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An abstract framework in the numerical solution of boundary value problems for neutral functional differential equations

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Abstract We consider the numerical solution of boundary value problems for general neutral functional differential equations. The problems are restated in an abstract form and, then, a general discretization of the abstract form is introduced and a convergence analysis of this discretization is developed.

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1 Introduction

Let V be the space of the continuous functions $[a, b] \to \mathbb{R}^d$ and let U be a Banach space of integrable functions $[a, b] \to \mathbb{R}^d$. We deal with the numerical solution of the *functional differential equation boundary value problem* (BVP)

$$\begin{cases} y'(t) = F\left(t, y, y', p\right), & t \in [a, b], \\ B\left(y, y', p\right) = 0, \end{cases} \tag{1}$$

where the functionals $F\colon [a, b] \times V \times U \times \mathbb{R}^{d_0} \to \mathbb{R}^d$ and $B\colon V \times U \times \mathbb{R}^{d_0} \to \mathbb{R}^d \times \mathbb{R}^{d_0}$ are given and the pair $(y, p) \in V \times \mathbb{R}^{d_0}$ is unknown.

The reason to include $p \in \mathbb{R}^{d_0}$ as an unknown of the problem (1) is that, in many real applications, there are parameters to be determined along with the solution y. For example, the determination of periodic solutions for an autonomous functional differential equation reduces to a BVP, where the unknown period of the periodic solution appears as a parameter.

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The general functional differential equation

$$y'(t) = F(t, y, y', p), \quad t \in [a, b], \tag{2}$$

in (1) includes the two particular and important cases of *differential equations with deviating arguments*

$$y'(t) = f(t, y(t), y(\theta_1(t)), \ldots, y(\theta_k(t)), y'(\vartheta_1(t)), \ldots, y'(\vartheta_l(t)), p), \quad t \in [a, b], \tag{3}$$

and *integro-differential equations*

$$y'(t) = f\left(t, y(t), \int_{\alpha(t)}^{\beta(t)} k\left(t, s, y(s), y'(s)\right) ds, p\right), \quad t \in [a, b]. \tag{4}$$

In order to restate (3) and (4) in the form (2), it is necessary to have

$$\theta_r(t), \vartheta_s(t) \in [a, b], \quad t \in [a, b] \quad \text{and} \quad r = 1, \ldots, k \quad \text{and} \quad s = 1, \ldots, l, \tag{5}$$

for (3) and

$$\alpha(t), \beta(t) \in [a, b], \quad t \in [a, b]. \tag{6}$$

for (4). However, one often encounters Eq. (3), or Eq. (4), where the condition (5), or the condition (6), is not fulfilled. For example, this happens when some $\theta_r(t)$ or $\vartheta_s(t)$ in (3), or one of $\alpha(t)$ and $\beta(t)$ in (4), has the form $t \pm \tau$, where $\tau > 0$. For such equations, we need to specify the solution y and its derivative y' outside the interval $[a, b]$ by the *side condition*

$$y(t) = \phi(t) \quad \text{and} \quad y'(t) = \varphi(t), \quad t < a \text{ or } t > b, \tag{7}$$

where $\phi, \varphi\colon (-\infty, a) \cup (b, +\infty) \to \mathbb{R}^d$ are given functions (of course, it makes sense take $\varphi = \phi'$). Then, the equation can be still restated in the form (2) by incorporating the side condition in the functional F: we write the Eq. (3) as

$$y'(t) = f\left(t, y(t), \Theta\left(y, t; \phi\right), \Theta\left(y', t; \varphi\right), p\right), \quad t \in [a, b],$$

where

$$\Theta(y, t; \phi) := \left(\Theta\left(y, t; \phi\right)_1, \ldots, \Theta\left(y, t; \phi\right)_k\right)$$
$$\Theta\left(y, t; \phi\right)_r := \begin{cases} y\left(\theta_r(t)\right) \text{ if } \theta_r(t) \in [a, b] \\ \phi\left(\theta_r(t)\right) \text{ if } \theta_r(t) < a \text{ or } \theta_r(t) > b \end{cases}, \quad r = 1, \ldots, k,$$
$$\Theta(y', t; \varphi) := \left(\Theta\left(y', t; \varphi\right)_1, \ldots, \Theta\left(y', t; \varphi\right)_l\right)$$
$$\Theta\left(y', t; \varphi\right)_s := \begin{cases} y'\left(\vartheta_s(t)\right) \text{ if } \vartheta_s(t) \in [a, b] \\ \varphi\left(\vartheta_s(t)\right) \text{ if } \vartheta_s(t) < a \text{ or } \vartheta_s(t) > b \end{cases}, \quad s = 1, \ldots, l,$$

and the Eq. (4) as

$$y'(t) = f\left(t, y(t), \int_{\theta_1(t)}^{\theta_2(t)} K\left(t, s, y, y'; \phi, \varphi\right) ds, p\right), \quad t \in [a, b],$$

where

$$K(t, s, y, y'; \phi, \varphi) := \begin{cases} k(t, s, y(s), y'(s)) & \text{if } s \in [a, b] \\ k(t, s, \phi(s), \varphi(s)) & \text{if } s < a \text{ or } s > b. \end{cases}$$

Observe that the side condition (7) is considered as a part of the functional differential equation (2), not as a boundary condition.

We recall here that a particular and important case of the Eq. (3) is given by *delay differential equations*, where

$$\theta_r(t), \vartheta_s(t) \le t, \quad t \in [a, b] \quad \text{and} \quad r = 1, \ldots, k \quad \text{and} \quad s = 1, \ldots, l,$$

and two particular and important cases of the Eq. (4) are given by *Fredholm integro-differential equations*, where

$$\alpha(t) = a \quad \text{and} \quad \beta(t) = b, \quad t \in [a, b],$$

and *Volterra integro-differential equations*, where

$$\alpha(t) = a \quad \text{and} \quad \beta(t) = t, \quad t \in [a, b].$$

We also remark that the general form (2) includes integro-differential equations

$$y'(t) = f\left(t, y(t), y(\theta(t)), \int_{\alpha(t)}^{\beta(t)} k\left(t, s, y(s), y'\left(\vartheta(s)\right)\right) ds, p\right), \quad t \in [a, b],$$

which cannot be seen as Eq. (4).

The general boundary condition

$$B(y, y', p) = 0 \tag{8}$$

in (1) includes the *classical boundary condition*

$$g(y(a), y(b), p) = 0$$

and the more general *multipoint boundary condition*

$$g(y(a), y(b), y(t_1), \ldots, y(t_q), p) = 0, \tag{9}$$

where $t_i \in (a, b),\ i = 1, \ldots, q$, and *integral boundary condition*

$$g\left(y(a), y(b), \int_a^b w(t, y(t))dt, p\right) = 0. \tag{10}$$

Note that, in general, boundary conditions for first order differential equation do not involve the derivative y'. In this paper, we consider boundary conditions involving y' since our theory can deal with this situation without any further complication.

By following the usual terminology, the functional differential equation (2) can be called *neutral* since, in general, the values $F(t, y, y', p)$ depend on y'.

Of course, the neutral equation (2) also includes the case where the values of F do not depend on y'. In this case, we say that the Eq. (2) is *non-neutral*. Similarly, we say that the boundary condition (8) is non-neutral if the values of B do not depend on y' and that the BVP (1) is non-neutral if both (2) and (8) are non-neutral.

1.1 Numerical literature on BVPs for functional differential equations and aim of the paper

The papers dealing with the numerical solution of functional differential equation BVPs, apart from [13,30] described below, address special cases of the problem (1). Table 1 collects such papers according to the special case considered.

Instead, the papers [13,30] deal with BVPs for general non-neutral second order functional differential equations. The paper [13] deals with BVPs

$$\begin{cases} y''(t) = g\left(t, y(t)\right) + \mathcal{F}\left(y\right)(t), & t \in [0, 1], \\ y\left(0\right) = y(1) = 0, \end{cases}$$

where $\mathcal{F}$ is an operator acting on y, and considers a discretization of the second derivative by a central difference. As a consequence, a method of order two is obtained. The paper [30] deals with BVPs

$$\begin{cases} y''(t) = \mathcal{F}\left(y, y'\right)(t), & t \in [a, b], \\ y\left(a\right) = \alpha,\ \ y\left(b\right) = \beta, \end{cases}$$

where $\mathcal{F}$ is an operator acting on y and y', and uses special continuous (dense output) methods for second order differential equations. Such methods can reach an arbitrarily high order, if $\mathcal{F}\left(y, y'\right)$ is independent of y', and have order two at most, otherwise.

Regarding the theoretical (non-numerical) literature on BVPs for functional differential equations, among many papers, we mention here the monograph [26], which contains a collection of articles dealing with many aspects of the theory of such problems, and the book [2], which considers only the non-neutral case.

Aim of the present paper is to study the numerical solution of problems (1). We restate such problems in an abstract form and, then, we introduce a general type of discretization of the abstract form and develop a convergence analysis of this discretization.

Table 1 Numerical literature on functional differential equations BVPs

Special cases of (1)	Papers
BVPs for *non-neutral* delay differential equations	[8,12,28,29,37,38,49,52]
BVPs for *non-neutral* differential equations with deviating arguments	[1,3,9–11,19,20,50,51,53]
BVPs for *non-neutral* differential equations with a state-dependent deviating argument $\theta(t, y(t))$	[4]
BVPs for *non-neutral* singularly perturbed differential equations with deviating arguments	[31–35]
BVPs for the determination of periodic solutions of *non-neutral* delay differential equations	[15,16,18,39,54]
BVPs for the determination of periodic solutions of *non-neutral* delay differential equations with a state-dependent delay	[40]
BVPs for the determination of periodic solutions of *non-neutral* differential equations with deviating arguments	[6,7]
BVPs for *non-neutral* Fredholm integro-differential equations	[21,22,45]
BVPs for *non-neutral* Fredholm integro-differential equations with weakly singular kernels	[46–48]
BVPs for *non-neutral* Volterra integro-differential equations	[23]
BVPs for the determination of periodic solutions of *neutral* delay differential equations	[5,17]
BVPs for *neutral* differential equations with state-dependent deviating arguments arising in Wheeler–Feynman electrodynamics	[14]
BVPs for *neutral* Fredholm integro-differential equations	[25,27,55,56]
BVPs for *neutral* Volterra integro-differential equations	[24]

The general discretization studied in this paper includes the two particular discretizations of the problem (1) given by *collocation method* and the *Fourier series method*. However, in order to avoid having a very long paper, here we do not deal with these two discretizations. They are the subject of the papers [41,42] and the forthcoming papers [43,44]. The present paper contains the theoretical bases for the numerical solution of functional differential equations BVPs.

When compared to the current literature, the research started in this paper and continued in [41–44] contains the following advances.

1. The general form (1) of BVP has not ever been studied in literature, even confining to the non-neutral case. Also the abstract form and the general type of discretization considered in this paper are a novelty.
2. By confining to the non-neutral case, we consider a more general situation than that dealt in the papers [13,30]. Moreover, the methods that we introduce have

arbitrarily high order of convergence, unlike the methods in [13,30] which have order two only.
3. The study of the numerical solution of BVPs for neutral differential equations with deviating arguments is at a seminal stage. On this subject, there are only the three papers [5,14,17], where [17] introduces and proves the convergence of a method of order one and [5,14] are experimental works without convergence proofs. Moreover, regarding BVPs for neutral integro-differential equations, the literature is confined to Volterra and Fredholm equations. In our research, we introduce methods for general neutral functional differential equations BVPs of arbitrarily high order of convergence.

The plan of the paper is the following. In Sect. 2, we introduce the abstract form of the problem (1). In Sect. 3, we introduce the general type of discretization used for the abstract form. In Sect. 4, we analyze the convergence of this general discretization. In Sect. 5, we specialize the results obtained in Sect. 4, in preparation for their application in [41–44] to the problem (1) discretized by the collocation method and the Fourier series method.

1.2 Notations

We finish this section giving a list of conventions and notations used throughout the paper.

- The norm of a space Y is denoted by $\|\cdot\|_Y$.
- Cartesian product spaces are equipped with the norm given by the sum of the norms of the factor spaces.
- In the space Y, the closed ball of center $y \in Y$ and radius $r \geq 0$ is denoted by $\overline{B}(y, r)$.
- The identity operator of a space Y is denoted by I_Y.
- The norm of a bounded linear operator L from the space Y to the space Z is denoted by $\|L\|$, without any reference to the domain Y and the codomain Z.
- The Fréchet-derivative of the operator A at the point y is denoted by $DA(y)$.
- For an operator $A: Y \to Z_1 \times \cdots \times Z_k$, we define the operators $A_{Z_i}: Y \to Z_i$, $i = 1, \ldots, k$, by

  $$A(y) = (A_{Z_1}(y), \ldots, A_{Z_k}(y)), \quad y \in Y,$$

  and call them the *components* of A.

2 The abstract form

We assume that the functional F in (1) is such that $F(\cdot, v, u, \beta) \in U$, for any $(v, u, \beta) \in V \times U \times \mathbb{R}^{d_0}$. This assumption permits to introduce the operator $\mathcal{F}: V \times U \times \mathbb{R}^{d_0} \to U$ given by

$$\mathcal{F}(v, u, \beta) = F(\cdot, v, u, \beta), \quad (v, u, \beta) \in V \times U \times \mathbb{R}^{d_0},$$

and write the functional differential equation (2) as

$$y' = \mathcal{F}(y, y', p),$$

which is an equation in the space U.

The BVP (1) is now restated in abstract form, where we use the derivative y', instead of y, as the actual unknown of (1).

Consider the very simple linear differential equation

$$v'(t) = u(t), \quad t \in [a, b], \tag{11}$$

where $u \in U$ is given and $v \in V$ is the unknown. Each solution of this equation is determined by a parameter $\alpha \in \mathbb{R}^d$. Thus, we introduce a linear operator $\mathcal{G} \colon U \times \mathbb{R}^d \to V$ such that, for any $u \in U$,

$$\{v \in V \colon v \text{ is a solution of (11)}\} = \{\mathcal{G}(u, \alpha) \colon \alpha \in \mathbb{R}^d\}.$$

By following the usual terminology of the differential equations, the linear operator $\mathcal{G}$ can be called a *Green operator* for the Eq. (11). Of course, examples of a Green operator for (11) are

$$\mathcal{G}(u, \alpha)(t) = \int_c^t u(s)ds + \alpha, \quad t \in [a, b] \text{ and } (u, \alpha) \in U \times \mathbb{R}^d,$$

where $c \in [a, b]$.

The abstract form of the BVP (1) is based on the interpretation of the Eq. (11) as

$$v = \mathcal{G}(u, \alpha) \quad \text{for some } \alpha \in \mathbb{R}^d.$$

In other words, we replace the derivative operator with the Green operator.

Once a Green operator for (11) is given, the abstract form is introduced by defining what we mean for a solution of (1). Let $(y, p) \in V \times \mathbb{R}^{d_0}$. We say that (y, p) is a solution of (1) if $y = \mathcal{G}(u, \alpha)$ for some $u \in U$ and $\alpha \in \mathbb{R}^d$ such that

$$\begin{cases} u = \mathcal{F}(\mathcal{G}(u, \alpha), u, p) \\ B\,(\mathcal{G}(u, \alpha), u, p) = 0. \end{cases}$$

Hence, we reach the following abstract form of the problem (1).

PAF (**P**roblem in **A**bstract **F**orm). Given:

- a normed space $\mathbb{V}$ and Banach spaces $\mathbb{U}$, $\mathbb{A}$ and $\mathbb{B}$;
- operators $\mathfrak{F} \colon \mathbb{V} \times \mathbb{U} \times \mathbb{B} \to \mathbb{U}$ and $\mathfrak{B} \colon \mathbb{V} \times \mathbb{U} \times \mathbb{B} \to \mathbb{A} \times \mathbb{B}$;
- a linear operator $\mathfrak{G} \colon \mathbb{U} \times \mathbb{A} \to \mathbb{V}$;

find a pair $(v, \beta) \in \mathbb{V} \times \mathbb{B}$ such that $v = \mathfrak{G}(u, \alpha)$ for some $u \in \mathbb{U}$ and $\alpha \in \mathbb{A}$ satisfying

$$\begin{cases} u = \mathfrak{F}(\mathfrak{G}(u, \alpha), u, \beta) \\ \mathfrak{B}\,(\mathfrak{G}(u, \alpha), u, \beta) = 0. \end{cases}$$

Clearly, the BVPs (1) are in the form **PAF** with $\mathbb{V} = V$, $\mathbb{U} = U$, $\mathbb{A} = \mathbb{R}^d$, $\mathbb{B} = \mathbb{R}^{d_0}$, $\mathfrak{F} = \mathcal{F}$, $\mathfrak{B} = B$ and $\mathfrak{G} = \mathcal{G}$.

2.1 Other instances of the abstract form

Besides the BVPs (1), **PAF** includes other types of BVPs.

For example, consider a second order problem (not restated as a first order problem)

$$\begin{cases} y''(t) = F(t, y, p), \quad t \in [a, b], \\ B(y, p) = 0 \end{cases} \tag{12}$$

where $F\colon [a, b] \times V \times \mathbb{R}^{d_0} \to \mathbb{R}^d$ and $B\colon V \times \mathbb{R}^{d_0} \to \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d_0}$, V being, as above, the space of the continuous function $[a, b] \to \mathbb{R}^d$.

Let U be a Banach space of integrable function $[a, b] \to \mathbb{R}^d$ and consider the differential equation

$$v''(t) = u(t), \quad t \in [a, b], \tag{13}$$

where $u \in U$ is given and $v \in V$ is the unknown. A Green operator for (13) is the linear operator $\mathcal{G}\colon U \times \mathbb{R}^d \times \mathbb{R}^d \to V$ given by

$$\mathcal{G}(u, \alpha_1, \alpha_2) = \text{solution of } \begin{cases} v''(t) = u(t), \quad t \in [a, b], \\ v(a) = \alpha_1, v(b) = \alpha_2, \end{cases}$$
$$(u, \alpha_1, \alpha_2) \in U \times \mathbb{R}^d \times \mathbb{R}^d,$$

i.e.

$$\mathcal{G}(u, \alpha_1, \alpha_2)(t) = \int_a^t \int_a^s u(\sigma) d\sigma ds - \frac{t-a}{b-a} \int_a^b \int_a^s u(\sigma) d\sigma ds + \frac{b-t}{b-a}\alpha_1 + \frac{t-a}{b-a}\alpha_2$$
$$t \in [a, b].$$

Under the assumption that $F(\cdot, v, \beta) \in U$ for any $(v, \beta) \in V \times \mathbb{R}^{d_0}$, the problem (12) can be restated in the form **PAF** by introducing the operator $\mathcal{F}\colon V \times \mathbb{R}^{d_0} \to U$ given by

$$\mathcal{F}(v, \beta) = F(\cdot, v, \beta), \quad (v, \beta) \in V \times \mathbb{R}^{d_0}.$$

The BVPs (12) are in the form **PAF** with $\mathbb{V} = V$, $\mathbb{U} = U$, $\mathbb{A} = \mathbb{R}^d \times \mathbb{R}^d$, $\mathbb{B} = \mathbb{R}^{d_0}$, $\mathfrak{F} = \mathcal{F}$, $\mathfrak{B} = B$ and $\mathfrak{G} = \mathcal{G}$ (of course, here U, $\mathcal{F}$, B and $\mathcal{G}$ are those defined for the problems (12)).

Clearly, **PAF** also includes BVPs for general differential equations obtained by replacing the second derivative $y''(t)$ in (12) with a general linear differentiation operator.

PAF even includes BVPs for partial functional differential equations. In fact, consider the problem

$$\begin{cases} \Delta v(x) = F(x, v), & x \in \Omega, \\ B(v) = 0 \end{cases} \tag{14}$$

where Ω is an open set of $\mathbb{R}^d$ with boundary $\partial\Omega$, Δ is the Laplacian operator, $F\colon \Omega \times V \to \mathbb{R}$, with V the space of the continuous function $\overline{\Omega} = \Omega \cup \partial\Omega \to \mathbb{R}$, and $B\colon V \to A$, with A the space of the continuous function $\partial\Omega \to \mathbb{R}$.

Given a Banach space U of integrable functions $\overline{\Omega} \to \mathbb{R}$, we consider the differential equation

$$\Delta v(x) = u(x), \quad x \in \Omega, \tag{15}$$

where $u \in U$ is given and $v \in V$ is unknown. A Green operator for (15) is the linear operator $\mathcal{G}\colon U \times A \to V$ given by

$$\mathcal{G}(u, \alpha) = \text{ solution of } \begin{cases} \Delta v(x) = u(x), \; x \in \Omega, \\ v(x) = \alpha(x), \quad x \in \partial\Omega, \end{cases} \; (u, \alpha) \in U \times A.$$

Under the assumption that $F(\cdot, v) \in U$ for any $v \in V$, the problem (14) can be restated in the form **PAF** (without the space $\mathbb{B}$), by introducing the operator $\mathcal{F}\colon V \to U$ given by

$$\mathcal{F}(v) = F(\cdot, v), \quad v \in V.$$

The BVPs (14) are in the form **PAF** with $\mathbb{V} = V$, $\mathbb{U} = U$, $\mathbb{A} = A$, $\mathfrak{F} = \mathcal{F}$, $\mathfrak{B} = B$ and $\mathfrak{G} = \mathcal{G}$.

Note that BVPs (14) have the space $\mathbb{A} = A$ of infinite dimension. However, in the particular and important case of Dirichlet boundary conditions

$$B(v) = v|_{\partial\Omega} - g, \quad v \in V,$$

where $g \in A$, we can consider $\mathbb{V} = \{v \in V\colon v|_{\partial\Omega} \in \operatorname{span}(g)\}$, instead of $\mathbb{V} = V$, and $\mathbb{A} = \operatorname{span}(g)$, instead of $\mathbb{A} = A$, where $\operatorname{span}(g) = \{kg\colon k \in \mathbb{R}\}$, so to have the space $\mathbb{A}$ of finite dimension.

2.2 The abstract form as a fixed point problem

From now on we consider the problem **PAF** with $\mathbb{A}$ and $\mathbb{B}$ of finite dimension, rather than its particular instance (1). By introducing the product Banach space

$$X := \mathbb{U} \times \mathbb{A} \times \mathbb{B},$$

PAF can be seen as the search for fixed points of the operator $\Phi : X \to X$ given by

$$\Phi(x) = (\mathfrak{F}(\mathfrak{G}(u,\alpha),u,\beta), (\alpha,\beta) - \mathfrak{B}(\mathfrak{G}(u,\alpha),u,\beta)), \quad x = (u,\alpha,\beta) \in X.$$

We have that $(v,\beta) \in \mathbb{V} \times \mathbb{B}$ is a solution of **PAF** if and only if $v = \mathcal{G}(u,\alpha)$ for some fixed point $(u,\alpha,\beta) \in X$ of Φ.

Regarding the operators $\mathfrak{F}$ and $\mathfrak{B}$ and the linear operator $\mathfrak{G}$, we do the following assumptions.

A$\mathfrak{F}\mathfrak{B}$ (**A**ssumption $\mathfrak{F}\mathfrak{B}$). The operators $\mathfrak{F}$ and $\mathfrak{B}$ are Fréchet-differentiable at any point $(v_0,u_0,\beta_0) \in \mathbb{V} \times \mathbb{U} \times \mathbb{B}$.
A$\mathfrak{G}$. The linear operator $\mathfrak{G}$ is bounded.

Since **A$\mathfrak{F}\mathfrak{B}$** and **A$\mathfrak{G}$** hold, Φ is Fréchet-differentiable at any point $x_0 = (u_0,\alpha_0,\beta_0) \in X$ and the Fréchet-derivative $D\Phi(x_0)$ is given by

$$\begin{aligned} D\Phi(x_0)\,x &= (D\mathfrak{F}(v_0,u_0,\beta_0)(v,u,\beta), (\alpha,\beta) - D\mathfrak{B}(v_0,u_0,\beta_0)(v,u,\beta)) \\ x &= (u,\alpha,\beta) \in X, \end{aligned}$$

where $v_0 = \mathfrak{G}(u_0,\alpha_0)$ and $v = \mathfrak{G}(u,\alpha)$.

3 Discretization of the abstract form

Our aim is to numerically solve **PAF** and, in this section, we describe an its quite general discretization. In the following, the positive integer K denotes the level of the discretization: the larger K, the higher is the "quality" of the discretization.

There are two types of discretizations involved in the numerical solution of **PAF**, that we call *secondary discretization* and *primary discretization*.

3.1 The secondary discretization

Consider the BVP (1). In some cases, the values of the functional F cannot be exactly computed. For example, in case of integro-differential equation (4), F involves an integral which has to be replaced with a quadrature rule. Therefore, for any positive integer K, we have to replace F with a suitable functional $F_K : [a,b] \times V \times U \times \mathbb{R}^{d_0} \to \mathbb{R}^d$, whose values can be exactly computed. If the values of F can be exactly computed, as in case of differential equations with deviating arguments (3), we consider $F_K = F$.

As done for F, we require $F_K(\cdot,v,u,\beta) \in U$, for any positive integer K and $(v,u,\beta) \in V \times U \times \mathbb{R}^{d_0}$. Hence, for any positive integer K, we can replace the operator $\mathcal{F}$ with the operator $\mathcal{F}_K : V \times U \times \mathbb{R}^{d_0} \to U$ given by

$$\mathcal{F}_K(v,u,\beta) = F_K(\cdot,v,u,\beta), \quad (v,u,\beta) \in V \times U \times \mathbb{R}^{d_0}.$$

Analogously, for any positive integer K, we replace the functional B with a suitable functional $B_K : V \times U \times \mathbb{R}^{d_0} \to \mathbb{R}^d \times \mathbb{R}^{d_0}$, whose values can be exactly computed. If the values of B can be exactly computed, as in case of multipoint boundary conditions (9), we consider $B_K = B$.

The *secondary discretization* of **PAF** consists in replacing, for any positive integer K, the operators $\mathfrak{F}$ and $\mathfrak{B}$ with operators $\mathfrak{F}_K : \mathbb{V}\times\mathbb{U}\times\mathbb{B} \to \mathbb{U}$ and $\mathfrak{B}_K : \mathbb{V}\times\mathbb{U}\times\mathbb{B} \to \mathbb{A}\times\mathbb{B}$, respectively, whose values can be exactly computed. As done for $\mathfrak{F}$ and $\mathfrak{B}$, we assume what follows.

> $\mathbf{A}\boldsymbol{\mathfrak{F}_K}\boldsymbol{\mathfrak{B}_K}$. For any positive integer K, the operators $\mathfrak{F}_K$ and $\mathfrak{B}_K$ are Fréchet-differentiable at any point $(v_0, u_0, \beta_0) \in \mathbb{V}\times\mathbb{U}\times\mathbb{B}$.

For any positive integer K, the operator Φ is then replaced with the operator $\Phi_K : X \to X$ given by

$$\begin{aligned} \Phi_K(x) &= (\mathfrak{F}_K(\mathfrak{G}(u,\alpha),u,\beta), (\alpha,\beta) - \mathfrak{B}_K(\mathfrak{G}(u,\alpha),u,\beta)) \\ x &= (u,\alpha,\beta) \in X. \end{aligned}$$

Since $\mathbf{A}\boldsymbol{\mathfrak{F}_K}\boldsymbol{\mathfrak{B}_K}$ and $\mathbf{A}\boldsymbol{\mathfrak{G}}$ hold, Φ_K is Fréchet-differentiable at any point $x_0 = (u_0, \alpha_0, \beta_0) \in X$ and $D\Phi_K(x_0)$ is given by

$$\begin{aligned} D\Phi_K(x_0)\,x &= (D\mathfrak{F}_K(v_0,u_0,\beta_0)(v,u,\beta), (\alpha,\beta) - D\mathfrak{B}_K(v_0,u_0,\beta_0)(v,u,\beta)) \\ x &= (u,\alpha,\beta) \in X, \end{aligned} \tag{16}$$

where $v_0 = \mathfrak{G}(u_0, \alpha_0)$ and $v = \mathfrak{G}(u,\alpha)$.

3.2 The primary discretization

The *primary discretization* consists in the discretization of the space X into a finite dimensional space and of the operator Φ, actually replaced with Φ_K by the secondary discretization, into an operator acting on this finite dimensional space.

Let K be a positive integer (level of discretization). Given a finite dimensional space $\widehat{\mathbb{U}}_K$ and linear bounded operators $\pi_K : \widehat{\mathbb{U}}_K \to \mathbb{U}$ and $\rho_K : \mathbb{U} \to \widehat{\mathbb{U}}_K$, called *prolongation* to $\mathbb{U}$ and *restriction* to $\widehat{\mathbb{U}}_K$, respectively, we consider the finite-dimensional product space

$$\widehat{X}_K := \widehat{\mathbb{U}}_K \times \mathbb{A} \times \mathbb{B}$$

and the linear bounded operators $P_K : \widehat{X}_K \to X$ and $R_K : X \to \widehat{X}_K$ defined by

$$P_K\widehat{x} = (\pi_K\widehat{u}, \alpha, \beta), \quad \widehat{x} = (\widehat{u}, \alpha, \beta) \in \widehat{X}_K,$$

and

$$R_K x = (\rho_K u, \alpha, \beta), \quad x = (u, \alpha, \beta) \in X.$$

Note that if the spaces $\mathbb{A}$ and $\mathbb{B}$ were not finite-dimensional, restrictions and prolongations also for these spaces had to be introduced.

The finite-dimensional space $\widehat{X}_K$ is considered as the discretization of level K of X and the operator

$$\widehat{\Phi}_K := R_K \Phi_K P_K : \widehat{X}_K \to \widehat{X}_K$$

is considered as the discretization of level K of Φ. We have

$$\widehat{\Phi}_K(\widehat{x}) = \left(\rho_K \mathfrak{F}_K\left(\mathfrak{G}\left(\pi_K \widehat{u}, \alpha\right), \pi_K \widehat{u}, \beta\right), (\alpha, \beta) - \mathfrak{B}_K\left(\mathfrak{G}\left(\pi_K \widehat{u}, \alpha\right), \pi_K \widehat{u}, \beta\right)\right)$$
$$\widehat{x} = (\widehat{u}, \alpha, \beta) \in \widehat{X}. \tag{17}$$

Given a fixed point $\widehat{x}_K^* = \left(\widehat{u}_K^*, \alpha_K^*, \beta_K^*\right) \in \widehat{X}_K$ of $\widehat{\Phi}_K$, which can be found by a standard numerical method for solving nonlinear systems of algebraic equations, we consider

$$P_K \widehat{x}_K^* = \left(\pi_K \widehat{u}_K^*, \alpha_K^*, \beta_K^*\right) \in X \tag{18}$$

as an approximation of a fixed point of Φ and

$$(v_K^*, \beta_K^*), \quad \text{where } v_K^* = \mathfrak{G}(\pi_K \widehat{u}_K^*, \alpha_K^*), \tag{19}$$

as an approximation of a solution of **PAF**.

The papers [41–44], for the particular instance of **PAF** given by a BVP (1), deal with two types of primary discretization falling in the previous abstract general description, namely the *collocation method* and the *Fourier series method*.

Remark 1 Note that, unlike the operators $\mathfrak{F}$ and $\mathfrak{B}$, we do not replace the linear operator $\mathfrak{G}$ with an approximation $\mathfrak{G}_K$, whose values can be exactly computed. The reason for this is that we assume, as it happens for the primary discretizations dealt in [41–44], the possibility to compute exactly $\mathfrak{G}(u, \alpha)$ for any $u \in \pi_K(\widehat{\mathbb{U}}_K)$ and $\alpha \in \mathbb{A}$ (see (17)).

4 Convergence analysis

Let $x^* = (u^*, \alpha^*, \beta^*)$ be a fixed point of Φ and let (v^*, β^*), where $v^* = \mathfrak{G}(u^*, \alpha^*)$, be the relevant solution of **PAF**.

We set $D^*\Phi := D\Phi(x^*)$ and we make the following two assumptions regarding x^*.

A*x1**. There exist $r_0 > 0$ and $L \geq 0$ such that

$$\|D\Phi(x) - D^*\Phi\| \leq L\|x - x^*\|_X, \quad x \in \overline{B}(x^*, r_0).$$

A*x2**. The linear bounded operator $I_X - D^*\Phi$ is invertible, i.e. for any $(u_0, \alpha_0, \beta_0) \in X$ the linear problem

$$\begin{cases} u = D^*\mathfrak{F}(\mathfrak{G}(u, \alpha), u, \beta) + u_0 \\ D^*\mathfrak{B}(\mathfrak{G}(u, \alpha), u, \beta) = (\alpha_0, \beta_0), \end{cases}$$

where $D^*\mathfrak{F} := D\mathfrak{F}(v^*, u^*, \beta^*)$ and $D^*\mathfrak{B} := D\mathfrak{B}(v^*, u^*, \beta^*)$, has a unique solution $(u, \alpha, \beta) \in X$.

Observe that **Ax*2** says that x^* is a simple zero of $I_X - \Phi$ and implies that x^* is an isolated fixed point of Φ.

In this section, we study how x^* and (v^*, β^*) can be approximated by the approximations (18) and (19), respectively, obtained by some fixed point $\widehat{x}_K^*$ of $\widehat{\Phi}_K$.

Our analysis is based on studying of how x^* is approximated by fixed points of the operator

$$P_K R_K \Phi_K : X \to X.$$

Unlike $\widehat{\Phi}_K = R_K \Phi_K P_K$, this operator has the advantage to be defined on the space X as Φ.

Clearly, the operator $P_K R_K \Phi_K$ is Fréchet-differentiable at any point $x_0 \in X$ and its Fréchet-derivative at x_0 is $P_K R_K D\Phi_K(x_0)$, where $D\Phi_K(x_0)$ is given in (16). We set $D^*\Phi_K := D\Phi_K(x^*)$.

For notational convenience, we also introduce the operator

$$\Psi_K := I_X - P_K R_K \Phi_K,$$

whose zeros are the fixed points of $P_K R_K \Phi_K$. Note that Ψ_K is Fréchet-differentiable at any point $x_0 \in X$ and

$$D\Psi_K(x_0) = I_X - P_K R_K D\Phi_K(x_0).$$

We set

$$D^*\Psi_K := D\Psi_K(x^*) = I_X - P_K R_K D^*\Phi_K. \tag{20}$$

Since we consider the operator $P_K R_K \Phi_K$ as an approximation of the operator Φ, it is expected that $\Psi_K x^*$ has a small norm. We call $\Psi_K x^*$ the *consistency error*.

Now, we introduce the following two *stability conditions*.

CS1 (**C**ondition **S**tability **1**) There exist $r_1 > 0$ and, for any positive integer K, $L_K \geq 0$ such that

$$\begin{aligned} &\|D\Psi_K(x) - D^*\Psi_K\| = \|P_K R_K \left(D\Phi_K(x) - D^*\Phi_K\right)\| \leq L_K \|x - x^*\|_X \\ &x \in \overline{B}(x^*, r_1) \end{aligned} \tag{21}$$

(compare with **Ax*1**).

CS2. There exists a positive integer K_2 such that, for any positive integer $K \geq K_2$, $D^*\Psi_K$ is invertible and

$$\lim_{K \to \infty} \frac{1}{r_2(K)} \cdot \|(D^*\Psi_K)^{-1}\| \cdot \|\Psi_K x^*\|_X = 0, \tag{22}$$

where

$$r_2(K) := \min\left\{r_1, \frac{1}{2\|(D^*\Psi_K)^{-1}\| \cdot L_K}\right\}$$

with r_1 and L_K given in **CS1**.

By using the Lemma 1 on the zeros of Fréchet-differentiable operators given in Appendix, we obtain the next theorem.

Theorem 1 *Let* ***CS1*** *and* ***CS2*** *hold. Then, there exists a positive integer* $\overline{K}$ *such that, for any positive integer* $K \geq \overline{K}$*,* $P_K R_K \Phi_K$ *has a unique fixed point* x_K^* *in* $\overline{B}(x^*, r_2(K))$ *and*

$$\left\| x_K^* - x^* \right\|_X \leq 2\|(D^*\Psi_K)^{-1}\| \cdot \|\Psi_K x^*\|_X \tag{23}$$

holds. Moreover, we have the expansion

$$x_K^* - x^* = -(D^*\Psi_K)^{-1}\Psi_K x^* + \delta_K, \tag{24}$$

where

$$\|\delta_K\|_X \leq 4L_K \cdot \|(D^*\Psi_K)^{-1}\|^3 \cdot \|\Psi_K x^*\|_X^2. \tag{25}$$

Here, L_K *is defined in* ***CS1*** *and* $r_2(K)$ *is defined in* ***CS2****.*

Proof The proof is an application of the Lemma 1 with $Y = X$, $A = \Psi_K$ and $y^* = x^*$. Note that $Y = X$ is a Banach space since $\mathbb{U}$, $\mathbb{A}$ and $\mathbb{B}$ are Banach spaces. For $K \geq K_2$, where K_2 is defined in **CS2**, it is immediate to verify that

$$q(r_2(K)) \leq \frac{1}{2},$$

where the quantity $q(r)$ is defined in Lemma 1. Now, let $\overline{K} \geq K_2$ be such that, for $K \geq \overline{K}$,

$$\frac{1}{r_2(K)} \cdot \|(D^*\Psi_K)^{-1}\| \cdot \|\Psi_K x^*\|_X \leq \frac{1}{2} \tag{26}$$

(recall (22)). For $K \geq \overline{K}$, we have

$$\|(D^*\Psi_K)^{-1}\Psi_K x^*\|_X \leq \frac{1}{2} \cdot r_2(K) \leq (1 - q(r_2(K))) \cdot r_2(K)$$

and so, since (71) in Lemma 1 is fulfilled for $r = r_2(K)$, Ψ_K has a unique zero x_K^* in $\overline{B}(x^*, r_2(K))$ and (23) holds by (72).

As for the second part of the theorem, for $K \geq \overline{K}$, take

$$r = 2\|(D^*\Psi_K)^{-1}\| \cdot \|\Psi_K x^*\|_X$$

in the second part of Lemma 1. Since $r \leq r_2(K)$ holds (recall (26)), we have

$$q(r) \leq q(r_2(K)) \leq \frac{1}{2}$$

and

$$\|(D^*\Psi_K)^{-1}\Psi_K x^*\|_X \leq \frac{1}{2} r \leq (1 - q(r))r$$

and so the condition (71) is fulfilled. Then, we obtain (24) by (73) and

$$\|\delta_K\|_X \leq 2q(r) \cdot \|(D^*\Psi_K)^{-1}\| \cdot \|\Psi_K x^*\|_X$$

by (74). Now, since

$$q(r) \leq \|(D^*\Psi_K)^{-1}\| \cdot L_K \cdot r = 2L_K \cdot \|(D^*\Psi_K)^{-1}\|^2 \cdot \|\Psi_K x^*\|_X$$

holds, we have (25). □

Next result is a consequence of the Theorem 1 and says how x^* and (v^*, β^*) can be approximated by (18) and (19), respectively.

Theorem 2 *Let **CS1** and **CS2** hold. Then, there exists a positive integer $\widehat{K}$ such that, for any positive integer $K \geq \widehat{K}$, the operator $\widehat{\Phi}_K$ has a fixed point $\widehat{x}_K^*$ and*

$$\left\| P_K \widehat{x}_K^* - x^* \right\|_X \leq 2\|(D^*\Psi_K)^{-1}\| \cdot \|\Psi_K x^*\|_X \tag{27}$$

and

$$P_K \widehat{x}_K^* - x^* = -(D^*\Psi_K)^{-1}\Psi_K x^* + \delta_K, \tag{28}$$

where δ_K is defined in (24) and satisfies (25), hold. Moreover, if $\widehat{x}_K$ is a fixed point of $\widehat{\Phi}_K$ different from $\widehat{x}_K^*$, then

$$\|P_K \widehat{x}_K - x^*\|_X > r_2(K) \tag{29}$$

and

$$\|\widehat{x}_K - \widehat{x}_K^*\|_{\widehat{X}_K} > \frac{r_2(K)}{2\max\{\|\pi_K\|, 1\}}. \tag{30}$$

*Here, $r_2(K)$ is defined in **CS2**. Finally, regarding the approximation (v_K^*, β_K^*) of (v^*, β^*), we have*

$$\|(v_K^*, \beta_K^*) - (v^*, \beta^*)\|_{\mathbb{V}\times\mathbb{B}} \leq 2\max\{\|\mathfrak{G}\|, 1\} \cdot \|(D^*\Psi_K)^{-1}\| \cdot \|\Psi_K x^*\|_X. \tag{31}$$

Proof By recalling Theorem 1, for $K \geq \overline{K}$, let x_K^* be the unique fixed point of $P_K R_K \Phi_K$ in $\overline{B}(x^*, r_2(K))$. It is immediate to verify that $\widehat{x}_K^* = R_K \Phi_K x_K^*$ is a fixed point of $\widehat{\Phi}_K$. Moreover, we have

$$P_K \widehat{x}_K^* = P_K R_K \Phi_K x_K^* = x_K^*.$$

Therefore, (27) and (28) follow by (23) and (24) in Theorem 1, respectively.

Now, we prove the second part. Let $\widehat{x}_K$ be a fixed point of $\widehat{\Phi}_K$ different from $\widehat{x}_K^*$. It is immediate to verify that $P_K\widehat{x}_K$ is a fixed point of $P_K R_K \Phi_K$. Since x_K^* is the unique fixed point of $P_K R_K \Phi_K$ in $\overline{B}(x^*, r_2(K))$, we have (29). As for the inequality (30), observe that

$$\|P_K\widehat{x}_K - P_K\widehat{x}_K^*\|_X \geq \|P_K\widehat{x}_K - x^*\|_X - \|x_K^* - x^*\|_X > r_2(K) - \|x_K^* - x^*\|_X.$$

Since (22) and (23) hold, we have

$$\lim_{K\to\infty} \frac{1}{r_2(K)} \cdot \|x_K^* - x^*\|_X = 0.$$

Hence, there exists $\widehat{K} \geq \overline{K}$ such that, for $K \geq \widehat{K}$, we have

$$\|x_K^* - x^*\|_X \leq \frac{r_2(K)}{2}$$

and then

$$\|P_K \widehat{x}_K - P_K \widehat{x}_K^*\|_X > \frac{r_2(K)}{2}. \tag{32}$$

Now, (30) follows by (32),

$$\|P_K \widehat{x}_K - P_K \widehat{x}_K^*\|_X \leq \|P_K\| \cdot \|\widehat{x} - \widehat{x}_K^*\|_{\widehat{X}_K}$$

and

$$\|P_K\| = \max\{\|\pi_K\|, 1\}.$$

Finally, the estimate (31) is obtained by

$$\begin{aligned}
&\left\|\left(v_K^*, \beta_K^*\right) - \left(v^*, \beta^*\right)\right\|_{\mathbb{V}\times\mathbb{B}} \\
&\quad = \left\|\mathfrak{G}\left(\pi_K \widehat{u}_K^*, \alpha_K^*\right) - \mathfrak{G}\left(u^*, \alpha^*\right)\right\|_{\mathbb{V}} + \left\|\beta_K^* - \beta^*\right\|_{\mathbb{B}} \\
&\quad \leq \|\mathfrak{G}\| \left(\left\|\pi_K \widehat{u}_K^* - u^*\right\|_{\mathbb{U}} + \left\|\alpha_K^* - \alpha^*\right\|_{\mathbb{A}}\right) + \left\|\beta_K^* - \beta^*\right\|_{\mathbb{B}} \\
&\quad \leq \max\{\|\mathfrak{G}\|, 1\} \cdot \left\|P_K \widehat{x}_K^* - x^*\right\|_X.
\end{aligned}$$

□

In the next subsection, we will give an estimate of the error of the approximation (v_K^*, β_K^*), better than (31) in some situations.

Remark 2 Regarding the consistency error $\Psi_K x^*$, which appears in (27) and (31), we have

$$\|\Psi_K x^*\|_X \leq \|(P_K R_K - I_X)x^*\|_X + \|P_K R_K(\Phi_K - \Phi)x^*\|_X,$$

where we have separated the contributions of the primary and secondary discretizations. If only a primary discretization is used, i.e. $\Phi_K = \Phi$, then

$$-\Psi_K x^* = (P_K R_K - I_X)x^*.$$

Remark 3 Suppose there exists a sequence $\{\widehat{x}_K\}$ of fixed points of $\widehat{\Phi}_K$ such that $\widehat{x}_K$ is eventually different from $\widehat{x}_K^*$. By (29) and (30), we obtain

$$\frac{1}{\|P_K\widehat{x}_K - x^*\|_X} = O\left(\frac{1}{r_2(K)}\right), \quad K \to \infty, \tag{33}$$

and

$$\frac{1}{\|\widehat{x}_K - \widehat{x}_K^*\|_{\widehat{X}_K}} = O\left(\max\{\|\pi_K\|, 1\} \cdot \frac{1}{r_2(K)}\right), \quad K \to \infty, \tag{34}$$

respectively. Note that, by (33), (27) and (22), we have

$$\|P_K\widehat{x}_K^* - x^*\|_X = o\left(\|P_K\widehat{x}_K - x^*\|_X\right), \ K \to \infty. \tag{35}$$

The estimates (33)–(35) give informations on how much the fixed point $\widehat{x}_K^*$ is isolated from other fixed points of $\widehat{\Phi}_K$.

4.1 The simple case

Let us introduce the space

$$Z := \mathbb{V} \times \mathbb{A} \times \mathbb{B}$$

and the linear operator $\Lambda\colon X \to Z$ given by

$$\Lambda x = (\mathfrak{G}(u, 0), \alpha, \beta), \quad x = (u, \alpha, \beta) \in X.$$

Clearly, the linear operator Λ is bounded and

$$\|\Lambda\| = \max\{\|\mathfrak{G}(\cdot, 0)\|, 1\}$$

holds.

In this subsection, we consider the situation where, for any $x \in X$, we can factorize $D\Phi(x)$ as

$$D\Phi(x) = \Sigma(x)\Lambda, \tag{36}$$

and, for any positive integer K, $D\Phi_K(x)$ as

$$D\Phi_K(x) = \Sigma_K(x)\Lambda, \tag{37}$$

where $\Sigma(x), \Sigma_K(x)\colon Z \to X$ are linear bounded operators. We call this situation *the simple case*.

In the following, we set $\Sigma^* := \Sigma(x^*)$ and $\Sigma_K^* := \Sigma_K(x^*)$.

Note that the simple case holds if $\mathfrak{F}(v, u, \beta) = \mathfrak{F}(v, \beta)$, $\mathfrak{F}_K(v, u, \beta) = \mathfrak{F}_K(v, \beta)$, $\mathfrak{B}(v, u, \beta) = \mathfrak{B}(v, \beta)$ and $\mathfrak{B}_K(v, u, \beta) = \mathfrak{B}_K(v, \beta)$. In fact, for $x_0 = (u_0, \alpha_0, \beta_0) \in X$, factorizations (36) and (37) hold with $\Sigma(x_0)$, $\Sigma_K(x_0)\colon Z \to X$ given by

$$\Sigma(x_0)z = (D^0\mathfrak{F}(v + \mathfrak{G}(0,\alpha), \beta), (\alpha, \beta) - D^0\mathfrak{B}(v + \mathfrak{G}(0,\alpha), \beta))$$
$$\Sigma_K(x_0)z = (D^0\mathfrak{F}_K(v + \mathfrak{G}(0,\alpha), \beta), (\alpha, \beta) - D^0\mathfrak{B}_K(v + \mathfrak{G}(0,\alpha), \beta))$$
$$z = (v, \alpha, \beta) \in Z$$

where $D^0\mathfrak{F} := D\mathfrak{F}(v_0, \beta_0)$, $D^0\mathfrak{F}_K := D\mathfrak{F}_K(v_0, \beta_0)$, $D^0\mathfrak{B} := D\mathfrak{B}(v_0, \beta_0)$ and $D^0\mathfrak{B}_K := D\mathfrak{B}_K(v_0, \beta_0)$, with $v_0 = \mathfrak{G}(u_0, \alpha_0)$.

Therefore, the simple case holds for the particular instance of **PAF** given by a non-neutral BVP (1). As it is shown in [43], the simple case can hold also in case of BVP for neutral functional differential equations. In particular, it holds in case of BVPs for neutral integro-differential equation (4) and non-neutral boundary conditions.

Now, we present two theorems for the simple case. The first result is a condition under which the invertibility of the linear bounded operator $D^*\Psi_K$, and the uniform boundedness with respect to K of the norm of its inverse, are guaranteed. We recall that $D^*\Psi_K$ is defined in (20) and the norm of its inverse appears in **CS2** and in the error estimates of Theorem 2.

Theorem 3 *Assume the simple case. If*

$$\lim_{K\to\infty} \|(P_K R_K \Sigma_K^* - \Sigma^*)\Lambda\| = 0, \tag{38}$$

then there exists a positive integer K_2 such that, for any positive integer $K \geq K_2$, $D^*\Psi_K$ is invertible and

$$\|(D^*\Psi_K)^{-1}\| \leq 2\|(I_X - D^*\Phi)^{-1}\|.$$

Note that the previous theorem implicitly requires the invertibility of $I_X - D^*\Phi$, which is assumed in **A*x**2**.

Proof By recalling (20), we have

$$D^*\Psi_K = I_X - D^*\Phi - (P_K R_K \Sigma_K^* - \Sigma^*)\Lambda.$$

The theorem now follows by an application of the Banach perturbation lemma. □

Note that in (38) we can take advantage of the fact that the error operator $P_K R_K \Sigma_K^* - \Sigma^*$ is applied to elements that have been regularized by the operator Λ.

Remark 4 By separating the contributions of the primary and secondary discretizations, we have that (38) holds if

$$\lim_{K\to\infty} \|(P_K R_K - I_X)\Sigma^*\Lambda\| = 0$$

and

$$\lim_{K\to\infty} \|P_K R_K(\Sigma_K^* - \Sigma^*)\Lambda\| = 0.$$

The second result for the simple case is an estimate of the error of $\left(v_K^*, \beta_K^*\right)$ different from (31).

Theorem 4 *Let* ***CS1*** *and* ***CS2*** *hold. Assume the simple case. We have*

$$\begin{aligned} &\| \left(v_K^*, \beta_K^*\right) - \left(v^*, \beta^*\right) \|_{\mathbb{V}\times\mathbb{B}} \\ &\quad \le \max\{\|\mathfrak{G}(0,\cdot)\|, 1\} \cdot \left(\|\Xi_K\| \cdot \|\Lambda\Psi_K x^*\|_Z + \|\Lambda\| \cdot \|\delta_K\|_X\right), \end{aligned} \tag{39}$$

where

$$\Xi_K := I_Z + \Lambda(D^*\Psi_K)^{-1} P_K R_K \Sigma_K^* \tag{40}$$

and $\|\delta_K\|_X$ *satisfies (25).*

Proof Of course, all that is stated in Theorem 2 holds. Now, consider the expansion (28):

$$P_K \widehat{x}_K^* - x^* = -w_k + \delta_K,$$

where we set

$$w_K := (D^*\Psi_K)^{-1}\Psi_K x^*.$$

We have

$$\begin{aligned} &\left\|\left(v_K^*, \beta_K^*\right) - \left(v^*, \beta^*\right)\right\|_{\mathbb{V}\times\mathbb{B}} \\ &\quad = \left\|\mathfrak{G}\left(\pi_K \widehat{u}_K^*, \alpha_K^*\right) - \mathfrak{G}\left(u^*, \alpha^*\right)\right\|_{\mathbb{V}} + \left\|\beta_K^* - \beta^*\right\|_{\mathbb{B}} \\ &\quad = \left\|\mathfrak{G}\left(\pi_K \widehat{u}_K^* - u^*, 0\right) + \mathfrak{G}\left(0, \alpha_K^* - \alpha^*\right)\right\|_{\mathbb{V}} + \left\|\beta_K^* - \beta^*\right\|_{\mathbb{B}} \\ &\quad \le \max\{\|\mathfrak{G}(0,\cdot)\|, 1\} \cdot \left(\left\|\mathfrak{G}\left(\pi_K \widehat{u}_K^* - u^*, 0\right)\right\|_{\mathbb{V}} + \left\|\alpha_K^* - \alpha^*\right\|_{\mathbb{A}} + \left\|\beta_K^* - \beta^*\right\|_{\mathbb{B}}\right) \\ &\quad = \max\{\|\mathfrak{G}(0,\cdot)\|, 1\} \cdot \left\|\Lambda\left(\pi_K \widehat{u}_K^* - u^*, \alpha_K^* - \alpha^*, \beta_K^* - \beta^*\right)\right\|_Z \\ &\quad = \max\{\|\mathfrak{G}(0,\cdot)\|, 1\} \cdot \|\Lambda(-w_K + \delta_K)\|_Z \\ &\quad \le \max\{\|\mathfrak{G}(0,\cdot)\|, 1\} \cdot (\|\Lambda w_K\|_Z + \|\Lambda\| \, \|\delta_K\|_X). \end{aligned} \tag{41}$$

From (20), we obtain

$$\begin{aligned} \Lambda(D^*\Psi_K)^{-1} &= \Lambda(D^*\Psi_K)^{-1}(D^*\Psi_K + P_K R_K D^*\Phi_K) \\ &= \Lambda + \Lambda(D^*\Psi_K)^{-1} P_K R_K D^*\Phi_K \end{aligned}$$

and then

$$\begin{aligned} \Lambda w_K &= \Lambda(D^*\Psi_K)^{-1}\Psi_K x^* \\ &= \Lambda\Psi_K x^* + \Lambda(D^*\Psi_K)^{-1} P_K R_K D^*\Phi_K \Psi_K x^*. \end{aligned}$$

Now, since $D^*\Phi_K = \Sigma_K^* \Lambda$ holds, we have

$$\Lambda w_K = \Xi_K \Lambda \Psi_K x^*,$$

with Ξ_K defined in (40), and the estimate (39) follows by (41). □

This result is indeed useful since it can happen, as it is illustrated in [41] in case of the collocation method (version finite element method), that $\|\Lambda\Psi_K x^*\|_Z$ has an order of convergence to zero, as $K \to \infty$, higher than $\|\Psi_K x^*\|_X$.

Remark 5 Regarding the *regularized consistency error* $\Lambda\Psi_K x^*$, we have

$$\|\Lambda\Psi_K x^*\|_Z \leq \|\Lambda(P_K R_K - I_X)x^*\|_Z + \|\Lambda P_K R_K(\Phi_K - \Phi)x^*\|_Z,$$

where we have separated the contributions of the primary and secondary discretizations. If only a primary discretization is used, i.e. $\Phi_K = \Phi$, then

$$-\Lambda\Psi_K x^* = \Lambda(P_K R_K - I_X)x^*.$$

4.2 Invertibility of $D^*\Phi_K$

In the previous subsection, in Theorem 3, it has been presented a condition under which the invertibility of $D^*\Phi$ is guaranteed in the simple case. In this subsection, we study the invertibility of $D^*\Phi$ in the general case.

We consider a splitting

$$D^*\Phi = \Gamma^* + \Sigma^*\Lambda \tag{42}$$

of $D^*\Phi$, where $\Gamma^*\colon X \to X$ and $\Sigma^*\colon Z \to X$ are linear bounded operators. (Recall that Z and Λ have been introduced at the beginning of the previous subsection). Similarly, for any positive integer K, we consider a splitting

$$D^*\Phi_K = \Gamma_K^* + \Sigma_K^*\Lambda, \tag{43}$$

of $D^*\Phi_K$, where $\Gamma_K^*\colon X \to X$ and $\Sigma_K^*\colon Z \to X$ are linear bounded operators.

Note that in the simple case, described in the previous subsection, we have splittings (42) and (43) with $\Gamma^* = \Gamma_K^* = 0$.

In this subsection, by using splittings (42) and (43), we give a theorem concerning the invertibility of $D^*\Psi_K$ and the norm of its inverse. This theorem is an extension of the Theorem 3 (which is valid only for the simple case) and it is based on the Lemma 2 in Appendix.

Theorem 5 *Assume that there exist a splitting* (42) *such that* $I_X - \Gamma^*$ *is invertible and, for any positive integer* K*, a splitting* (43) *such that* $I_X - P_K R_K \Gamma_K^*$ *is invertible. If*

$$\lim_{K\to\infty} \|(I_X - P_K R_K \Gamma_K^*)^{-1}\| \cdot \|(P_K R_K \Gamma_K^* - \Gamma^*)(I_X - \Gamma^*)^{-1}\Sigma^*\Lambda\| = 0 \tag{44}$$

and

$$\lim_{K\to\infty} \|(I_X - P_K R_K \Gamma_K^*)^{-1}\| \cdot \|(P_K R_K \Sigma_K^* - \Sigma^*)\Lambda\| = 0, \tag{45}$$

then there exists a positive integer K_2 *such that, for any positive integer* $K \geq K_2$*,* $D^*\Psi_K$ *is invertible and*

$$\|(D^*\Psi_K)^{-1}\| \leq 2\|(I_X - D^*\Phi)^{-1}(I_X - \Gamma^*)\| \cdot \|(I_X - P_K R_K \Gamma_K^*)^{-1}\|. \tag{46}$$

Proof The proof is an application of the Lemma 2 with $Y = X$, $A = I_X - D^*\Phi$ (recall **A*x**2**), $B = I_X - \Gamma^*$, $C = -\Sigma^*\Lambda$ and, for any positive integer K, $A_K = D^*\Psi_K = I_X - P_K R_K D^*\Phi_K$, $B_K = I_X - P_K R_K \Gamma_K^*$ and $C_K = -P_K R_K \Sigma_K^* \Lambda$. □

The previous theorem reduces the invertibility of $D^*\Psi_K = I_X - P_K R_K D^*\Phi_K$ to the invertibility of $I_X - P_K R_K \Gamma_K^*$. Note that in (44) and (45), we can take advantage of the fact that the error operators $P_K R_K \Gamma_K^* - \Gamma^*$ and $P_K R_K \Sigma_K^* - \Sigma^*$ are applied to elements that have been regularized by Λ.

Remark 6 By separating the contributions of the primary and secondary discretizations, we have that (44) holds if

$$\lim_{K\to\infty} \|(I_X - P_K R_K \Gamma_K^*)^{-1}\| \cdot \|(P_K R_K - I_X)\Gamma^*(I_X - \Gamma^*)^{-1}\Sigma^*\Lambda\| = 0$$

and

$$\lim_{K\to\infty} \|(I_X - P_K R_K \Gamma_K^*)^{-1}\| \cdot \|P_K R_K (\Gamma_K^* - \Gamma^*)(I_X - \Gamma^*)^{-1}\Sigma^*\Lambda\| = 0,$$

and (45) holds if

$$\lim_{K\to\infty} \|(I_X - P_K R_K \Gamma_K^*)^{-1}\| \cdot \|(P_K R_K - I_X)\Sigma^*\Lambda\| = 0$$

and

$$\lim_{K\to\infty} \|(I_X - P_K R_K \Gamma_K^*)^{-1}\| \cdot \|P_K R_K (\Sigma_K^* - \Sigma^*)\Lambda\| = 0.$$

4.3 The nilpotency case and the splitting case

In view of Theorem 5, it remains to study the invertibility of $I_X - P_K R_K \Gamma_K^*$. To this aim, we consider the situation where there exist a positive integer c, a splitting (42) such that $(\Gamma^*)^c = 0$ and, for any positive integer K, a splitting (43) such that $(P_K R_K \Gamma_K^*)^c = 0$. We call this situation *the nilpotency case*.

Remark 7 If the nilpotency case holds, then $I_X - \Gamma^*$ is invertible and, for any positive integer K, $I_X - P_K R_K \Gamma_K^*$ is invertible and

$$\|(I_X - P_K R_K \Gamma_K^*)^{-1}\| \leq \sum_{i=0}^{c-1} \|P_K R_K\|^i \cdot \|\Gamma_K^*\|^i.$$

holds.

On the other hand, we call the *splitting case* the more general situation (including the nilpotency case) described in the premise in Theorem 5, namely there exist a splittings (42) such that $I_X - \Gamma^*$ is invertible and, for any positive integer K, a splitting (43) such that $I_X - P_K R_K \Gamma_K^*$ is invertible.

4.4 The operators $P_K R_K$ and $P_K R_K - I_X$

Regarding the primary discretization, the previous subsections have shown that the role played by the linear operators $P_K R_K : X \to X$ and $P_K R_K - I_X : X \to X$ given by

$$P_K R_K x = (\pi_K \rho_K u, \alpha, \beta), \quad x = (u, \alpha, \beta) \in X,$$

and

$$(P_K R_K - I_X)x = ((\pi_K \rho_K - I_{\mathbb{U}})u, 0, 0), \quad x = (u, \alpha, \beta) \in X, \tag{47}$$

is crucial. In this subsection, we list some simple facts about them to be used in the next section.

We have

$$\|(P_K R_K - I_X)x\|_X = \|(\pi_K \rho_K - I_{\mathbb{U}})u\|_{\mathbb{U}}, \quad x = (u, \alpha, \beta) \in X,$$

and, for a linear bounded operator $A : X \to X$,

$$\|(P_K R_K - I_X)A\| = \|(\pi_K \rho_K - I_{\mathbb{U}})A_{\mathbb{U}}\|, \tag{48}$$

where $A_{\mathbb{U}}$ is the $\mathbb{U}$-component of A defined in Sect. 1.2.

Moreover, note that

$$\lambda_K := \|P_K R_K\| = \max\{\|\pi_K \rho_K\|, 1\} \tag{49}$$

and

$$\|(P_K R_K - I_X)x^*\|_X = \|e_K^*\|_{\mathbb{U}}, \tag{50}$$

where $x^* = (u^*, \alpha^*, \beta^*)$ is the fixed point of Φ and

$$e_K^* := (\pi_K \rho_K - I_{\mathbb{U}})u^* \tag{51}$$

can be called *the consistency error of the primary discretization* (see Remark 2).

5 Specialization of the convergence results

In the convergence analysis presented above, we have considered the general situation where, beside a primary discretization, also a secondary discretization is introduced. This means that approximations $\mathfrak{F}_K$ of $\mathfrak{F}$ and $\mathfrak{B}_K$ of $\mathfrak{B}$ are used and then the operator Φ is actually replaced by Φ_K. However, in the papers [41,44], where the results of this paper are specialized to the problem (1) for two particular primary discretizations, we do not consider a secondary discretization, in order to avoid giving results with too many assumptions and details.

As previously remarked, in case of the problem (1), approximations $\mathfrak{F}_K = \mathcal{F}_K$ of $\mathfrak{F} = \mathcal{F}$ and $\mathfrak{B}_K = B_K$ of $\mathfrak{B} = B$ are used for integro-differential equation (4) and integral boundary conditions (10), respectively, where the involved integrals are approximated by quadrature rules. Convergence results, when quadrature rules are used in integro-differential equations BVPs, can be deduced from the general theory given above and they are addressed in [43]. We also remark that the choice of considering the exact computation of integrals in integro-differential equations BVPs is adopted in the papers [21,22,25,27].

The case of BVPs (1) for differential equations with deviating arguments (3) and multipoint boundary conditions (9), where a secondary discretization is not necessary, is dealt in [42].

In this section, for the situation where only a primary discretization is used (i.e., for any positive integer K, we have $\mathfrak{F}_K = \mathfrak{F}$ and $\mathfrak{B}_K = \mathfrak{B}$ and then $\Phi_K = \Phi$), we give two convergence theorems for the problem **PAF**, less abstract than Theorem 2. The first is for the simple case and the second is for the splitting case, which includes the nilpotency case. Such theorems are used in [41,44], in case of the problem (1), for the two particular primary discretizations given by the collocation method and the Fourier series method.

As already remarked, the simple case holds for non-neutral BVPs (1) and for BVPs given by neutral integro-differential equation (4) and non-neutral boundary conditions. Moreover, for the collocation method and the Fourier series method, the nilpotency case holds for BVPs given by neutral differential equations with deviating arguments (3) and non-neutral boundary conditions, whenever the *neutral* deviating arguments ϑ_s, $s = 1, \ldots, l$, are such that

$$\vartheta_s(t) \leq t - \tau, \quad s = 1, \ldots, l \quad \text{and} \quad t \in [a, b],$$

or

$$\vartheta_s(t) \geq t + \tau, \quad s = 1, \ldots, l \quad \text{and} \quad t \in [a, b],$$

for some $\tau > 0$. This is shown in [42,44].

Below, since we are considering only a primary discretization, we have that:

- the simple case reduces to the sole factorization (36);
- the splitting case uses the sole splitting (42) and requires the invertibility of $I_X - \Gamma^*$ and, for any positive integer K, of $I_X - P_K R_K \Gamma^*$;
- the nilpotency case uses the sole splitting (42) and requires $(\Gamma^*)^c = 0$ and, for any positive integer K, $(P_K R_K \Gamma^*)^c = 0$, for some positive integer c.

Moreover, we use diffusely the notation $A_{\mathbb{U}}$ of the $\mathbb{U}$-component of an operator A introduced in Sect. 1.2. Finally, we remark that the quantities λ_K and $\|e_K^*\|_{\mathbb{U}}$ (see (49)–(51)) play a crucial role. In particular, we have

$$\|\Psi_K x^*\|_X = \|(P_K R_K - I_X)x^*\|_X = \|e_K^*\|_{\mathbb{U}} \tag{52}$$

(see Remark 2).

5.1 The simple case

Here is the theorem for the simple case. Before to present it, we introduce the following condition, which is formulated only for the simple case.

CSC (**C**ondition **S**imple **C**ase) There exist $r_2 > 0$ and, for any positive integer K, $\sigma_K \geq 0$ such that

$$\begin{aligned}\|(\pi_K\rho_K - I_{\mathbb{U}})(D\Phi(x) - D^*\Phi)_{\mathbb{U}}\| &= \|(\pi_K\rho_K - I_{\mathbb{U}})((\Sigma(x) - \Sigma^*)\Lambda)_{\mathbb{U}}\| \\ &\leq \sigma_K\|x - x^*\|_X, \quad x \in \overline{B}(x^*, r_2),\end{aligned}$$

and

$$\sigma_K = O(1), \quad K \to \infty.$$

Theorem 6 *Assume that only a primary discretization is used. Moreover, assume the simple case,*

$$\lim_{K\to\infty}\|(\pi_K\rho_K - I_{\mathbb{U}})(D^*\Phi)_{\mathbb{U}}\| = \lim_{K\to\infty}\|(\pi_K\rho_K - I_{\mathbb{U}})(\Sigma^*\Lambda)_{\mathbb{U}}\| = 0 \tag{53}$$

and

$$\lim_{K\to\infty}\begin{cases}\lambda_K \cdot \|e_K^*\|_{\mathbb{U}} & \\ \|e_K^*\|_{\mathbb{U}} & \text{if } \mathbf{CSC} \text{ holds}\end{cases} = 0. \tag{54}$$

(*One has to read the lower row after* {, *instead of the upper one, if* **CSC** *holds*). *Then, there exists a positive integer $\widehat{K}$ such that, for any positive integer $K \geq \widehat{K}$, $\widehat{\Phi}_K$ has a fixed point $\widehat{x}_K^*$ such that*

$$\left\|P_K\widehat{x}_K^* - x^*\right\|_X = O(\|e_K^*\|_{\mathbb{U}}), \quad K \to \infty. \tag{55}$$

Moreover, for the approximation (v_K^*, β_K^*) of (v^*, β^*), we have the two estimates

$$\|(v_K^*, \beta_K^*) - (v^*, \beta^*)\|_{\mathbb{V}\times\mathbb{B}} = O(\|e_K^*\|_{\mathbb{U}}), \quad K \to \infty, \tag{56}$$

and

$$\begin{aligned}&\|(v_K^*, \beta_K^*) - (v^*, \beta^*)\|_{\mathbb{V}\times\mathbb{B}} \\ &= O\left(\lambda_K \cdot \|\mathfrak{G}(e_K^*, 0)\|_{\mathbb{V}}\right) + \begin{cases}O(\lambda_K \cdot \|e_K^*\|_{\mathbb{U}}^2) & \\ O(\|e_K^*\|_{\mathbb{U}}^2) & \text{if } \mathbf{CSC} \text{ holds}\end{cases}, \quad K \to \infty.\end{aligned} \tag{57}$$

Finally, suppose there exists a sequence $\{\widehat{x}_K\}$ of fixed points of $\widehat{\Phi}_K$ such that $\widehat{x}_K$ is eventually different from $\widehat{x}_K^*$. Then

$$\frac{1}{\|P_K\widehat{x}_K - x^*\|_X} = \begin{cases}O(\lambda_K) & \\ O(1) & \text{if } \mathbf{CSC} \text{ holds}\end{cases}, \quad K \to \infty, \tag{58}$$

and

$$\frac{1}{\|\widehat{x}_K - \widehat{x}_K^*\|_{\widehat{X}_K}} = \begin{cases} O(\max\{\|\pi_K\|, 1\} \cdot \lambda_K) \\ O(\max\{\|\pi_K\|, 1\}) \qquad \textit{if } \textbf{CSC} \textit{ holds} \end{cases}, \quad K \to \infty. \tag{59}$$

Note that, in **CSC** and (53), the error operator $\pi_K \rho_K - I_{\mathbb{U}}$ is applied to elements regularized by means of Λ.

Proof Since **Ax*1** holds, **CS1** is fulfilled with a constant

$$L_K = \begin{cases} O(\lambda_K) \\ O(1) \qquad \text{if } \textbf{CSC} \text{ holds} \end{cases}, \quad K \to \infty.$$

Now, we show that **CS2** holds. By (53), (48), Remark 4 and Theorem 3, we obtain that there exists a positive integer K_2 such that, for any positive integer $K \geq K_2$, $D^*\Psi_K$ is invertible and

$$\|(D^*\Psi_K)^{-1}\| = O\,(1)\,, \quad K \to \infty. \tag{60}$$

Now, we have

$$\frac{1}{r_2(K)} = \begin{cases} O(\lambda_K) \\ O(1) \qquad \text{if } \textbf{CSC} \text{ holds} \end{cases}, \quad K \to \infty. \tag{61}$$

By (61), (60), (52) and (54), we conclude that **CS2** is fulfilled.

Then, Theorem 2 says that there exists a positive integer $\widehat{K}$ such that, for any positive integer $K \geq \widehat{K}$, $\widehat{\Phi}_K$ has a fixed point $\widehat{x}_K^*$ such that (55) and (56) hold: see (60) and (52). Moreover, by Remark 3, we obtain (58) and (59): see (61).

It remains to prove (57). Since

$$\|\Lambda\Psi_K x^*\|_Z = \|\Lambda(P_K R_K - I_X)x^*\|_Z = \|\mathfrak{G}(e_K^*, 0)\|_{\mathbb{V}}$$

(see Remark 5, (47) and (51)) and

$$\|\Xi_K\| = O(\lambda_K), \quad K \to \infty,$$

(see (40) with $\Sigma_K^* = \Sigma^*$) and

$$\|\delta\|_X = \begin{cases} O(\lambda_K \cdot \|e_K^*\|_{\mathbb{U}}^2) \\ O(\|e_K^*\|_{\mathbb{U}}^2) \qquad \text{if } \textbf{CSC} \text{ holds} \end{cases}, \quad K \to \infty,$$

(see (25)), we obtain (57) by Theorem 4. □

5.2 The splitting case

Now, we give the theorem for the splitting case. If the splitting case holds, we set, for any positive integer K,

$$\mu_K := \| \left(I_X - P_K R_K \Gamma^*\right)^{-1} \|$$

If the nilpotency case holds, then, by Remark 7, we have

$$\mu_K = O\left(\lambda_K^{c-1}\right), \quad K \to \infty.$$

Theorem 7 *Assume that only a primary discretization is used. Moreover, assume the splitting case,*

$$\lim_{K\to\infty} \mu_K \cdot \|(\pi_K \rho_K - I_{\mathbb{U}})(\Gamma^*(I_X - \Gamma^*)^{-1}\Sigma^* \Lambda)_{\mathbb{U}}\| = 0, \tag{62}$$

$$\lim_{K\to\infty} \mu_K \cdot \|(\pi_K \rho_K - I_{\mathbb{U}})(\Sigma^* \Lambda)_{\mathbb{U}}\| = 0, \tag{63}$$

and

$$\lim_{K\to\infty} \mu_K^2 \lambda_K \cdot \|e_K^*\|_{\mathbb{U}} = 0. \tag{64}$$

Then, there exists a positive integer $\widehat{K}$ *such that, for any positive integer* $K \geq \widehat{K}$, $\widehat{\Phi}_K$ *has a fixed point* $\widehat{x}_K^*$ *such that*

$$\left\| P_K \widehat{x}_K^* - x^* \right\|_X = O(\mu_K \cdot \|e_K^*\|_{\mathbb{U}}), \quad K \to \infty. \tag{65}$$

Moreover, for the approximation (v_K^*, β_K^*) *of* (v^*, β^*)*, we have the estimate*

$$\|(v_K^*, \beta_K^*) - (v^*, \beta^*)\|_{\mathbb{V}\times\mathbb{B}} = O(\mu_K \cdot \|e_K^*\|_{\mathbb{U}}), \quad K \to \infty. \tag{66}$$

Finally, suppose there exists a sequence $\{\widehat{x}_K\}$ *of fixed points of* $\widehat{\Phi}_K$ *such that* $\widehat{x}_K$ *is eventually different from* $\widehat{x}_K^*$*. Then*

$$\frac{1}{\|P_K \widehat{x}_K - x^*\|_X} = O(\mu_K \lambda_K), \quad K \to \infty, \tag{67}$$

and

$$\frac{1}{\|\widehat{x}_K - \widehat{x}_K^*\|_{\widehat{X}_K}} = O\left(\max\{\|\pi_K\|, 1\} \cdot \mu_K \lambda_K\right), \quad K \to \infty. \tag{68}$$

Proof Since **A*x**1** holds, **CS1** is fulfilled with

$$L_K = O(\lambda_K), \quad K \to \infty.$$

Now, we show that **CS2** holds. By (62), (63), (48), Remark 6 and Theorem 5, we obtain that there exists a positive integer K_2 such that, for any positive integer $K \geq K_2$, $D^*\Psi_K$ is invertible and

$$\|(D^*\Psi_K)^{-1}\| = O\left(\mu_K\right), \quad K \to \infty. \tag{69}$$

Now, we have

$$\frac{1}{r_2(K)} = O(\mu_K \lambda_K), \quad K \to \infty. \tag{70}$$

By (70), (69), (52) and (64), we conclude that **CS2** is fulfilled.

Then, Theorem 2 says that there exists a positive integer $\widehat{K}$ such that, for any positive integer $K \geq \widehat{K}$, $\widehat{\Phi}_K$ has a fixed point $\widehat{x}_K^*$ such that (65) and (66) hold: see (69) and (52). Moreover, by Remark 3 we obtain (67) and (68): see (70). □

6 Conclusions

In this paper we have studied the numerical solution of **PAF**, introduced in Sect. 2, in case of spaces $\mathbb{A}$ and $\mathbb{B}$ of finite dimension. **PAF** has been discretized by a primary discretization and a secondary discretization, as explained in Sect. 3. A convergence analysis has been carried out in Sect. 4. In Sect. 4, we have also addressed the two particular situations of the simple case and the splitting case (which includes the nilpotency case). In Sect. 5, under the assumption that only a primary discretization is used, the convergence results have been specialized to these two particular situations.

The functional differential equation BVP (1) is a particular instance of **PAF**. The results of Sect. 5 are applied in [41,44] to this particular instance for the two particular primary discretizations given by the collocation method and the Fourier series method. The present paper provides the theoretical basis for the analysis of such methods.

We finish observing that **PAF** also includes BVPs for partial functional differential equations, as it has been illustrated in Sect. 2.1. Apart from the possible infinite-dimensionality of the space $\mathbb{A}$, a numerical study of such problems in the context of **PAF** has to take into account the use of approximations $\mathfrak{G}_K$ of the linear operator $\mathfrak{G}$.

Appendix

Lemma 1 *Let Y be a Banach space with norm $\|\cdot\|_Y$, let $A\colon \Omega \subseteq Y \to Y$, where Ω is open, be a Fréchet-differentiable operator and let $y^* \in \Omega$ such that $DA\left(y^*\right)$ is invertible. For any $r > 0$ such that $\overline{B}\left(y^*, r\right) \subseteq \Omega$, define*

$$q\left(r\right) := \sup_{y \in \overline{B}(y^*, r)} \left\| DA\left(y^*\right)^{-1}\left(DA(y) - DA\left(y^*\right)\right)\right\|.$$

Now, let $r > 0$ be such that $\overline{B}\left(y^*, r\right) \subseteq \Omega$. If

$$q\left(r\right) < 1 \quad \textit{and} \quad \left\| DA\left(y^*\right)^{-1} Ay^*\right\|_Y \leq \left(1 - q\left(r\right)\right) r, \tag{71}$$

then A has a unique zero $\overline{y}^*$ *in* $\overline{B}(y^*, r)$ *and*

$$\left\| \overline{y}^* - y^* \right\|_Y \leq \frac{\left\| DA\left(y^*\right)^{-1} Ay^* \right\|_Y}{1 - q\left(r\right)}. \tag{72}$$

Moreover, we have

$$\overline{y}^* - y^* = -DA\left(y^*\right)^{-1} Ay^* + \delta, \tag{73}$$

where

$$\|\delta\|_Y \leq \frac{q\left(r\right) \left\| DA\left(y^*\right)^{-1} Ay^* \right\|_Y}{1 - q\left(r\right)}. \tag{74}$$

The proof of the first part is more or less similar to the proof of [36, Lemma 19.1, page 293]. The proof of the second part (73) is clear once the proof of first part is understood.

Lemma 2 *Let Y be a Banach space with norm* $\|\cdot\|_Y$*. Let* $A, B, C\colon Y \to Y$ *be linear bounded operators such that* $A = B + C$ *and B is invertible. Let* $\{A_K\}$*,* $\{B_K\}$ *and* $\{C_K\}$ *be sequences of linear bounded operators* $Y \to Y$ *such that, for any positive integer K,* $A_K = B_K + C_K$ *and* B_K *is invertible.*

If A is invertible,

$$\lim_{K\to\infty} \|B_K^{-1}\| \cdot \|\left(B_K - B\right) B^{-1} C\| = 0 \tag{75}$$

and

$$\lim_{K\to\infty} \|B_K^{-1}\| \cdot \|C_K - C\| = 0, \tag{76}$$

then there exists a positive integer K_2 *such that, for any positive integer* $K \geq K_2$*,* A_K *is invertible and*

$$\|A_K^{-1}\| \leq 2\|A^{-1}B\| \cdot \|B_K^{-1}\|.$$

Proof Assume that A is invertible and (75) and (76) hold. For any positive integer K, we have

$$A_K = B_K + C_K = B_K\left(I_Y + B_K^{-1}C_K\right)$$

and then A_K is invertible if $I_Y + B_K^{-1}C_K$ is invertible. In this case, we have

$$A_K^{-1} = \left(I_Y + B_K^{-1}C_K\right)^{-1} B_K^{-1}.$$

Now, since

$$\begin{aligned} I_Y + B_K^{-1} C_K &= I_Y + B^{-1}C + B_K^{-1} C_K - B^{-1}C \\ &= I_Y + B^{-1}C + B_K^{-1}\left(C_K - C\right) + \left(B_K^{-1} - B^{-1}\right) C \\ &= I_Y + B^{-1}C + B_K^{-1}\left(C_K - C\right) - B_K^{-1}\left(B_K - B\right) B^{-1} C \end{aligned}$$

and

$$I_Y + B^{-1}C = B^{-1}\left(B + C\right) = B^{-1}A$$

is invertible with inverse

$$\left(I_Y + B^{-1}C\right)^{-1} = A^{-1}B,$$

the thesis follows by the Banach perturbation Lemma. □

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