$

Similarly,

$$|J_{1,2}(y)| \lesssim d^{2-n}.$$

A straightforward calculation yields that

$$|J_{1,3}(y)| \lesssim d^{2-n} |\ln d|,$$

and

$$|J_{1,4}| \lesssim \int_{\Re^n \backslash \Omega_0} \left|\frac{\partial^2 \overline{G}_z(s)}{\partial s_j \partial s_t}\right| \left|\frac{\partial G_y(s)}{\partial s_i}\right| ds \lesssim \int_{\Re^n \setminus \Omega_0} |s - z|^{-n} |y - s|^{1-n} ds$$
$$\lesssim d_0^{1-n} \lesssim \min\{d^{2-n}, 1\}. \tag{2.20}$$

Then by (2.19), we have

$$|J_1(y)| \lesssim d^{2-n}|\ln d|.$$

Similarly,

$$|J_2(y)| \lesssim d^{2-n}|\ln d|, \quad \text{and} \quad |J_3(y)| \lesssim d^{2-n}|\ln d|.$$

Then, by (2.18), we get the desired result (2.13) for the case $l=1$. We turn now to the proof of (2.13) for $l \geq 2$. From (2.17) it follows that, for any positive integer m,

$$\begin{aligned}
|\nabla^m \mathcal{L}\chi_t(s)| &\leq \left|\nabla^m \left(\frac{\partial}{\partial s_i}\left[(a_{ij}(z)-a_{ij}(s))\frac{\partial^2 \overline{G}_z(s)}{\partial s_j \partial s_t}\right]\right)\right| \\
&+ \left|\nabla^m \left(\frac{\partial}{\partial s_i}\left[\frac{\partial a_{ij}(s)}{\partial s_t}\frac{\partial \overline{G}_z(s)}{\partial s_j}\right]\right)\right| + \left|\nabla^m \left(\frac{\partial}{\partial s_i}(\frac{\partial a_{ij}(s)}{\partial s_t}\frac{\partial E(s)}{\partial s_j})\right)\right| \\
&\lesssim |s-z||\nabla^{m+3}\overline{G}_z(s)| + |\nabla^{m+2}\overline{G}_z(s)| + |\nabla^{m+2}G_z(s)| \\
&\lesssim |s-z|^{-m-n} + |s-z|^{-m-n} + |s-z|^{-m-n} \lesssim |s-z|^{-m-n}, \qquad (2.21)
\end{aligned}$$

where we have used the estimate (see [11] and [12])

$$|\nabla^{m'}\overline{G}_z(s)| + |\nabla^{m'}G_z(s)| \lesssim |s-z|^{2-m'-n},$$

where m' is a positive integer. Assume that the weighted Sobolev space $\kappa_a^T(B(s,R))$ is a normed linear spaces if equipped with the norms $\|\mu\|_{\kappa_a^T(B(s,R))} = \sum_{|\alpha|\leq T}\left(\int_{B(s,R)}\left(|D^\alpha \mu(x)|\rho(x,M)^{|\alpha|-a}\right)^2 dx\right)^{\frac{1}{2}}$. By Theorem 6.5 (see [3]), (2.21) and (2.13) for the case $l=1$, we obtain, for all positive integer $m \geq 2$,

$$\begin{aligned}
\|\chi_t\|_{\kappa_{m-2}^m(B(s,d/2))} &\lesssim \|\mathcal{L}E\|_{\kappa_{m-2}^{m-2}(B(s,d/2))} + \|\chi_t\|_{\kappa_{-2}^0(B(s,d/2))} \\
&\lesssim d^{m-2}d^{2-m-n+\frac{n}{2}} + d^{-2}d^{2-n+\frac{n}{2}}|\ln d| \\
&\lesssim d^{-\frac{n}{2}}|\ln d|.
\end{aligned}$$

This implies

$$\|\chi_t\|_{H^m(B(s,d/2))} \lesssim d^{2-m}\|\chi_t\|_{\kappa_{m-2}^m(B(s,d/2))} \lesssim d^{2-m-\frac{n}{2}}|\ln d|. \qquad (2.22)$$

By (2.22), we have

$$\begin{aligned}
\|\chi_t\|_{W^{l-1,\infty}(B(s,d/2))} &\lesssim \|\chi_t\|_{W^{l+n-1,1}(B(s,d/2))} \lesssim d^{\frac{n}{2}}\|\chi_t\|_{H^{l+n-1}(B(s,d/2))} \\
&\lesssim d^{\frac{n}{2}}d^{3-l-n-\frac{n}{2}}|\ln d| \lesssim d^{3-l-n}|\ln d|. \qquad (2.23)
\end{aligned}$$

Consequently,

$$\|E\|_{W^{l,\infty}(\Re^n \setminus B(z,d))} \lesssim d^{3-l-n}|\ln d|.$$

This gives the desired result (2.13).
Next we show (2.14). For any $y \in \Re^n$,

$$\begin{aligned}(E - R_h E)(y) &= a(E - R_h E, G_y) = a(E - R_h E, G_y - R_h G_y) \\ &= a(E - I_h^k E, G_y - R_h G_y),\end{aligned}$$

where I_h^k is the standard interpolating operator from $H_0^1(\Re^n)$ to S_h^0. Letting

$$B_1 = \int_{B(z,d)} a_{ij}(s) \frac{\partial(E - I_h^k E)(s)}{\partial s_i} \frac{\partial(G_y - R_h G_y)(s)}{\partial s_j} ds,$$
$$B_2 = \int_{\Re^n \setminus B(z,d)} a_{ij}(s) \frac{\partial(E - I_h^k E)(s)}{\partial s_i} \frac{\partial(G_y - R_h G_y)(s)}{\partial s_j} ds,$$

we have

$$(E - R_h E)(y) = B_1 + B_2. \tag{2.24}$$

We next estimate B_1 and B_2 for $y \in \Re^n \setminus B(z, 2d)$ for some $d \geq c_1 h > h$. Since $y \in \Re^n \setminus B(z, 2d)$, we have $s \in \Re^n \setminus B(y, d)$ if $s \in B(z, d)$. Note that

$$\mathcal{L}E = \frac{\partial}{\partial s_i}\left[(a_{ij}(z) - a_{ij}(s))\frac{\partial \overline{G}_z(s)}{\partial s_j}\right].$$

Therefore, by (2.3), we have

$$\begin{aligned}|B_1| &\lesssim \|E - I_h^k E\|_{W^{1,1}(B(z,d))}\|G_y - R_h G_y\|_{W^{1,\infty}(\Re^n \setminus B(y,d))} \\ &\lesssim h^k d^{1-k-n}|\ln h|\left[\|E - I_h^k E\|_{W^{1,1}(B(z,h))} + h^k\|E\|_{W^{k+1,1}(B(z,d)\setminus B(z,h))}\right] \\ &\lesssim h^{k+2}|\ln h| d^{1-k-n},\end{aligned} \tag{2.25}$$

where we have used (2.3), (2.13) for the case $l = k + 1$ and the fact that $d > h$ in the last inequality.
Similarly,

$$|B_2| \lesssim h|\ln h| h^k \|E\|_{W^{k+1,\infty}(\Re^n \setminus B(z,d))} \lesssim h^{k+1}|\ln h| d^{2-k-n}. \tag{2.26}$$

Substituting (2.25) and (2.26) into (2.24) and noticing $h < d$, we obtain

$$\|E - R_h E\|_{L^\infty(\Re^n \setminus B(z,2d))} \lesssim h^{k+1}|\ln h| d^{2-k-n}. \tag{2.27}$$

Thus

$$\begin{aligned}
&\|E - R_h E\|_{W^{1,\infty}(\Re^n \setminus B(z,2d))} \\
&\quad \le \|E - I_h^k E\|_{W^{1,\infty}(\Re^n \setminus B(z,2d))} + \|I_h^k E - R_h E\|_{W^{1,\infty}(\Re^n \setminus B(z,2d))} \\
&\quad \lesssim h^k \|E\|_{W^{k+1,\infty}(\Re^n \setminus B(z,2d))} + h^{-1}\|E - R_h E\|_{L^\infty(\Re^n \setminus B(z,2d))} \\
&\quad \lesssim h^k |\ln h| d^{2-k-n}.
\end{aligned}$$

This implies

$$\|E - R_h E\|_{W^{1,1}(\Re^n \setminus B(z,2c_1 h))} \lesssim h^k |\ln h| h^{2-k} |\ln h| \lesssim h^2 |\ln h|^2. \tag{2.28}$$

We turn now to the estimation of $\|E - R_h E\|_{W^{1,1}(B(z,2c_1 h))}$. One observes that, for all $y \in B(z, 2c_1 h)$,

$$\|R_h G_y\|_{H^1(\Re^n)} \lesssim a(R_h G_y, R_h G_y)^{\frac{1}{2}} \lesssim \|R_h G_y\|_{L^\infty(\Re^n)}^{\frac{1}{2}}.$$

Using the inverse estimate, we have, if $n = 2$,

$$\begin{aligned}
\|R_h G_y\|_{L^\infty(\Re^n)} &= \sup_{e \in \mathcal{T}_h} \|R_N G_{x_0}\|_{L^\infty(e)} \lesssim \sup_{e \in \mathcal{T}_h} \|R_N G_{x_0}\|_{L^{|\ln h|}(e)} |\ln h|^{\frac{-1}{2}} \\
&\lesssim \|R_N G_{x_0}\|_{L^{|\ln h|}(\Re^n)} |\ln h|^{\frac{-1}{2}} \lesssim |\ln h|^{\frac{-1}{2}} \|R_N G_{x_0}\|_{H^1(\Re^n)},
\end{aligned}$$

and if $n = 3$,

$$\begin{aligned}
\|R_h G_y\|_{L^\infty(\Re^n)} &= \sup_{e \in \mathcal{T}_h} \|R_N G_{x_0}\|_{L^\infty(e)} \lesssim \sup_{e \in \mathcal{T}_h} \|R_N G_{x_0}\|_{L^6(e)} h^{\frac{-1}{2}} \\
&\lesssim \|R_N G_{x_0}\|_{L^6(\Re^n)} h^{\frac{-1}{2}} \lesssim h^{\frac{-1}{2}} \|R_N G_{x_0}\|_{H^1(\Omega)}.
\end{aligned}$$

Combining the above three estimates, we have

$$\|R_h G_y\|_{L^\infty(\Re^n)} \le h^{2-n} |\ln h|^{-1}. \tag{2.29}$$

Furthermore, by (2.29) and the inverse estimate, we have, for $n = 2, 3$,

$$\|R_h G_y\|_{W^{1,\infty}(\Re^n)} \lesssim h^{-1} \|R_h G_y\|_{L^\infty(\Re^n)} \lesssim h^{1-n} |\ln h|^{-1}.$$

This implies

$$\begin{aligned}
|(I_h^k E - R_h E)(y)| = |a(I_h^k E - E, R_h G_y)| &\lesssim \|I_h^k E - E\|_{W^{1,1}(\Re^n)} \|R_h G_y\|_{W^{1,\infty}(\Re^n)} \\
&\lesssim h^{3-n} |\ln h|.
\end{aligned} \tag{2.30}$$

Consequently,

$$\begin{aligned}\|E - R_h E\|_{W^{1,1}(B(z,2c_1h))} &\leq \|E - I_h^k E\|_{W^{1,1}(B(z,2c_1h))} + \|I_h^k E - R_h E\|_{W^{1,1}(B(z,2c_1h))} \\ &\lesssim h^2|\ln h| + h^n h^{-1}\|I_h^k E - R_h E\|_{L^\infty(B(z,2c_1h))} \lesssim h^2|\ln h|.\end{aligned} \tag{2.31}$$

The estimate (2.14) is a direct result of (2.28) and (2.31). □

Lemma 2.3 *If all $a_{ij} \in C^\infty(\Re^n)$, $1 \leq i, j \leq n$, then*

$$\|(R_h - \overline{R}_h^z)\overline{G}_z\|_{W^{1,1}(\Re^n)} \lesssim h^2|\ln h|^3. \tag{2.32}$$

Proof By the definition of R_h and $\overline{R}_h^z$, we have that for any $v \in S_0^h(\Re^n)$,

$$\int_{\Re^n} a_{ij}(s)\frac{\partial(R_h\overline{G}_z - \overline{G}_z)(s)}{\partial s_i}\frac{\partial v(s)}{\partial s_j}ds = 0,$$
$$\int_{\Re^n} a_{ij}(z)\frac{\partial(\overline{R}_h^z\overline{G}_z - \overline{G}_z)(s)}{\partial s_i}\frac{\partial v(s)}{\partial s_j}ds = 0.$$

Then,

$$\begin{aligned}&\int_{\Re^n} a_{ij}(s)\frac{\partial((R_h - \overline{R}_h^z)\overline{G}_z)(s)}{\partial s_i}\frac{\partial v(s)}{\partial s_j}ds \\ &= \int_{\Re^n}[a_{ij}(s) - a_{ij}(z)]\frac{\partial(\overline{R}_h^z\overline{G}_z - \overline{G}_z)(s)}{\partial s_i}\frac{\partial v(s)}{\partial s_j}ds.\end{aligned} \tag{2.33}$$

Consequently,

$$\frac{\partial((R_h - \overline{R}_h^z)\overline{G}_z)(y)}{\partial y_l} = \int_{\Re^n}(a_{ij}(s) - a_{ij}(z))\frac{\partial(\overline{G}_z - \overline{R}_h^z\overline{G}_z)(s)}{\partial s_i}\frac{\partial R_h g_l(s)}{\partial s_j}ds, \tag{2.34}$$

where for $1 \leq l \leq n$, $R_h g_l$ is the discrete derivative of Green's function defined by

$$a(R_h g_l, w) = \frac{\partial w(y)}{\partial y_l} \quad \forall w \in H_0^1(\Re^n) \cap W^{2,q}(\Re^n), q > n.$$

There exists the following estimates (see [35,36])

$$\begin{aligned}\|R_h g_l\|_{W^{1,1}(\Re^n)} &\lesssim |\ln h|, \|R_h g_l\|_{W^{1,\infty}(\Re^n)} \lesssim h^{-n}|\ln h|, \|R_h g_l\|_{W^{1,\infty}(\Re^n \setminus B(y,d))} \\ &\lesssim d^{-n}|\ln h|,\end{aligned} \tag{2.35}$$

where $d > c_0 h$. In the following, we use (2.3), (2.34), (2.35) to prove (2.32). Let $d_0 = c_0 h$. Note that

$$\begin{aligned}
&\int_{\Re^n} |(a_{ij}(s) - a_{ij}(z))\nabla(\overline{G}_z - \overline{R}_h^z \overline{G}_z)(s)|ds \\
&= \int_{B(z,d_0)} + \int_{\Re^n \setminus B(z,d_0)} |(a_{ij}(s) - a_{ij}(z))\nabla(\overline{G}_z - \overline{R}_h^z \overline{G}_z)(s)|ds \\
&\lesssim h\|\overline{G}_z - \overline{R}_h^z \overline{G}_z\|_{W^{1,1}(\Re^n)} + \int_{\Re^n \setminus B(z,d_0)} |s - z| \times h^k |s - z|^{1-k-n} |\ln h| ds \\
&\lesssim h^2 |\ln h|^2. \qquad (2.36)
\end{aligned}$$

Combining this estimate and (2.35), we have

$$\|(R_h - \overline{R}_h^z)\overline{G}_z\|_{W^{1,\infty}(\Re^n)} \lesssim h^2 h^{-n} |\ln h|^2 \lesssim h^{2-n} |\ln h|^2. \qquad (2.37)$$

We next turn to an improved estimate of $\|(R_h - \overline{R}_h^z)\overline{G}_z\|_{W^{1,\infty}(\Re^n \setminus B(z,d))}$ for all $d \geq d_0 = c_0 h$. By (2.34), we have that for any $y \in \Re^n \setminus B(z, d)$,

$$\frac{\partial((R_h - \overline{R}_h^z)\overline{G}_z)(y)}{\partial y_l} = I_1 + I_2, \qquad (2.38)$$

with

$$\begin{aligned}
I_1 &= \int_{B(z,d/2)} (a_{ij}(s) - a_{ij}(z)) \frac{\partial(\overline{G}_z - \overline{R}_h^z \overline{G}_z)(s)}{\partial s_i} \frac{\partial R_h g_l(s)}{\partial s_j} ds, \\
I_2 &= \int_{\Re^n \setminus B(z,d/2)} (a_{ij}(s) - a_{ij}(z)) \frac{\partial(\overline{G}_z - \overline{R}_h^z \overline{G}_z)(s)}{\partial s_i} \frac{\partial R_h g_l(s)}{\partial s_j} ds.
\end{aligned}$$

By (2.35) and (2.36), we have

$$|I_1| \lesssim h^2 |\ln h| \|R_h g_l\|_{W^{1,\infty}(\Re^n \setminus B(y,d/2))} \lesssim h^2 |\ln h|^2 d^{-n}. \qquad (2.39)$$

On the other hand, by the fact of $a_{ij} \in C^\infty$ and (2.3),

$$\left\| (a_{ij}(s) - a_{ij}(z)) \frac{\partial(\overline{G}_z - \overline{R}_h^z \overline{G}_z)(s)}{\partial s_i} \right\|_{L^\infty(\Re^n \setminus B(z,d/2))} \lesssim h^k d^{2-k-n} |\ln h|.$$

This estimate, together with (2.35), gives

$$|I_2| \lesssim h^k d^{2-k-n} |\ln h|^2. \qquad (2.40)$$

Inserting (2.39) and (2.40) into (2.38),

$$\|(R_h - \overline{R}_h^z)\overline{G}_z\|_{W^{1,\infty}(\Re^n \setminus B(z,d))} \lesssim h^2 d^{-n} |\ln h|^2.$$

Consequently,

$$\|(R_h - \overline{R}_h^z)\overline{G}_z\|_{W^{1,1}(\Re^n \setminus B(z,d_0))} \lesssim h^2|\ln d_0||\ln h|^2 \lesssim h^2|\ln h|^3. \tag{2.41}$$

As an immediate consequence of (2.37) and (2.41), we obtain

$$\begin{aligned}
&\|(R_h - \overline{R}_h^z)\overline{G}_z\|_{W^{1,1}(\Re^n)} \\
&\quad = \|(R_h - \overline{R}_h^z)\overline{G}_z\|_{W^{1,1}(B(z,d_0))} + \|(R_h - \overline{R}_h^z)\overline{G}_z\|_{W^{1,1}(\Re^n \setminus B(z,d_0))} \\
&\quad \lesssim h^k h^{2-k-n}|\ln h|^3 h^n + h^2|\ln h|^3 \lesssim h^2|\ln h|^3
\end{aligned}$$

from which the proof is completed. □

Based on Lemmas 2.2 and 2.3, we are now ready to prove Theorem 2.1.

Proof of Theorem 2.1 First, noticing the definition of α, the estimate (2.9) is a direct consequence of the Lemmas 2.3 and 2.4.

Next we show (2.10). Let $E_0(y) = G_z(y+z)$, $E_1(y) = G_0(y) - E_0(y)$, $y \in \Re^n$. Noticing the fact that $\tilde{R}_h^z E_0(y) = (R_h G_z)(y+z)$, one observes that

$$\alpha_1(y,z) = (E_1 - R_h E_1)(y) - (R_h - \tilde{R}_h^z)E_0(y). \tag{2.42}$$

By the same arguments in the proof of (2.13), we have that, for all $|z| \lesssim h$ and all $d > 0$,

$$\|E_1\|_{L^\infty(\Re^n \setminus B(z,d))} \lesssim hd^{2-n}|\ln d|.$$

Then by the same reasoning to show (2.14), we can prove

$$\int_{\Re^n} |\nabla(E_1 - R_h E_1)(y)||y| dy \lesssim h^3|\ln h|^3. \tag{2.43}$$

Moreover, similarly to the proof of (2.32), we can show that

$$\int_{\Re^n} |y||\nabla((R_h - \tilde{R}_h^z)E_0(y))| dy \lesssim h^3|\ln h|^3. \tag{2.44}$$

Plugging the estimates (2.43) and (2.44) into the equality (2.42), we get the desired result (2.10).

By the same reasoning, we obtain the estimate (2.11) .

Next we show (2.12). Note that

$$\begin{aligned}
\alpha(y,0) &= [(G_0 - R_h G_0) - (\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)](y) \\
&= [(G_0 - \overline{G}_0)(y) - R_h(G_0 - \overline{G}_0)(y)] - (R_h\overline{G}_0 - \overline{R}_h^0\overline{G}_0)(y),
\end{aligned}$$

and

$$\alpha(y+z,z) = [(G_z - R_h G_z) - (\overline{G}_z - \overline{R}_h^z \overline{G}_z)](y+z)$$
$$= [(G_z - \overline{G}_z)(y+z) - R_h(G_z - \overline{G}_z)(y+z)] - (R_h \overline{G}_z - \overline{R}_h^z \overline{G}_z)(y+z).$$

Then by letting

$$\beta_t(y) = (G_t - \overline{G}_t)(y+t), \quad \bar{\beta}_z(y) = \beta_0(y) - \beta_z(y),$$
$$\omega(y) = (R_h - \overline{R}_h^0)\overline{G}_0(y) - ((R_h - \overline{R}_h^z)\overline{G}_z)(y+z),$$

and noticing the fact that $\tilde{R}_h^z \beta_z(y) = R_h(G_z - \overline{G}_z)(y+z)$, we have

$$\alpha(y,0) - \alpha(y+z,z)$$
$$= (\bar{\beta}_z - R_h \bar{\beta}_z)(y) - (R_h - \tilde{R}_h^z)\beta_z(y) - [(R_h \overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y) - (R_h \overline{G}_z - \overline{R}_h^z \overline{G}_z)(y+z)]$$
$$= (\bar{\beta}_z - R_h \bar{\beta}_z)(y) + \omega(y) - (R_h - \tilde{R}_h^z)\beta_z(y). \tag{2.45}$$

We next estimate the three terms of the right-hand of the above equality. First, note that

$$\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial \beta_0(y)}{\partial y_j}\right) = \frac{\partial}{\partial y_i}\left[(a_{ij}(0) - a_{ij}(y))\frac{\partial \overline{G}_0(y)}{\partial y_j}\right],$$
$$\frac{\partial}{\partial y_i}\left(a_{ij}(y+z)\frac{\partial \beta_z(y+z)}{\partial y_j}\right) = \frac{\partial}{\partial y_i}\left[(a_{ij}(z) - a_{ij}(y+z))\frac{\partial \overline{G}_z(y+z)}{\partial y_j}\right],$$

we have

$$\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial \bar{\beta}_z(y)}{\partial y_j}\right) = \frac{\partial}{\partial y_i}(a_{ij}(y) - a_{ij}(y+z))\frac{\partial \beta_z(y+z)}{\partial y_j})$$
$$+ \frac{\partial}{\partial y_i}\left[((a_{ij}(0) - a_{ij}(y)) - (a_{ij}(z) - a_{ij}(y+z))\frac{\partial \overline{G}_0(y)}{\partial y_j}\right]$$
$$+ \frac{\partial}{\partial y_i}\left[(a_{ij}(z) - a_{ij}(y+z))\frac{\partial(\overline{G}_0(y) - \overline{G}_z(y+z))}{\partial y_j}\right]. \tag{2.46}$$

Note that

$$|(a_{ij}(0) - a_{ij}(y)) - (a_{ij}(z) - a_{ij}(y+z))| \lesssim h|y|, \quad |a_{ij}(z) - a_{ij}(y+z)| \lesssim |y|. \tag{2.47}$$

Similarly to (2.13), it follows from (2.47) that, for all $d > 0$,

$$\|\beta_z\|_{W^{1,\infty}(\Re^n \setminus B(z,d))} \lesssim d^{2-n}|\ln d|,$$

and

$$\|\bar{\beta}_z\|_{W^{1,\infty}(\Re^n \setminus B(z,d))} \lesssim hd^{2-n}|\ln d|.$$

Furthermore, by the same arguments in the proof of (2.14), we have from the above estimate that

$$\int_{\Re^n} |\nabla(\bar{\beta}_z - R_h\bar{\beta}_z)(y)|dy \lesssim h^3|\ln h|^3. \tag{2.48}$$

We now estimate the second term of the right-hand side of (2.45). Note that, for any $v \in S_h^0(\Re^n)$,

$$\begin{aligned}
&\int_{\Re^n} a_{ij}(y)\frac{\partial(R_h\overline{G}_0(y) - \overline{R}_h^0\overline{G}_0)(y)}{\partial y_i}\frac{\partial v(y)}{\partial y_j}dy \\
&= -\int_{\Re^n}(a_{ij}(0) - a_{ij}(y))\frac{\partial(\overline{G}_0 - \overline{R}_h^0\overline{G}_0)(y)}{\partial y_i}\frac{\partial v(y)}{\partial y_j}dy,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Re^n} a_{ij}(y+z)\frac{\partial(R_h\overline{G}_z - \overline{R}_h^z\overline{G}_z)(y+z)}{\partial y_i}\frac{\partial v(y)}{\partial y_j}dy \\
&= -\int_{\Re^n}(a_{ij}(z) - a_{ij}(y+z))\frac{\partial(\overline{G}_z - \overline{R}_h^z\overline{G}_z)(y+z)}{\partial y_i}\frac{\partial v(y)}{\partial y_j}dy.
\end{aligned}$$

The above two equalities yields

$$\begin{aligned}
&\int_{\Re^n} a_{ij}(y)\frac{\partial\omega(y)}{\partial y_i}\frac{\partial v(y)}{\partial y_j}dy \\
&= -\int_{\Re^n}[(a_{ij}(0) - a_{ij}(y)) - (a_{ij}(z) - a_{ij}(y+z))]\frac{\partial(\overline{G}_0 - \overline{R}_h^0\overline{G}_0)(y)}{\partial y_i}\frac{\partial v(y)}{\partial y_j}dy \\
&- \int_{\Re^n}(a_{ij}(z) - a_{ij}(y+z))\frac{\partial\alpha_2(y,z)}{\partial y_i}\frac{\partial v(y)}{\partial y_j}dy.
\end{aligned} \tag{2.49}$$

Similarly to (2.32), by (2.3), (2.10), (2.47) and (2.49), we have

$$\int_{\Re^n} |\nabla\omega(y)|dy \lesssim h^3|\ln h|^3. \tag{2.50}$$

Similarly, we have

$$\int_{\Re^n} |\nabla(R_h - \tilde{R}_h^z)\beta_z(y)|dy \lesssim h^3|\ln h|^3. \tag{2.51}$$

Then (2.12) follows by combining (2.45), (2.48), (2.50) and (2.51). □

3 Ultraconvergence in the case *k* is even

In this section, we discuss the ultraconvergence of the finite element solution of (1.1). It is known that the weak solution $u \in H_0^1(\Re^n)$ of (1.1) satisfies the following variational form

$$a(u, w) = (f, w), \forall\, w \in H_0^1(\Re^n) \tag{3.1}$$

and the finite element solution $u^h \in S_h^0$ satisfies

$$a(u^h, w^h) = (f, w^h), \forall\, w^h \in S_h^0. \tag{3.2}$$

Note that by the definition (2.2), we actually have $u^h = R_h u$.

Let τ_0 be a parallelogram or parallelepiped constituting of the elements in the $\mathcal{T}_h$ such that each edge of τ_0 contains $2k+1$ vertices of T_h. We denote by y_0 the center of τ_0. Note that the fact $\mathcal{T}_h$ yields $y_0 \in \mathcal{N}_h$. We introduce a $2k-$degree interpolation operator Π_{2kh}^{2k} over τ_0 by letting $\Pi_{2kh}^{2k} v \in (P_{2k})^n$ satisfying

$$\Pi_{2kh}^{2k} v(y) = v(y) \quad \forall\, y \in \tau_0 \cap \mathcal{N}_h. \tag{3.3}$$

Next we present the main result of this section.

Theorem 3.1 *Let $k \geq 2$ be even. If $a_{ij} \in C^\infty(\Re^n), 1 \leq i, j \leq n$ and $u \in W^{k+3,\infty}(\Re^n)$ with a compact support, then*

$$|\nabla(u - \Pi_{2kh}^{2k} u^h)(y_0)| \lesssim h^{k+2}|\ln h|^3 \|u\|_{W^{k+3,\infty}(\Re^n)}. \tag{3.4}$$

The rest of this section is dedicated to the proof of Theorem 3.1. Without loss of generality, we assume $y_0 = 0$. Assume that $\Delta y \in \tau_0 \cap \mathcal{N}_h$. In the process of proving (3.4), we mainly apply the following estimate

$$|(u - R_h u)(0) - (u - R_h u)(\Delta y)| \lesssim h^{k+3}|\ln h|^3 \|u\|_{W^{k+3,\infty}(\Re^n)}.$$

Let $d > 0$ be a constant such that $\mathrm{Supp} u \subset B(0, 2d)$. Let $\mu \in C_0^\infty(\Re^n)$ satisfy

$$\mu = 1 \quad \text{in } B(0, d).$$

Moreover, let $u^J \in \mathbb{P}_{k+1}$ satisfy

$$\nabla^l (u^J - u)(0) = 0, \quad 0 \le |l| \le k+1$$

We split $u(y)$ as

$$u = u_1 + u_2$$

with $u_1 = \mu u^J$. It is easy to check that both u_1, u_2 have a compact support and $\nabla^l u_1(0) = 0, \forall y \in B(0,d), |l| \ge k+2$ and $u_1(y) = 0$ and that $\nabla^{k+1} u_2(0) = 0$. Next, we estimate $|\nabla(u_1 - \Pi_{2kh}^{2k}(R_h u_1))(0)|$ and $|\nabla(u_2 - \Pi_{2kh}^{2k}(R_h u_2))(0)|$ separately.

In the following, we estimate $|(u_1 - R_h u_1)(0) - (u_1 - R_h u_1)(\Delta y)|$. For all $\Delta y \in \tau_0 \cap \mathcal{N}_h$,

$$\begin{aligned}(u_1 - R_h u_1)(0) - (u_1 - R_h u_1)(\Delta y) &= \left[(u_1 - \overline{R}_h^0 u_1)(0) - (u_1 - \overline{R}_h^{\Delta y} u_1)(\Delta y)\right] \\ &\quad + \left[(\overline{R}_h^0 u_1 - R_h u_1)(0) - (\overline{R}_h^{\Delta y} u_1 - R_h u_1)(\Delta y)\right].\end{aligned} \tag{3.5}$$

The following Lemmas 3.2 and 3.3 estimate the two terms of the right-hand side of the above equality, respectively.

Lemma 3.2 *Under the assumptions of* Theorem 3.1, *we have that for all* $\Delta y \in \tau_0 \cap \mathcal{N}_h$,

$$|(u_1 - \overline{R}_h^0 u_1)(0) - (u_1 - \overline{R}_h^{\Delta y} u_1)(\Delta y)| \lesssim h^{k+3} |\ln h|^3 \|u_1\|_{W^{k+3,\infty}(\Re^n)}. \tag{3.6}$$

Proof We first present $(u_1 - \overline{R}_h^0 u_1)(0)$ in its integral form. Let the linear operators ζ and ζ_1 be defined for all $v \in C^0(\Re^n)$ by

$$\zeta(v)(y) = \frac{1}{2}[v(y) + v(-y)], \quad \zeta_1(v)(y) = \frac{1}{2}[v(y + \Delta y) + v(\Delta y - y)], \; y \in \Re^n. \tag{3.7}$$

Apparently,

$$\zeta(u_1)(0) = u_1(0), \quad \overline{R}_h^0 \zeta(u_1)(0) = \zeta(\overline{R}_h^0 u_1)(0) = \overline{R}_h^0 u_1(0). \tag{3.8}$$

Moreover, the fact u_1 is a polynomial of order $k+1$ in $B(0,d)$ and that k is even yield

$$\nabla^{k+1} \zeta(u_1)(0) = 0, \tag{3.9}$$

we conclude that $\zeta(u_1)$ is a polynomial of order k in $B(0,d)$. That is

$$\zeta(u_1) = I_h^k \zeta(u_1) \quad \text{in} \quad B(0,d).$$

This equality, together with (3.8), implies

$$
\begin{aligned}
&(u_1 - \overline{R}_h^0 u_1)(0) \\
&= (\zeta(u_1) - \overline{R}_h^0 \zeta(u_1))(0) = \int_{\Re^n} a_{ij}(0) \frac{\partial(\zeta(u_1) - I_h^k \zeta(u_1))(y)}{\partial y_i} \frac{\partial(\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y)}{\partial y_j} dy \\
&= \int_{\Re^n \setminus B(0,d/2)} a_{ij}(0) \frac{\partial(\zeta(u_1) - I_h^k \zeta(u_1))(y)}{\partial y_i} \frac{\partial(\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y)}{\partial y_j} dy. \qquad (3.10)
\end{aligned}
$$

Next we present $(u_1 - \overline{R}_h^{\Delta y} u_1)(\Delta y)$. We also have

$$
\zeta_1(u_1)(0) = u_1(\Delta y), \quad \overline{R}_h^{\Delta y} \zeta_1(u_1)(0) = \overline{R}_h^{\Delta y} u_1(\Delta y). \qquad (3.11)
$$

Since $|\Delta y| \lesssim h$, we have that $y + \Delta y \in B(0, d)$ for all $y \in B(0, d/2)$. That is, for all given $\Delta y \in \tau_0 \cap N_h$, $\zeta_1(u)$ is also a polynomial of degree $k + 1$. On the other hand, the fact that k is even yields

$$
\nabla^{k+1} \zeta_1(u_1)(\Delta y) = 0. \qquad (3.12)
$$

Then we have

$$
\zeta_1(u_1) - I_h^k \zeta_1(u_1) = 0, \quad \text{in} \quad B(0, d/2).
$$

This equality, together with (3.11), shows

$$
\begin{aligned}
&(u_1 - \overline{R}_h^{\Delta y} u_1)(\Delta y) = (\zeta_1(u_1) - \overline{R}_h^{\Delta y} \zeta_1(u_1))(0) \\
&= \int_{\Re^n} a_{ij}(\Delta y) \frac{\partial(\zeta_1(u_1) - I_h^k \zeta_1(u_1))(y)}{\partial y_i} \frac{\partial(\overline{G}_{\Delta y} - \overline{R}_h^{\Delta y} \overline{G}_{\Delta y})(y)}{\partial y_j} dy \\
&= \int_{\Re^n} a_{ij}(\Delta y) \frac{\partial(\zeta_1(u_1) - I_h^k \zeta_1(u_1))(y + \Delta y)}{\partial y_i} \frac{\partial(\overline{G}_{\Delta y} - \overline{R}_h^{\Delta y} \overline{G}_{\Delta y})(y + \Delta y)}{\partial y_j} dy \\
&= \int_{\Re^n \setminus B(0,d/2)} a_{ij}(\Delta y) \frac{\partial(\zeta_1(u_1) - I_h^k \zeta_1(u_1))(y+\Delta y)}{\partial y_i} \frac{\partial(\overline{G}_{\Delta y} - \overline{R}_h^{\Delta y} \overline{G}_{\Delta y})(y+\Delta y)}{\partial y_j} dy.
\end{aligned}
\qquad (3.13)
$$

Set $\varsigma(y) = \frac{1}{2}(u_1(y) - u_1(y + 2\Delta y))$. One observes that

$$
\begin{aligned}
&\zeta(u_1)(y) - \zeta_1(u_1)(y + \Delta y) \\
&\quad = \frac{1}{2}(u_1(y) + u_1(-y)) - \frac{1}{2}(u_1((y + \Delta y) + \Delta y) + u_1(\Delta y - (y + \Delta y))) \\
&\quad = \frac{1}{2}(u_1(y) + u_1(-y)) - \frac{1}{2}(u_1(y + 2\Delta y) + u_1(-y)) \\
&\quad = \frac{1}{2}(u_1(y) - u_1(y + 2\Delta y)) = \varsigma(y). \qquad (3.14)
\end{aligned}
$$

Using (3.10), (3.13) and (3.14), one observes that $(u_1 - \overline{R}_h^0 u_1)(0) - (u_1 - \overline{R}_h^{\Delta y} u_1)(\Delta y)$ can be split into

$$
\begin{aligned}
&(u_1 - \overline{R}_h^0 u_1)(0) - (u_1 - \overline{R}_h^{\Delta y} u_1)(\Delta y) \\
&= \int_{\Re^n \setminus B(0,d/2)} (a_{ij}(0) - a_{ij}(\Delta y)) \frac{\partial(\zeta(u_1) - I_h^k \zeta(u_1))(y)}{\partial y_i} \frac{\partial(\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y)}{\partial y_j} dy \\
&\quad + \int_{\Re^n \setminus B(0,d/2)} a_{ij}(\Delta y) \frac{\partial(\varsigma - I_h^k \varsigma)(y)}{\partial y_i} \frac{\partial(\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y)}{\partial y_j} dy \\
&\quad + \int_{\Re^n \setminus B(0,d/2)} a_{ij}(\Delta y) \frac{\partial(\zeta_1(u_1) - I_h^k \zeta_1(u_1))(y + \Delta y)}{\partial y_i} \frac{\partial \alpha_2(y, \Delta y)}{\partial y_j} dy \\
&= W_1 + W_2 + W_3,
\end{aligned} \tag{3.15}
$$

where $\alpha_2(y, z)$ is defined as (2.8). We next estimate $W_i, i = 1, 2, 3$ separately. To estimate W_1, using (2.3) and the following estimate

$$|a_{ij}(\Delta y) - a_{ij}(0)| \lesssim h, \tag{3.16}$$

we have

$$|W_1| \lesssim h^{2k+1} |\ln h|^3 \|u_1\|_{W^{k+1,\infty}(\Re^n)}. \tag{3.17}$$

We turn now to the estimation of W_2. Note that

$$\|\nabla^{k+1} \varsigma\|_{L^\infty(\Re^n)} \lesssim h \|u_1\|_{W^{k+2,\infty}(\Re^n)}. \tag{3.18}$$

Inserting (2.3), (3.14) and (3.18) into (3.15), we obtain

$$
\begin{aligned}
|W_2| &= \left| \int_{\Re^n \setminus B(0,d/2)} a_{ij}(\Delta y) \frac{\partial(\varsigma - I_h^k \varsigma)(y)}{\partial y_i} \frac{\partial(\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y)}{\partial y_j} dy \right| \\
&\lesssim h^{k+1} \|u_1\|_{W^{k+2,\infty}(\Re^n)} \|\overline{G}_0 - \overline{R}_h^0 \overline{G}_0\|_{W^{1,1}(\Re^n \setminus B(0,d/2))} \\
&\lesssim h^{2k+1} |\ln h|^3 \|u_1\|_{W^{k+2,\infty}(\Re^n)}.
\end{aligned} \tag{3.19}
$$

Similarly, by (2.11), we obtain

$$|W_3| \lesssim h^{k+3} |\ln h|^3 \|u_1\|_{W^{k+1,\infty}(\Re^n)}. \tag{3.20}$$

Summing up (3.15), (3.17), (3.19) and (3.20), we have the desired result (3.6). □

Lemma 3.3 *Assume that $u \in W^{k+3,\infty}(\Re^n)$. Then*

$$|(\overline{R}_h^0 u_1 - R_h u_1)(0) - (\overline{R}_h^{\Delta y} u_1 - R_h u_1)(\Delta y)| \lesssim h^{k+3} |\ln h|^3 \|u_1\|_{W^{k+3,\infty}(\Re^n)}. \tag{3.21}$$

Proof Note that

$$(u_1 - \overline{R}_h^0)(0) = \int_{\Re^n} a_{ij}(0) \frac{\partial(u_1 - I_h^k u_1)(y)}{\partial y_i} \frac{\partial(\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y)}{\partial y_j} dy,$$

and

$$(u_1 - R_h u_1)(0) = \int_{\Re^n} a_{ij}(y) \frac{\partial(u_1 - I_h^k u_1)(y)}{\partial y_i} \frac{\partial(G_0 - R_h G_0)(y)}{\partial y_j} dy.$$

We observe that $(\overline{R}_h^0 u_1 - R_h u_1)(0)$ can be split into

$$\begin{aligned}(\overline{R}_h^0 u_1 - R_h u_1)(0) &= (u_1 - R_h u_1)(0) - (u_1 - \overline{R}_h^0 u_1)(0) \\ &= \int_{\Re^n} [a_{ij}(y) - a_{ij}(0)] \frac{\partial(u_1 - I_h^k u_1)(y)}{\partial y_i} \frac{\partial(G_0 - R_h G_0)(y)}{\partial y_j} dy \\ &\quad + \int_{\Re^n} a_{ij}(0) \frac{\partial(u_1 - I_h^k u_1)(y)}{\partial y_i} \frac{\partial \alpha(y, 0)}{\partial y_j} dy \\ &= I_1 + I_2, \end{aligned} \tag{3.22}$$

where $\alpha(y, z)$ is defined as (2.6). Moreover, since

$$\begin{aligned}&(u_1 - R_h u_1)(\Delta y) \\ &\quad = \int_{\Re^n} a_{ij}(y + \Delta y) \frac{\partial(u_1 - I_h^k u_1)(y + \Delta y)}{\partial y_i} \frac{\partial(G_{\Delta y} - R_h G_{\Delta y})(y + \Delta y)}{\partial y_j} dy,\end{aligned}$$

and

$$\begin{aligned}&(u_1 - \overline{R}_h^{\Delta y} u_1)(\Delta y) \\ &\quad = \int_{\Re^n} a_{ij}(\Delta y) \frac{\partial(u_1 - I_h^k u_1)(y + \Delta y)}{\partial y_i} \frac{\partial(\overline{G}_{\Delta y} - \overline{R}_h^{\Delta y} \overline{G}_{\Delta y})(y + \Delta y)}{\partial y_j} dy.\end{aligned}$$

We obtain

$$\begin{aligned}&(\overline{R}_h^{\Delta y} u_1 - R_h u_1)(\Delta y) \\ &= \int_{\Re^n} [a_{ij}(y+\Delta y) - a_{ij}(\Delta y)] \frac{\partial(u_1 - I_h^k u_1)(y+\Delta y)}{\partial y_i} \frac{\partial(G_{\Delta y} - R_h G_{\Delta y})(y+\Delta y)}{\partial y_j} dy \\ &\quad + \int_{\Re^n} a_{ij}(\Delta y) \frac{\partial(u_1 - I_h^k u_1)(y + \Delta y)}{\partial y_i} \frac{\partial \alpha(y + \Delta y, \Delta y)}{\partial y_j} dy = J_1 + J_2. \end{aligned} \tag{3.23}$$

Then by (3.22) and (3.23),

$$(\overline{R}_h^0 u_1 - R_h u_1)(0) - (\overline{R}_h^{\Delta y} u_1 - R_h u_1)(\Delta y) = (I_1 - J_1) + (I_2 - J_2). \tag{3.24}$$

We first estimate $I_1 - J_1$. Let $\alpha_1(y, z)$ be defined by (2.7). We have that

$$\begin{aligned}
&I_1 - J_1\\
&= \int_{\Re^n} [(a_{ij}(y) - a_{ij}(0)) - (a_{ij}(y+\Delta y) - a_{ij}(\Delta y))] \frac{\partial(u_1 - I_h^k u_1)(y)}{\partial y_i} \frac{\partial(G_0 - R_h G_0)(y)}{\partial y_j} dy\\
&\quad + \int_{\Re^n} (a_{ij}(y+\Delta y) - a_{ij}(\Delta y)) \frac{\partial[(u_1 - I_h^k u_1)(y) - (u_1 - I_h^k u_1)(y+\Delta y)]}{\partial y_i} \frac{\partial(G_0 - R_h G_0)(y)}{\partial y_j} dy\\
&\quad + \int_{\Re^n} (a_{ij}(y+\Delta y) - a_{ij}(\Delta y)) \frac{\partial(u_1 - I_h^k u_1)(y+\Delta y)}{\partial y_i} \frac{\partial \alpha_1(y, \Delta y)}{\partial y_j} dy\\
&= Z_1^1 + Z_1^2 + Z_1^3.
\end{aligned} \tag{3.25}$$

We first estimate Z_1^1. Note that

$$\begin{aligned}
|(a_{ij}(y) - a_{ij}(0)) - (a_{ij}(y+\Delta y) - a_{ij}(\Delta y))| &\lesssim h|y|,\\
|a_{ij}(y+\Delta y) - a_{ij}(\Delta y)| &\lesssim |y|.
\end{aligned} \tag{3.26}$$

Combining (2.3) and (3.26) gives

$$\begin{aligned}
|Z_1^1| &\lesssim h^{k+1} \|u_1\|_{W^{k+1,\infty}(\Re^n)} \int_{\Re^n} |y||\nabla(G_0 - R_h G_0)(y)| dy\\
&\lesssim h^{k+3} |\ln h|^3 \|u_1\|_{W^{k+1,\infty}(\Re^n)}.
\end{aligned} \tag{3.27}$$

We turn now to the estimation of Z_1^2. Set $\varsigma_1(y) = u_1(y) - u_1(y + \Delta y)$. One observes that

$$\|\varsigma_1 - I_h^k \varsigma_1\|_{W^{1,\infty}(\Re^n)} \lesssim h^{k+1} \|u_1\|_{W^{k+2,\infty}(\Re^n)}. \tag{3.28}$$

By (2.3), (3.26) and (3.28), we arrive at

$$|Z_1^2| \lesssim h^{k+3} |\ln h|^3 \|u_1\|_{W^{k+3,\infty}(\Re^n)}.$$

Note that the combination of (2.10) and (3.26) implies

$$|Z_1^3| \lesssim h^{k+3} |\ln h|^3 \|u_1\|_{W^{k+3,\infty}(\Re^n)}.$$

Inserting the above two estimates and (3.27) into (3.25) yields

$$|I_1 - J_1| \lesssim h^{k+3} |\ln h|^3 \|u_1\|_{W^{k+3,\infty}(\Re^n)}. \tag{3.29}$$

Next we estimate $I_2 - J_2$. One observes that $I_2 - J_2$ can be decomposed into

$$\begin{aligned} I_2 - J_2 &= \int_{\Re^n} (a_{ij}(0) - a_{ij}(\Delta y)) \frac{\partial (u_1 - I_h^k u_1)(y)}{\partial y_i} \frac{\partial \alpha(y,0)}{\partial y_j} dy \\ &\quad + \int_{\Re^n} a_{ij}(\Delta y) \frac{\partial [(u_1 - I_h^k u_1)(y) - (u_1 - I_h^k u_1)(y + \Delta y)]}{\partial y_i} \frac{\partial \alpha(y,0)}{\partial y_j} dy \\ &\quad + \int_{\Re^n} a_{ij}(\Delta y) \frac{\partial (u_1 - I_h^k u_1)(y + \Delta y)}{\partial y_i} \frac{\partial (\alpha(y,0) - \alpha(y + \Delta y, \Delta y))}{\partial y_j} dy \\ &= Z_2^1 + Z_2^2 + Z_2^3. \end{aligned} \tag{3.30}$$

We need estimate the three items of the right-hand side. Summing up (2.9), (2.12), (3.26) and (3.28), we also get, for $l = 1, 2, 3$,

$$|Z_2^l| \lesssim h^{k+3} |\ln h|^3 \|u_1\|_{W^{k+3,\infty}(\Re^n)}.$$

Substituting the above estimate into (3.30), we obtain

$$|I_2 - J_2| \lesssim h^{k+3} |\ln h|^3 \|u_1\|_{W^{k+3,\infty}(\Re^n)}. \tag{3.31}$$

The desired result (3.21) follows from (3.24), (3.29) and (3.31). □

We are now in a perfect position to give an estimate for $\nabla(u_1 - \Pi_{2kh}^{2k}(R_h u_1))(0)$.

Theorem 3.4 *Under the assumptions of Theorem 3.1, we have*

$$|\nabla(u_1 - \Pi_{2kh}^{2k}(R_h u_1))(0)| \lesssim h^{k+2} |\ln h|^3 \|u_1\|_{W^{k+3,\infty}(\Re^n)}. \tag{3.32}$$

Proof The estimate of $|\nabla(u_1 - \Pi_{2kh}^{2k}(R_h u_1))(0)|$ can be reduced to the boundedness of $|(u_1 - R_h u_1)(0) - (u_1 - R_h u_1)(\Delta y)|$ for all $\Delta y \in \tau_0 \cap \mathcal{N}_h$. In fact,

$$\begin{aligned} |\nabla(u_1 - \Pi_{2kh}^{2k}(R_h u_1))(0)| &\le |\nabla(u_1 - \Pi_{2kh}^{2k}(I_h u_1))(0)| \\ &\quad + |\nabla \Pi_{2kh}^{2k}(R_h u_1 - (I_h u_1))(0)|, \end{aligned}$$

where $I_h u_1 \in S_h$ is the interpolation of u_1. We first estimate $\nabla(u_1 - \Pi_{2kh}^{2k}(I_h u_1))(0)$. One has

$$|\nabla(u_1 - \Pi_{2kh}^{2k}(I_h u_1))(0)| \lesssim h^{2k} \|u_1\|_{W^{2k+1,\infty}(\Re^n)} \lesssim h^{2k} \|u_1\|_{W^{2k+1,\infty}(\Re^n)}. \tag{3.33}$$

Next we estimate $\nabla \Pi_{2kh}^{2k}(R_h u_1 - I_h u_1)(0)$. Combining (3.6) and (3.21) gives

$$|(u_1 - R_h u_1)(0) - (u_1 - R_h u_1)(\Delta y)| \lesssim h^{k+3} |\ln h|^3 \|u_1\|_{W^{k+3,\infty}(\Re^n)}.$$

Set $\psi(y) = (R_h u_1 - I_h u_1)(y) - (R_h u_1 - I_h u_1)(0)$ for all $y \in \tau_0$. Then, by the inverse estimate,

$$\begin{aligned} |\nabla \Pi_{2kh}^{2k}(R_h u_1 - I_h u_1)(0)| &= |\nabla \Pi_{2kh}^{2k}\psi(0)| \lesssim h^{-1}\|\psi\|_{L^\infty(\tau_0)} \\ &\lesssim h^{-1}\max_{\Delta y\in\tau_0}|\psi(\Delta y)| \\ &\lesssim h^{-1}\max_{\Delta y\in\tau_0}|(u_1 - R_h u_1)(0) - (u_1 - R_h u_1)(\Delta y)| \\ &\lesssim h^{k+2}|\ln h|^3\|u_1\|_{W^{k+3,\infty}(\Re^n)}. \end{aligned} \tag{3.34}$$

Then (3.32) can be obtained by combining the estimates (3.33) and (3.34). □

Next, we turn to the estimation of $|\nabla(u_2 - \Pi_{2kh}^{2k}(R_h u_2))(0)|$.

Theorem 3.5 *Under the same assumptions of Theorem 3.1,*

$$|\nabla(u_2 - \Pi_{2kh}^{2k}(R_h u_2))(0)| \lesssim h^{k+2}|\ln h|^3\|u_2\|_{W^{k+3,\infty}(\Re^n)}. \tag{3.35}$$

Proof By the same arguments in the proof of Theorem 3.4, the estimate of $|\nabla(u_2 - \Pi_{2kh}^{2k}(R_h u_2))(0)|$ can be reduced to the boundedness of $|(u_2 - R_h u_2)(0) - (u_2 - R_h u_2)(\Delta y)|$ for all $\Delta y \in \tau_0 \cap \mathcal{N}_h$. We decompose

$$\begin{aligned} &(u_2 - R_h u_2)(0) - (u_2 - R_h u_2)(\Delta y) \\ &= [(u_2 - \overline{R}_h^0 u_2)(0) - (u_2 - \overline{R}_h^{\Delta y} u_2)(\Delta y)] \\ &\quad + [(\overline{R}_h^0 u_2 - R_h u_2)(0) - (\overline{R}_h^{\Delta y} u_2 - R_h u_2)(\Delta y)], \end{aligned} \tag{3.36}$$

and we will estimate the above two terms separately. We first present $(u_2 - \overline{R}_h^0 u_2)(0) - (u_2 - \overline{R}_h^{\Delta y} u_2)(\Delta y)$ in its integral form. Similarly to Theorem 3.1, we have

$$(u_2 - \overline{R}_h^0 u_2)(0) = \int_{\Re^n} a_{ij}(0)\frac{\partial(u_2 - I_h^k u_2)(y)}{\partial y_i}\frac{\partial(\overline{G}_0 - \overline{R}_h^0 G_0)(y)}{\partial y_j}dy,$$

and

$$\begin{aligned} &(u_2 - \overline{R}_h^{\Delta y} u_2)(\Delta y) \\ &= \int_{\Re^n} a_{ij}(\Delta y)\frac{\partial(u_2 - I_h^k u_2)(y+\Delta y)}{\partial y_i}\frac{\partial(\overline{G}_{\Delta y} - \overline{R}_h^{\Delta y}\overline{G}_{\Delta y})(y+\Delta y)}{\partial y_j}dy. \end{aligned}$$

Then

$$
\begin{aligned}
&(u_2-\overline{R}_h^0 u_2)(0)-(u_2-\overline{R}_h^{\Delta y}u_2)(\Delta y)\\
&=\int_{\Re^n}[a_{ij}(0)-a_{ij}(\Delta y)]\frac{\partial(u_2-I_h^k u_2)(y)}{\partial y_i}\frac{\partial(\overline{G}_0-\overline{R}_h^0\overline{G}_0)(y)}{\partial y_j}dy\\
&\quad+\int_{\Re^n}a_{ij}(\Delta y)\frac{\partial[(u_2-I_h^k u_2)(y)-(u_2-I_h^k u_2)(y+\Delta y)]}{\partial y_i}\frac{\partial(\overline{G}_0-\overline{R}_h^0\overline{G}_0)(y)}{\partial y_j}dy\\
&\quad+\int_{\Re^n}a_{ij}(\Delta y)\frac{\partial(u_2-I_h^k u_2)(y+\Delta y)}{\partial y_i}\frac{\partial\alpha_2(y,\Delta y)}{\partial y_j}dy\\
&=K_1+K_2+K_3. \qquad (3.37)
\end{aligned}
$$

Noticing the facts that $\nabla^{k+1}u_2(0)=0$, Supp$u_2\subset B(0,2d)$, (2.3) and (3.26), we have

$$
\begin{aligned}
|K_1| &\lesssim h^{k+1}\|u_2\|_{W^{k+2,\infty}(\Re^n)}\int_{B(0,2d)}|y||\nabla(\overline{G}_0-\overline{R}_h^0\overline{G}_0)(y)|dy\\
&\lesssim h^{k+3}|\ln h|\|u_2\|_{W^{k+2,\infty}(\Re^n)}. \qquad (3.38)
\end{aligned}
$$

By $\nabla^{k+1}u_2(0)=0$ and (2.10), we have

$$
\begin{aligned}
|K_3| &\lesssim h^{k}\|u_2\|_{W^{k+2,\infty}(\Re^n)}\int_{B(0,2d)}|y||\nabla_y\alpha_2(y,\Delta y)|dy\\
&\lesssim h^{k+3}|\ln h|^3\|u_2\|_{W^{k+2,\infty}(\Re^n)}. \qquad (3.39)
\end{aligned}
$$

We next estimate K_2. For any $y\in\Re^n$,

$$
\overline{G}_0(y)=\overline{G}_0(-y),\qquad \overline{R}_h^0\overline{G}_0(y)=\overline{R}_h^0\overline{G}_0(-y).
$$

This equality implies

$$
\frac{\partial(\overline{G}_0-\overline{R}_h^0\overline{G}_0)(-y)}{\partial y_j}=-\frac{\partial(\overline{G}_0-\overline{R}_h^0\overline{G}_0)(y)}{\partial y_j}.
$$

Then, we have

$$
\begin{aligned}
K_2&=\int_{\Re^n}a_{ij}(\Delta y)\frac{\partial[(u_2-I_h^k u_2)(-y)-(u_2-I_h^k u_2)(-y+\Delta y)]}{\partial y_i}\frac{\partial(\overline{G}_0-\overline{R}_h^0\overline{G}_0)(-y)}{\partial y_j}dy\\
&=-\int_{\Re^n}a_{ij}(\Delta y)\frac{\partial[(u_2-I_h^k u_2)(-y)-(u_2-I_h^k u_2)(-y+\Delta y)]}{\partial y_i}\frac{\partial(\overline{G}_0-\overline{R}_h^0\overline{G}_0)(y)}{\partial y_j}dy. \qquad (3.40)
\end{aligned}
$$

Let $\mu(y) = (u_2(y) - u_2(y + \Delta y)) - (u_2(-y) - u_2(-y + \Delta y))$. Combining (3.37) and (3.40) gives

$$2K_2 = \int_{\Re^n} a_{ij}(\Delta y) \frac{\partial(\mu - I_h^k \mu)(y)}{\partial y_i} \frac{\partial(\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y)}{\partial y_j} dy. \tag{3.41}$$

Again by $\nabla^{k+1} u_2(0) = 0$, we have

$$|\nabla^{k+1} \mu(y)| \lesssim h|y| \|u_2\|_{W^{k+3,\infty}(\Re^n)}. \tag{3.42}$$

Plugging (2.3) and (3.42) into (3.41) and noticing that Supp$u_2 \subset B(0, 2d)$, we have

$$|K_2| \lesssim h^{k+3} |\ln h|^3 \|u_2\|_{W^{k+2,\infty}(\Re^n)}. \tag{3.43}$$

Combining (3.38), (3.39) and (3.43), we have that

$$|(u_2 - R_h u_2)(\Delta y) - (u_2 - R_h u_2)(0)| \lesssim h^{k+3} |\ln h|^3 \|u_2\|_{W^{k+2,\infty}(\Re^n)}.$$

Similarly to (3.21), we have

$$|(\overline{R}_h^0 u_2 - R_h u_2)(0) - (\overline{R}_h^{\Delta y} u_2 - R_h u_2)(\Delta y)| \lesssim h^{k+3} |\ln h|^3 \|u_2\|_{W^{k+3,\infty}(\Re^n)}.$$

Therefore,

$$|(u_2 - R_h u_2)(0) - (u_2 - R_h u_2)(\Delta y)| \lesssim h^{k+3} |\ln h|^3 \|u_2\|_{W^{k+2,\infty}(\Re^n)}. \tag{3.44}$$

By the same arguments in the proof of (3.32), using (3.44), we get the desired result (3.35).

Finally, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. By the definition of u_1 and u_2, we have that

$$\|u_1\|_{W^{k+3,\infty}(\Re^n)}, \|u_2\|_{W^{k+3,\infty}(\Re^n)} \lesssim \|u\|_{W^{k+3,\infty}(\Re^n)}.$$

Then (3.4) is an immediate consequence of Theorems 3.4 and 3.5.

4 Ultraconvergence in the case k is odd

When k is odd, the equality (3.9) is not necessary valid. Therefore, the reasoning in the previous section can not be generalized to an arbitrary integer k. In other words, the inequality (3.4) is not necessary valid. To obtain the similar result for the case k is odd, we first need to extrapolate the finite element solution. Precisely, let $\mathcal{T}_{h/2}$ be obtained by decomposing each element of $\mathcal{T}_h$ into 2^n equal-sized elements. We assume that $\mathcal{T}_{h/2}$ is symmetric in the sense that each vertex $y \in \mathcal{N}_{h/2}$ is a *symmetric center* of the mesh $\mathcal{T}_{h/2}$. We denote by $u_{h/2}$ the finite element solution corresponding to the mesh $\mathcal{T}_{h/2}$. We define the Richardson extrapolating function $Pu^h \in S_h$ by letting

$$Pu^h(y) = \frac{2^{k+1}u^{h/2}(y) - u^h(y)}{2^{k+1}-1}, \; y \in \mathcal{N}_h. \tag{4.1}$$

Theorem 4.1 *Assume that $k \geq 3$ is odd. Let $a_{ij} \in C^\infty(\Re^n), 1 \leq i, j \leq n$ and $u \in W^{k+3,\infty}(\Re^n)$. Then*

$$|\nabla(u - \Pi_{2kh}^{2k} Pu^h)(y_0)| \lesssim h^{k+2}|\ln h|^3 \|u\|_{W^{k+3,\infty}(\Re^n)}. \tag{4.2}$$

As in the previous section, we also decompose $u = u_1 + u_2$ and estimate the errors for u_1 and u_2 separately.

Lemma 4.2 *Under the assumptions of* Theorem 4.1,

$$|(u_1 - P(\overline{R}_h^0 u_1))(0) - (u_1 - P(\overline{R}_h^{\Delta y} u_1))(\Delta y)| \lesssim h^{k+3}|\ln h|^3 \|u\|_{W^{k+3,\infty}(\Re^n)}. \tag{4.3}$$

Proof We first present $(u_1 - P(\overline{R}_h^0 u_1))(0)$ in its integral form. One observes that

$$(u_1 - \overline{R}_h^0 u_1)(0) = \int_{\Re^n} a_{ij}(0) \frac{\partial(u_1 - I_h^k u_1)(y)}{\partial y_i} \frac{\partial(\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y)}{\partial y_j} dy. \tag{4.4}$$

Set

$$\chi(y) = 2^{k+1} u_1(\frac{y}{2}), \quad \chi_1(y) = u_1(y) - \chi(y). \tag{4.5}$$

Similarly to (4.4), by (4.5), we have

$$\begin{aligned} 2^{k+1}(u_1 - \overline{R}_{h/2}^0 u_1)(0) &= (\chi - \overline{R}_h^0 \chi)(0) \\ &= \int_{\Re^n} a_{ij}(0) \frac{\partial(\chi - I_h^k \chi)(y)}{\partial y_i} \frac{\partial(\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y)}{\partial y_j} dy. \end{aligned} \tag{4.6}$$

Since u_1 is a polynomial of order $k+1$ in $B(0,d)$. One observes that χ_1 is a polynomial of order k in $B(0,d)$. By (4.4) and (4.6), we have

$$\begin{aligned} &(u_1 - P(\overline{R}_h^0 u_1))(0) \\ &\quad = \frac{1}{2^{k+1}-1}[2^{k+1}(u_1 - \overline{R}_{h/2}^0 u_1)(0) - (u_1 - \overline{R}_h^0 u_1)(0)] \end{aligned}$$

$$= \frac{-1}{2^{k+1}-1} \int_{\Re^n} a_{ij}(0) \frac{\partial(\chi_1 - I_h^k \chi_1)(y)}{\partial y_i} \frac{\partial(\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y)}{\partial y_j} dy$$

$$= \frac{-1}{2^{k+1}-1} \int_{\Re^n \setminus B(0,d/2)} a_{ij}(0) \frac{\partial(\chi_1 - I_h^k \chi_1)(y)}{\partial y_i} \frac{\partial(\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y)}{\partial y_j} dy.$$

Similarly, we have

$$(u_1 - P\overline{R}_h^{\Delta y} u_1)(\Delta y)$$
$$= \frac{-1}{2^{k+1}-1} \int_{\Re^n \setminus B(0,d/2)} a_{ij}(\Delta y) \frac{\partial(\chi_3 - I_h^k \chi_3)(y+\Delta y)}{\partial y_i} \frac{\partial(\overline{G}_{\Delta y} - \overline{R}_h^{\Delta y} \overline{G}_{\Delta y})(y+\Delta y)}{\partial y_j} dy,$$

where $\chi_2(y) = 2^{k+1} u_1(\frac{y+\Delta y}{2})$, $\chi_3(y) = u_1(y) - \chi_2(y)$. Combining the above two estimates, we obtain

$$\begin{aligned}
&(1-2^{k+1})[(u_1 - P(\overline{R}_h^0 u_1))(0) - (u_1 - P(\overline{R}_h^{\Delta y} u_1))(\Delta y)] \\
&= \int_{\Re^n \setminus B(0,d/2)} (a_{ij}(0) - a_{ij}(\Delta y)) \frac{\partial(\chi_1 - I_h^k \chi_1)(y)}{\partial y_i} \frac{\partial(\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y)}{\partial y_j} dy \\
&\quad + \int_{\Re^n \setminus B(0,d/2)} a_{ij}(\Delta y) \frac{\partial(\chi_4 - I_h^k \chi_4)(y)}{\partial y_i} \frac{\partial(\overline{G}_0 - \overline{R}_h^0 \overline{G}_0)(y)}{\partial y_j} dy \\
&\quad + \int_{\Re^n \setminus B(0,d/2)} a_{ij}(\Delta y) \frac{\partial(\chi_3 - I_h^k \chi_3)(y+\Delta y)}{\partial y_i} \frac{\partial \alpha_2(y, \Delta y)}{\partial y_j} dy \\
&= S_1 + S_2 + S_3,
\end{aligned} \tag{4.7}$$

where $\chi_4(y) = \chi_1(y) - \chi_3(y+\Delta y)$ and $\alpha_2(y,z)$ is defined as (2.8).
Next we estimate S_i, $i = 1, 2, 3$. First, noticing (2.3) and (3.16), we obtain

$$|S_1| \lesssim h^{2k+1} |\ln h| \|u\|_{W^{k+1,\infty}(\Re^n)}. \tag{4.8}$$

Secondly, the estimate (2.3) and the fact that

$$|\nabla^{k+1} \chi_4(y)| \lesssim h \|u\|_{W^{k+2,\infty}(\Re^n)}$$

yield

$$|S_2| \lesssim h^{2k+1} \|u\|_{W^{k+2,\infty}(\Re^n)}. \tag{4.9}$$

Finally, by (2.11),

$$|S_3| \lesssim h^k \|u\|_{W^{k+1,\infty}(\Re^n)} \int_{\Re^n \setminus B(0,d/2)} |\nabla_y \alpha_2(y, \Delta y)| dy \lesssim h^{k+3} |\ln h|^3 \|u\|_{W^{k+1,\infty}(\Re^n)}. \tag{4.10}$$

Then (4.3) follows by substituting (4.7), (4.8), (4.9) into (4.10). □

Based on Lemmas 3.3, 4.2 and Theorem 3.4, we are now in a position to show Theorem 4.1.

Proof of Theorem 4.1. A straightforward calculation yields that

$$(u - P(R_h u))(0) - (u - P(R_h u))(\Delta y) = Y_1 + Y_2 + Y_3, \tag{4.11}$$

with

$$\begin{aligned}
Y_1 &= [(u_1 - P(\overline{R}_h^0 u_1))(0) - (u_1 - P(\overline{R}_h^{\Delta y} u_1))(\Delta y)],\\
Y_2 &= [(P(\overline{R}_h^0 u_1) - P(R_h u_1))(0) - (P(\overline{R}_h^{\Delta y} u_1) - P(R_h u_1))(\Delta y)],\\
Y_3 &= [(u_2 - P(R_h u_2))(0) - (u_2 - P(R_h u_2))(\Delta y)].
\end{aligned}$$

First by (4.3),

$$|Y_1| \lesssim h^{k+3} |\ln h|^3 \|u\|_{W^{k+3,\infty}(\Re^n)}.$$

Secondly, by Lemma 3.3,

$$\begin{aligned}
|Y_2| &\le \left|(\overline{R}_h^0 u_1 - R_h u_1)(0) - (\overline{R}_h^{\Delta y} u_1 - R_h u_1)(\Delta y)\right|\\
&\quad + \left|(\overline{R}_{h/2}^0 u_1 - R_{h/2} u_1)(0) - (\overline{R}_{h/2}^{\Delta y} u_1 - R_{h/2} u_1)(\Delta y)\right|\\
&\lesssim h^{k+3} |\ln h|^3 \|u\|_{W^{k+3,\infty}(\Re^n)}.
\end{aligned}$$

Finally, it follows from (3.35) that

$$\begin{aligned}
|Y_3| &\lesssim |(u_2 - R_h u_2)(\Delta y) - (u_2 - R_h u_2)(0)|\\
&\quad + |(u_2 - R_{h/2} u_2)(\Delta y) - (u_2 - R_{h/2} u_2)(0)|\\
&\lesssim h^{k+3} |\ln h|^3 \|u\|_{W^{k+3,\infty}(\Re^n)}.
\end{aligned}$$

Inserting the above three estimates into the equality (4.11), we have

$$|(u - P(R_h u)(\Delta y) - (u - P(R_h u))(0)| \lesssim h^{k+3} |\ln h|^3 \|u\|_{W^{k+3,\infty}(\Re^n)}.$$

Combining this estimate and the inverse inequality, we get the desired result (4.2) by the same arguments in the proof of (3.32). □

5 Ultraconvergence in a bounded domain

In this section, we apply the previous theory to the problem (1.1). Let $\mathcal{T}_h^\Omega$ be a quasi-uniform conforming partition of Ω satisfying the following property: there exists a parallelogram or parallelepipe $\tau \subset \Omega$ such that : 1) τ is the union of some elements in

$\mathcal{T}_h^{\Omega}$, 2) τ has a size $h_\tau \simeq 1$, 3) $\mathcal{T}_h^{\Omega} \cap \tau = \mathcal{T}_h \cap \tau$. Here $\mathcal{T}_h$ is a mesh in the whole domain $\Re^n$ defined in Sect. 2, $\mathcal{T}_h^{\Omega} \cap \tau = \{\tau' \in \mathcal{T}_h^{\Omega} \colon \tau' \subset \tau\}$ and $\mathcal{T}_h \cap \tau = \{\tau' \in \mathcal{T}_h \colon \tau' \subset \tau\}$. Let

$$S_h(\Omega) = \{v_h \in C(\Omega) : v_h|_{\tau'} \in P_k, \forall \tau' \in \mathcal{T}_h^{\Omega}\}$$

be the associated standard finite element space of degree k and let $S_h^0(\Omega) = S_h(\Omega) \cap H_0^1(\Omega)$, we introduce the finite element projector $R_h^{\Omega} \colon H_0^1(\Omega) \to S_h^0(\Omega)$ for all $\psi \in S_h^0(\Omega)$ by

$$a^{\Omega}(w - R_h^{\Omega} w, \psi) = 0, \tag{5.1}$$

where the associated bilinear form is defined by

$$a^{\Omega}(\phi, \psi) = \int_{\Omega} a_{ij}(y) \frac{\partial \phi(y)}{\partial y_i} \frac{\partial \psi(y)}{\partial y_j} dy \quad \phi, \psi \in H^1(\Omega).$$

Based on Theorem 3.1, we have the following result.

Theorem 5.1 *Let $a_{ij} \in C^{\infty}(\Omega)$, $1 \le i, j \le n$ and Π_{2kh}^{2k} be defined as in Section 3, and $\mathcal{N}_h^{\tau}$ be the set of all vertices of $\overline{\mathcal{T}}_h^{\Omega} \cap \tau$. Let $k \ge 2$ be an even. Assume that $y_0 \in \mathcal{N}_h^{\Omega}$ is away from the boundary of τ with a fixed distance and $u \in W^{k+3,\infty}(\tau)$, then*

$$|\nabla(u - \Pi_{2kh}^{2k}(R_h^{\Omega} u))(y_0)| \lesssim h^{k+2} |\ln h|^3 \|u\|_{W^{k+3,\infty}(\tau)} + \|u - R_h^{\Omega} u\|_{W^{1-k,2}(\Omega)}. \tag{5.2}$$

Proof Assume that $d \simeq 1$ satisfies $B(y_0, 2d) \subset \tau$. Let $\phi \in C^{\infty}(\Re^n)$ satisfy $0 \le \phi \le 1$, and $\phi = 1$ in $\mathrm{B}(y_0, d)$, $\phi = 0$ in $\Re^n \setminus B(y_0, 2d)$, and $\|\phi\|_{W^{k+3,\infty}(\Re^n)} \lesssim 1$. Let $\upsilon(y) = \phi(y)u(y)$. One observes that $\nabla(u - \Pi_{2kh}^{2k}(R_h^{\Omega} u))(y_0)$ can be decomposed into

$$\begin{aligned} &\nabla(u - \Pi_{2kh}^{2k}(R_h^{\Omega} u))(y_0) \\ &\quad = \nabla(\upsilon - \Pi_{2kh}^{2k}(R_h \upsilon))(y_0) + \nabla(\Pi_{2kh}^{2k}(R_h \upsilon) - \Pi_{2kh}^{2k}(R_h^{\Omega} u))(y_0). \end{aligned} \tag{5.3}$$

Using (3.4), we have

$$|\nabla(\upsilon - \Pi_{2kh}^{2k}(R_h \upsilon))(y_0)| \lesssim h^{k+2} |\ln h|^3 \|\upsilon\|_{W^{k+3,\infty}(\Re^n)} \lesssim h^{k+2} |\ln h|^3 \|u\|_{W^{k+3,\infty}(\tau)}. \tag{5.4}$$

We turn now to the estimation of $\nabla \Pi_{2kh}^{2k}(R_h \upsilon - R_h^{\Omega} u)(y_0)$. We denote that $S_h^0(B(y_0, d)) = \{v^h \in C(\Omega) \colon v^h|_e \in P_k, \forall e \in \overline{\mathcal{T}}_h^{\Omega}\} \cap H_0^1(B(y_0, d))$. Note that $\overline{\mathcal{T}}_h^{\Omega} \cap \tau = \overline{\mathcal{T}}_h \cap \tau$, we have, for any $w \in S_h^0(B(y_0, d))$,

$$\begin{aligned} a(R_h \upsilon - R_h^{\Omega} u, w) &= a(R_h \upsilon - \upsilon, w) + a(\upsilon - u, w) + a(u - R_h^{\Omega} u, w) \\ &= 0 + 0 + 0 = 0. \end{aligned} \tag{5.5}$$

Then by the arguments of Schatz et al. in [22,23], we have

$$\|R_h\upsilon - R_h^{\Omega}u\|_{W^{1,\infty}(B(y_0,\frac{d}{2}))} \lesssim \|u - R_h^{\Omega}u\|_{W^{1-k,2}(B(y_0,d))} \lesssim \|u - R_h^{\Omega}u\|_{W^{1-k,2}(\Omega)}.$$

Furthermore, we get

$$|\nabla(\Pi_{2kh}^{2k}(R_h\upsilon) - \Pi_{2kh}^{2k}(R_h^{\Omega}u))(y_0)| \lesssim \|u - R_h^{\Omega}u\|_{W^{1-k,2}(\Omega)}. \tag{5.6}$$

Inserting the estimates (5.4) and (5.6) into the equality (5.3), we get the desired result (5.2). □

Next we consider the case that k is odd. Let $\overline{\mathcal{T}}_{h/2}^{\Omega}$ be a quasi-uniform partition obtained by decomposing each element of $\overline{\mathcal{T}}_h^{\Omega}$ into 2^n elements. Furthermore, we assume that $\overline{\mathcal{T}}_{h/2}^{\Omega} \cap \tau = \overline{\mathcal{T}}_{h/2} \cap \tau$ where $\overline{\mathcal{T}}_{h/2}$ is defined as in Section 4. We denote by $R_{h/2}^{\Omega}u$ the finite element solution corresponding to the mesh $\overline{\mathcal{T}}_{h/2}^{\Omega}$. We define the Richardson extrapolating function $P(R_{h/2}^{\Omega}u) \in S_h(\Omega)$ by letting

$$P(R_h^{\Omega}u)(y) = \frac{2^{k+1}R_{h/2}^{\Omega}u(y) - R_h^{\Omega}u(y)}{2^{k+1}-1}, \; y \in \mathcal{N}_h^{\tau}. \tag{5.7}$$

Similarly to Theorem 5.1, we have the following result.

Theorem 5.2 *Under the assumptions of Theorem 5.1, if $k \geq 3$ is odd, then*

$$|\nabla(u - \Pi_{2kh}^{2k}(P(R_h^{\Omega}u)))(y_0)| \lesssim h^{k+2}|\ln h|^3\|u\|_{W^{k+3,\infty}(\tau)} + \|u - R_h^{\Omega}u\|_{W^{1-k,2}(\Omega)}. \tag{5.8}$$

Based on Theorems 5.1 and 5.2, we have the following corollary.

Corollary 5.3 *Under the assumptions of Theorem 5.1, if k is even, then*

$$|\nabla(u - \Pi_{2kh}^{2k}(R_h^{\Omega}u))(y_0)| \lesssim h^{k+2}|\ln h|^3\|u\|_{W^{k+3,\infty}(\Omega)}, \tag{5.9}$$

and, if $k \geq 3$ is odd, then

$$|\nabla(u - \Pi_{2kh}^{2k}(P(R_h^{\Omega}u)))(y_0)| \lesssim h^{k+2}|\ln h|^3\|u\|_{W^{k+3,\infty}(\Omega)}. \tag{5.10}$$

Proof Recall a classical result [21]

$$\|u - R_h^{\Omega}u\|_{W^{1-k,2}(\Omega)} \lesssim h^{k+l}\|u\|_{W^{k+l,2}(\Omega)},$$

when $u \in W^{k+l,2}(\Omega)$ for any $2 \leq l \leq k$. Then (5.9) and (5.10) follows from (5.2) and (5.8), respectively.

6 Numerical examples

We consider (1.1) with $\Omega = [0, 1]^2$ and the coefficients

$$a_{11}(x_1, x_2) = 1 + x_1, \quad a_{22}(x_1, x_2) = 1 + x_2,$$
$$a_{12}(x_1, x_2) = a_{21}(x_1, x_2) = 0.25(x_1 + x_2).$$

The problem admits the exact solution

$$u(x) = 16x_1(1 - x_1)x_2(1 - x_2)e^{x_1+x_2}.$$

We will validate (5.9) and (5.10) with numerical experiments. For simplicity, the underlying mesh is chosen as a uniform one which consists of equal-sized isosceles right-angled triangles.

Since the estimates (5.9) and (5.10) are only valid for an interior vertex y_0, without loss of generality, we test our results in the following vertices set

$$\mathcal{N}_h^0 = \mathcal{N}_h^\Omega \cap [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}].$$

Correspondingly, we define the discrete norm $\|v\|_{\infty,h} = \max_{x_j \in \mathcal{N}_h^0} |v(x_j)|$.

We will test our estimates for different orders $k = 2, 3, 4$. Note that once k and the mesh size h are given, the corresponding finite element solution $R_h^\Omega u$ can be computed with the standard finite element method. Let $y_0 \in \mathcal{N}_h^0$, $\Delta y_1 = (h, 0)$ and $\Delta y_2 = (0, h)$. One observes that $\Pi_{2kh}^{2k} R_h^\Omega u(y)$ is a polynomial of degree $2k$ along each direction y_i, $i = 1, 2$.

In the following, we explain how to compute $\Pi_{2kh}^{2k} R_h^\Omega u(y_0)$ and $\Pi_{2kh}^{2k}(P R_h^\Omega)u(y_0)$.

When $k = 2$, we have

$$\frac{\partial \Pi_{2kh}^{2k} R_h^\Omega u(y_0)}{\partial y_i} = \frac{8[R_h^\Omega u(y_0 + \Delta y_i) - R_h^\Omega u(y_0 - \Delta y_i)] - [R_h^\Omega u(y_0 + 2\Delta y_i) - R_h^\Omega u(y_0 - 2\Delta y_i)]}{12h}. \tag{6.1}$$

When $k = 4$, we use

$$\begin{aligned}\frac{\partial \Pi_{2kh}^{2k} R_h^\Omega u(y_0)}{\partial y_i} &\simeq \frac{0.8[R_h^\Omega u(y_0 + \Delta y_i) - R_h^\Omega u(y_0 - \Delta y_i)] - 0.2[R_h^\Omega u(y_0 + 2\Delta y_i) - R_h^\Omega u(y_0 - 2\Delta y_i)]}{h} \\ &+ \frac{0.038095238095[R_h^\Omega u(y_0 + 3\Delta y_i) - R_h^\Omega u(y_0 - 3\Delta y_i)]}{h} \\ &- \frac{0.003571428571[R_h^\Omega u(y_0 + 4\Delta y_i) - R_h^\Omega u(y_0 - 4\Delta y_i)]}{h}.\end{aligned} \tag{6.2}$$

to compute $\frac{\partial \Pi_{2kh}^{2k} R_h^\Omega u(y_0)}{\partial y_i}$.

Table 1 $k = 2$

	$h = 1/20$	$h = 1/24$	$h = 1/28$	$h = 1/32$
$\left\Vert\nabla(u - R_h^\Omega u)\right\Vert_{\infty,h}$	$3.09e{-}002$	$2.16e{-}002$	$1.59e{-}002$	$1.22e{-}002$
$\frac{\left\Vert\nabla(u - R_h^\Omega u)\right\Vert_{\infty,h}}{h^2}$	12.4	12.5	12.5	12.5
$\left\Vert\nabla(u - \Pi_{2kh}^{2k}(R_h^\Omega u))\right\Vert_{\infty,h}$	$2.25e{-}004$	$1.08e{-}004$	$0.583e{-}004$	$0.34e - 004$
$\frac{\left\Vert\nabla(u - \Pi_{2kh}^{2k}(R_h^\Omega u))\right\Vert_{\infty,h}}{h^4}$	36.0	35.9	35.8	35.8

Table 2 $k = 4$

	$h = 1/16$	$h = 1/20$	$h = 1/24$	$h = 1/28$
$\left\Vert\nabla(u - R_h^\Omega u)\right\Vert_{\infty,h}$	$0.56e{-}004$	$0.22e{-}004$	$0.98e{-}005$	$0.51{-}005$
$\frac{\left\Vert\nabla(u - R_h^\Omega u)\right\Vert_{\infty,h}}{h^4}$	3.67	3.45	3.28	3.15
$\left\Vert\nabla(u - \Pi_{2kh}^{2k}(R_h^\Omega u))\right\Vert_{\infty,h}$	$0.26e{-}006$	$0.64e{-}007$	$0.22e{-}007$	$0.86e{-}008$
$\frac{\left\Vert\nabla(u - \Pi_{2kh}^{2k}(R_h^\Omega u))\right\Vert_{\infty,h}}{h^6}$	4.3	4.1	4.1	4.1

Table 3 $k = 3$

	$h = 1/16$	$h = 1/20$	$h = 1/24$
$\left\Vert\nabla(u - R_h^\Omega u)\right\Vert_{\infty,h}$	$2.02e{-}003$	$1.01e{-}003$	$5.8e{-}004$
$\frac{\left\Vert\nabla(u - R_h^\Omega u)\right\Vert_{\infty,h}}{h^3}$	8.28	8.09	7.98
$\left\Vert\nabla(u - P(R_h^\Omega u))\right\Vert_{\infty,h}$	$1.5e{-}005$	$5.8e{-}006$	$2.6e{-}006$
$\frac{\left\Vert\nabla(u - P(R_h^\Omega u))\right\Vert_{\infty,h}}{h^4}$	0.98	0.93	0.89
$\left\Vert\nabla(u - \Pi_{2kh}^{2k}(P(R_h^\Omega u)))\right\Vert_{\infty,h}$	$7.97e{-}008$	$2.28e{-}008$	$0.78e{-}008$
$\frac{\left\Vert\nabla(u - \Pi_{2kh}^{2k}(P(R_h^\Omega u)))\right\Vert_{\infty,h}}{h^5}$	0.084	0.073	0.062

When $k = 3$, we first use (5.7) to compute $P(R_h^\Omega u)(y_0)$ and we obtain

$$\frac{\partial \Pi_{2kh}^{2k} P R_h^\Omega u(y_0)}{\partial y_i} = \frac{45[R_h^\Omega u(y_0 + \Delta y_i) - R_h^\Omega u(y_0 - \Delta y_i)] - 9[R_h^\Omega u(y_0 + 2\Delta y_i) - R_h^\Omega u(y_0 - 2\Delta y_i)]}{60h} + \frac{R_h^\Omega u(y_0 + 3\Delta y_i) - R_h^\Omega u(y_0 - 3\Delta y_i)}{60h}. \tag{6.3}$$

Depicted in Tables 1, 2, 3 are our numerical ultraconvergence results corresponding to the finite element degree $k = 2, 4, 3$ respectively.

From Tables 1, 2, 3, we observe that the gradients of the post-processed FE solutions approximate the gradient of u with orders $\mathcal{O}(h^4)$, $\mathcal{O}(h^6)$ and $\mathcal{O}(h^5)$, respectively, which validate the estimates (5.9) and (5.10). Moreover, it is interesting to find that the hidden constant is independent of the mesh size h which indicates that maybe the 'lnh' appeared in the right-hand side of the estimates (5.9) and (5.10) can be removed.

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References

1. Asadzadeh, M., Schatz, A., Wendland, W.: Asymptotic error expansions for the finite element method for second order elliptic problems in $R^N (N \geq 2)$, I: Local interior expansions. SIAM J. Numer. Anal. **48**, 2000–2017 (2010)
2. Asadzadeh, M., Schatz, A., Wendland, W.: A non-standard approach to Richardson extrapolation in the finite element method for second order elliptic problems. Math. Comp. **78**, 1951–1973 (2009)
3. Bacuta, C., Nistor, V., Zikatanov, L.T.: Improving the rate of convergence of 'high order finite elements on polyhedra I: a priori estimates. Numer. Funct. Anal. Optim. **26**, 613–639 (2005)
4. Blum, H., Lin, Q., Rannacher, R.: Asymptotic error expansions and Richardson extrapolation for linear finite elements. Numer. Math. **49**, 11–37 (1986)
5. Bank, R.E., Xu, J.: Asymptotic exact a posteriori error estimates, Part I: grids with superconvergence. SIAM J. Numer. Anal. **41**(6), 2294–2312 (2003)
6. Chen, C., Huang, Y.: High accuracy theory of finite element methods. Hunan Science and Technology Press, People's Republic of China (1995). (in Chinese)
7. Chen, C., Lin, Q.: Extrapolation of finite element approximations in a rectangular domain. J. Comput. Math. **7**, 235–255 (1989)
8. Chen, L., Holst, M., Xu, J.: Convergence and optimality of adaptive mixed finite element methods. Math. Comput. **78**, 35–53 (2009)
9. Chen, L., Sun, P., Xu, J.: Optimal anisotropic meshes for minimizing interpolation errors in L^p-norm. Math. Comput. **76**, 179–204 (2007)
10. Grisvard, P.: Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain, numerical solution of partial differential equations III (Hubbard, B., ed.), Academic Press, New York (1976)
11. John F.: General properties of solutions of linear elliptic partial differential equations. Proc. Sympos. on Spectral Theory and Differential Problems, Oklahoma A & M College, Stillwater, Okla., pp. 113–175. MR 13, 349 (1951)
12. Miranda, C.: Partial differential equations of elliptic type, 2nd edn. Springer, Berlin (1970)
13. He, W., Guan, X., Cui, J.: The Local Superconvergence of the trilinear element for the three-dimensional Poisson problem. J. Math. Anal. Appl. **388**, 863–872 (2012)
14. Chen, C., Hu, S.: The highest order superconvergence for bi-k degree rectangular elements at nodes-a proof of $2k$-conjecture. Math. Comp. **82**, 1337–1355 (2013)
15. Huang, Y., Xu, J.: Superconvergence of quadratic finite elements on mildly structured grids. Math. Comp. **77**(263), 1253–1268 (2008)
16. Krasovskii, J.P.: Isolation of singularities of the Green's function. Math. USSR-IZV **1**, 935–966 (1967)

17. Lin, Q.: Fourth order eigenvalue approximation by extrapolation on domains with reentrant corners. Numer. Math. **58**, 631–640 (1991)
18. Lin, Q., Yan, N.: Construction and analysis for finite element methods, Hebei University (1996)
19. Lin, Q., Zhou, J.: Superconvergence in high-order Galerkin finite element methods. Comput. Method Appl. Mech. Eng. **196**, 3779–3784 (2007)
20. Lin, Q., Zhu, Q.: The preprocessing and postprocessing for finite element methods. Shanghai Scientific and Technical Publishers, Shanghai (1994) (in Chinese)
21. Nitsche, J., Schatz, A.H.: Interior estimates for Ritz-Galerkin methods. Math. Comp. **28**, 937–955 (1974)
22. Schatz, A.H., Wahlbin, L.B.: Interior maximum norm estimates for finite element methods, part II. Math. Comp. **64**, 907–928 (1995)
23. Schatz, A.H., Sloan, L.H., Wahlbin, L.B.: Superconvergence in finite element methods and meshes that are locally symmetric with respect to a point. SIAM J. Numer. Anal. **33**, 505–521 (1996)
24. Schatz, A.H., Wahlbin, L.B.: Interior maximum norm estimates for finite element methods. Math. Comp. **31**, 414–442 (1977)
25. Schatz, A.H., Wahlbin, L.B.: Asymptotically exact a posterior estimators for the pointwise gradient error on each element in irregular meshes. part II: The piecewise linear case. Math. Comp. **73**, 517–523 (2004)
26. Wahlbin, L.B.: Superconvergence in Galerkin finite element methods. Springer, Berlin (1995)
27. Wahlbin, L.B.: General principles of superconvergence in Galerkin finite element methods. Lect. Notes Pure Appl. Math. **198**, 269–285 (1998)
28. Xu, J., Zhang, Z.: Analysis of recovery type a posteriori error estimates for mildly structured grids. Math. Comp. **73**, 1139–1152 (2004)
29. Zhang, T.: The derivative patch interpolating recovery technique and superconvergence. Numer. Math. Appl. **2**, 1–10 (2001)
30. Zhang, Z.: Recovery technique in finite element methods. In: Adaptive computations: theory and algorithm, Edited by Tao Tang, Jinchao Xu, Mathematics Monograph Series 6. Science Publisher, Beijing, People's Republic of China, pp. 333–412 (2007)
31. Zhang, Z.: Polynomial preserving recovery for anisotropic and irregular grids. J. Comp. Math. **22**, 331–340 (2004)
32. Zhang, Z.: Ultraconvergence of the patch recovery technique II. Math. Comp. **69**, 141–158 (2000)
33. Zhang, Z., Naga, A.: A new finite element gradient recovery method: superconvergence Property. SIAM J. on Sci. Comput. **26**, 1192–1213 (2005)
34. Zienkiewicz, O.C., Zhu, J.: The superconvergence patch recovery (SPR) and adaptive finite element refinement. Comput. Methods Appl. Mech. Eng. **101**, 207–224 (1992)
35. Zhu, Q., Lin, Q.: Theory of superconvergence of finite elements. Hunan Science and Technology Press, Hunan (1989). (in Chinese)
36. Zhu, Q.: High precision and postprocessing theory of finite element method. Science Publisher, Beijing (2008). (in Chinese)
37. Zienkiewicz, O.C., Zhu, J.: The superconvergence patch recovery and a posteriori estimates Part I: the recovery technique. Int. J. Num. Methods Eng. **33**, 1331–1364 (1992)