Numer. Math. (2017) 135:711–732
DOI 10.1007/s00211-016-0814-1

Numerische Mathematik

A fictitious domain approach with Lagrange multiplier for fluid-structure interactions

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Received: 16 October 2015 / Revised: 29 April 2016 / Published online: 4 June 2016

Abstract We study a recently introduced formulation for fluid-structure interaction problems which makes use of a distributed Lagrange multiplier in the spirit of the fictitious domain method. The time discretization of the problem leads to a mixed problem for which a rigorous stability analysis is provided. The finite element space discretization is discussed and optimal convergence estimates are proved.

Mathematics Subject Classification 65N30 · 65N12 · 74F10

1 Introduction

Numerical schemes for fluid-structure interaction problems include interface fitted meshes (thus requiring suitable remeshing in order to keep the fluid computational grid aligned with the interface) or interface non-fitted meshes (allowing to keep the fluid computational grid fixed and independent from the position of the solid).

The immersed boundary method (see [21] for a review) is a typical example of non-fitted schemes. It has been introduced in the 70's for the simulation of biological problems related to the blood flow in the heart and it has been extended to finite elements in a series of papers starting from [5] by using a variational approach (FE- IBM)

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and [22] where finite elements and reproducing kernel particle methods are combined. The FE- IBM allows for *thick* (i.e., of codimension zero) or *thin* (i.e., of codimension one) structures. In particular, in [7] the original fiber-like description of the structure has been abandoned in favor of a more natural and intrinsically *thick* modeling of the solid domain. With this representation, a unified treatment of immersed structures is possible in any combination of dimensions. Two novelties have been introduced in [18]: a compressible model for the structure has been considered, and the motion of the solid has been taken care with a fully variational approach.

In [4] a new formulation (DLM- IBM) for fluid-structure interaction problems has been introduced based on the FE- IBM which makes use of a distributed Lagrange multiplier in the spirit of the fictitious domain method (see, for instance, [14–16]). The DLM- IBM in the codimension one case has some similarities with the so called "immersogeometric" method recently introduced in [19,20]. An important feature of the DLM- IBM, as it has been shown in [4], is that its semi-implicit time discretization results to be unconditionally stable as opposed to the standard FE- IBM where a suitable CFL condition has to be satisfied (see [3,6,17]).

The time discretization of the problem leads to a saddle point problem (see Problem 5 and its discrete counterpart Problem 6). The main contribution of this paper is the rigorous analysis of Problems 5 and 6: it is shown that the saddle point problems are stable and that the discrete solution converges optimally towards the continuous one. Suitable conditions on the solid mesh are stated: in the case of codimension zero structures, the mesh is assumed to allow H^1 stability for the L^2 projection (more detailed description of this assumption is given in the discussion after Proposition 7); in the case of codimension one structures, the solid meshsize is assumed to satisfy a suitable compatibility condition with respect to the fluid one.

The structure of the paper is the following: in Sect. 2 we introduce the problem and derive the DLM- IBM in the case of solids of codimension zero. This is a new approach and provides an interesting result, since the DLM- IBM, previously derived as a modification of the FE- IBM, is now seen as a natural fictitious domain formulation originating from a strong form of a fluid-structure interaction problem. Section 3 recalls the semi-implicit time discretization of the DLM- IBM and the known energy estimates. Section 4 is the core of our paper in the case of thick structures, presenting the analysis of the mixed problem and of its numerical approximation. Finally, Sect. 5 performs the same analysis of the mixed problem in the case where thin structures are considered.

2 Fictitious domain approach in the case of a thick solid immersed in a fluid

The fluid-structure interaction system that we are going to analyze in this paper consists of a solid elastic body immersed in a fluid. We refer to a *thick* solid when it occupies a domain of codimension zero, and to a *thin* solid when the corresponding domain can be reduced to a region of codimension one in the fluid by using standard assumptions on the behavior of the involved physical quantities. This case will be treated in Sect. 5.

Let $\Omega_t^f \subset \mathbb{R}^d$ and $\Omega_t^s \subset \mathbb{R}^d$ with $d = 2, 3$ be the time dependent regions occupied by the fluid and the structure, respectively. We set Ω the interior of $\overline{\Omega}_t^f \cup \overline{\Omega}_t^s$ and assume that Ω is a fixed domain. We denote by $\Gamma_t = \partial\Omega_t^f \cap \partial\Omega_t^s$ the moving interface between the fluid and the solid regions. For simplicity, we assume that the structure is immersed in the fluid so that $\partial\Omega_t^s \cap \partial\Omega = \emptyset$.

Assuming that both the fluid and the solid material are incompressible, the fluid-structure interaction problem can be written in a very general form as follows:

$$\begin{array}{ll} \rho_f \dot{\mathbf{u}}_f = \mathrm{div}\boldsymbol{\sigma}_f & \text{in } \Omega_t^f \\ \mathrm{div}\mathbf{u}_f = 0 & \text{in } \Omega_t^f \\ \rho_s \dot{\mathbf{u}}_s = \mathrm{div}\boldsymbol{\sigma}_s & \text{in } \Omega_t^s \\ \mathrm{div}\mathbf{u}_s = 0 & \text{in } \Omega_t^s \\ \mathbf{u}_f = \mathbf{u}_s & \text{on } \Gamma_t \\ \boldsymbol{\sigma}_f \mathbf{n}_f = -\boldsymbol{\sigma}_s \mathbf{n}_s & \text{on } \Gamma_t. \end{array} \tag{1}$$

The system can be complemented with the following initial and boundary conditions on $\partial\Omega$:

$$\begin{array}{ll} \mathbf{u}_f(0) = \mathbf{u}_{f0} & \text{on } \Omega_0^f, \\ \mathbf{u}_s(0) = \mathbf{u}_{s0} & \text{on } \Omega_0^s, \\ \mathbf{u}_f(t) = 0 & \text{on } \partial\Omega. \end{array} \tag{2}$$

In (1) $\mathbf{u}$, $\boldsymbol{\sigma}$, and ρ denote velocity, stress tensor, and mass density, respectively. The subscript f or s refers to fluid or solid. We assume moreover that ρ_f and ρ_s are positive constants.

In the following we introduce the constitutive laws for fluid and solid materials and derive the variational formulation of (1)–(2).

First of all, let us define some functional spaces we shall work with. For a domain ω we denote by $L^2(\omega)$ the space of square integrable functions in ω, endowed with the norm $\|v\|_{0,\omega}^2 = \int_\omega |v|^2 dx$ and the corresponding scalar product denoted by $(\cdot,\cdot)_\omega$. Then $H^1(\omega)$ is the space of functions belonging to $L^2(\omega)$ together with their gradient; then $\|v\|_{1,\omega}^2 = \|v\|_{0,\omega}^2 + \|\nabla v\|_{0,\omega}^2$ defines the norm in $H^1(\omega)$. We denote by $H_0^1(\omega)$ the subspace of $H^1(\omega)$ of functions vanishing on the boundary of ω and by $L_0^2(\omega)$ the subspace of $L^2(\omega)$ of functions with zero mean value. When $\omega = \Omega$ we omit the subscripts Ω.

The equations in (1) are written using the Eulerian description, but the deformation of the solid is usually described in the Lagrangian framework. For this, we consider Ω_t^s as the image of a reference domain $\mathcal{B} \subset \mathbb{R}^d$. For every $t \in [0, T]$, we associate points $\mathbf{s} \in \mathcal{B}$ and $\mathbf{x} \in \Omega_t^s$ via a family of mappings $\mathbf{X}(t) : \mathcal{B} \to \Omega_t^s$. We refer to $\mathbf{s} \in \mathcal{B}$ as the material or Lagrangian coordinate and to $\mathbf{x} = \mathbf{X}(\mathbf{s}, t)$ as the spatial or Eulerian coordinate with $\mathbf{x} \in \Omega_t^s$. We assume that $\mathbf{X}$ fulfills the following conditions: $\mathbf{X}(t) \in W^{1,\infty}(\mathcal{B})$, $\mathbf{X}(t)$ is one to one, and there exists a constant γ such that for all $t \in [0, T]$ $\|\mathbf{X}(\mathbf{s}_1, t) - \mathbf{X}(\mathbf{s}_2, t)\| \geq \gamma\|\mathbf{s}_1 - \mathbf{s}_2\|$ for all $\mathbf{s}_1, \mathbf{s}_2 \in \mathcal{B}$. Note that this requirements imply that $\mathbf{X}(t)$ is invertible with Lipschitz inverse. This in particular implies that $\mathbf{Y} \in H^1(\mathcal{B})^d$ if and only if $\mathbf{v} = \mathbf{Y}(\mathbf{X}^{-1}(t)) \in H^1(\Omega_t^s)^d$. The deformation gradient is defined as $\mathbb{F} = \nabla_s \mathbf{X}$ and we indicate with $|\mathbb{F}|$ its determinant. In (1), the dot over the velocity denotes the material time derivative. In the fluid, using the Eulerian description, we have $\dot{\mathbf{u}}_f = \partial \mathbf{u}_f/\partial t + \mathbf{u}_f \cdot \nabla \mathbf{u}_f$. In the solid, the Lagrangian framework is preferred and the spatial description of the material velocity reads

$$\mathbf{u}_s(\mathbf{x}, t) = \frac{\partial \mathbf{X}(\mathbf{s}, t)}{\partial t}\Big|_{\mathbf{x}=\mathbf{X}(\mathbf{s},t)} \tag{3}$$

so that $\dot{\mathbf{u}}_s(\mathbf{x}, t) = \partial^2 \mathbf{X}(\mathbf{s}, t)/\partial t^2|_{\mathbf{x}=\mathbf{X}(\mathbf{s},t)}$. Thanks to the incompressibility condition for fluid and solid, expressed by the divergence free condition in (1), it results that $|\mathbb{F}|$ is constant in time and equals its initial value. In particular, if the reference domain $\mathcal{B}$ coincides with the initial position of the solid Ω_0^s one has that $|\mathbb{F}| = 1$ for all t.

Let us introduce now the constitutive laws for fluid and solid materials, in order to model the stress tensor. We consider a Newtonian fluid characterized by the usual Navier–Stokes stress tensor

$$\boldsymbol{\sigma}_f = -p_f \mathbb{I} + \nu_f \nabla_{\text{sym}} \mathbf{u}_f, \tag{4}$$

where $\nabla_{\text{sym}} \mathbf{u} = (1/2)\left(\nabla \mathbf{u}_f + (\nabla \mathbf{u}_f)^\top\right)$ is the symmetric gradient and ν_f represents the viscosity of the fluid. The solid material is assumed to be viscous-hyperelastic, so that the Cauchy stress tensor can be represented as the sum $\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s^f + \boldsymbol{\sigma}_s^s$ of a fluid-like part, with viscosity ν_s,

$$\boldsymbol{\sigma}_s^f = -p_s \mathbb{I} + \nu_s \nabla_{\text{sym}} \mathbf{u}_s \tag{5}$$

and an elastic part $\boldsymbol{\sigma}_s^s$. By changing variable from Eulerian to Lagrangian, we express $\boldsymbol{\sigma}_s^s$ in term of the first Piola–Kirchhoff stress tensor $\mathbb{P}$:

$$\mathbb{P}(\mathbb{F}(\mathbf{s}, t)) = |\mathbb{F}(\mathbf{s}, t)| \boldsymbol{\sigma}_s^s(\mathbf{x}, t) \mathbb{F}^{-\top}(\mathbf{s}, t) \qquad \text{for } \mathbf{x} = \mathbf{X}(\mathbf{s}, t). \tag{6}$$

On the other hand, hyperelastic materials are characterized by a positive energy density $W(\mathbb{F})$ which is related to the Piola–Kirchhoff stress tensor as follows:

$$(\mathbb{P}(\mathbb{F}(\mathbf{s}, t))_{\alpha i} = \frac{\partial W}{\partial \mathbb{F}_{\alpha i}}(\mathbb{F}(\mathbf{s}, t)) = \left(\frac{\partial W}{\partial \mathbb{F}}(\mathbb{F}(\mathbf{s}, t))\right)_{\alpha i}, \tag{7}$$

where $i = 1, \ldots, m$ and $\alpha = 1, \ldots, d$. The elastic potential energy of the body is given by:

$$E\left(\mathbf{X}(t)\right) = \int_{\mathcal{B}} W(\mathbb{F}(s, t)) d\mathbf{s}. \tag{8}$$

Let $\mathbf{v} \in H_0^1(\Omega)^d$ be given. We multiply the first equation in (1) by $\mathbf{v}|_{\Omega_t^f}$, integrate over Ω_t^f, and integrate by parts; analogously, we multiply the third equation by $\mathbf{v}|_{\Omega_t^s}$ and integrate over Ω_t^s, and integrate by parts. Summing up the two equations and taking into account the transmission conditions on Γ_t, we obtain the following equation which corresponds to the principle of virtual work:

$$\int_{\Omega_t^f} \rho_f \dot{\mathbf{u}}_f \ \mathbf{v} d\mathbf{x} + \int_{\Omega_t^s} \rho_s \dot{\mathbf{u}}_s \ \mathbf{v} d\mathbf{x} + \int_{\Omega_t^f} \boldsymbol{\sigma}_f : \nabla_{\text{sym}} \mathbf{v} d\mathbf{x} + \int_{\Omega_t^s} \boldsymbol{\sigma}_s : \nabla_{\text{sym}} \mathbf{v} d\mathbf{x} = 0.$$

Introducing the models (4)–(5) and taking into account (3) and (6), we arrive to the following equation

$$\begin{aligned}
&\int_{\Omega_t^f} \rho_f \dot{\mathbf{u}}_f \ \mathbf{v} d\mathbf{x} + \int_{\mathcal{B}} \rho_s \frac{\partial^2 \mathbf{X}}{\partial t^2} \mathbf{v}(\mathbf{X}(\mathbf{s},t)) d\mathbf{s} + \int_{\Omega_t^f} \nu_f \nabla_{\text{sym}} \mathbf{u}_f : \nabla_{\text{sym}} \mathbf{v} d\mathbf{x} \\
&- \int_{\Omega_t^f} p_f \operatorname{div} \mathbf{v} d\mathbf{x} + \int_{\Omega_t^s} \nu_s \nabla_{\text{sym}} \mathbf{u}_s : \nabla_{\text{sym}} \mathbf{v} d\mathbf{x} \\
&+ \int_{\mathcal{B}} \mathbb{P}(\mathbb{F}(\mathbf{s},t)) : \nabla_s \mathbf{v}(\mathbf{X}(\mathbf{s},t)) d\mathbf{s} - \int_{\Omega_t^s} p_s \operatorname{div} \mathbf{v} d\mathbf{x} = 0 \quad \forall \mathbf{v} \in H_0^1(\Omega)^d, \quad (9)
\end{aligned}$$

where we used the standard notation $\mathbb{D} : \mathbb{E} = \sum_{\alpha,i=1}^{d} \mathbb{D}_{\alpha i} \mathbb{E}_{\alpha i}$ for all tensors $\mathbb{D}$ and $\mathbb{E}$.

We observe that in (9) p_f and p_s are not uniquely determined. In fact, if we take $p_f + c_f$ and $p_s + c_s$ instead of p_f and p_s, respectively, the left hand side of (9) does not change if $c_f = c_s$. To avoid this situation we impose that

$$\int_{\Omega_t^f} p_f d\mathbf{x} + \int_{\Omega_t^s} p_s d\mathbf{x} = 0. \quad (10)$$

At the end, the incompressibility condition for both materials can be written in variational form as:

$$\int_{\Omega_t^f} \operatorname{div} \mathbf{u}_f q d\mathbf{x} + \int_{\Omega_t^s} \operatorname{div} \mathbf{u}_s q d\mathbf{x} = 0 \quad \forall q \in L_0^2(\Omega). \quad (11)$$

Then the fluid-structure interaction problem can be written in the following form.

Problem 1 *For* $t \in]0, T]$ *find* $\mathbf{u}_f(t) \in H^1(\Omega_t^f)^d$, $p_f(t) \in L^2(\Omega_t^f)$, $\mathbf{u}_s(t) \in H^1(\Omega_t^s)^d$, $p_s(t) \in L^2(\Omega_t^s)$, *and* $\mathbf{X}(t) \in H^1(\mathcal{B})^d$ *such that* $\mathbf{u}_f(t) = \mathbf{u}_s(t)$ *on* Γ_t, *and equations* (9), (10), (11), (3), *and* (2) *are satisfied together with* $\mathbf{X}(0) = \mathbf{X}_0$ *on* $\mathcal{B}$, *where* $\mathbf{X}_0 : \mathcal{B} \to \Omega_0^s$.

Remark Thanks to (3), the initial condition for $\mathbf{u}_s$ provides also an initial condition for $\partial \mathbf{X}/\partial t$.

In the following, we use a *fictitious domain* approach with a distributed Lagrange multiplier in order to rewrite the variational formulation of the problem. Namely, we extend the fluid velocity and pressure into the solid domain by introducing new unknowns with the following meaning:

$$\mathbf{u} = \begin{cases} \mathbf{u}_f & \text{in } \Omega_t^f \\ \mathbf{u}_s & \text{in } \Omega_t^s \end{cases}, \quad p = \begin{cases} p_f & \text{in } \Omega_t^f \\ p_s & \text{in } \Omega_t^s \end{cases} \quad (12)$$

with the condition that the material velocity of the solid is equal to the velocity of the fictitious fluid, that is

$$\frac{\partial \mathbf{X}(\mathbf{s},t)}{\partial t} = \mathbf{u}(\mathbf{X}(\mathbf{s},t),t) \quad \text{for } \mathbf{s} \in \mathcal{B}. \tag{13}$$

This equation, which governs the evolution of the immersed solid, represents a constraint for the problem, therefore we enforce it in variational form by introducing a Lagrange multiplier as follows. Let $\mathbf{\Lambda}$ be a functional space to be defined later on and $\mathbf{c} : \mathbf{\Lambda} \times H^1(\mathcal{B})^d \to \mathbb{R}$ a bilinear form such that

$$\begin{aligned} &\mathbf{c} \text{ is continuous on } \mathbf{\Lambda} \times H^1(\mathcal{B})^d \\ &\mathbf{c}(\boldsymbol{\mu}, \mathbf{Z}) = 0 \text{ for all } \boldsymbol{\mu} \in \mathbf{\Lambda} \text{ implies } \mathbf{Z} = 0. \end{aligned} \tag{14}$$

For example we can take as $\mathbf{\Lambda}$ the dual space of $H^1(\mathcal{B})^d$ and define $\mathbf{c}$ as the duality pairing between $H^1(\mathcal{B})^d$ and $(H^1(\mathcal{B})^d)'$, that is:

$$\mathbf{c}(\boldsymbol{\mu}, \mathbf{Y}) = \langle \boldsymbol{\mu}, \mathbf{Y} \rangle \quad \forall \boldsymbol{\mu} \in (H^1(\mathcal{B})^d)', \ \mathbf{Y} \in H^1(\mathcal{B})^d. \tag{15}$$

Alternatively, one can set $\mathbf{\Lambda} = H^1(\mathcal{B})^d$ and define

$$\mathbf{c}(\boldsymbol{\mu}, \mathbf{Y}) = (\nabla_s \boldsymbol{\mu}, \nabla_s \mathbf{Y})_{\mathcal{B}} + (\boldsymbol{\mu}, \mathbf{Y})_{\mathcal{B}} \quad \forall \boldsymbol{\mu}, \ \mathbf{Y} \in H^1(\mathcal{B})^d. \tag{16}$$

Relation (13) can now be written in variational form as:

$$\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}(\mathbf{X}(\cdot,t),t) - \frac{\partial \mathbf{X}}{\partial t}(t)\right) = 0 \quad \forall \boldsymbol{\mu} \in \mathbf{\Lambda}. \tag{17}$$

Then the problem can be formulated in the following weak form.

Problem 2 *Given* $\mathbf{u}_0 \in H^1_0(\Omega)^d$ *and* $\mathbf{X}_0 \in W^{1,\infty}(\mathcal{B})^d$, *for almost every* $t \in]0,T]$ *find* $(\mathbf{u}(t), p(t)) \in H^1_0(\Omega)^d \times L^2_0(\Omega)$, $\mathbf{X}(t) \in H^1(\mathcal{B})^d$, *and* $\boldsymbol{\lambda}(t) \in \mathbf{\Lambda}$ *such that it holds*

$$\begin{aligned} &\rho_f \frac{d}{dt}(\mathbf{u}(t), \mathbf{v}) + b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) + a(\mathbf{u}(t), \mathbf{v}) \\ &\qquad - (\mathrm{div}\mathbf{v}, p(t)) + \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{v}(\mathbf{X}(\cdot,t))) = 0 \qquad \forall \mathbf{v} \in H^1_0(\Omega)^d \end{aligned} \tag{18a}$$

$$(\mathrm{div}\mathbf{u}(t), q) = 0 \qquad \forall q \in L^2_0(\Omega) \tag{18b}$$

$$\delta_\rho \left(\frac{\partial^2 \mathbf{X}}{\partial t^2}(t), \mathbf{Y}\right)_{\mathcal{B}} + (\mathbb{P}(\mathbb{F}(t)), \nabla_s \mathbf{Y})_{\mathcal{B}} - \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{Y}) = 0 \qquad \forall \mathbf{Y} \in H^1(\mathcal{B})^d \tag{18c}$$

$$\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}(\mathbf{X}(\cdot,t),t) - \frac{\partial \mathbf{X}}{\partial t}(t)\right) = 0 \qquad \forall \boldsymbol{\mu} \in \mathbf{\Lambda} \tag{18d}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega, \qquad \mathbf{X}(0) = \mathbf{X}_0 \quad \text{in } \mathcal{B}. \tag{18e}$$

Here $\delta_\rho = \rho_s - \rho_f$ and

$$a(\mathbf{u},\mathbf{v}) = (\nu\boldsymbol{\nabla}_{\mathrm{sym}}\mathbf{u}, \boldsymbol{\nabla}_{\mathrm{sym}}\mathbf{v}) \quad \text{with } \nu = \begin{cases} \nu_f & \text{in } \Omega_t^f \\ \nu_s & \text{in } \Omega_t^s \end{cases},$$
$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{\rho_f}{2}\left((\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{v},\mathbf{w}) - (\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{w},\mathbf{v})\right).$$

We assume that $\nu \in L^\infty(\Omega)$ and that there exists a positive constant $\nu_0 > 0$ such that $\nu \geq \nu_0 > 0$ in Ω.

Remark In the literature of the Immersed Boundary Method, it is generally assumed that the fluid and the solid visco-hyperelastic materials have the same viscosity. If this is not the case, the integral in the definition of a has to be decomposed into the integral over Ω_t^f and Ω_t^s. Therefore, in the finite element discretization, the associated stiffness matrix has to be recomputed at each time step as ν is discontinuous along the moving interface Γ_t. This could be avoided if we treat this term using the fictitious domain method as it is done for the first integral containing the time derivative of $\mathbf{u}$. Then Problem 2 takes the following form.

Problem 3 *Given* $\mathbf{u}_0 \in H_0^1(\Omega)^d$ *and* $\mathbf{X}_0 \in W^{1,\infty}(\mathcal{B})^d$, *find* $(\mathbf{u}(t), p(t)) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$, $\mathbf{X}(t) \in H^1(\mathcal{B})^d$, *and* $\boldsymbol{\lambda}(t) \in \boldsymbol{\Lambda}$, *such that for almost every* $t \in]0,T]$ *it holds*

$$\begin{aligned}
&\rho_f \frac{d}{dt}(\mathbf{u}(t),\mathbf{v}) + b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}) + \tilde{a}(\mathbf{u}(t),\mathbf{v}) \\
&\quad -(\mathrm{div}\mathbf{v}, p(t)) + \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{v}(\mathbf{X}(\cdot,t))) = 0 && \forall \mathbf{v} \in H_0^1(\Omega)^d \\
&(\mathrm{div}\mathbf{u}(t), q) = 0 && \forall q \in L_0^2(\Omega) \\
&\delta_\rho \left(\frac{\partial^2 \mathbf{X}}{\partial t^2}(t), \mathbf{Y}\right)_{\mathcal{B}} + d\left(\frac{\partial \mathbf{X}}{\partial t}(t), \mathbf{Y}\right) \\
&+(\mathbb{P}(\mathbb{F}(t)), \boldsymbol{\nabla}_s \mathbf{Y})_{\mathcal{B}} - \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{Y}) = 0 && \forall \mathbf{Y} \in H^1(\mathcal{B})^d \\
&\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}(\mathbf{X}(\cdot,t),t) - \frac{\partial \mathbf{X}}{\partial t}(t)\right) = 0 && \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda} \\
&\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega, \qquad \mathbf{X}(0) = \mathbf{X}_0 \quad \text{in } \mathcal{B},
\end{aligned}$$

where $\tilde{a}(\mathbf{u},\mathbf{v}) = \nu_f(\boldsymbol{\nabla}_{\mathrm{sym}}\mathbf{u}, \boldsymbol{\nabla}_{\mathrm{sym}}\mathbf{v})$ and

$$d(\mathbf{X},\mathbf{Y}) = \frac{1}{2}\int_{\mathcal{B}} (\nu_s - \nu_f)\left(\boldsymbol{\nabla}_s\mathbf{X}\mathbb{F}^{-1} + \mathbb{F}^{-\top}\boldsymbol{\nabla}_s\mathbf{X}^\top\right) : \left(\boldsymbol{\nabla}_s\mathbf{Y}\mathbb{F}^{-1} + \mathbb{F}^{-\top}\boldsymbol{\nabla}_s\mathbf{Y}^\top\right)|\mathbb{F}|d\mathbf{s}.$$

For the sake of simplicity, in the rest of this paper we are going to consider a constant viscosity throughout the domain.

First of all we show that Problems 1 and 2 are equivalent.

Theorem 1 *Let* $(\mathbf{u}_f, \mathbf{u}_s, p_f, p_s, \mathbf{X})$ *be a solution of Problem 1, such that* $\mathbf{X}(t) \in W^{1,\infty}(\mathcal{B})^d$ *and* $\mathbf{X}(t) : \mathcal{B} \to \Omega_t^s$ *is one to one, then setting* $(\mathbf{u}, p)$ *as in (12), there exists* $\boldsymbol{\lambda}(t) \in \boldsymbol{\Lambda}$ *such that* $(\mathbf{u}, p, \mathbf{X}, \boldsymbol{\lambda})$ *is a solution of Problem 2.*

Conversely, let $(\mathbf{u}, p, \mathbf{X}, \boldsymbol{\lambda})$ *be a solution of Problem 2 such that* $\mathbf{X}(t) \in W^{1,\infty}(\mathcal{B})^d$ *and* $\mathbf{X}(t) : \mathcal{B} \to \Omega_t^s$ *is one to one. Set* $\mathbf{u}_f(t) = \mathbf{u}(t)|_{\Omega_t^f}$, $p_f(t) = p(t)|_{\Omega_t^f}$, $\mathbf{u}_s(t) = \mathbf{u}(t)|_{\Omega_t^s}$, $p_s(t) = p(t)|_{\Omega_t^s}$, *then* $(\mathbf{u}_f, \mathbf{u}_s, p_f, p_s, \mathbf{X})$ *is a solution of Problem 1.*

Proof Let $(\mathbf{u}_f, \mathbf{u}_s, p_f, p_s, \mathbf{X})$ be a solution of Problem 1. Taking into account that $\partial\Omega_t^s \cap \partial\Omega = \emptyset$, the third condition in (2) gives $\mathbf{u} = 0$ on $\partial\Omega$. From (12), we obtain that $\mathbf{u} \in H_0^1(\Omega)^d$, since $\mathbf{u}_f(t) \in H^1(\Omega_t^f)^d$, $\mathbf{u}_s(t) \in H^1(\Omega_t^s)^d$, and $\mathbf{u}_f(t) = \mathbf{u}_s(t)$ on Γ_t, and that $p \in L_0^2(\Omega)$ thanks to (10). Next (11) implies (18b), while (14), and (17) gives that (18d) holds true. Setting

$$\mathbf{u}_0 = \begin{cases} \mathbf{u}_{f0} & \text{in } \Omega_0^f \\ \mathbf{u}_{s0} & \text{in } \Omega_0^s, \end{cases}$$

the initial conditions (18e) are satisfied. It remains to prove (18a) and (18c). For this, we introduce $\boldsymbol{\lambda}(t) \in \boldsymbol{\Lambda}$ such that (18a) is satisfied. Differentiating condition (13) with respect to time gives $\dot{\mathbf{u}}(\mathbf{x}, t) = \partial^2\mathbf{X}(\mathbf{s}, t)/\partial t^2|_{\mathbf{x}=\mathbf{X}(\mathbf{s},t)}$, hence recalling the incompressibility of the structure, we have the following equality

$$\int_{\Omega_t^f} \rho_f \dot{\mathbf{u}}\, \mathbf{v} d\mathbf{x} = \int_{\mathcal{B}} \rho_f \frac{\partial^2\mathbf{X}(\mathbf{s}, t)}{\partial t^2} \mathbf{v}(\mathbf{X}(\mathbf{s}, t)) d\mathbf{s}.$$

Then taking into account the definition of the forms a and b and (17), we have from (9):

$$\int_{\mathcal{B}} \delta_\rho \frac{\partial^2\mathbf{X}}{\partial t^2} \mathbf{v}(\mathbf{X}(\mathbf{s}, t)) d\mathbf{s} + \int_{\mathcal{B}} \mathbb{P}(\mathbb{F}(\mathbf{s}, t)) : \nabla_s \mathbf{v}(\mathbf{X}(\mathbf{s}, t)) d\mathbf{s} = \mathbf{c}(\boldsymbol{\lambda}(t), \mathbf{v}(\mathbf{X}(\cdot, t)))$$

for all $\mathbf{v} \in H_0^1(\Omega)^d$. Since $\mathbf{X}(t) : \mathcal{B} \to \Omega_t^s$ is one to one and belongs to $W^{1,\infty}(\mathcal{B})$, $\mathbf{Y} = \mathbf{v}(\mathbf{X}(\cdot, t))$ is an arbitrary element of $H^1(\mathcal{B})^d$ and (18c) holds true.

Let us now prove the converse. Let $(\mathbf{u}, p, \mathbf{X}, \boldsymbol{\lambda})$ be a solution of Problem 2 and set $\mathbf{u}_f(t) = \mathbf{u}(t)|_{\Omega_t^f}$, $p_f(t) = p(t)|_{\Omega_t^f}$, $\mathbf{u}_s(t) = \mathbf{u}(t)|_{\Omega_t^s}$, $p_s(t) = p(t)|_{\Omega_t^s}$. From (18d) and (14) we have that (13) is fulfilled. Using again the fact that $\mathbf{X}(t) : \mathcal{B} \to \Omega_t^s$ is one to one, we take $\mathbf{Y} = \mathbf{v}(\mathbf{X}(t))$ in (18c) and sum it to (18a). Equations (9) and (11) follow from the definition of $\mathbf{u}_f$, $\mathbf{u}_s$, p_f and p_s. Moreover, we have that the condition (10) holds true since $p \in L_0^2(\Omega)$. It is easy to verify that the initial and boundary conditions are fulfilled. □

Thanks to the elastic properties of the viscous-hyperelastic material, (see (7) and (8)), we have the following energy estimate (see [4] for the details).

Proposition 2 *Let us assume that $\delta_\rho \geq 0$, that the potential energy density W is a C^1 convex function over the set of second order tensors and that for almost every $t \in [0, T]$, the solution of Problem 2 is such that $\mathbf{X}(t) \in (W^{1,\infty}(\mathcal{B}))^d$ with $\frac{\partial \mathbf{X}}{\partial t}(t) \in L^2(\mathcal{B})^d$, then the following equality holds true*

$$\frac{\rho_f}{2} \frac{d}{dt} ||\mathbf{u}(t)||_0^2 + \nu ||\nabla_{\text{sym}} \mathbf{u}(t)||_0^2 + \frac{\delta_\rho}{2} \frac{d}{dt} \left\| \frac{\partial \mathbf{X}(t)}{\partial t} \right\|_{0,\mathcal{B}}^2 + \frac{d}{dt} E(\mathbf{X}(t)) = 0. \quad (19)$$

Proof The proof is quite simple. Take $\mathbf{v} = \mathbf{u}(t)$, $q = p(t)$, $\mathbf{Y} = \partial\mathbf{X}(t)/\partial t$, and $\boldsymbol{\mu} = \boldsymbol{\lambda}(t)$ in equations (18a)–(18d) respectively and sum. The inequality (19) is achieved using (7) and (8) to estimate the second term arising from (18c). □

3 Time semi-discretization

In this subsection we briefly recall the results of [4] related to the time discretization of Problem 2 and to the analysis of the stability of the resulting scheme. The presented results are valid not only for thick structures but also for thin ones, according to the formulation that will be presented in Sect. 5.

Given an integer $N > 0$, set $\Delta t = T/N$ the time step and $t_n = n\Delta t$. For a given function z depending on t we denote by z^n the approximation of $z(t_n)$.

Problem 2 presents several nonlinear terms whose time discretization by an implicit method would require the solution of a nonlinear stationary system with non trivial computational cost. Therefore we adopt the following semi-implicit time advancing scheme:

Problem 4 *Given* $\mathbf{u}_0 \in H_0^1(\Omega)^d$ *and* $\mathbf{X}_0 \in W^{1,\infty}(\mathcal{B})$*, for* $n = 0, \dots, N-1$ *find* $(\mathbf{u}^n, p^n) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$*,* $\mathbf{X}^n \in H^1(\mathcal{B})^d$*, and* $\boldsymbol{\lambda}^n \in \boldsymbol{\Lambda}$*, such that*

$$\begin{aligned}\rho_f\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t},\mathbf{v}\right) &+ b(\mathbf{u}^n,\mathbf{u}^{n+1},\mathbf{v}) + a(\mathbf{u}^{n+1},\mathbf{v})\\ &- (\mathrm{div}\mathbf{v}, p^{n+1}) + \mathbf{c}(\boldsymbol{\lambda}^{n+1},\mathbf{v}(\mathbf{X}^n)) = 0 \qquad \forall \mathbf{v} \in H_0^1(\Omega)^d \end{aligned} \tag{20a}$$

$$(\mathrm{div}\mathbf{u}^{n+1}, q) = 0 \qquad \forall q \in L_0^2(\Omega) \tag{20b}$$

$$\begin{aligned}\delta_\rho\left(\frac{\mathbf{X}^{n+1}-2\mathbf{X}^n+\mathbf{X}^{n-1}}{\Delta t^2},\mathbf{Y}\right)_{\mathcal{B}} &+ (\mathbb{P}(\mathbb{F}^{n+1}), \nabla_s\mathbf{Y})_{\mathcal{B}}\\ &- \mathbf{c}(\boldsymbol{\lambda}^{n+1},\mathbf{Y}) = 0 \qquad \forall \mathbf{Y} \in H^1(\mathcal{B})^d\end{aligned} \tag{20c}$$

$$\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}^{n+1}(\mathbf{X}^n) - \frac{\mathbf{X}^{n+1}-\mathbf{X}^n}{\Delta t}\right) = 0 \qquad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda} \tag{20d}$$

$$\mathbf{u}^0 = \mathbf{u}_0, \qquad \mathbf{X}^0 = \mathbf{X}_0 \tag{20e}$$

In the second term of (20c) the implicit quantity $\mathbb{P}(\mathbb{F}^{n+1})$ might be difficult to compute, in such case different choices can be made. In particular, if $\mathbb{P}$ is linear with respect to $\mathbb{F}$ this term does not cause any trouble; otherwise it can be linearized.

In order to initialize equation (20c) we need to know the first two values $\mathbf{X}^0$ and $\mathbf{X}^1$. These can be obtained from the initial conditions taking into account (13) and (14) as follows:

$$\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}_0(\mathbf{X}^0) - \frac{\mathbf{X}^1-\mathbf{X}^0}{\Delta t}\right) = 0 \quad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}.$$

Following the same lines of the proof of Proposition 2, we can show the following unconditional stability for the time advancing scheme, (see [4] for the details).

Proposition 3 *Under the same assumptions as in Proposition 2, if* $\mathbf{u}^n \in H_0^1(\Omega)^d$ *and* $\mathbf{X}^n \in H^1(\mathcal{B})^d$ *for* $n = 0, \dots, N$ *satisfy Problem 4 with* $\mathbf{X}^n \in (W^{1,\infty}(\mathcal{B}))^d$*, then the following estimate holds true for all* $n = 0, \dots, N-1$

$$\begin{aligned}&\tfrac{\rho_f}{2\Delta t}\left(\|\mathbf{u}^{n+1}\|_0^2 - \|\mathbf{u}^n\|_0^2\right) + \nu\|\nabla_{\mathrm{sym}}\mathbf{u}^{n+1}\|_0^2\\ &\quad+\tfrac{\delta_\rho}{2\Delta t}\left(\left\|\tfrac{\mathbf{X}^{n+1}-\mathbf{X}^n}{\Delta t}\right\|_{0,\mathcal{B}}^2 - \left\|\tfrac{\mathbf{X}^n-\mathbf{X}^{n-1}}{\Delta t}\right\|_{0,\mathcal{B}}^2\right) + \tfrac{E(\mathbf{X}^{n+1})-E(\mathbf{X}^n)}{\Delta t} \le 0.\end{aligned} \tag{21}$$

4 Analysis of the stationary problem

We now focus on the stationary problem that we resolve at each time step and we analyze its well-posedness and finite element discretization. We consider a linear model for the Piola–Kirchhoff stress tensor, that is

$$\mathbb{P}(\mathbb{F}) = \kappa\mathbb{F} = \kappa\nabla_s\mathbf{X}. \tag{22}$$

In this case, we have that the energy density is $W(\mathbb{F}) = (\kappa/2)\mathbb{F} : \mathbb{F}$ and the elastic potential energy is given by

$$E(\mathbf{X}) = \frac{\kappa}{2}\int_{\mathcal{B}} \mathbb{F} : \mathbb{F} d\mathbf{s} = \frac{\kappa}{2}\int_{\mathcal{B}} |\nabla_s\mathbf{X}|^2 d\mathbf{s}.$$

With this simplification it is possible to apply the results on existence, uniqueness, stability and error estimates of linear saddle point problems, see [2]. We think that the results can be extended to the nonlinear case with additional assumptions on the nonlinear terms. Hence we have the following saddle point problem.

Problem 5 *Let* $\overline{\mathbf{X}} \in W^{1,\infty}(\mathcal{B})^d$ *be invertible with Lipschitz inverse and* $\overline{\mathbf{u}} \in L^\infty(\Omega)$*. Given* $\mathbf{f} \in L^2(\Omega)^d$*,* $\mathbf{g} \in L^2(\mathcal{B})^d$*, and* $\mathbf{d} \in L^2(\mathcal{B})^d$*, find* $\mathbf{u} \in H_0^1(\Omega)^d$*,* $p \in L_0^2(\Omega)$*,* $\mathbf{X} \in H^1(\mathcal{B})^d$*, and* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ *such that*

$$\begin{aligned}&\mathbf{a}_f(\mathbf{u},\mathbf{v}) - (\mathrm{div}\mathbf{v}, p) + \mathbf{c}(\boldsymbol{\lambda}, \mathbf{v}(\overline{\mathbf{X}})) = (\mathbf{f},\mathbf{v}) && \forall \mathbf{v} \in H_0^1(\Omega)^d\\ &(\mathrm{div}\mathbf{u}, q) = 0 && \forall q \in L_0^2(\Omega)\\ &\mathbf{a}_s(\mathbf{X},\mathbf{Y}) - \mathbf{c}(\boldsymbol{\lambda},\mathbf{Y}) = (\mathbf{g},\mathbf{Y})_{\mathcal{B}} && \forall \mathbf{Y} \in H^1(\mathcal{B})^d\\ &\mathbf{c}(\boldsymbol{\mu}, \mathbf{u}(\overline{\mathbf{X}}) - \mathbf{X}) = \mathbf{c}(\boldsymbol{\mu}, \mathbf{d}) && \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}\end{aligned} \tag{23}$$

where

$$\begin{aligned}&\mathbf{a}_f(\mathbf{u},\mathbf{v}) = \alpha(\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + b(\overline{\mathbf{u}},\mathbf{u},\mathbf{v}) && \forall \mathbf{u},\mathbf{v} \in H_0^1(\Omega)^d\\ &\mathbf{a}_s(\mathbf{X},\mathbf{Y}) = \beta(\mathbf{X},\mathbf{Y})_{\mathcal{B}} + \gamma(\nabla_s\mathbf{X}, \nabla_s\mathbf{Y})_{\mathcal{B}} && \forall \mathbf{X},\mathbf{Y} \in H^1(\mathcal{B})^d\end{aligned}$$

It is easy to see that Problem 5 corresponds to one step of Problem 4 if we take:

$$\begin{aligned}&\mathbf{u}=\mathbf{u}^{n+1},\ p=p^{n+1},\ \mathbf{X}=\mathbf{X}^{n+1}/\Delta t,\ \boldsymbol{\lambda}=\boldsymbol{\lambda}^{n+1}\\&\mathbf{f}=\frac{\rho_f}{\Delta t}\mathbf{u}^n\\&\mathbf{g}=\frac{\delta_\rho}{\Delta t^2}\left(2\mathbf{X}^n-\mathbf{X}^{n-1}\right)\\&\mathbf{d}=-\frac{1}{\Delta t}\mathbf{X}^n\\&\alpha=\rho_f/\Delta t,\ \beta=\delta_\rho/\Delta t,\ \gamma=\kappa\Delta t\end{aligned}$$

and $\overline{\mathbf{X}}=\mathbf{X}^n$ and $\overline{\mathbf{u}}=\mathbf{u}^n$ in the nonlinear terms.

We remark that while α and γ are strictly positive, the constant β might vanish when the densities in the solid and in the fluid are equal.

4.1 Well-posedness of Problem 5

Problem 5 fits in the framework of saddle point problems with the following functional setting. Let us introduce the Hilbert space

$$\mathbb{V}=H_0^1(\Omega)^d\times L_0^2(\Omega)\times H^1(\mathcal{B})^d \tag{24}$$

endowed with the graph norm

$$|||\mathbf{V}|||_{\mathbb{V}}=\left(\|\mathbf{v}\|_1^2+\|q\|_0^2+\|\mathbf{Y}\|_{1,\mathcal{B}}^2\right)^{1/2},$$

where $\mathbf{V}=(\mathbf{v},q,\mathbf{Y})$ is a generic element of $\mathbb{V}$.

We define the bilinear forms $\mathbb{A}:\mathbb{V}\times\mathbb{V}\to\mathbb{R}$ and $\mathbb{B}:\mathbb{V}\times\boldsymbol{\Lambda}\to\mathbb{R}$

$$\begin{aligned}&\mathbb{A}(\mathbf{U},\mathbf{V})=\mathbf{a}_f(\mathbf{u},\mathbf{v})-(p,\mathrm{div}\mathbf{v})+(\mathrm{div}\mathbf{u},q)+\mathbf{a}_s(\mathbf{X},\mathbf{Y})\\&\mathbb{B}(\mathbf{V},\boldsymbol{\mu})=\mathbf{c}(\boldsymbol{\mu},\mathbf{v}(\overline{\mathbf{X}})-\mathbf{Y}).\end{aligned} \tag{25}$$

Then Problem 5 can be reformulated as follows: given $\mathbf{f}\in L^2(\Omega)$, $\mathbf{g}\in L^2(\mathcal{B})$, and $\mathbf{d}\in H^1(\mathcal{B})^d$, find $(\mathbf{U},\boldsymbol{\lambda})\in\mathbb{V}\times\boldsymbol{\Lambda}$ such that

$$\begin{aligned}&\mathbb{A}(\mathbf{U},\mathbf{V})+\mathbb{B}(\mathbf{V},\boldsymbol{\lambda})=(\mathbf{f},\mathbf{v})+(\mathbf{g},\mathbf{Y})_{\mathcal{B}} && \forall\mathbf{V}\in\mathbb{V}\\&\mathbb{B}(\mathbf{U},\boldsymbol{\mu})=\mathbf{c}(\boldsymbol{\mu},\mathbf{d}) && \forall\boldsymbol{\mu}\in\boldsymbol{\Lambda}.\end{aligned} \tag{26}$$

In order to verify the well-posedness of (26) we have to check the two inf-sup conditions, see [2, Chapt. 4]. The kernel of the operator associated with the bilinear form $\mathbb{B}$ is given by:

$$\mathbb{K}=\{\mathbf{V}\in\mathbb{V}:\ \mathbb{B}(\mathbf{V},\boldsymbol{\mu})=0\ \forall\boldsymbol{\mu}\in\boldsymbol{\Lambda}\}. \tag{27}$$

Then $\mathbf{V}\in\mathbb{K}$ if and only if $\mathbf{v}(\overline{\mathbf{X}})-\mathbf{Y}=0$ in the sense of $H^1(\mathcal{B})^d$.

For the sake of simplicity, in the sequel we neglect the convective term associated with the trilinear form b.

Proposition 4 (First inf-sup condition) *There exists* $\alpha_0 > 0$ *such that*

$$\inf_{\mathbf{U}\in\mathbb{K}} \sup_{\mathbf{V}\in\mathbb{K}} \frac{\mathbb{A}(\mathbf{U},\mathbf{V})}{|||\mathbf{U}|||_{\mathbb{V}}|||\mathbf{V}|||_{\mathbb{V}}} \geq \alpha_0. \tag{28}$$

Proof We show that there exists a constant $\alpha_0 > 0$ such that for any $\mathbf{U} = (\mathbf{u}, p, \mathbf{X}) \in \mathbb{K}$ it holds

$$\sup_{\mathbf{V}\in\mathbb{K}} \frac{\mathbb{A}(\mathbf{U},\mathbf{V})}{|||\mathbf{V}|||_{\mathbb{V}}} \geq \alpha_0 |||\mathbf{U}|||_{\mathbb{V}}. \tag{29}$$

It is well known that for any $p \in L^2_0(\Omega)$ there exists an element $\hat{\mathbf{v}} \in H^1_0(\Omega)^d$ such that $\mathrm{div}\hat{\mathbf{v}} = -p$, see [2], with $\|\hat{\mathbf{v}}\|_1 \leq C\|p\|_0$.

Given $\mathbf{U} = (\mathbf{u}, p, \mathbf{X})$ we take $\mathbf{V} = (\mathbf{v}, q, \mathbf{Y})$ with $\mathbf{v} = a\mathbf{u} + b\hat{\mathbf{v}}$, $q = ap$, and $\mathbf{Y} = \mathbf{X}$. Then

$$|||\mathbf{V}|||^2_{\mathbb{V}} = \|\mathbf{v}\|^2_1 + \|q\|^2_0 + \|\mathbf{Y}\|^2_{1,\mathcal{B}} \leq a^2\|\mathbf{u}\|^2_1 + (a^2 + C^2b^2)\|p\|^2_0 + \|\mathbf{X}\|^2_{1,\mathcal{B}}. \tag{30}$$

Since $\mathbf{a}_f$ is coercive and continuous on $H^1_0(\Omega)^d$ there exist two positive constants such that for all $\mathbf{u}, \mathbf{v} \in H^1_0(\Omega)^d$ it holds

$$\begin{aligned} \mathbf{a}_f(\mathbf{u},\mathbf{u}) &\geq c_1\|\mathbf{u}\|^2_1 \\ \mathbf{a}_f(\mathbf{u},\mathbf{v}) &\leq c_2\|\mathbf{u}\|_1\|\mathbf{v}\|_1. \end{aligned}$$

Hence we have

$$\begin{aligned} \mathbb{A}(\mathbf{U},\mathbf{V}) &= a\mathbf{a}_f(\mathbf{u},\mathbf{u}) + b\mathbf{a}_f(\mathbf{u},\hat{\mathbf{v}}) - a(p, \mathrm{div}\mathbf{u}) \\ &\qquad -b(p, \mathrm{div}\hat{\mathbf{v}}) + a(\mathrm{div}\mathbf{u}, p) + \mathbf{a}_s(\mathbf{X},\mathbf{X}) \\ &\geq ac_1\|\mathbf{u}\|^2_1 + b\|p\|^2_0 - bc_2\|\mathbf{u}\|_1\|\hat{\mathbf{v}}\|_1 + \mathbf{a}_s(\mathbf{X},\mathbf{X}) \\ &\geq ac_1\|\mathbf{u}\|^2_1 + b\|p\|^2_0 - \frac{bc_2^2}{2\varepsilon}\|\mathbf{u}\|_1 - \frac{\varepsilon}{2}bC^2\|p\|^2_0 + \mathbf{a}_s(\mathbf{X},\mathbf{X}) \\ &= \left(ac_1 - \frac{bc_2^2}{2\varepsilon}\right)\|u\|^2_1 + b\left(1 - \frac{\varepsilon}{2}C^2\right)\|p\|^2_0 + \mathbf{a}_s(\mathbf{X},\mathbf{X}). \end{aligned}$$

We can choose ε, a, and b so that $1 - \varepsilon C^2/2 > \delta > 0$ and $ac_1 - bc_2^2/(2\varepsilon) > \delta > 0$ and we arrive at

$$\mathbb{A}(\mathbf{U},\mathbf{V}) \geq \delta\left(\|\mathbf{u}\|^2_1 + \|p\|^2_0\right) + \mathbf{a}_s(\mathbf{X},\mathbf{X}).$$

It remains to bound by below the last term $\mathbf{a}_s(\mathbf{X},\mathbf{X})$. By definition we have

$$\mathbf{a}_s(\mathbf{X},\mathbf{X}) = \beta\|\mathbf{X}\|^2_{0,\mathcal{B}} + \gamma\|\nabla_s\mathbf{X}\|^2_{0,\mathcal{B}} \geq \min(\beta,\gamma)\|\|\mathbf{X}\|^2_{1,\mathcal{B}}.$$

We observe that if $\beta = 0$ we still obtain the desired estimate, since $\mathbf{U} \in \mathbb{K}$ and $\mathbf{X} = \mathbf{u}(\overline{\mathbf{X}})$ so that $\|\mathbf{X}\|_{0,\mathcal{B}} = \|\mathbf{u}(\overline{\mathbf{X}})\|_{0,\mathcal{B}} \leq \|\mathbf{u}\|_0 \leq C_\Omega\|\mathbf{u}\|_1$ where C_Ω is the Poincaré constant.

Therefore we can determine a constant $C > 0$ such that

$$\mathbb{A}(\mathbf{U},\mathbf{V}) \ge C\left(\|\mathbf{u}\|_1^2 + \|p\|_0^2 + \|\mathbf{X}\|_{1,\mathcal{B}}^2\right).$$

This inequality together with (30) gives the desired estimate (29). □

Remark In the proof of the above proposition, the assumption (22) has been used in order to have the coerciveness of the bilinear form $\mathbf{a}_s$. The result extends easily to nonlinear cases whenever the elastic potential energy satisfies the following bound for some positive constant γ_0

$$E(\mathbf{X}) \ge \gamma_0 \|\mathbf{X}\|_{1,\mathcal{B}}^2.$$

Proposition 5 (Second inf-sup condition) *There exists a constant* $\beta_0 > 0$ *such that for all* $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ *it holds*

$$\sup_{\mathbf{V}\in\mathbb{V}} \frac{\mathbb{B}(\mathbf{V},\boldsymbol{\mu})}{\|\mathbf{V}\|_{\mathbb{V}}} \ge \beta_0 \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}. \tag{31}$$

Proof Recalling the definition of $\mathbb{B}$ we have to show that

$$\sup_{(\mathbf{v},q,\mathbf{Y})\in\mathbb{V}} \frac{\mathbf{c}(\boldsymbol{\mu},\mathbf{v}(\overline{\mathbf{X}})-\mathbf{Y})}{\|(\mathbf{v},q,\mathbf{Y})\|_{\mathbb{V}}} \ge \beta_0 \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}.$$

We give the proof for the choices of $\mathbf{c}$ given in (15) and (16). In the first case $\mathbf{c}$ is the duality pairing between $H^1(\mathcal{B})^d$ and $(H^1(\mathcal{B})^d)'$. By definition of the norm in the dual space $(H^1(\mathcal{B})^d)'$ we have

$$\begin{aligned}
\|\boldsymbol{\mu}\|_{(H^1(\mathcal{B})^d)'} &= \sup_{\mathbf{Y}\in H^1(\mathcal{B})^d} \frac{\langle \boldsymbol{\mu},\mathbf{Y}\rangle}{\|\mathbf{Y}\|_{H^1(\mathcal{B})^d}} \\
&\le \sup_{(\mathbf{v},q,\mathbf{Y})\in\mathbb{V}} \frac{\langle \boldsymbol{\mu},\mathbf{v}(\overline{\mathbf{X}})-\mathbf{Y}\rangle}{(\|\mathbf{v}\|_1^2 + \|\mathbf{Y}\|_{1,\mathcal{B}}^2)^{1/2}}
\end{aligned}$$

which gives (31).

The proof is exactly the same for the case of $\mathbf{c}$ given by the scalar product in $H^1(\mathcal{B})^d$. □

4.2 Finite element discretization

Let us consider a family $\mathcal{T}_h$ of regular meshes in Ω and a family $\mathcal{T}_h^{\mathcal{B}}$ of regular meshes in $\mathcal{B}$. We denote by h_x and h_s the meshsize of $\mathcal{T}_h$ and $\mathcal{T}_h^{\mathcal{B}}$, respectively. Let $V_h \subseteq H_0^1(\Omega)^d$ and $Q_h \subseteq L_0^2(\Omega)$ be finite element spaces which satisfy the usual discrete ellipticity on the kernel and the discrete inf-sup conditions for the Stokes problem [2]. Moreover, we set

$$S_h = \{\mathbf{Y} \in H^1(\mathcal{B})^d : \mathbf{Y}|_T \in \mathbf{P}^1(T)\ \forall T \in \mathcal{T}_h^{\mathcal{B}}\}. \tag{32}$$

The natural choice for $\mathbf{\Lambda}_h$, corresponding to the case of $\mathbf{c}$ given by (16), is to take $\mathbf{\Lambda}_h = S_h$. This is actually reasonable also when $\mathbf{c}$ is defined by (15), since in this case the duality pairing can be represented as a scalar product in $L^2(\mathcal{B})$, that is:

$$\mathbf{c}(\boldsymbol{\mu}, \mathbf{Y}) = (\boldsymbol{\mu}, \mathbf{Y}) \quad \forall \boldsymbol{\mu} \in \mathbf{\Lambda}_h, \mathbf{Y} \in S_h. \tag{33}$$

Of course, several other choices for $\mathbf{\Lambda}_h$ might be made; we are not going to investigate them in this paper.

Then the finite element counterpart of Problem 4 reads.

Problem 6 *Let $\overline{\mathbf{X}} \in W^{1,\infty}(\mathcal{B})^d$ be invertible with Lipschitz inverse and $\overline{\mathbf{u}} \in L^\infty(\Omega)$. Given $\mathbf{f} \in L^2(\Omega)^d$, $\mathbf{g} \in L^2(\mathcal{B})^d$, and $\mathbf{d} \in L^2(\mathcal{B})^d$, find $\mathbf{u}_h \in V_h$, $p_h \in Q_h$, $\mathbf{X}_h \in S_h$, and $\boldsymbol{\lambda}_h \in \mathbf{\Lambda}_h$ such that*

$$\begin{aligned}
&\mathbf{a}_f(\mathbf{u}_h, \mathbf{v}) - (\mathrm{div}\mathbf{v}, p_h) + \mathbf{c}(\boldsymbol{\lambda}_h, \mathbf{v}(\overline{\mathbf{X}})) = (\mathbf{f}, \mathbf{v}) && \forall \mathbf{v} \in V_h\\
&(\mathrm{div}\mathbf{u}_h, q) = 0 && \forall q \in Q_h\\
&\mathbf{a}_s(\mathbf{X}_h, \mathbf{Y}) - \mathbf{c}(\boldsymbol{\lambda}_h, \mathbf{Y}) = (\mathbf{g}, \mathbf{Y})_{\mathcal{B}} && \forall \mathbf{Y} \in S_h\\
&\mathbf{c}(\boldsymbol{\mu}, \mathbf{u}_h(\overline{\mathbf{X}}) - \mathbf{X}_h) = \mathbf{c}(\boldsymbol{\mu}, \mathbf{d}) && \forall \boldsymbol{\mu} \in \mathbf{\Lambda}_h.
\end{aligned} \tag{34}$$

Using the same notation as in the previous subsection, we set

$$\mathbb{V}_h = V_h \times Q_h \times S_h,$$

then the finite element counterpart of (26) reads: given $\mathbf{f} \in L^2(\Omega)$, $\mathbf{g} \in L^2(\mathcal{B})$, and $\mathbf{d} \in H^1(\mathcal{B})^d$, find $(\mathbf{U}_h, \boldsymbol{\lambda}_h) \in \mathbb{V}_h \times \mathbf{\Lambda}_h$ such that

$$\begin{aligned}
&\mathbb{A}(\mathbf{U}_h, \mathbf{V}) + \mathbb{B}(\mathbf{V}, \boldsymbol{\lambda}_h) = (\mathbf{f}, \mathbf{v}) + (\mathbf{g}, \mathbf{Y})_{\mathcal{B}} && \forall \mathbf{V} \in \mathbb{V}_h\\
&\mathbb{B}(\mathbf{U}_h, \boldsymbol{\mu}) = \mathbf{c}(\boldsymbol{\mu}, \mathbf{d}) && \forall \boldsymbol{\mu} \in \mathbf{\Lambda}_h.
\end{aligned} \tag{35}$$

It is well-known that sufficient conditions for existence and uniqueness of the solution of (35) are the discrete versions of the two inf-sup conditions (28) and (31).

Proposition 6 (First discrete inf-sup condition) *Assume that $V_h \times Q_h$ is stable for the Stokes equation. There exists $\alpha_1 > 0$ independent of h_x and h_s such that*

$$\inf_{\mathbf{U}_h \in \mathbb{K}_h} \sup_{\mathbf{V}_h \in \mathbb{K}_h} \frac{\mathbb{A}(\mathbf{U}_h, \mathbf{V}_h)}{|||\mathbf{U}_h|||_{\mathbb{V}} |||\mathbf{V}_h|||_{\mathbb{V}}} \geq \alpha_1, \tag{36}$$

where the discrete kernel $\mathbb{K}_h$ is given by

$$\mathbb{K}_h = \left\{ \mathbf{V}_h \in \mathbb{V}_h : \mathbb{B}(\mathbf{V}_h, \boldsymbol{\mu}_h) = 0 \, \forall \boldsymbol{\mu}_h \in \mathbf{\Lambda}_h \right\}.$$

Proof We show that there exists a constant $\alpha_0 > 0$ such that for any $\mathbf{U}_h = (\mathbf{u}_h, p_h, \mathbf{X}_h) \in \mathbb{K}_h$ it holds

$$\sup_{\mathbf{V}_h \in \mathbb{K}_h} \frac{\mathbb{A}(\mathbf{U}_h, \mathbf{V}_h)}{|||\mathbf{V}_h|||_{\mathbb{V}}} \geq \alpha_1 |||\mathbf{U}_h|||_{\mathbb{V}}. \tag{37}$$

Since the couple (V_h, Q_h) is stable for Stokes equations, for any $p_h \in Q_h$ there exists an element $\hat{\mathbf{v}}_h \in V_h$ such that $(\mathrm{div}\hat{\mathbf{v}}_h, q) = -(p_h, q)$ for all $q \in Q_h$, see [2], with $\|\hat{\mathbf{v}}_h\|_1 \le C\|p_h\|_0$.

Given $\mathbf{U}_h = (\mathbf{u}_h, p_h, \mathbf{X}_h)$ we take $\mathbf{V}_h = (\mathbf{v}_h, q_h, \mathbf{Y}_h)$ with $\mathbf{v}_h = a\mathbf{u}_h + b\hat{\mathbf{v}}_h$, $q_h = ap_h$ and $\mathbf{Y}_h = \mathbf{X}_h$. Then as in the continuous case

$$|||\mathbf{V}_h|||_{\mathbb{V}}^2 \le a^2\|\mathbf{u}_h\|_1^2 + (a^2 + C^2b^2)\|p_h\|_0^2 + \|\mathbf{X}_h\|_{1,\mathcal{B}}^2. \tag{38}$$

Since $\mathbf{a}_f$ is coercive and continuous on $H_0^1(\Omega)^d$, we have

$$\begin{aligned}
\mathbb{A}(\mathbf{U}_h, \mathbf{V}_h) &= a\mathbf{a}_f(\mathbf{u}_h, \mathbf{u}_h) + b\mathbf{a}_f(\mathbf{u}_h, \hat{\mathbf{v}}_h) - a(p_h, \mathrm{div}\mathbf{u}_h) - b(p_h, \mathrm{div}\hat{\mathbf{v}}_h)\\
&\quad + a(\mathrm{div}\mathbf{u}_h, p_h) + \mathbf{a}_s(\mathbf{X}_h, \mathbf{X}_h)\\
&\ge ac_1\|\mathbf{u}_h\|_1^2 + b\|p_h\|_0^2 - bc_2\|\mathbf{u}_h\|_1\|\hat{\mathbf{v}}_h\|_1 + \mathbf{a}_s(\mathbf{X}_h, \mathbf{X}_h)\\
&\ge ac_1\|\mathbf{u}_h\|_1^2 + b\|p_h\|_0^2 - \frac{bc_2^2}{2\varepsilon}\|\mathbf{u}_h\|_1 - \frac{\varepsilon}{2}bC^2\|p_h\|_0^2 + \mathbf{a}_s(\mathbf{X}_h, \mathbf{X}_h)\\
&= \left(ac_1 - \frac{bc_2^2}{2\varepsilon}\right)\|\mathbf{u}_h\|_1^2 + b\left(1 - \frac{\varepsilon}{2}C^2\right)\|p_h\|_0^2 + \mathbf{a}_s(\mathbf{X}_h, \mathbf{X}_h).
\end{aligned}$$

Choosing again ε, a and b so that $1 - \varepsilon C^2/2 > \delta > 0$ and $ac_1 - bc_2^2/(2\varepsilon) > \delta > 0$, we obtain

$$\mathbb{A}(\mathbf{U}_h, \mathbf{V}_h) \ge \delta\left(\|\mathbf{u}_h\|_1^2 + \|p_h\|_0^2\right) + \mathbf{a}_s(\mathbf{X}_h, \mathbf{X}_h).$$

It remains to bound by below the last term $\mathbf{a}_s(\mathbf{X}_h, \mathbf{X}_h)$. By definition we have for $\beta > 0$

$$\mathbf{a}_s(\mathbf{X}_h, \mathbf{X}_h) = \beta\|\mathbf{X}_h\|_{0,\mathcal{B}}^2 + \gamma\|\nabla_s\mathbf{X}_h\|_{0,\mathcal{B}}^2 \ge \min(\beta, \gamma)\|\|\mathbf{X}_h\|_{1,\mathcal{B}}^2,$$

and $\mathbf{a}_s(\mathbf{X}_h, \mathbf{X}_h) = \gamma\|\nabla_s\mathbf{X}_h\|_{0,\mathcal{B}}^2$ if $\beta = 0$. In this case we need to estimate $\|\mathbf{X}_h\|_{0,\mathcal{B}}$ by means of the other terms appearing in the right hand side of the last two inequalities. This part of the proof depends on the definition of $\mathbf{c}$.

First case. Let us assume first that $\mathbf{c}$ is given by (15). Taking into account (33), we have that $\mathbf{U}_h \in \mathbb{K}_h$ is characterized by $\mathbf{X}_h = P_0(\mathbf{u}_h(\overline{\mathbf{X}}))$ where P_0 represents the L^2 projections onto S_h. Hence we obtain

$$\begin{aligned}
\|\mathbf{X}_h\|_{0,\mathcal{B}} &= \|P_0(\mathbf{u}_h(\overline{\mathbf{X}}))\|_{0,\mathcal{B}}\\
&\le \|\mathbf{u}_h(\overline{\mathbf{X}})\|_{0,\mathcal{B}} + \|\mathbf{u}_h(\overline{\mathbf{X}}) - P_0(\mathbf{u}_h(\overline{\mathbf{X}}))\|_{0,\mathcal{B}}\\
&\le \|\mathbf{u}_h(\overline{\mathbf{X}})\|_{0,\mathcal{B}} + Ch_s|\mathbf{u}_h(\overline{\mathbf{X}})|_{1,\mathcal{B}} \le \|\mathbf{u}_h\|_{0,\Omega} + Ch_s|\mathbf{u}_h|_{1,\Omega}.
\end{aligned}$$

Therefore we can determine a constant $C > 0$ such that

$$\mathbb{A}(\mathbf{U}, \mathbf{V}) \ge C\left(\|\mathbf{u}_h\|_1^2 + \|p_h\|_0^2 + \|\mathbf{X}_h\|_{1,\mathcal{B}}^2\right).$$

Second case. In the case of $\mathbf{c}$ given by (16), the fact that $\mathbf{U}_h \in \mathbb{K}_h$ implies that $\mathbf{X}_h = P_1(\mathbf{u}_h(\overline{\mathbf{X}}))$ where P_1 stands for the H^1 projections onto S_h, so that we obtain

$$\begin{aligned}\|\mathbf{X}_h\|_{0,\mathcal{B}} = \|P_1(\mathbf{u}_h(\overline{\mathbf{X}}))\|_{0,\mathcal{B}} &\le \|P_1(\mathbf{u}_h(\overline{\mathbf{X}}))\|_{1,\mathcal{B}} \\ &\le \|\mathbf{u}_h(\overline{\mathbf{X}})\|_{1,\mathcal{B}} \le \|\mathbf{u}_h\|_1.\end{aligned}$$

These inequalities together with (38) give the desired estimate (37). □

If $\mathbf{c}$ is given by (15) (which has the discrete counterpart (33)), we make the additional assumption that the mesh sequence $\mathcal{T}_h^{\mathcal{B}}$ is such that the L^2-projection P_0 from $H^1(\mathcal{B})^d$ onto S_h is H^1-stable, that is

$$|P_0\mathbf{v}|_{1,\mathcal{B}} \le c|\mathbf{v}|_{1,\mathcal{B}} \quad \forall \mathbf{v} \in H^1(\mathcal{B})^d, \tag{39}$$

where $|\cdot|_{1,\mathcal{B}}$ is the H^1-seminorm.

Proposition 7 (Second discrete inf-sup condition) *There exists a constant $\beta_1 > 0$ independent of h_x and h_s such that for all $\mu_h \in \Lambda_h$ it holds true*

$$\sup_{\mathbf{V}_h\in\mathbb{V}_h} \frac{\mathbb{B}(\mathbf{V}_h,\boldsymbol{\mu}_h)}{\|\mathbf{V}_h\|_{\mathbb{V}}} \ge \beta_1\|\boldsymbol{\mu}_h\|_{\mathbf{\Lambda}}. \tag{40}$$

Proof *First case*. Let $\mathbf{c}$ be given by (15), so that (33) holds true. Then we have to show that there exists $\beta_1 > 0$ independent of h such that

$$\sup_{\mathbf{V}_h\in\mathbb{V}_h} \frac{(\boldsymbol{\mu}_h, \mathbf{v}_h(\overline{\mathbf{X}}) - \mathbf{Y}_h)}{\|\mathbf{V}_h\|_{\mathbb{V}}} \ge \beta_1\|\boldsymbol{\mu}_h\|_{\mathbf{\Lambda}}.$$

By definition of the norm in the dual space $\mathbf{\Lambda} = (H^1(\mathcal{B})^d)'$, there exists $\widetilde{\mathbf{Y}} \in H^1(\mathcal{B})^d$ such that

$$\|\boldsymbol{\mu}_h\|_{\mathbf{\Lambda}} = \sup_{\mathbf{Y}\in H^1(\mathcal{B})^d} \frac{(\boldsymbol{\mu}_h,\mathbf{Y})}{\|\mathbf{Y}\|_{1,\mathcal{B}}} = \frac{(\boldsymbol{\mu}_h,\widetilde{\mathbf{Y}})}{\|\widetilde{\mathbf{Y}}\|_{1,\mathcal{B}}} = \frac{(\boldsymbol{\mu}_h,P_0\widetilde{\mathbf{Y}})}{\|\widetilde{\mathbf{Y}}\|_{1,\mathcal{B}}}. \tag{41}$$

where P_0 denotes the projection operator from $H^1(\mathcal{B})^d$ into $\mathbf{\Lambda}_h = S_h$.

Well-known properties of P_0 and (39) imply

$$\begin{aligned}&\|P_0\widetilde{\mathbf{Y}}\|_{0,\mathcal{B}} \le \|\widetilde{\mathbf{Y}}\|_{0,\mathcal{B}}\\ &|P_0\widetilde{\mathbf{Y}}|_{1,\mathcal{B}} \le c|\widetilde{\mathbf{Y}}|_{1,\mathcal{B}}\\ &\|\widetilde{\mathbf{Y}} - P_0\widetilde{\mathbf{Y}}\|_{0,\mathcal{B}} \le Ch_s|\widetilde{\mathbf{Y}}|_{1,\mathcal{B}}.\end{aligned}$$

Therefore there exists a constant C such that $\|P_0\widetilde{\mathbf{Y}}\|_{1,\mathcal{B}} \le C\|\widetilde{\mathbf{Y}}\|_{1,\mathcal{B}}$. This last inequality inserted in (41) gives

$$\begin{aligned}\|\boldsymbol{\mu}_h\|_{\mathbf{\Lambda}} &\le C\frac{(\boldsymbol{\mu}_h,P_0\widetilde{\mathbf{Y}})}{\|P_0\widetilde{\mathbf{Y}}\|_{1,\mathcal{B}}} \le C\sup_{\mathbf{Y}_h\in S_h}\frac{(\boldsymbol{\mu}_h,\mathbf{Y}_h)}{\|\mathbf{Y}_h\|_{1,\mathcal{B}}}\\ &\le C\sup_{\mathbf{V}_h\in\mathbb{V}_h}\frac{\mathbf{c}(\boldsymbol{\mu}_h,\mathbf{v}_h(\overline{\mathbf{X}})-\mathbf{Y}_h)}{\|\mathbf{V}_h\|_{\mathbb{V}}} = C\sup_{\mathbf{V}_h\in\mathbb{V}_h}\frac{\mathbb{B}(\mathbf{V}_h,\boldsymbol{\mu}_h)}{\|\mathbf{V}_h\|_{\mathbb{V}}}.\end{aligned}$$

Second case. Let us now consider $\mathbf{c}$ given by (16), hence it is the scalar product in $H^1(\mathcal{B})^d$. By definition of the norm in $H^1(\mathcal{B})^d$ and of the H^1-projection operator P_1 we have:

$$\|\boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}} = \sup_{\mathbf{Y}\in H^1(\mathcal{B})^d} \frac{\mathbf{c}(\boldsymbol{\mu},\mathbf{Y})}{\|\mathbf{Y}\|_{1,\mathcal{B}}} \le \sup_{\mathbf{Y}\in H^1(\mathcal{B})^d} \frac{\mathbf{c}(\boldsymbol{\mu},P_1\mathbf{Y})}{\|P_1\mathbf{Y}\|_{1,\mathcal{B}}}$$
$$\le \sup_{\mathbf{Y}\in S_h} \frac{\mathbf{c}(\boldsymbol{\mu},\mathbf{Y})}{\|\mathbf{Y}\|_{1,\mathcal{B}}} \le \sup_{\mathbf{V}_h\in\mathbb{V}_h} \frac{\mathbb{B}(\mathbf{V}_h,\boldsymbol{\mu}_h)}{\|\mathbf{V}_h\|_{\mathbb{V}}}.$$

□

Remark Condition (39) has been widely studied in the literature. It can be easily obtained by using an inverse inequality on quasi-uniform meshes. Weaker assumptions than the quasi-uniformity of the mesh have been investigated in several papers, see for example [1,8,10–12]. In particular, the stability of the L^2-projection in H^1 has been proved in [12] under the assumption that neighboring element-sizes obey a global growth-condition and in [8] in the case of locally quasiuniform meshes. These conditions have been weakened in [10], while [11] extends the result to meshes generated by red-green-blue refinements in adaptive procedures for piecewise linear finite elements. Recently [1] has improved the previous results to cover the case of many commonly used adaptive meshing strategies. More general mesh refinements are considered in [13].

From the theory of the discretization of saddle point problems, the above propositions yield the following error estimate theorem (see [2, Th. 5.2.1]).

Theorem 8 *Let* $(\mathbf{u}, p, \mathbf{X}, \boldsymbol{\lambda})$ *and* $(\mathbf{u}_h, p_h, \mathbf{X}_h, \boldsymbol{\lambda}_h)$ *be solutions of Problems 5 and 6. Under the assumption* (39) *if* $\mathbf{c}$ *is given by* (15)*, the following optimal error estimate holds true:*

$$\|\mathbf{u}-\mathbf{u}_h\|_1 + \|p-p_h\|_0 + \|\mathbf{X}-\mathbf{X}_h\|_{1,\mathcal{B}} + \|\boldsymbol{\lambda}-\boldsymbol{\lambda}_h\|_{\boldsymbol{\Lambda}}$$
$$\le C \inf_{\substack{\mathbf{v}\in V_h\\ q\in Q_h\\ \mathbf{Y}\in S_h\\ \boldsymbol{\mu}\in S_h}} \left(\|\mathbf{u}-\mathbf{v}\|_1 + \|p-q\|_0 + \|\mathbf{X}-\mathbf{Y}\|_{1,\mathcal{B}} + \|\boldsymbol{\lambda}-\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}\right).$$

5 The case of a thin solid immersed in a fluid

In this section we consider the case of thin structures with very small constant thickness t_s, so that we assume that the physical quantities depend only on variables along the middle section of the structure and are constant in the normal direction. In order to maintain the same notation in the final formulation of the problem, the region occupied by the solid is $\Omega_t^s\times]-t_s,t_s[$, where Ω_t^s is a subset of Ω of codimension one (a surface in the 3D case or a curve in the 2D one). Therefore we have that the reference domain $\mathcal{B}$ is a subset of $\mathbb{R}^{d-1}$ and the deformation gradient $\mathbb{F}:\mathcal{B}\to\mathbb{R}^{d\times(d-1)}$ is such that

$$|\mathbb{F}| = \left|\frac{\partial\mathbf{X}}{\partial s}\right| \text{ if } d=2, \qquad |\mathbb{F}| = \left|\frac{\partial\mathbf{X}}{\partial s_1}\times\frac{\partial\mathbf{X}}{\partial s_2}\right| \text{ if } d=3,$$

s, s_1 and s_2 being the parametric variables in $\mathcal{B}$.

Following the same arguments as in [3], Equations (9)–(11) can be written in the following form:

$$\begin{aligned}
&\rho_f \frac{d}{dt}(\mathbf{u}(t),\mathbf{v}) + b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}) + a(\mathbf{u}(t),\mathbf{v}) \\
&\quad -(\operatorname{div}\mathbf{v}, p(t)) = \langle \mathbf{F}_1(t),\mathbf{v}\rangle + \langle \mathbf{F}_2(t),\mathbf{v}\rangle && \forall \mathbf{v}\in H^1_0(\Omega)^d \\
&(\operatorname{div}\mathbf{u}(t), q) = 0 && \forall q\in L^2_0(\Omega) \\
&\langle \mathbf{F}_1(t),\mathbf{v}\rangle = -\delta_\rho \int_{\mathcal{B}} \frac{\partial^2\mathbf{X}}{\partial t^2}\mathbf{v}(\mathbf{X}(\mathbf{s},t))\,d\mathbf{s} && \forall \mathbf{v}\in H^1_0(\Omega)^d \\
&\langle \mathbf{F}_2(t),\mathbf{v}\rangle = -\int_{\mathcal{B}} \mathbb{P}(\mathbb{F}(\mathbf{s},t)) : \nabla_s \mathbf{v}(\mathbf{X}(\mathbf{s},t))\,d\mathbf{s} && \forall \mathbf{v}\in H^1_0(\Omega)^d.
\end{aligned} \tag{42}$$

Here $\mathbf{u}$ and p represent velocity and pressure of the fluid, respectively, $\delta_\rho = (\rho_s - \rho_f)t_s$, and $\mathbb{P} = t_s\tilde{\mathbb{P}}$ where $\tilde{\mathbb{P}}$ is obtained from (6) with the necessary modifications to cover the present situation. Moreover, the motion of the thin structure is governed by the following condition

$$\mathbf{u}(\mathbf{x},t) = \frac{\partial \mathbf{X}(\mathbf{s},t)}{\partial t}\Big|_{\mathbf{x}=\mathbf{X}(\mathbf{s},t)}.$$

Then the problem has the same form as Problem 2 by rewriting the above body motion constraint variationally as

$$\mathbf{c}\left(\boldsymbol{\mu}, \mathbf{u}(\mathbf{X}(\cdot,t),t) - \frac{\partial \mathbf{X}}{\partial t}(t)\right) = 0$$

for all $\boldsymbol{\mu}$ in a suitably defined functional space $\boldsymbol{\Lambda}$. Assuming that $\mathbf{X}(t)\in W^{1,\infty}(\mathcal{B})$ is one to one, $\mathbf{u}(\mathbf{X}(\cdot,t),t)$ represents the trace of $\mathbf{u}$ along Ω^s_t. Therefore $\mathbf{u}(\mathbf{X}(\cdot,t),t)\in H^{1/2}(\mathcal{B})^d$; we set $\boldsymbol{\Lambda} = (H^{1/2}(\mathcal{B})^d)'$ and $\mathbf{c} : \boldsymbol{\Lambda}\times H^{1/2}(\mathcal{B})^d \to \mathbb{R}$ given by

$$\mathbf{c}(\boldsymbol{\mu},\mathbf{z}) = \langle \boldsymbol{\mu},\mathbf{z}\rangle \quad \forall \boldsymbol{\mu}\in\boldsymbol{\Lambda}, \mathbf{z}\in H^{1/2}(\mathcal{B})^d, \tag{43}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $H^{1/2}(\mathcal{B})^d$ and $\boldsymbol{\Lambda} = (H^{1/2}(\mathcal{B})^d)'$.

With this definition, we can perform the same stability analysis as in Sect. 2 and show that Propositions 2 and 3 hold true also in this case. In the following we analyze the well-posedness of Problem 5 and its finite element discretization. The discussion will be carried on using the same arguments as in Sects. 4.1 and 4.2 relying on the formulation (26).

The following inf-sup conditions ensure existence and uniqueness of the solution (26) in the case of thin structures.

Proposition 9 *Let* $\mathbf{c}$ *be given by* (43)*, then there exists* $\alpha_0 > 0$ *such that*

$$\inf_{\mathbf{U}\in\mathbb{K}} \sup_{\mathbf{V}\in\mathbb{K}} \frac{\mathbb{A}(\mathbf{U},\mathbf{V})}{|||\mathbf{U}|||_{\mathbb{V}}|||\mathbf{V}|||_{\mathbb{V}}} \ge \alpha_0. \tag{44}$$

Proof The proof is the same except for the case $\overline{\beta} = 0$. The fact that $\mathbf{U}\in\mathbb{K}$ implies again that $\mathbf{X} = \mathbf{u}(\overline{\mathbf{X}})$, that is $\mathbf{X}$ is the trace of $\mathbf{u}$ on $\overline{\Gamma} = \overline{\mathbf{X}}(\mathcal{B})$ and the bound $\|\mathbf{X}\|_{0,\mathcal{B}} \le C\|\mathbf{u}\|_1$ is a consequence of the trace theorem in $H^1_0(\Omega)^d$. □

Proposition 10 *Let* $\mathbf{c}$ *be given by* (43)*, then there exists a constant* $\beta_0 > 0$ *such that for all* $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ *it holds true*

$$\sup_{\mathbf{V}\in\mathbb{V}} \frac{\mathbb{B}(\mathbf{V},\boldsymbol{\mu})}{\|\mathbf{V}\|_{\mathbb{V}}} \geq \beta_0 \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}. \tag{45}$$

Proof Since $\boldsymbol{\Lambda} = (H^{1/2}(\mathcal{B})^d)'$, we have the following definition of the norm in $\boldsymbol{\Lambda}$:

$$\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} = \sup_{\mathbf{z}\in H^{1/2}(\mathcal{B})^d} \frac{\langle \boldsymbol{\mu}, \mathbf{z}\rangle}{\|\mathbf{z}\|_{H^{1/2}(\mathcal{B})^d}} = \sup_{\mathbf{z}\in H^{1/2}(\mathcal{B})^d} \frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{z})}{\|\mathbf{z}\|_{H^{1/2}(\mathcal{B})^d}}.$$

Let us consider a maximizing sequence $\{\mathbf{z}_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{z}_n)}{\|\mathbf{z}_n\|_{H^{1/2}(\mathcal{B})^d}} = \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}.$$

Thanks to the surjectivity of the trace operator from $H^1_0(\Omega)^d$ to $(H^{1/2}(\overline{\Gamma}))^d$, there exists $\mathbf{u}_n \in H^1_0(\Omega)^d$ such that $\mathbf{u}_n(\overline{\mathbf{X}}) = \mathbf{z}_n$ with $\|\mathbf{u}_n\|_1 \leq c\|z_n\|_{H^{1/2}(\mathcal{B})^d}$ for some $c > 1$. Hence we obtain the desired inequality (45) as follows

$$\begin{aligned}
\sup_{\mathbf{V}\in\mathbb{V}} \tfrac{\mathbb{B}(\mathbf{V},\boldsymbol{\mu})}{\|\mathbf{V}\|_{\mathbb{V}}} &= \sup_{\mathbf{V}\in\mathbb{V}} \frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{v}(\overline{\mathbf{X}}) - \mathbf{Y})}{\|\mathbf{V}\|_{\mathbb{V}}} \geq \sup_{\mathbf{v}\in H^1_0(\Omega)^d} \frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{v}(\overline{\mathbf{X}}))}{\|\mathbf{v}\|_1} \\
&\geq \frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{u}_n(\overline{\mathbf{X}}))}{\|\mathbf{u}_n\|_1} \geq \frac{1}{c} \frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{z}_n)}{\|\mathbf{z}_n\|_{H^{1/2}(\mathcal{B})^d}} \geq \frac{1}{2c} \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}.
\end{aligned}$$

□

Let us now introduce a finite element discretization of Problem 2 with $\mathbf{c}$ given by (43). With the same notation as in Sect. 4.2, we set $\boldsymbol{\Lambda}_h = S_h \subset \boldsymbol{\Lambda}$. Then we have again that the duality pairing of regular elements in $\boldsymbol{\Lambda}$ can be computed as the scalar product in $L^2(\mathcal{B})$. Let us show the discrete inf-sup conditions which ensure existence and uniqueness of the discrete solution together with optimal error estimate.

Proposition 11 (Discrete first inf-sup condition) *Assume that* $V_h \times Q_h$ *is stable for the Stokes equation. There exists* $\alpha_1 > 0$ *independent of* h_x *and* h_s *such that*

$$\inf_{\mathbf{U}_h\in\mathbb{K}_h} \sup_{\mathbf{V}_h\in\mathbb{K}_h} \frac{\mathbb{A}(\mathbf{U}_h, \mathbf{V}_h)}{|||\mathbf{U}_h|||_{\mathbb{V}} |||\mathbf{V}_h|||_{\mathbb{V}}} \geq \alpha_1, \tag{46}$$

where the discrete kernel $\mathbb{K}_h$ *is given by*

$$\mathbb{K}_h = \left\{ \mathbf{V}_h \in \mathbb{V}_h : \mathbb{B}(\mathbf{V}_h, \boldsymbol{\mu}_h) = 0 \; \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h \right\}.$$

Proof Using the same arguments as in the proof of Proposition 6 we arrive to show that

$$\mathbb{A}(\mathbf{U}_h, \mathbf{V}_h) \geq \delta(\|\mathbf{u}_h\|_1^2 + \|p_h\|_0^2) + \mathbf{a}_s(\mathbf{X}_h, \mathbf{X}_h),$$

with $\mathbf{V}_h$ bounded as in (38). The definition of $\mathbf{a}_s$ yields

$$\mathbf{a}_s(\mathbf{X}_h, \mathbf{X}_h) = \overline{\beta}\|\mathbf{X}_h\|_{0,\mathcal{B}}^2 + \overline{\gamma}\|\nabla_s \mathbf{X}_h\|_{0,\mathcal{B}}^2$$

and in the case $\overline{\beta} = 0$ we need to bound $\|\mathbf{X}_h\|_{0,\mathcal{B}}^2$. Since $\mathbf{U}_h \in \mathbb{K}_h$, we have by definition that $\mathbf{c}(\boldsymbol{\mu}, \mathbf{u}_h(\overline{\mathbf{X}}) - \mathbf{X}_h) = (\boldsymbol{\mu}, \mathbf{u}_h(\overline{\mathbf{X}}) - \mathbf{X}_h)_{\mathcal{B}} = 0$ for all $\boldsymbol{\mu} \in S_h$. Therefore $\mathbf{X}_h = P_0(\mathbf{u}_h(\overline{\mathbf{X}}))$ where P_0 is the L^2-projection onto S_h, so that

$$\begin{aligned}\|\mathbf{X}_h\|_{0,\mathcal{B}} &= \|P_0(\mathbf{u}_h(\overline{\mathbf{X}}))\|_{0,\mathcal{B}} \leq \|\mathbf{u}_h(\overline{\mathbf{X}})\|_{0,\mathcal{B}} + \|P_0(\mathbf{u}_h(\overline{\mathbf{X}})) - \mathbf{u}_h(\overline{\mathbf{X}})\|_{0,\mathcal{B}} \\ &\leq \|\mathbf{u}_h(\overline{\mathbf{X}})\|_{0,\mathcal{B}} + Ch_s^{1/2}\|\mathbf{u}_h(\overline{\mathbf{X}})\|_{1/2,\mathcal{B}} \leq \|\mathbf{u}_h(\overline{\mathbf{X}})\|_{0,\mathcal{B}} + Ch_s^{1/2}\|\mathbf{u}_h\|_{1/2,\overline{\Gamma}} \\ &\leq C\|\mathbf{u}_h\|_1.\end{aligned}$$

The last three inequalities together with (38) give the desired result. □

Proposition 12 (Second discrete inf-sup condition) *If h_x/h_s is sufficiently small and the mesh $\mathcal{T}_h^{\mathcal{B}}$ is quasi-uniform, then there exists a constant $\beta_1 > 0$ independent of h_x and h_s such that for all $\mu_h \in \Lambda_h$ it holds true*

$$\sup_{\mathbf{V}_h \in \mathbb{V}_h} \frac{\mathbb{B}(\mathbf{V}_h, \boldsymbol{\mu}_h)}{\|\mathbf{V}_h\|_{\mathbb{V}}} \geq \beta_1\|\boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}}. \tag{47}$$

Proof In the proof of Proposition 10 we have shown that

$$\sup_{\mathbf{v} \in H_0^1(\Omega)^d} \frac{\mathbf{c}(\boldsymbol{\mu}, \mathbf{v}(\overline{\mathbf{X}}))}{\|\mathbf{v}\|_1} \geq \frac{1}{2c}\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}$$

for all $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$. Let us fix $\boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h$. Since $\boldsymbol{\Lambda}_h \subset \boldsymbol{\Lambda}$, the above inequality holds true also for $\boldsymbol{\mu}_h$. Let $\bar{\mathbf{u}} \in H_0^1(\Omega)^d$ be the element in $H_0^1(\Omega)^d$ where the above supremum is attained, hence

$$\mathbf{c}(\boldsymbol{\mu}_h, \bar{\mathbf{u}}(\overline{\mathbf{X}})) \geq \frac{1}{2c}\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}\|\bar{\mathbf{u}}\|_1.$$

Let $\bar{\mathbf{u}}_h = \Pi\bar{\mathbf{u}} \in V_h$ be the Clément interpolation of $\bar{\mathbf{u}}$, which satisfies

$$\|\bar{\mathbf{u}} - \Pi\bar{\mathbf{u}}\|_{r,\Omega} \leq C\left(\sum_{K \in \mathcal{T}_h} h_K^{2-2r}|\bar{\mathbf{u}}|_{1,K}^2\right)^{1/2},$$

then, we write

$$\mathbf{c}(\boldsymbol{\mu}_h, \bar{\mathbf{u}}_h(\overline{\mathbf{X}})) = \mathbf{c}(\boldsymbol{\mu}_h, \bar{\mathbf{u}}(\overline{\mathbf{X}})) + \mathbf{c}(\boldsymbol{\mu}_h, \bar{\mathbf{u}}_h(\overline{\mathbf{X}}) - \bar{\mathbf{u}}(\overline{\mathbf{X}})).$$

We bound the second term on the right hand side, using a trace theorem [9, Th.1.6.6] as follows

$$\|\bar{\mathbf{u}}_h(\overline{\mathbf{X}}) - \bar{\mathbf{u}}(\overline{\mathbf{X}})\|_{0,\mathcal{B}} \le C(\|\bar{\mathbf{u}}_h - \bar{\mathbf{u}}\|_{0,\Omega}\|\bar{\mathbf{u}}_h - \bar{\mathbf{u}}\|_{1,\Omega})^{1/2} \le Ch_x^{1/2}\|\bar{\mathbf{u}}\|_1$$

and the first one by means of the following inverse inequality

$$\|\boldsymbol{\mu}_h\|_{0,\mathcal{B}} \le Ch_s^{-1/2}\|\boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}}.$$

Hence we obtain

$$\begin{aligned}\mathbf{c}(\boldsymbol{\mu}_h, \bar{\mathbf{u}}_h(\overline{\mathbf{X}})) &= \mathbf{c}(\boldsymbol{\mu}_h, \bar{\mathbf{u}}(\overline{\mathbf{X}})) + \mathbf{c}(\boldsymbol{\mu}_h, \bar{\mathbf{u}}_h(\overline{\mathbf{X}}) - \bar{\mathbf{u}}(\overline{\mathbf{X}}))\\ &\ge \tfrac{1}{2c}\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}\|\bar{\mathbf{u}}\|_1 - C\|\boldsymbol{\mu}_h\|_{0,\mathcal{B}}h_x^{1/2}\|\bar{\mathbf{u}}\|_1\\ &\ge \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}}\|\bar{\mathbf{u}}\|_1\left(\tfrac{1}{2c} - C\left(\tfrac{h_x}{h_s}\right)^{1/2}\right).\end{aligned}$$

The desired inequality follows easily from

$$\sup_{\mathbf{v}\in V_h}\frac{\mathbf{c}(\boldsymbol{\mu}_h, \mathbf{v}(\overline{\mathbf{X}}))}{\|\mathbf{v}\|_1} \ge \frac{\mathbf{c}(\boldsymbol{\mu}, \bar{\mathbf{u}}_h(\overline{\mathbf{X}}))}{\|\bar{\mathbf{u}}_h\|_1} \ge \left(\frac{1}{2c} - C\left(\frac{h_x}{h_s}\right)^{1/2}\right)\frac{\|\bar{\mathbf{u}}\|_1}{\|\bar{\mathbf{u}}_h\|_1}\|\boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}}$$

if $(h_x/h_s)^{1/2} \le 1/(2cC)$. □

Acknowledgements The authors were partially supported by PRIN/MIUR, GNCS/INDAM, and IMATI/C-NR.

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