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Vertex-centered finite volume schemes of any order over quadrilateral meshes for elliptic boundary value problems

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Abstract A family of any order finite volume (FV) schemes over quadrilateral meshes is analyzed under the framework of Petrov–Galerkin method. By constructing a special mapping from the trial space to the test space, a unified proof for the inf–sup condition of any order FV schemes is provided under a weak condition that the underlying mesh is an $h^{1+\gamma}, \gamma > 0$ parallelogram mesh. The optimal convergence rate of FV solutions is then obtained with well-known techniques.

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1 Introduction

Due to its local conservation property and other advantages, the finite volume method (FVM) has a wide range of applications in scientific and engineering computations see,

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e.g., [12,16,17,21–27]. Comparing to its wide applications, the mathematical theory of FVM (cf., [1–7,9,11,13–15,18,19]) has not been fully developed, especially for high order schemes.

A main difficulty in the theoretical analysis is the establishment of the stability result (or inf–sup condition in general) for higher order FV schemes. Earlier approaches (see, e.g. [6,19,20,28]) in the literature often adopt the so-called *element stiffness matrix analysis* by calculating eigenvalues of the stiffness matrix. The stability is established if all eigenvalues of the stiffness matrix are positive. This technique has been successful for linear, quadratic, and cubic elements under different triangular mesh conditions. However, generalization to higher-order elements must be done case-by-case under more and more restrictive mesh conditions. On the other hand, those mesh conditions required for stability are usually sufficient but might not be necessary. Therefore, it is necessary to develop a general framework in analyzing FV schemes of arbitrary order with a common and yet less restrictive mesh condition.

In this paper, we provide a unified analysis for vertex-centered FV schemes of any order over quadrilateral meshes, which is completely different from the classical *element stiffness matrix analysis*. An essential idea behind our analysis is the following: (1) construct a special mapping from the trial space to the test space. This mapping transfers the bilinear form defined on the trial-test spaces to a bilinear form on the trial space only, and thereby changes the the analysis framework from a Petrov–Galerkin method to a Galerkin finite element method. (2) The transferred bilinear form can be expressed on each element as a summation of function values with weights at those discrete points of the dual partition. With some proper selection of the dual partition points, e.g, the Gauss points, the summation can be viewed as a numerical quadrature (such as the Gauss-quadrature) of a finite element bilinear form. Consequently, coercivity proof of the transferred bilinear form becomes possible.

It is a challenging task to construct the *from-trial-to-test-space* mapping over an arbitrary unstructured quadrilateral mesh. In this paper, to guarantee the FV bilinear form to be expressed as the summation of function values at Gauss points, we establish our mapping through constraining at all Gauss points. Since the total number of the Gauss points is often greater than the dimension of the test space, it is necessary to justify the existence and uniqueness of our construction. To this end, we first determine those redundant constraints and eliminate them to obtain linearly independent constraints. Then by a discussion of the relation between the number of Gauss points and the dimension of the test space, we prove that the number of linearly independent constraints equals the dimension of the test space. In this way, our operator is uniquely defined.

Another major difficulty in the analysis is: unlike the triangular mesh, the transformation from the reference square to an arbitrary quadrilateral is no longer an affine mapping. As a consequence, the integrand in the transferred bilinear form are not polynomials anymore. We have to take into account of residual of the numerical quadrature. Special care and new design must be taken for the underlying FVM to overcome this difficulty.

Our main result is: Under the shape regular assumption and $h^{1+\gamma}$ distortion (to be specified in Sect. 2) of the mesh, the inf–sup condition is valid for sufficiently small h, and hence the error under the H^1-norm has the optimal rate of convergence.

Different from previous case-by-case works on FVM for linear, quadratic, and cubic quadrilateral elements, our analysis is applicable to any order FVM. Furthermore, our mesh-condition is more relaxed in two aspects: (1) We only require that the minimum interior angle of any quadrilateral is bounded from below; and (2) the mesh distortion parameter $\gamma > 0$ can be arbitrarily small. Note that $\gamma = 0$ implies an arbitrary quadrilateral without any structure. Therefore, our mesh condition here is very much similar to the most relaxed mesh condition in finite element methods.

The rest of the paper is organized as follows. In Sect. 2, we present the properties of unstructured quadrilateral meshes. In Sect. 3, we present and analyze a family of FV schemes over quadrilateral meshes under the inf–sup condition. In Sect. 4, we provide a rigorous proof for the inf–sup condition. Several numerical examples illustrating our theory is presented in Sect. 5. A brief conclusion is given in the final and sixth section.

In the rest of this paper, "$A \lesssim B$" means that A can be bounded by B multiplied by a constant which is independent of the parameters which A and B may depend on. "$A \sim B$" means "$A \lesssim B$" and "$B \lesssim A$".

2 Quadrilateral meshes

Let $\Omega \subset R^2$ be a simply connected polygon. We partition Ω into the union of a finite number of convex quadrilaterals and denote this quadrilateral mesh by $\mathcal{T}_h$, where h is the largest diameter of all quadrilaterals. We denote by $\mathcal{N}_h$ and $\mathcal{E}_h$ respectively the set of all vertices and all edges of $\mathcal{T}_h$. Moreover, let $\mathcal{N}_h^\circ = \mathcal{N}_h \backslash \partial\Omega$, $\mathcal{E}_h^\circ = \mathcal{E}_h \backslash \partial\Omega$, $\mathcal{N}_h^b = \mathcal{N}_h \cap \partial\Omega$, $\mathcal{E}_h^b = \mathcal{E}_h \backslash \partial\Omega$ be the set of interior vertices, internal edges, boundary vertices and boundary edges, respectively.

We call $\mathcal{T}_h$ *conforming* if different quadrilaterals in $\mathcal{T}_h$ have no common interior points and a vertex of any quadrilateral does not lie on the interior of a side of any other quadrilateral. We call $\mathcal{T}_h$ *shape regular* if there exist a positive constant c_1 and some angle $0 < \theta_0 < \pi/3$ such that

$$\frac{h_\tau}{\rho_\tau} \le c_1, \quad \theta_\tau \ge \theta_0, \quad \forall \tau \in \mathcal{T}_h,$$

where ρ_τ, θ_τ are the maximum diameter of circles contained in τ and the minimal interior angle of τ, respectively. For a quadrilateral $\tau \in \mathcal{T}_h$, let d_τ be the distance between midpoints of two diagonals of τ. If $d_\tau = \mathcal{O}(h^{1+\gamma})$, $\gamma \ge 0$, we call τ an $h^{1+\gamma}$ *parallelogram* (cf., [8,26]). If all quadrilaterals in $\mathcal{T}_h$ are $h^{1+\gamma}$-parallelograms, we call $\mathcal{T}_h$ an $h^{1+\gamma}$*-parallelogram mesh*. Note that $\gamma = 0$ represents arbitrary quadrilateral meshes, $\gamma = \infty$ represents parallelogram meshes.

For an arbitrary quadrilateral mesh, there exist some relations between the cardinalities of vertices, edges and elements. First, we observe that for a quadrilateral mesh $\mathcal{T}_h$,

$$\#\mathcal{N}_h^b = \#\mathcal{E}_h^b \tag{2.1}$$

and $\#\mathcal{E}_h^b$ must be an even integer, where $\#S$ is the cardinality of some set S. Moreover, we have the followingrelationship

$$\#\mathcal{T}_h = \frac{1}{2}\#\mathcal{E}_h^\circ + \frac{1}{4}\#\mathcal{E}_h^b = \#\mathcal{N}_h^\circ + \frac{1}{2}\#\mathcal{N}_h^b - 1. \tag{2.2}$$

In fact, since each quadrilateral in $\mathcal{T}_h$ has four edges, each interior edge belongs to two quadrilaterals while each boundary edge only belongs to one quadrilateral, we have

$$4\#\mathcal{T}_h = 2\#\mathcal{E}_h^\circ + \#\mathcal{E}_h^b.$$

The first equality in (2.2) is verified. We next show the second equality of (2.2) by induction. When $\#\mathcal{T}_h = 1$, we have $\#\mathcal{N}_h^\circ = 0$, $\#\mathcal{N}_h^b = 4$. Therefore the second equality of (2.2) is valid when $\#\mathcal{T}_h = 1$. Now we suppose the above equality holds for any mesh $\mathcal{T}_h$ with $\#\mathcal{T}_h = k$. Let $\tilde{\mathcal{T}}_h$ be some quadrilateral mesh with $\#\tilde{\mathcal{T}}_h = k+1$. The mesh $\tilde{\mathcal{T}}_h$ can be obtained by adding a boundary quadrilateral τ to a mesh $\mathcal{T}_h$ with $\#\mathcal{T}_h = k$. Since both the union of the elements in $\mathcal{T}_h$ and $\tilde{\mathcal{T}}_h$ are simply-connected, there are only three cases: (1) One edge of τ coincides with a boundary edge of $\mathcal{T}_h$, in this case, $\tilde{\mathcal{N}}_h^\circ = \mathcal{N}_h^\circ, \tilde{\mathcal{N}}_h^b = \mathcal{N}_h^b + 2$. (2) Two edges of τ coincide with two boundary edges of $\mathcal{T}_h$, in this case, $\tilde{\mathcal{N}}_h^\circ = \mathcal{N}_h^\circ + 1, \tilde{\mathcal{N}}_h^b = \mathcal{N}_h^b$. (3) Three edges of τ coincide with three boundary edges of $\mathcal{T}_h$, in this case, $\tilde{\mathcal{N}}_h^\circ = \mathcal{N}_h^\circ + 2, \tilde{\mathcal{N}}_h^b = \mathcal{N}_h^b - 2$. We find that in all above 3 cases, we have $\#\tilde{\mathcal{N}}_h^\circ + \frac{1}{2}\#\tilde{\mathcal{N}}_h^b = k+2$. In other words, we have

$$\#\tilde{\mathcal{T}}_h = \#\tilde{\mathcal{N}}_h^\circ + \frac{1}{2}\#\tilde{\mathcal{N}}_h^b - 1.$$

Therefore, the second equality of (2.2) is also valid.

The mesh discussed in this paper maybe completely unstructured. We next explain how to label the boundary edges and the four vertices of an element $\tau \in \mathcal{T}_h$. If two edges $E_1, E_2 \in \mathcal{E}_h$ belong to the same quadrilateral τ and they have no intersections, we call E_2 an opposite edge of E_1 and denote $E_2 = op(E_1, \tau)$. For two edges $E, F \in \mathcal{E}_h$, if there exists a finite number of quadrilaterals $\tau_1, \ldots, \tau_{m-1}$ and associated edges $E_1, \ldots, E_m \in \mathcal{E}_h, m \geq 2$ such that $E_1 = E, E_m = F, E_{i+1} = op(E_i, \tau_i), i = 1, \ldots, m-1$, then we call F as a (far) opposite edge of E and we denote $F = op(E, \tau_1, \ldots, \tau_{m-1})$ or simply denote $F = op(E)$. If $F_1 = op(E)$, $F_2 = op(E)$, the number of opposite edges between E and F_1 is smaller than that between E and F_2, we say that F_1 is *closer* to E than F_2. The boundary edge set $\mathcal{E}_h^b$ can be split into two subsets $\mathcal{E}_h^{b1}$ and $\mathcal{E}_h^{b2}$ such that $\mathcal{E}_h^{b2} = \{F = op(E) | E \in \mathcal{E}_h^{b1}\}$. We list edges in $\mathcal{E}_h^{b1}$ by $\mathcal{E}_h^{b1} = \{E_i | i = 1, \ldots, \frac{\#\mathcal{E}_h^b}{2}\}$, then $\mathcal{E}_h^{b2} = \{F_i = op(E_i) | i = 1, \ldots, \frac{\#\mathcal{E}_h^b}{2}\}$, see Fig. 1 for an example in which $\mathcal{E}_h^{b1} = \{E_1, \ldots, E_6\}$ and $\mathcal{E}_h^{b2} = \{F_1, \ldots, F_6\}$ with $F_i = op(E_i), i = 1, \ldots, 6$. For a quadrilateral $\tau \in \mathcal{T}_h$, its four vertices should be labeled such that: (1) P_1, P_2, P_3, P_4 are arranged in counter-clock order. (2) If $E_i, E_j \in \mathcal{E}_h^{b1}$, are the boundary edges such that $P_1P_2 = E_i$ or $op(E_i)$, $P_3P_4 = op(E_i)$ and $P_1P_4 = E_j$ or $op(E_j)$, $P_3P_4 = op(E_j)$, we require that P_1P_2 is *closer* to E_i than P_3P_4, P_1P_4 is closer to $E_j \in \mathcal{E}_h^{b1}$ than P_2P_3. See also Fig. 1 for an example where the four vertices of a τ are labeled.

It is well-known that the geometry of a quadrilateral $\tau = \Box P_1P_2P_3P_4 \in \mathcal{T}_h$ is determined by a transformation from the reference square $\tau_0 = [-1, 1]^2$ to τ (cf., e.g. [29,30]). For all $(\xi, \eta) \in \tau_0$, let $(x, y) = F_\tau(\xi, \eta)$ be defined by

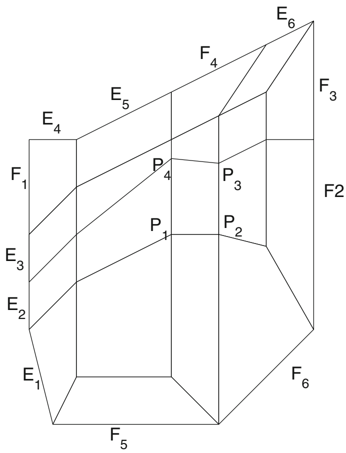


Fig. 1 Labeling boundary edges and 4 vertices of a quadrilateral

$$x = x_1 + \alpha_2^\tau \frac{1+\xi}{2} + \alpha_4^\tau \frac{1+\eta}{2} + \alpha_3^\tau \frac{(1+\xi)(1+\eta)}{4},$$
$$y = y_1 + \beta_2^\tau \frac{1+\xi}{2} + \beta_4^\tau \frac{1+\eta}{2} + \beta_3^\tau \frac{(1+\xi)(1+\eta)}{4},$$

where (x_i, y_i) is the coordinate of the vertices $P_i, i = 1, 2, 3, 4$ and

$$\alpha_2^\tau = x_2 - x_1, \alpha_4^\tau = x_4 - x_1, \alpha_3^\tau = x_1 - x_2 + x_3 - x_4,$$
$$\beta_2^\tau = y_2 - y_1,\ \beta_4^\tau = y_4 - y_1,\ \beta_3^\tau = y_1 - y_2 + y_3 - y_4.$$

Obviously, we have $F_\tau(-1,-1) = P_1$, $F_\tau(1,-1) = P_2$, $F_\tau(1,1) = P_3$, $F_\tau(-1,1) = P_4$. Moreover, noticing $d_\tau = \frac{1}{2}\sqrt{|\alpha_3^\tau|^2 + |\beta_3^\tau|^2}$, we have $\alpha_3^\tau = \beta_3^\tau = 0$ if τ is a parallelogram; $\alpha_3^\tau = \beta_3^\tau = \alpha_4^\tau = \beta_2^\tau = 0$ if τ is a rectangle; $\sqrt{|\alpha_3^\tau|^2 + |\beta_3^\tau|^2} = \mathcal{O}(h^{1+\gamma})$, if τ is an $h^{1+\gamma}$-parallelogram.

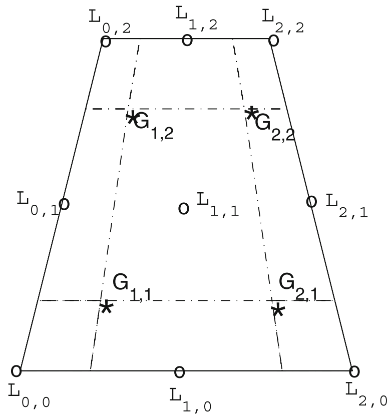
We next introduce Gauss and Lobatto points in $\mathcal{T}_h$. For any integer $n \geq 1$, we denote $\mathbb{Z}_n = \{1, \ldots, n\}$, $\mathbb{Z}_n^0 = \{0, 1, \ldots, n\}$. Let $\{l_m | m \in \mathbb{Z}_r^0\}$ be $r+1$ Lobatto points of degree r in the interval $[-1, 1]$, that is, $l_0 = -1$, $l_r = 1$ and $\{l_m | m \in \mathbb{Z}_{r-1}\}$ are the $r-1$ zeros of L_r', where L_r is the Legendre polynomial of degree r in $[-1, 1]$. We denote the set of Lobatto points in τ as

$$\mathcal{N}_\tau = \{L_{i,j}^\tau | i, j \in \mathbb{Z}_r^0\},$$

where $L_{i,j}^\tau = F_\tau(l_i, l_j)$. Moreover let $\mathcal{N} = \cup_{\tau \in \mathcal{T}_h} \mathcal{N}_\tau$ be the set of all Lobatto points in $\mathcal{T}_h$. Let $\{g_i | i \in \mathbb{Z}_r\}$ be the r Gauss points, i.e., zeros of L_r, the Legendre polynomial of degree r, on the interval $[-1, 1]$. We denote the set of Gauss points in τ by

$$\mathcal{G}_\tau = \{G_{i,j}^\tau | i, j \in \mathbb{Z}_r\},$$

Fig. 2 Gauss and Lobatto points in a quadrilateral ($r = 2$)



where $G^\tau_{i,j} = F_\tau(g_i, g_j)$. We denote the set of all Gauss points in $\mathcal{T}_h$ by $\mathcal{G} = \cup_{\tau\in\mathcal{T}_h}\mathcal{G}_\tau$. As an example, the Gauss and Lobatto points of a quadrilateral τ for $r = 2$ are depicted in Fig. 2.

We next compare the cardinality of the set of Gauss points $\mathcal{G}$ and the cardinality of the set of the interior Lobatto points $\mathcal{N}^\circ = \mathcal{N}\backslash\partial\Omega$. Obviously

$$\#\mathcal{G} = \#\mathcal{T}_h r^2,$$

where $\#\mathcal{T}_h$ is the number of quadrilaterals in $\mathcal{T}_h$. We next calculate $\#\mathcal{N}^\circ$. Since $\mathcal{N}^\circ$ consists of the interior vertices, the Lobatto points in the interior of quadrilaterals and the Lobatto points in the interior of internal edges, we have

$$\#\mathcal{N}^\circ = \#\mathcal{N}_h^\circ + \#(\mathcal{E}_h^\circ)(r-1) + \#\mathcal{T}_h(r-1)^2.$$

Substituting (2.2) and (2.1) into the above formula, we obtain

$$\#\mathcal{N}^\circ = (\#\mathcal{T}_h)r^2 - \frac{1}{2}(\#\mathcal{E}_h^b)r + 1. \tag{2.3}$$

In other words,

$$\#\mathcal{N}^\circ = \#\mathcal{G} - \frac{1}{2}(\#\mathcal{E}_h^b)r + 1. \tag{2.4}$$

We next study further properties of the transformation $F_\tau, \tau \in \mathcal{T}_h$. It is easy to calculate the Jacobi matrix

$$DF_\tau(\xi,\eta) = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_2^\tau}{2} + \frac{\alpha_3^\tau(1+\eta)}{4} & \frac{\beta_2^\tau}{2} + \frac{\beta_3^\tau(1+\eta)}{4} \\ \frac{\alpha_4^\tau}{2} + \frac{\alpha_3^\tau(1+\xi)}{4} & \frac{\beta_4^\tau}{2} + \frac{\beta_3^\tau(1+\xi)}{4} \end{pmatrix},$$

and its inverse

$$DF_\tau^{-1}(\xi,\eta) = J_\tau^{-1}\begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{pmatrix},$$

where the determinant of the Jacobi matrix DF_τ is

$$J_\tau = J_{\tau,p} + J_{\tau,\xi}(1+\xi) + J_{\tau,\eta}(1+\eta),$$

with

$$J_{\tau,p} = \frac{1}{4}(\alpha_2^\tau\beta_4^\tau - \alpha_4^\tau\beta_2^\tau),\ J_{\tau,\xi} = \frac{1}{8}(\alpha_2^\tau\beta_3^\tau - \alpha_3^\tau\beta_2^\tau),\ J_{\tau,\eta} = \frac{1}{8}(\alpha_3^\tau\beta_4^\tau - \alpha_4^\tau\beta_3^\tau).$$

Since $\tau = \Box P_1P_2P_3P_4$ is convex and its vertices P_1, P_2, P_3, P_4 are labeled counter-clockwisely, we have

$$J_{\tau,p} = \frac{1}{2}S_{\Delta P_1P_2P_4} > 0.$$

Since τ is a shape regular $h^{1+\gamma}$ parallelogram, we have

$$J_{\tau,p} = \mathcal{O}(h^2),\ J_{\tau,\xi},\ J_{\tau,\eta} = \mathcal{O}(h^{2+\gamma}). \tag{2.5}$$

We close this section by a study of the relationship of directional derivatives of a function on τ and that on τ_0. Let v be a differentiable function defined on τ and $\hat{v}_\tau = v \circ F_\tau$ be a function defined on τ_0. We denote the gradient of v on τ by $\triangledown v = (\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y})^T$ and the gradient of $\hat{v}_\tau$ on τ_0 by $\hat{\triangledown}\hat{v}_\tau = (\frac{\partial \hat{v}_\tau}{\partial \xi}, \frac{\partial \hat{v}_\tau}{\partial \eta})^T$. A straightforward calculation yields that

$$\hat{\triangledown}^{\hat{v}_\tau} = DF_\tau \triangledown v$$

Let $\mathbf{n} = (n_1, n_2)$ be a unit direction, we have the directional derivative

$$\begin{aligned}\frac{\partial v}{\partial \mathbf{n}} &= \mathbf{n}\cdot\triangledown v = \mathbf{n}\cdot DF_\tau^{-1}\hat{\triangledown}^{\hat{v}_\tau}\\ &= J_\tau^{-1}\left(\left(\frac{\partial y}{\partial\eta}n_1 - \frac{\partial x}{\partial\eta}n_2\right)\frac{\partial\hat{v}_\tau}{\partial\xi} + \left(-\frac{\partial y}{\partial\xi}n_1 + \frac{\partial x}{\partial\xi}n_2\right)\frac{\partial\hat{v}_\tau}{\partial\eta}\right).\end{aligned} \tag{2.6}$$

We next derive the directional derivatives for some specific $\mathbf{n}$.

For all $(\xi,\eta)\in\tau_0$, $F_\tau([-1,1]\times\{\eta\})$ and $F_\tau(\{\xi\}\times[-1,1]\})$ are two segments in τ which passing the point $F_\tau(\xi,\eta)$. We let

$$r_{\xi,\eta}^{\tau,1} = |F_\tau([-1,1]\times\{\eta\})|/2,\quad r_{\xi,\eta}^{\tau,2} = |F_\tau(\{\xi\}\times[-1,1]\})|/2$$

be the length of two vectors $\frac{1}{2}\overrightarrow{F_\tau(-1,\eta)F_\tau(1,\eta)}$ and $\frac{1}{2}\overrightarrow{F_\tau(\xi,-1)F_\tau(\xi,1)}$, respectively. A straightforward calculation yields

$$(r_{\xi,\eta}^{\tau,1})^2 = \left(\frac{\alpha_2^\tau}{2} + \frac{\alpha_3^\tau(1+\eta)}{4}\right)^2 + \left(\frac{\beta_2^\tau}{2} + \frac{\beta_3^\tau(1+\eta)}{4}\right)^2,$$

and

$$(r^{\tau,2}_{\xi,\eta})^2 = \left(\frac{\alpha_4^\tau}{2} + \frac{\alpha_3^\tau(1+\xi)}{4}\right)^2 + \left(\frac{\beta_4^\tau}{2} + \frac{\beta_3^\tau(1+\xi)}{4}\right)^2.$$

We also let

$$\begin{aligned} s^\tau_{\xi,\eta} = &\left(\frac{\alpha_2^\tau}{2} + \frac{\alpha_3^\tau(1+\eta)}{4}\right)\left(\frac{\alpha_4^\tau}{2} + \frac{\alpha_3^\tau(1+\xi)}{4}\right) \\ &+ \left(\frac{\beta_2^\tau}{2} + \frac{\beta_3^\tau(1+\eta)}{4}\right)\left(\frac{\beta_4^\tau}{2} + \frac{\beta_3^\tau(1+\xi)}{4}\right) \end{aligned}$$

be the inner product of two vectors $\frac{1}{2}\overrightarrow{F_\tau(-1,\eta)F_\tau(1,\eta)}$ and $\frac{1}{2}\overrightarrow{F_\tau(\xi,-1)F_\tau(\xi,1)}$. Since $\mathcal{T}_h$ is shape regular, there exists a minimal angle θ_0 such that each angle of the quadrilateral τ is larger than θ_0. Thus we have

$$|s^\tau_{\xi,\eta}| \le \cos\theta_0 r^{\tau,1}_{\xi,\eta} r^{\tau,2}_{\xi,\eta}. \tag{2.7}$$

For a fixed η, the normal direction on the edge $\overrightarrow{F_\tau(-1,\eta)F_\tau(1,\eta)}$ is

$$\mathbf{n}^1_{\xi,\eta} = \left(-\frac{\beta_2^\tau}{2} - \frac{\beta_3^\tau(1+\eta)}{4}, \frac{\alpha_2^\tau}{2} + \frac{\alpha_3^\tau(1+\eta)}{4}\right) \Big/ r^{\tau,1}_{\xi,\eta}.$$

Similarly, for a fixed ξ, the normal direction on the edge $\overrightarrow{F_\tau(\xi,-1)F_\tau(\xi,1)}$ is

$$\mathbf{n}^2_{\xi,\eta} = \left(\frac{\beta_4^\tau}{2} + \frac{\beta_3^\tau(1+\xi)}{4}, -\frac{\alpha_4^\tau}{2} - \frac{\alpha_3^\tau(1+\xi)}{4}\right) \Big/ r^{\tau,2}_{\xi,\eta}.$$

Therefore, on the edge $\overrightarrow{F_\tau(-1,\eta)F_\tau(1,\eta)}$,

$$\frac{\partial v}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}^1_{\xi,\eta}} = \frac{1}{r^{\tau,1}_{\xi,\eta} J_\tau}\left(-s^\tau_{\xi,\eta}\frac{\partial \hat{v}_\tau}{\partial \xi} + (r^{\tau,1}_{\xi,\eta})^2 \frac{\partial \hat{v}_\tau}{\partial \eta}\right). \tag{2.8}$$

And on the edge $\overrightarrow{F_\tau(\xi,-1)F_\tau(\xi,1)}$,

$$\frac{\partial v}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}^2_{\xi,\eta}} = \frac{1}{r^{\tau,2}_{\xi,\eta} J_\tau}\left((r^{\tau,2}_{\xi,\eta})^2 \frac{\partial \hat{v}_\tau}{\partial \xi} - s^\tau_{\xi,\eta}\frac{\partial \hat{v}_\tau}{\partial \eta}\right). \tag{2.9}$$

From the above formulae, we find that even $\hat{v}_\tau$ is a polynomial, the directional derivative $\frac{\partial v}{\partial \mathbf{n}}$ may not be a polynomial since J_τ is not a constant for an arbitrary quadrilateral.

3 Finite volume schemes of any order

We consider finite volume schemes of any order for the elliptic boundary value problem

$$-\nabla\cdot(\alpha\nabla u) = f \text{ in } \Omega, \tag{3.1}$$
$$u = 0 \text{ on } \Gamma, \tag{3.2}$$

where $\Omega \subset \mathbb{R}^2$ is a simply connected polygon, $\Gamma = \partial\Omega, \alpha \in L^\infty(\Omega)$ and it is bounded from below: There exists a constant $\alpha_0 > 0$ such that $\alpha(x) \geq \alpha_0$ for almost all $x \in \Omega$, and $f \in L^2(\Omega)$ is a real-valued function defined on Ω.

We will present our finite volume schemes under the framework of Petrov–Galerkin method. We first choose the trial space as the standard FEM space of any degree $r \geq 1$ defined by

$$\mathcal{U}_h^r = \{v \in C(\Omega) | \hat{v}_\tau = v \circ F_\tau \in \mathbb{Q}_r(\tau_0), \quad \forall\tau \in \mathcal{T}_h, \text{and } v|_{\partial\Omega} = 0\},$$

where $\mathbb{Q}_r(\tau_0)$ is the set of all bi-polynomials of degree no more than r. By the standard approximation theory, we have

$$\dim \mathcal{U}_h^r = \#\mathcal{N}^\circ,$$

where $\mathcal{N}^\circ = \mathcal{N}\backslash\partial\Omega$ is the set of all internal Lobatto points.

We next present the dual mesh and the corresponding test space. The dual mesh is constructed as follows. For any segment E, we denote $G_{E,j}\, j \in \mathbb{Z}_r$ the Gauss points on E. For a quadrilateral $\tau = \square P_1P_2P_3P_4 \in \mathcal{T}_h$, the dual mesh in τ is obtained by connecting $G_{\overline{P_1P_2},j}$ and $G_{\overline{P_4P_3},j}$, $j \in \mathbb{Z}_r$, $G_{\overline{P_1P_4},j}$ and $G_{\overline{P_2P_3},j}$, $j \in \mathbb{Z}_r$ with lines, see Fig. 2 for an example $r = 2$. Since F_τ is a bilinear transformation, for a fixed $j \in \mathbb{Z}_{r+1}^0$, all Gauss points $G_{i,j}^\tau$, $i \in \mathbb{Z}_{r+1}^0$ are located on one same line and for a fixed $i \in \mathbb{Z}_{r+1}^0$, all points $G_{i,j}^\tau$, $j \in \mathbb{Z}_{r+1}^0$ are on another same line. In other words, we construct a control volume V_p for each Lobatto point $p \in \mathcal{N}$. The contribution from a quadrilateral $\tau \ni p$ is

$$V_{\tau,p} = \square G_{ij}^\tau G_{i+1j}^\tau G_{i+1j+1}^\tau G_{ij+1}^\tau$$

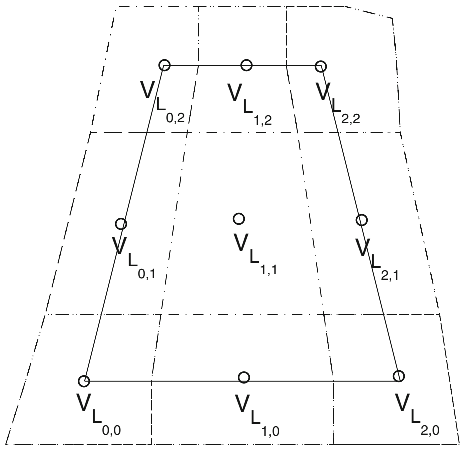
where i, j is chosen such that $p = L_{ij}^\tau$. Here, the notation of Gauss points has been extended to all $i, j \in \mathbb{Z}_{r+1}^0$ by letting $g_0 = -1, g_{r+1} = 1$, see Fig. 3 for a simple case where $r = 2$. Note that in this simple case, the generalized Gauss points $G_{0,0}^\tau = L_{0,0}^\tau$, $G_{3,0}^\tau = L_{2,0}^\tau$, $G_{3,3}^\tau = L_{2,2}^\tau$, $G_{0,2}^\tau = L_{0,2}^\tau$.

The whole control volume surrounding P is then defined as

$$V_P = \bigcup_{\tau\ni P} V_{\tau,P}.$$

For the simple case $r = 2$, whole control volumes surrounding Lobatto points in a quadrilateral are plotted in Fig. 3. In this figure, each control volume is a polygon

Fig. 3 Control volumes associated with a quadrilateral τ ($r = 2$)



surrounded by dash lines. The dual mesh $\mathcal{T}_h'$ consists of all control volumes V_p, $p \in \mathcal{N}$. That is,

$$\mathcal{T}_h' = \{V_p | p \in \mathcal{N}\}.$$

The corresponding test space is defined as

$$\mathcal{V}_h = \mathrm{Span}\{\psi_{V_p} | p \in \mathcal{N}^\circ\},$$

where ψ_A is the characteristic function of some set $A \subset \Omega$. Obviously, we have

$$\dim \mathcal{V}_h = \#\mathcal{N}_0 = \dim \mathcal{U}_h^r.$$

We are now ready to present our finite volume schemes. The finite volume solution of (3.1) and (3.2) is a function $u_h \in \mathcal{U}_h^r$ which satisfies conservation laws

$$-\int_{\partial V_p} \alpha \frac{\partial u_h}{\partial \mathbf{n}} ds = \int_{V_p} f dxdy \tag{3.3}$$

on each control volume V_p, $p \in \mathcal{N}^\circ$, where $\mathbf{n}$ is the unit outward normal on the boundary curve ∂V_p. Let $w_h \in \mathcal{V}_h$, w_h can be written as

$$w_h = \sum_{p \in \mathcal{N}^\circ} w_p \psi_{V_p}$$

where the coefficients w_p, $p \in \mathcal{N}^\circ$ are constants, ψ_S is the characteristic function of the subset $S \subset \Omega$. Multiplying (3.3) with w_p and then summing up for all $p \in \mathcal{N}^\circ$, we obtain

$$-\sum_{p\in\mathcal{N}^\circ} w_p \int_{\partial V_p} \alpha \frac{\partial u_h}{\partial \mathbf{n}} ds = \int_\Omega f w_h dxdy.$$

Defining the FVM bilinear form for all $v \in H_0^1(\Omega)$, $w_h \in \mathcal{V}_h$ as

$$a_h(v, w_{\mathcal{P}'}) = -\sum_{p\in\mathcal{N}^\circ} w_p \int_{\partial V_p} \alpha \frac{\partial v}{\partial \mathbf{n}} ds, \tag{3.4}$$

the finite volume method for solving Eqs. (3.1) and (3.2) reads as: Find $u_h \in \mathcal{U}_h^r$ such that

$$a_h(u_h, w_h) = (f, w_h), \quad \forall w_h \in \mathcal{V}_h. \tag{3.5}$$

We next study of the continuity of the bilinear form $a_h(\cdot,\cdot)$ defined in (3.4). We denote by $\mathcal{E}_h'$ the set of interior edges of the dual partition $\mathcal{T}_h'$. Moreover, we define a semi-norm in the test space $\mathcal{V}_h$ for all $w_h \in \mathcal{V}_h$ by

$$|w_h|_h' = \left(\sum_{E\in\mathcal{E}_h'} h_E^{-1} \int_E [w_h]^2 \, ds\right)^{\frac{1}{2}},$$

where h_E is the diameter of an edge E, and a semi-norm in the so-called *broken H^2 space*

$$H_h^2(\Omega) = \{v \in C(\Omega) : v|_\tau \in H^2, \quad \forall \tau \in \mathcal{T}_h\}$$

for all $v \in H_h^2(\Omega)$ by

$$|v|_h = \left(\sum_{\tau\in\mathcal{T}_h} |v|_{1,\tau}^2 + h_\tau^2 |v|_{2,\tau}^2\right)^{\frac{1}{2}},$$

where h_τ is the diameter of τ. The mesh dependent semi-norm $|\cdot|_h$ has also been used in [28].

The bilinear form $a_h(\cdot,\cdot)$ can be rewritten for all $v \in H_0^1(\Omega)$, $w_h \in \mathcal{V}_h$ as

$$a_h(v, w_h) = \sum_{E\in\mathcal{E}_h'} [w_h]_E \int_E \alpha \frac{\partial v}{\partial \mathbf{n}} ds, \tag{3.6}$$

where $[w_h]_E = w_h|_{V_2} - w_h|_{V_1}$ denotes the jump of the w_h across the common edge $E = V_1 \cap V_2$ of two volumes $V_1, V_2 \in \mathcal{T}_h'$ and $\mathbf{n}$ denotes the normal vector on E pointing from V_1 to V_2.

Theorem 3.1 *The finite volume bilinear form $a_h(\cdot,\cdot)$ is variationally exact:*

$$a_h(u, w_h) = (f, w_h) \quad \forall\, w_h \in \mathcal{V}_h \tag{3.7}$$

and continuous: for all $v \in H_0^1(\Omega) \cap H_h^2(\Omega)$, $w_h \in \mathcal{V}_h$,

$$|a_h(v, w_h)| \lesssim |v|_h |w_h|_h', \tag{3.8}$$

where the hidden constant is independent of h.

Furthermore, let u be the solution of (3.1) and (3.2), u_h the solution of (3.5). If there holds the following inf–sup condition

$$\inf_{v_h \in \mathcal{U}_h^r} \sup_{w_h \in \mathcal{V}_h^r} \frac{a_h(v_h, w_h)}{|v_h|_h |w_h|_h'} \gtrsim 1, \tag{3.9}$$

then

$$|u - u_h|_h \lesssim \inf_{v_h \in \mathcal{U}_h^r} |u - v_h|_h. \tag{3.10}$$

Consequently, if $u \in H^{r+1}(\Omega)$,

$$|u - u_h|_1 \lesssim h^r |u|_{r+1}. \tag{3.11}$$

Proof First, (3.7) follows by multiplying (3.1) with an arbitrary function $w_h \in \mathcal{V}_h$ and then using Green's formula in each control volume $\tau \in \mathcal{T}_h'$.

Secondly we prove (3.8). By the Cauchy–Schwartz inequality, for all $v \in H_0^1(\Omega)$ and all $w_h \in \mathcal{V}_h$, there holds

$$a_h(v, w_h) \le \|\alpha\|_\infty |w_h|_h' \left(\sum_{E \in \mathcal{E}_h'} h_E \int_E \left(\frac{\partial v}{\partial \mathbf{n}} \right)^2 ds \right)^{\frac{1}{2}}.$$

By the trace inequality and the shape regularity of $\mathcal{T}_h$,

$$\left(h_E \int_{E \cap \tau} \left(\frac{\partial v}{\partial \mathbf{n}} \right)^2 ds \right)^{\frac{1}{2}} \lesssim |v|_{1,\tau} + h_\tau |v|_{2,\tau}$$

where $\tau \in \mathcal{T}_h$ and $\tau \cap E \neq \emptyset$. Since for any given $E \in \mathcal{E}'_h$, there are at most two elements $\tau \in \mathcal{T}_h$ such that $\tau \cap E \neq \emptyset$, we have

$$a_{\mathcal{P}}(v, w_h) \lesssim |w_h|'_h \left(\sum_{E \in \mathcal{E}_h} \sum_{\tau \in \mathcal{T}_h, \tau \cap E \neq \emptyset} |v|^2_{1,\tau} + h^2_\tau |v|^2_{2,\tau} \right)^{\frac{1}{2}}$$
$$\lesssim |w_h|'_h \left(\sum_{\tau \in \mathcal{T}_h} |v|^2_{1,\tau} + h^2_\tau |v|^2_{2,\tau} \right)^{\frac{1}{2}}.$$

Then there exists a positive M which depends only on α and r such that (3.8) holds.

We next show (3.10). By (3.7) and the inf–sup condition (3.9), for all $v_h \in \mathcal{U}^r_h$, there holds

$$|u_h - v_h|_h \lesssim \sup_{w_h \in \mathcal{V}'_h} \frac{a_h(u_h - v_h, w_h)}{|w_h|'_h} = \sup_{w_h \in \mathcal{V}'_h} \frac{a_h(u - v_h, w_h)}{|w_h|'_h}.$$

Then by the triangle inequality and the continuity (3.8), we have

$$|u - u_h|_h \leq |u - v_h|_h + |v_h - u_h|_h \lesssim |u - v_h|_h.$$

That is, (3.10) holds.

We conclude from the definition of $|\cdot|_h$ and (3.10) that

$$|u - u_h|_1 \leq |u - u_h|_h \lesssim \inf_{v_h \in \mathcal{U}^r_h} |u - v_h|_h.$$

Note that

$$\inf_{v_h \in \mathcal{U}^r_h} |u - v_h|_h \leq |u - u_I|_1 + h|u - u_I|_2,$$

where $u_I \in \mathcal{U}^r_h$ is the interpolation of u which will be introduced precisely in next subsection. By the standard approximation theory, we obtain the estimate (3.11). □

Remark 3.2 We observe from the above theorem that the inf–sup condition (3.9) plays a critical role in the proof of the optimal convergence rates of the FV solutions. The proof of (3.9) is the task of next section.

4 Inf–sup property

This section is devoted to a rigorous proof for (3.9) which is the core of the paper. Since by the inverse inequality, we have the norm equivalence

$$|v_h|_h \sim |v_h|_1, \quad \forall v_h \in \mathcal{U}^r_h,$$

the inf–sup property (3.9) is equivalent to

$$\inf_{v_h\in\mathcal{U}_h^r}\sup_{w_h\in\mathcal{V}_h'}\frac{a_h(v_h,w_h)}{|v_h|_1|w_h|_h'}\gtrsim 1. \tag{4.1}$$

We begin with a simple observation that we only need to prove (4.1) for the case that α is piecewise constant with respect to the partition $\mathcal{T}_h$. In fact, for a piecewise continuous coefficient α, let

$$\bar{\alpha}(x,y)=\frac{1}{|\tau|}\int_\tau \alpha(x,y)dxdy,\quad \forall (x,y)\in\tau\in\mathcal{T}_h$$

and denote its piecewise modulus of continuity by

$$m_{\mathcal{T}_h}(\alpha,h)=\sup\{|\alpha(\mathbf{x}_1)-\alpha(\mathbf{x}_2)|:|\mathbf{x}_1-\mathbf{x}_2|\le h,\quad \forall\,\mathbf{x}_1,\mathbf{x}_2\in\tau,\forall\,\tau\in\mathcal{T}_h\}.$$

The fact that α is piecewise continuous implies that $m_{\mathcal{T}_h}(\alpha,h)$ converges to 0 when h goes to 0. We define a new bilinear form $\bar{a}_h(\cdot,\cdot)$ for all $v_h\in\mathcal{U}_h^r$, $w_h\in\mathcal{V}_h$ by

$$\bar{a}_h(v_h,w_h):=\sum_{E\in\mathcal{E}_h'}[w_h]\int_E\bar{\alpha}\frac{\partial v_h}{\partial\mathbf{n}}ds.$$

If (4.1) is valid for a piecewise constant coefficient, then for all $v_h\in\mathcal{U}_h^r$,

$$\sup_{w_h\in\mathcal{V}_h}\frac{\bar{a}_h(v_h,w_h)}{|w_h|_h'}\gtrsim|v_h|_1.$$

On the other hand, by the same arguments in Theorem 3.1, we have

$$|a_h(v_h,w_h)-\bar{a}_h(v_h,w_h)|\lesssim m_{\mathcal{T}_h}(\alpha,h)|v_h|_1|w_h|_h',$$

Then when h is sufficiently small,

$$\sup_{w_h\in\mathcal{V}_h}\frac{a_h(v_h,w_h)}{|w_h|_h'}\gtrsim(1-m_{\mathcal{T}_h}(\alpha,h))|v_h|_1\ge\frac{1}{2}|v_h|_1$$

which implies the inf–sup condition (4.1) for arbitrary piecewise continuous α.

In the rest analysis of this section, unless specifically mentioned, we always suppose that α is piecewise constant respect to $\mathcal{T}_h$.

4.1 An equivalent discrete form

In this subsection, we transfer the FV bilinear form (3.6) to a summation of function values at Gauss points.

To this end, we first describe the dual edges in a quadrilateral $\tau \in \mathcal{T}_h$. Let

$$E^{\xi}_{ij} = \overline{G^{\tau}_{i,j} G^{\tau}_{i+1,j}}, \quad \forall (i, j) \in \mathbb{Z}^0_r \times \mathbb{Z}_r,$$

be the edges along $\xi-$direction and

$$E^{\eta}_{ij} = \overline{G^{\tau}_{i,j} G^{\tau}_{i,j+1}}, \quad \forall (i, j) \in \mathbb{Z}_r \times \mathbb{Z}^0_r,$$

the edges along η-direction in τ. The set of dual edges in τ is the union of edges along two directions, that is:

$$\mathcal{E}'_h \cap \tau = \{E^{\eta}_{ij} : i \in \mathbb{Z}_r, j \in \mathbb{Z}^0_r\} \cup \{E^{\xi}_{ij}, i \in \mathbb{Z}^0_r, j \in \mathbb{Z}_r\}.$$

Secondly, we define the jumps of a test function $w_h \in \mathcal{V}_h$. Note that a function $w_h \in \mathcal{V}_h$ can be represented as $w_h = \sum_{p \in \mathcal{N}^\circ} (w_h)_p \psi_{V_p}$, where $(w_h)_p \in \mathbb{R}$ is a constant. Letting $(w_h)_p = 0$ for all $p \in \mathcal{N} \cap \partial\Omega$, we write $w_h = \sum_{p \in \mathcal{N}} (w_h)_p \psi_{V_p}$. For all $\tau \in \mathcal{T}_h$ and all $i, j \in \mathbb{Z}^0_r$, the contribution from τ to the control volume $V_{L^{\tau}_{i,j}}$ is $F_\tau([g_i, g_{i+1}] \times [g_j, g_{j+1})]$. Therefore in the quadrilateral τ,

$$w_h = \sum_{(i,j) \in \mathbb{Z}^0_r \times \mathbb{Z}^0_r} w^{\tau}_{ij} \psi_{F_\tau([g_i, g_{i+1}] \times [g_j, g_{j+1})]},$$

where $w^{\tau}_{i,j} = (w_h)_{L^{\tau}_{i,j}}$. For all $(i, j) \in \mathbb{Z}^0_r \times \mathbb{Z}_r$, we denote the jump of w_h on the edge E^{ξ}_{ij} by

$$[w_h]^{\tau}_{\xi,ij} = w^{\tau}_{ij} - w^{\tau}_{ij-1},$$

and for all $(i, j) \in \mathbb{Z}_r \times \mathbb{Z}^0_r$, we denote the jump on the edge E^{η}_{ij} by

$$[w_h]^{\tau}_{\eta,ij} = w^{\tau}_{ij} - w^{\tau}_{i-1j}.$$

For all $(i, j) \in \mathbb{Z}_r \times \mathbb{Z}_r$, we define the (double) jump of w_h at Gauss point $G^{\tau}_{i,j}$ as

$$\lfloor w_h \rfloor^{\tau}_{i,j} = w^{\tau}_{i,j} + w^{\tau}_{i-1,j-1} - w^{\tau}_{i-1,j} - w^{\tau}_{i,j-1}.$$

Obviously, we have

$$\lfloor w_h \rfloor^{\tau}_{i,j} = [w_h]^{\tau}_{\xi,ij} - [w_h]^{\tau}_{\xi,i-1j} = [w_h]^{\tau}_{\eta,ij} - [w_h]^{\tau}_{\eta,ij-1}.$$

With these notations, for all $v \in H^1_0(\Omega), \ w_h \in \mathcal{V}'_h$, we have

$$a_h(v, w_h) = \sum_{\tau \in \mathcal{T}_h} a_{h,\tau}(v, w_h)$$

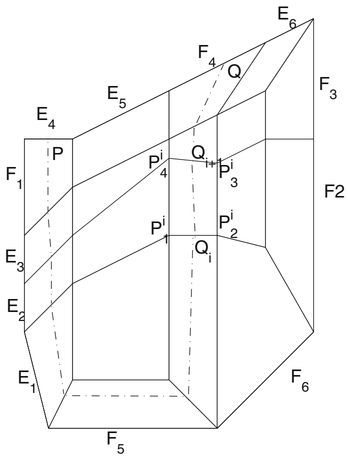


Fig. 4 A path starts from $P \in E_4$ and ends at $Q \in F_4$. The segment $\overline{Q_i Q_{i+1}}$ is the portion of this pass in the element $\tau_i = \Box P_1^i P_2^i P_3^i P_4^i$. The two points can be expressed as $Q_i = F_{\tau_i}(\xi, -1)$, $Q_{i+1} = F_{\tau_i}(\xi, 1)$

where the element-wise bilinear form

$$
\begin{aligned}
a_{h,\tau}(v, w_h) &= \sum_{E \in \tau \cap \mathcal{E}_h'} [w_h] \int_E \alpha \frac{\partial v}{\partial \mathbf{n}} ds, \\
&= \sum_{(i,j) \in \mathbb{Z}_r^0 \times \mathbb{Z}_r} [w_h]_{\xi,ij}^{\tau} \int_{E_{ij}^{\xi}} \alpha \frac{\partial v}{\partial \mathbf{n}} ds + \sum_{(i,j) \in \mathbb{Z}_r \times \mathbb{Z}_r^0} [w_h]_{\eta,ij}^{\tau} \int_{E_{ij}^{\eta}} \alpha \frac{\partial v}{\partial \mathbf{n}} ds.
\end{aligned}
$$

To transform the above formulae to a summation of function values at Gauss points, we next introduce a *primal* function of the function $\alpha \frac{\partial v}{\partial \mathbf{n}}$ along a *path* which is defined as below. For a boundary edge $E \in \mathcal{E}_h^{b1}$, let $E_i, i = 1, \ldots, m, m \geq 2$ be the sequence of edges in $\mathcal{E}_h$ satisfying: (1) $E_1 = E \in \mathcal{E}_h^{b1}$, $E_m = F \in \mathcal{E}_h^{b2}$, (2) The edges E_i, E_{i+1} belong to the same quadrilateral τ_i and $E_{i+1} = op(E_i), i = 1, \ldots, m-1$. Given some point $P \in E$, let $Q_i \in E_i, i = 1, \ldots, m$ be a sequence of points such that: (1) $Q_1 = P \in E = E_1$, (2) Let the four vertices of τ_i be labeled as $P_j^i, j = 1, \ldots, 4$. If $E_i = P_1^i P_2^i$, then $Q_i = F_{\tau_i}(\xi, -1)$ and $P_{i+1} = F_{\tau_i}(\xi, 1)$ for some $-1 < \xi < 1$; If $E_i = P_1^i P_4^i$, then $Q_i = F_{\tau_i}(-1, \eta)$ and $Q_{i+1} = F_{\tau_i}(1, \eta)$ for some $-1 < \eta < 1$,. We connect $Q_i Q_{i+1}, i = 1, \ldots m-1$ with segments, then we obtain the *pass* starting from the point $P \in E = E_1$ and ending at the point $Q = Q_m \in E_m = F \in \mathcal{E}_h^{b2}$. We denote this pass by $S_{E,P}$, see Fig. 4 for a path (depicted by dash line) which starts from $P \in E_4$ and ends at a point Q in the opposite boundary edge F_4.

For all $E \in \mathcal{E}_h^{b1}$, $P \in E$ and for any point $Q \in S_{E,P}$, we define

$$
V_{E,P}(Q) = \int_{\widehat{PQ}} \alpha \frac{\partial v}{\partial \mathbf{n}} ds, \tag{4.2}
$$

where the curve $\widehat{PQ} \subset S_{E,P}$ is a portion of the path $S_{E,P}$ which starts from P and ends at the point $\tilde{P}$ which belongs to the opposite boundary edge $F \in \mathcal{E}_h^{b2}$.

For $\tau = \Box P_1 P_2 P_3 P_4 \in\in \mathcal{T}_h$, let $E_1^\tau, E_2^\tau \in \mathcal{E}_h^{b1}$ be two boundary edges such that $P_1 P_2 = op(E_1^\tau)$, $P_1 P_4 = op(E_2^\tau)$. We notice that for a fixed $j \in \mathbb{Z}_r$, all the Gauss points $G_{i,j}^\tau$, $i \in \mathbb{Z}_{r+1}^0$ are situated on the same pass derived from some point $Q_j^1 \in E_1^\tau$. Similarly, for a fixed $i \in \mathbb{Z}_r$, all the Gauss points $G_{i,j}^\tau$, $j \in \mathbb{Z}_{r+1}^0$ are situated on the same pass derived from some point $Q_i^2 \in E_2^\tau$. Therefore, for all $j \in \mathbb{Z}_r$, $i \in \mathbb{Z}_r^0$,

$$\int_{E_{ij}^\xi} \alpha \frac{\partial v}{\partial \mathbf{n}} = V_{E_1^\tau, Q_j^1}(G_{i+1,j}^\tau) - V_{E_1^\tau, Q_j^1}(G_{i,j}^\tau);$$

and for all $i \in \mathbb{Z}_r$, $j \in \mathbb{Z}_r^0$,

$$\int_{E_{ij}^\eta} \alpha \frac{\partial v}{\partial \mathbf{n}} = V_{E_2^\tau, Q_i^2}(G_{i,j+1}^\tau) - V_{E_2^\tau, Q_i^2}(G_{i,j}^\tau).$$

Then, for all $v \in H_0^1(\Omega)$, $w_h \in \mathcal{V}_h'$,

$$\begin{aligned} a_{h,\tau}(v, w_h) = &\sum_{(i,j)\in \mathbb{Z}_r^0 \times \mathbb{Z}_r} [w_h]_{\xi,ij}^\tau (V_{E_1^\tau, Q_j^1}(G_{i+1,j}^\tau) - V_{E_1^\tau, Q_j^1}(G_{i,j}^\tau)) \\ &+ \sum_{(i,j)\in \mathbb{Z}_r \times \mathbb{Z}_r^0} [w_h]_{\eta,ij}^\tau (V_{E_2^\tau, Q_i^2}(G_{i,j+1}^\tau) - V_{E_2^\tau, Q_i^2}(G_{i,j}^\tau)) \\ = &\, a_\tau(v, w_h) + b_\tau(v, w_h), \end{aligned}$$

where the element-wise bilinear form

$$a_\tau(v, w_h) = - \sum_{(i,j)\in \mathbb{Z}_r \times \mathbb{Z}_r} \lfloor w_h \rfloor_{i,j}^\tau (V_{E_1^\tau, Q_j^1}(G_{i,j}^\tau) + V_{E_2^\tau, Q_i^2}(G_{i,j}^\tau)), \tag{4.3}$$

and the boundary term

$$\begin{aligned} b_\tau(v, w_h) = &\sum_{j\in \mathbb{Z}_r} ([w_h]_{\xi,rj}^\tau V_{E_1^\tau, Q_j^1}(G_{r+1,j}^\tau) - [w_h]_{\xi,0j}^\tau V_{E_1^\tau, Q_j^1}(G_{0,j}^\tau)) \\ &+ \sum_{i\in \mathbb{Z}_r} ([w_h]_{\eta,ir}^\tau V_{E_2^\tau, Q_i^2}(G_{i,r+1}^\tau) - [w_h]_{\eta,i0}^\tau V_{E_2^\tau, Q_i^2}(G_{i,0}^\tau)). \end{aligned}$$

Since the function $V_{E,P}$ is continuous across each internal edge in $\mathcal{T}_h$, we have

$$\sum_{\tau\in \mathcal{T}_h} b_\tau(v, w_h) = 0.$$

Finally, we obtain

$$a_h(v, w_h) = \sum_{\tau\in \mathcal{T}_h} a_\tau(v, w_h)$$

with $a_\tau(\cdot, \cdot)$ defined by (4.3).

4.2 A novel mapping from the trial space to test space

A key step to establish (4.1) is the construction of a novel mapping from the trial space to the test space.

Let Π be a mapping which maps $v_h \in \mathcal{U}_h^r$ to $w_h \in \mathcal{V}_h$ such that the coefficients of w_h satisfy

$$\lfloor w_h \rfloor_{i,j}^{\tau} = A_i A_j \frac{\partial^2 \hat{v}_\tau}{\partial \xi \partial \eta}(g_i, g_j), \quad \forall \tau \in \mathcal{T}_h, (i,j) \in \mathbb{Z}_r \times \mathbb{Z}_r, \tag{4.4}$$

where $\hat{v}_\tau = v_h \circ F_\tau \in \mathbb{Q}_r$ for all $\tau \in \mathcal{T}_h$, and A_j, $j \in \mathbb{Z}_r$ are the weights of the Gauss quadrature $\sum_{j=1}^r A_j v(g_j)$ for computing the integral $\int_{-1}^1 v(x)dx$.

We next explain that for a given $v_h \in \mathcal{U}_h^r$, (4.4) determine a unique $w_h \in \mathcal{V}_h$. First if $v_h = 0$, then $\forall \tau \in \mathcal{T}_h$, $(i,j) \in \mathbb{Z}_r \times \mathbb{Z}_r$,

$$\lfloor w_h \rfloor_{i,j}^{\tau} = w_{i,j}^{\tau} + w_{i-1,j-1}^{\tau} - w_{i-1,j}^{\tau} - w_{i,j-1}^{\tau} = 0.$$

Noticing the fact that $w_h = 0$ on $\partial\Omega$, we obtain that $w_h = 0$ on the whole domain Ω. The uniqueness of Π is proved.

We next explain that there exists a w_h which satisfies all constraints (4.4). Since (4.4) is a linear system with $\#\mathcal{G} = \#\mathcal{T}_h r^2$ equations, while the degree of freedom of a w_h is $\dim \mathcal{V}_h = \#\mathcal{N}^\circ = \#\mathcal{G} - \frac{1}{2}\#\mathcal{G}_h^b + 1$, one may doubt if the linear system is over-determined. The following lemma explains there exists a lot of redundant equations in (4.4).

Lemma 4.1 *Let $v_h \in \mathcal{U}_h^r$, then for any $E_i \in \mathcal{E}_h^{b1}$ and any $j \in \mathbb{Z}^r$, we have*

$$\sum_{l=1}^{m} \sum_{k \in \mathbb{Z}_r} A_k \frac{\partial^2 \hat{v}_{\tau_l}}{\partial \xi \partial \eta}(g_k, g_j) = 0, \tag{4.5}$$

where $\tau_l, l = 1, \ldots m$ are quadrilaterals which begin at $E_i \in \mathcal{E}_h^{b1}$ and end at $F_i = op(E_i) \in \mathcal{E}_h^{b2}$. Consequently, for all $E_i \in \mathcal{E}_h^{b1}$ and all $j \in \mathbb{Z}^r$ and all $w_h \in \mathcal{V}_h$,

$$\sum_{l=1}^{m} \sum_{k \in \mathbb{Z}_r} \lfloor w_h \rfloor_{k,j}^{\tau_l} = \sum_{l=1}^{m} \sum_{k \in \mathbb{Z}_r} A_k A_j \frac{\partial^2 \hat{v}_{\tau_l}}{\partial \xi \partial \eta}(g_k, g_j). \tag{4.6}$$

Proof For fixed $l \in \mathbb{Z}_m$, $j \in \mathbb{Z}_r$, the function $\frac{\partial^2 \hat{v}_{\tau_l}}{\partial \xi \partial \eta}(\cdot, g_j) \in \mathbb{P}_{r-1}(-1,1)$. Therefore, by the property of a Gauss quadrature

$$\sum_{k \in \mathbb{Z}_r} A_k \frac{\partial^2 \hat{v}_{\tau_l}}{\partial \xi \partial \eta}(g_k, g_j) = \int_{-1}^{1} \frac{\partial^2 \hat{v}_{\tau_l}}{\partial \xi \partial \eta}(\xi, g_j) d\xi = \frac{\partial \hat{v}_{\tau_l}}{\partial \eta}(1, g_j) - \frac{\partial \hat{v}_{\tau_l}}{\partial \eta}(-1, g_j).$$

Now since $v_h \in C(\Omega)$, we have for all $l = 1, \ldots, m-1$,

$$\frac{\partial \hat{v}_{\tau_l}}{\partial \eta}(1, g_j) = \frac{\partial \hat{v}_{\tau_{l+1}}}{\partial \eta}(-1, g_j).$$

Therefore,

$$\sum_{l=1}^{m} \sum_{k \in \mathbb{Z}_r} A_k \frac{\partial^2 \hat{v}_{\tau_l}}{\partial \xi \partial \eta}(g_k, g_j) = \frac{\partial \hat{v}_{\tau_m}}{\partial \eta}(1, g_j) - \frac{\partial \hat{v}_{\tau_1}}{\partial \eta}(-1, g_j).$$

Note that $E_i, F_i \subset \partial\Omega$, then $v_h = 0$ on E_i and F_i. That is $\hat{v}_{\tau_1}(-1, \eta) = 0 = \hat{v}_{\tau_m}(1, \eta), \forall \eta \in (-1, 1)$. Consequently, (4.5) hold for all $E_i \in \mathcal{E}_h^{b1}$ and all $j \in \mathbb{Z}_r$.

We next show (4.6). Since $(w_h)_p = 0$ for all $p \in \mathcal{N} \backslash \mathcal{N}^\circ$, we have

$$\sum_{l=1}^{m} \sum_{k \in \mathbb{Z}_r} \lfloor w_h \rfloor_{k,j}^{\tau_l} = [w_h]_{\xi,r,j}^{\tau_m} - [w_h]_{\xi,0,j}^{\tau_1} = 0,$$

from which and (4.5), the Eq. (4.6) follows. □

This lemma indicates that equations in (4.4) are linear dependent. Obviously we have $\#\mathcal{E}_h^{b1} r = \frac{1}{2}\#\mathcal{E}_h^b r$ equations in (4.6). Moreover, if we choose $\frac{1}{2}\#\mathcal{E}_h^b r - 1$ equations of (4.6) hold, the rest 1 equation will hold automatically. In other words, we have $\frac{1}{2}\#\mathcal{E}_h^b r - 1$ linear independent equations in (4.6). Now we remove $\frac{1}{2}\#\mathcal{E}_h^b r - 1$ constraints from (4.4), we get a linear system with $\#\mathcal{N}^\circ$ unknowns and $\#\mathcal{N}^\circ$ equations. Using the fact $v_h = 0$ implies $w_h = 0$, this linear system has a unique solution. The existence of $w_h = \Pi v_h$ is also proved.

We next show that Π is bounded from the trial space to the test space.

Lemma 4.2 *If $\mathcal{T}_h$ is shape regular, then for any $v_h \in \mathcal{U}_h^r$,*

$$|\Pi v_h|_h' \lesssim |v_h|_1, \tag{4.7}$$

where the hidden constant depends only on r.

Proof First, the fact that $\mathcal{T}_h$ is a shape regular mesh yields that

$$|v_h|_{1,\tau} \sim |\hat{v}_\tau|_{1,\tau_0}, \quad \forall \tau \in \mathcal{T}_h.$$

Consequently,

$$|v_h|_1^2 = \sum_{\tau \in \mathcal{T}_h} |v_h|_{1,\tau}^2 \sim \sum_{\tau \in \mathcal{T}_h} |\hat{v}_\tau|_{1,\tau_0}^2.$$

On the other hand, by the definition of the semi-norm $|\cdot|_h'$,

$$(|\Pi v_h|_h')^2 = \sum_{E \in \mathcal{E}_h'} ([\Pi v_h]_E)^2 = \sum_{\tau \in \mathcal{T}_h} \sum_{E \in \mathcal{E}_h' \cap \tau} ([\Pi v_h]_E)^2.$$

Therefore, to show (4.7), we only need to show that for all $\tau \in \mathcal{T}_h$,

$$\sum_{E \in \mathcal{E}'_h \cap \tau} ([\Pi v_h]_E)^2 \lesssim |\hat{v}_\tau|^2_{1,\tau_0}. \tag{4.8}$$

Noticing $\frac{\partial^2 \hat{v}_\tau}{\partial \xi \partial \eta} \in Q_{r-1}(\tau_0)$, we have

$$\begin{aligned}\sum_{i \in \mathbb{Z}_r} \lfloor \Pi v_h \rfloor^\tau_{ij} &= A_j \sum_{i \in \mathbb{Z}_r} A_i \frac{\partial^2 \hat{v}_\tau}{\partial \xi \partial \eta}(g_i, g_j) = A_j \int_{-1}^1 \frac{\partial^2 \hat{v}_\tau}{\partial \xi \partial \eta}(\xi, g_j) d\xi \\ &= A_j \frac{\partial \hat{v}_\tau}{\partial \eta}(1, g_j) - A_j \frac{\partial \hat{v}_\tau}{\partial \eta}(-1, g_j).\end{aligned}$$

Then, by the fact that $\lfloor \Pi v_h \rfloor^\tau_{i,j} = [\Pi v_h]^\tau_{\xi,i,j} - [\Pi v_h]^\tau_{\xi,i-1,j}$, we have

$$[\Pi v_h]^\tau_{\xi,r,j} - [\Pi v_h]^\tau_{\xi,0,j} = A_j \frac{\partial \hat{v}_\tau}{\partial \eta}(1, g_j) - A_j \frac{\partial \hat{v}_\tau}{\partial \eta}(-1, g_j).$$

Since $v_h = 0$ and $w_h = 0$ on $\partial\Omega$, for all $\tau \in \mathcal{T}_h$, $j \in \mathbb{Z}_r$, we have

$$[\Pi v_h]^\tau_{\xi,0,j} = A_j \frac{\partial \hat{v}_\tau}{\partial \eta}(-1, g_j).$$

Therefore, by the inverse inequality,

$$|[\Pi v_h]^\tau_{\xi,0,j}| \leq A_j \left\| \frac{\partial \hat{v}_\tau}{\partial \eta}(\cdot, g_j) \right\|_{L^\infty(-1,1)} \lesssim A_j \left\| \frac{\partial \hat{v}_\tau}{\partial \eta}(\cdot, g_j) \right\|_{L^2(-1,1)}.$$

Since for $i \in \mathbb{Z}_r$,

$$\begin{aligned}[\Pi v_h]^\tau_{\xi,i,j} &= [\Pi v_h]^\tau_{\xi,0,j} + \sum_{i'=1}^{i} \lfloor \Pi v_h \rfloor^\tau_{i',j} \\ &= [\Pi v_h]^\tau_{\xi,0,j} + A_j \sum_{i'=1}^{i} A_{i'} \frac{\partial^2 \hat{v}_\tau}{\partial \xi \partial \eta}(g_i, g_j),\end{aligned}$$

and by the inverse inequality,

$$\begin{aligned}\left| \sum_{i'=1}^{i} A_{i'} \frac{\partial^2 \hat{v}_\tau}{\partial \xi \partial \eta}(g_{i'}, g_j) \right| &\lesssim \left\| \frac{\partial^2 \hat{v}_\tau}{\partial \xi \partial \eta}(\cdot, g_j) \right\|_{L^\infty(-1,1)} \\ &\lesssim \left\| \frac{\partial^2 \hat{v}_\tau}{\partial \xi \partial \eta}(\cdot, g_j) \right\|_{L^2(-1,1)} \\ &\lesssim \left\| \frac{\partial \hat{v}_\tau}{\partial \eta}(\cdot, g_j) \right\|_{L^2(-1,1)},\end{aligned}$$

for all $i \in \mathbb{Z}_r^0$, there holds

$$|[\Pi v_h]^\tau_{\xi,i,j}| \lesssim A_j \left\| \frac{\partial \hat{v}_\tau}{\partial \eta}(\cdot, g_j) \right\|_{L^2(-1,1)}. \tag{4.9}$$

Since $\int_{-1}^1 (\frac{\partial \hat{v}_\tau}{\partial \eta}(\xi, \cdot))^2 d\xi$ is a polynomial (w.r.t η) of degree less than $2r-1$, we have

$$\sum_{j\in\mathbb{Z}_r} A_j \int_{-1}^1 \left(\frac{\partial \hat{v}_\tau}{\partial \eta}(\xi, g_j)\right)^2 d\xi = \int_{-1}^1 \int_{-1}^1 \left(\frac{\partial \hat{v}_\tau}{\partial \eta}(\xi, \eta)\right)^2 d\xi d\eta = \left\| \frac{\partial \hat{v}_\tau}{\partial \eta} \right\|^2_{L^2(\tau_0)}.$$

Noticing (4.9), we obtain

$$\sum_{i\in\mathbb{Z}_r^0, j\in\mathbb{Z}_r} ([\Pi v_h]^\tau_{\xi,i,j})^2 \lesssim \left\| \frac{\partial \hat{v}_\tau}{\partial \eta} \right\|^2_{L^2(\tau_0)}.$$

Similarly,

$$\sum_{i\in\mathbb{Z}_r, j\in\mathbb{Z}_r^0} ([\Pi v_h]^\tau_{\eta,i,j})^2 \lesssim \left\| \frac{\partial \hat{v}_\tau}{\partial \xi} \right\|^2_{L^2(\tau_0)}.$$

Consequently,

$$\sum_{E\in\mathcal{E}_h'\cap\tau} ([\Pi v_h]_E)^2 = \sum_{i\in\mathbb{Z}_r^0, j\in\mathbb{Z}_r} ([\Pi v_h]^\tau_{\xi,i,j})^2 + \sum_{i\in\mathbb{Z}_r, j\in\mathbb{Z}_r^0} ([\Pi v_h]^\tau_{\eta,i,j})^2 \lesssim |\hat{v}_\tau|^2_{1,\tau_0}.$$

That is, (4.8) is verified. The inequality (4.7) then follows. □

4.3 Coercivity of bilinear form $a_h(\cdot, \Pi\cdot)$

With the help of Π, we obtain a bilinear form $a_h(\cdot, \Pi\cdot)$ which is defined only on the trial space $\mathcal{U}_h^r$. In this subsection, we show the coercivity of $a_h(\cdot, \Pi\cdot)$.

By (4.3), for all $v \in \mathcal{U}_h^r$,

$$a_\tau(v, \Pi v) = I_1 + I_2.$$

where

$$I_1 = -\sum_{(i,j)\in\mathbb{Z}_r\times\mathbb{Z}_r} A_i A_j \frac{\partial^2 \hat{v}_\tau}{\partial\xi\partial\eta}(g_i, g_j) V_{E_1^\tau, Q_j^1}(G^\tau_{i,j}),$$

and

$$I_2 = -\sum_{(i,j)\in\mathbb{Z}_r\times\mathbb{Z}_r} A_i A_j \frac{\partial^2 \hat{v}_\tau}{\partial\xi\partial\eta}(g_i, g_j) V_{E_2^\tau, Q_i^2}(G_{i,j}^\tau).$$

We will only estimate the lower bound of I_1 since the lower bound of I_2 can be obtained by dual reasoning.

For all $j \in \mathbb{Z}^r$, we denote

$$\Theta_j(\xi) = \frac{\partial^2 \hat{v}_\tau}{\partial\xi\partial\eta}(\cdot, g_j) V_{E_1^\tau, Q_j^1}(F_\tau(\cdot, g_j)), \xi \in [-1, 1],$$

and let

$$Err_{\xi,j}^\tau = \int_{-1}^{1} \Theta_j(\xi)d\xi - \sum_{i\in\mathbb{Z}_r} A_i \Theta_j(g_i)$$

be the error of the Gauss quadrature.

Lemma 4.3 *If $\tau \in \mathcal{T}_h$ is an $h^{1+\gamma}$-parallelogram, then for sufficiently small h,*

$$Err_{\xi,j}^\tau \gtrsim -h_\tau^\gamma |\hat{v}_\tau|_{1,\tau_0}^2, \quad \forall j \in \mathbb{Z}_r, \tag{4.10}$$

where the hidden constant is independent of h.

Proof By [10, p98, (2.7.12)], for all $i \in \mathbb{Z}_N$

$$Err_{\xi,j}^\tau = \frac{2^{2r+1}(r!)^4}{(2r+1)[(2r)!]^3}(\Theta_j)^{(2r)}(\xi'),$$

where $\xi' \in (-1, 1)$.

We next calculate and estimate $(\Theta_j)^{(2r)}$. By the Definition (4.2) and the formula (2.8), we have that for all $\xi_0 \in (-1, 1)$ and all $j \in \mathbb{Z}_r$,

$$V_{E_1^\tau, Q_j^1}(F_\tau(\xi_0, g_j)) = V_{E_1^\tau, Q_j^1}(F_\tau(-1, g_j)) + \int_{-1}^{\xi_0} \frac{\alpha}{J_\tau}\left(-s_{\xi,g_j}^\tau \frac{\partial\hat{v}_\tau}{\partial\xi} + (r_{\xi,g_j}^{\tau,1})^2 \frac{\partial\hat{v}_\tau}{\partial\eta}\right) d\xi,$$

and consequently,

$$\frac{d}{d\xi} V_{E_1^\tau, Q_j^1}(F_\tau(\xi, g_j)) = \frac{\alpha}{J_\tau}\left(-s_{\xi,g_j}^\tau \frac{\partial\hat{v}_\tau}{\partial\xi} + (r_{\xi,g_j}^{\tau,1})^2 \frac{\partial\hat{v}_\tau}{\partial\eta}\right).$$

Then by the Leibnitz formula and noticing that the function $\frac{\partial^2 \hat{v}_\tau}{\partial\xi\partial\eta}(\cdot, g_j)$ is a polynomial of degree $r-1$, we have

$$\Theta_j^{(2r)}(\xi') = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.$$

where

$$\mathcal{J}_1 = \alpha\binom{2r}{r-1}\frac{(r_{\xi,j}^{\tau,1})^2}{J_\tau}\left(\frac{\partial^{r+1}\hat{v}_\tau}{\partial\xi^r\partial\eta}(\xi', g_j)\right)^2,$$
$$\mathcal{J}_2 = -\alpha\binom{2r}{r-1}\frac{(s_{\xi,j}^{\tau})'}{J_\tau}\frac{\partial^r\hat{v}_\tau}{\partial\xi^r}(\xi', g_j)\frac{\partial^{r+1}\hat{v}_\tau}{\partial\xi^r\partial\eta}(\xi', g_j),$$
$$\mathcal{J}_3 = \alpha\binom{2r}{r-1}\frac{\partial^{r+1}\hat{v}_\tau}{\partial\xi^r\partial\eta}(\xi', g_j)\sum_{l=1}^{r}\left(\frac{1}{J_\tau}\right)^{(l)}\left(-s_{\xi,g_j}^{\tau}\frac{\partial\hat{v}_\tau}{\partial\xi} + (r_{\xi,g_j}^{\tau,1})^2\frac{\partial\hat{v}_\tau}{\partial\eta}\right)^{(r-l)}(\xi')$$
$$+\sum_{k=0}^{r-2}\binom{2r}{k}\frac{\partial^{k+2}\hat{v}_\tau}{\partial\xi^{k+1}\partial\eta}(\xi')\left(\frac{\alpha}{J_\tau}\left(-s_{\xi,j}^{\tau}\frac{\partial\hat{v}_\tau}{\partial\xi} + (r_{\xi,j}^{\tau,1})^2\frac{\partial\hat{v}_\tau}{\partial\eta}\right)\right)^{(2r-k-1)}(\xi').$$

The fact that $J_{\tau,p} > 0$ and (2.5) yield that $J_\tau > 0$ for sufficiently small h. Consequently, when h is sufficiently small,

$$\mathcal{J}_1 \geq 0.$$

On the other hand, since $s_{\xi,j}^{\tau}$ is linear with respect to ξ, $(s_{\xi,j}^{\tau})' = (\frac{\beta_2^\tau}{2} + \frac{\beta_3^\tau(1+g_j)}{4})(\frac{\beta_3^\tau}{4}) = \mathcal{O}(h_\tau^{2+\gamma})$, by the inverse inequality

$$|\mathcal{J}_2| \lesssim h^\gamma|\hat{v}_\tau|_{1,\tau_0}^2.$$

Next we estimate $\mathcal{J}_3$. We observe that $\mathcal{J}_3$ is a summation of a finite number of terms which can be represented as

$$\alpha\frac{\partial^{k_1+1}\hat{v}_\tau}{\partial\xi^{k_1}\partial\eta}\left(\left(\frac{s_{\xi,j}^{\tau}}{J_\tau}\right)^{(l)}\frac{\partial^{k_2+1}\hat{v}_\tau}{\partial\xi^{k_2+1}} + \left(\frac{(r_{\xi,j}^{\tau,1})^2}{J_\tau}\right)^{(l)}\frac{\partial^{k_2+1}\hat{v}_\tau}{\partial\xi^{k_2}\partial\eta}\right)$$

for some $1 \leq l \leq 2r$, $0 \leq k_i, \leq 2r$, $i = 1, 2$. Recall that J_τ is linear with respect to ξ, $J_\tau' = J_{\tau,\xi} = \mathcal{O}(h_\tau^{2+\gamma})$. For all $l \geq 1$,

$$(J_\tau^{-1})^{(l)} = (-1)^l l!(J_\tau)^{-l-1}(J_\tau')^l = \mathcal{O}(h_\tau^{l\gamma-2}).$$

On the other hand, $(r_{\xi,j}^{\tau,1})^2$ is a constant with respect to ξ, $(r_{\xi,j}^{\tau,1})^2 = \mathcal{O}(h_\tau^2)$. $s_{\xi,j}^{\tau}$ is linear with respect to ξ, $s_{\xi,j}^{\tau} = \mathcal{O}(h^2)$, $(s_{\xi,j}^{\tau})' = \mathcal{O}(h_\tau^{2+\gamma})$. Therefore for all $l \geq 1$, we always have

$$\left(\frac{s_{\xi,j}^{\tau}}{J_\tau}\right)^{(l)} \lesssim h_\tau^\gamma, \left(\frac{(r_{\xi,j}^{\tau,1})^2}{J_\tau}\right)^{(l)} \lesssim h_\tau^\gamma.$$

Again by the inverse inequality, we have

$$|\mathcal{J}_3| \lesssim h_\tau^\gamma |\hat{v}_\tau|^2_{1,\tau_0}.$$

Consequently,

$$\Theta_j^{(2r)}(\xi') \gtrsim -h_\tau^\gamma |\hat{v}_\tau|^2_{1,\tau_0}.$$

The proof of (4.10) is completed. □

We are now ready to bound $a_\tau(\cdot,\cdot)$ from below.

Lemma 4.4 *If $\tau \in \mathcal{T}_h$ is a shape regular $h^{1+\gamma}$-parallelogram, then when h is sufficiently small,*

$$a_\tau(v_h, \Pi v_h) \gtrsim C|v_h|^2_{1,\tau} + \mathrm{bdy}_\tau, \tag{4.11}$$

where $C > 0$ depends only on the minimal angle θ_0, the parameters γ and r, and

$$\mathrm{bdy}_\tau = \sum_{j\in\mathbb{Z}_r} A_j \mathrm{bdy}_{\tau,1,j} + \sum_{i\in\mathbb{Z}_r} A_i \mathrm{bdy}_{\tau,2,i}.$$

with

$$\mathrm{bdy}_{\tau,1,j} = \frac{\partial \hat{v}_\tau}{\partial\eta}(-1, g_j) V_{E_1^\tau, g_j}(F_\tau(-1, g_j)) - \frac{\partial \hat{v}_\tau}{\partial\eta}(1, g_j) V_{E_1^\tau, g_j}(F_\tau(1, g_j)),$$

and

$$\mathrm{bdy}_{\tau,2,i} = \frac{\partial \hat{v}_\tau}{\partial\eta}(g_i, -1) V_{E_2^\tau, g_i}(F_\tau(g_i, -1)) - \frac{\partial \hat{v}_\tau}{\partial\eta}(g_i, 1) V_{E_2^\tau, g_i}(F_\tau(g_i, 1)).$$

Proof By (4.10), there exists a positive constant C_1 such that

$$I_1 \geq -\sum_{j\in\mathbb{Z}_r} A_j \int_{-1}^{1} \Theta_j(\xi)d\xi - C_1 h^\gamma |v|^2_{1,\tau}.$$

Using the integration by parts, we have

$$-\int_{-1}^{1} \Theta_j(\xi)d\xi = \int_{-1}^{1} \Phi_1(\xi, g_j)d\xi + \mathrm{bdy}_{\tau,1,j},$$

where

$$\Phi_1(\xi,\eta) = \frac{\partial \hat{v}_\tau}{\partial\eta}\left(\frac{\alpha}{J_\tau}\left(-s^\tau_{\xi,\eta}\frac{\partial \hat{v}_\tau}{\partial\xi} + (r^{\tau,1}_{\xi,\eta})^2 \frac{\partial \hat{v}_\tau}{\partial\eta}\right)\right).$$

Then

$$I_1 \geq \int_{-1}^{1} \sum_{j\in\mathbb{Z}_r} A_j \Phi_1(\xi, g_j) d\xi + \sum_{j\in\mathbb{Z}_r} A_j \mathrm{bdy}_{\tau,1,j} - Ch^{\gamma} |v_h|_{1,\tau}^2.$$

By the same arguments in Lemma 4.3, we obtain that for any fixed $\xi \in [-1, 1]$,

$$\left| \int_{-1}^{1} \Phi_1(\xi, \eta) d\eta - \sum_{j\in\mathbb{Z}_r} A_j \Phi_1(\xi, g_j) \right| \lesssim h^{\gamma} |v_h|_{1,\tau}^2.$$

Consequently,

$$I_1 \geq \int_{\tau_0} \Phi_1(\xi, \eta) d\xi d\eta + \sum_{j\in\mathbb{Z}_r} A_j \mathrm{bdy}_{\tau,1,j} - Ch^{\gamma} |v_h|_{1,\tau}^2.$$

By dual reasoning, we have

$$I_2 \geq \int_{\tau_0} \Phi_2(\xi, \eta) d\xi d\eta + \sum_{i\in\mathbb{Z}_r} A_i \mathrm{bdy}_{\tau,2,i} - C_1 h^{\gamma} |v_h|_{1,\tau}^2,$$

where

$$\Phi_2(\xi, \eta) = \frac{\partial \hat{v}_\tau}{\partial \xi} \left(\frac{\alpha}{J_\tau} \left(-s_{\xi,\eta}^{\tau} \frac{\partial \hat{v}_\tau}{\partial \eta} + (r_{\xi,\eta}^{\tau,2})^2 \frac{\partial \hat{v}_\tau}{\partial \xi} \right) \right).$$

Therefore, we have

$$a_\tau(v_h, \Pi v_h) \geq \int_{\tau_0} (\Phi_1(\xi, \eta) + \Phi_2(\xi, \eta)) d\xi d\eta + \mathrm{bdy}_\tau - Ch^{\gamma} |v_h|_{1,\tau}^2.$$

Note that

$$\Phi_1(\xi, \eta) + \Phi_2(\xi, \eta) = \frac{\alpha}{J_\tau} \left((r_{\xi,\eta}^{\tau,1})^2 \left(\frac{\partial \hat{v}_\tau}{\partial \eta} \right)^2 + (r_{\xi,\eta}^{\tau,2})^2 \left(\frac{\partial \hat{v}_\tau}{\partial \xi} \right)^2 - 2 s_{\xi,\eta}^{\tau} \frac{\partial \hat{v}_\tau}{\partial \xi} \frac{\partial \hat{v}_\tau}{\partial \eta} \right).$$

By (2.7) and Cauchy–Schwartz inequality, we have

$$\int_{\tau_0} \left| 2 s_{\xi,\eta}^{\tau} \frac{\partial \hat{v}_\tau}{\partial \xi} \frac{\partial \hat{v}_\tau}{\partial \eta} \right| d\xi d\eta \leq \cos\theta_0 \int_{\tau_0} \left((r_{\xi,\eta}^{\tau,1})^2 \left(\frac{\partial \hat{v}_\tau}{\partial \eta} \right)^2 + (r_{\xi,\eta}^{\tau,2})^2 \left(\frac{\partial \hat{v}_\tau}{\partial \xi} \right)^2 \right) d\xi d\eta.$$

Therefore,

$$\Phi_1(\xi, \eta) + \Phi_2(\xi, \eta) \geq (1 - \cos\theta_0) \frac{\alpha}{J_\tau} \left((r_{\xi,\eta}^{\tau,1})^2 \left(\frac{\partial \hat{v}_\tau}{\partial \eta} \right)^2 + (r_{\xi,\eta}^{\tau,2})^2 \left(\frac{\partial \hat{v}_\tau}{\partial \xi} \right)^2 \right).$$

Note that $r_{\xi,\eta}^{\tau,1}, r_{\xi,\eta}^{\tau,2} = \mathcal{O}(h_\tau)$, $J_\tau = \mathcal{O}(h_\tau^2)$, and $\alpha \geq \alpha_0$. Moreover, we have the equivalence $|\hat{v}_\tau|_{1,\tau_0} \sim |v_h|_{1,\tau}$. Therefore there exists a constant $C_2 > 0$ such that

$$a_\tau(v_h, \Pi v_h) \geq C_2 |v_h|_{1,\tau}^2 + \mathrm{bdy}_\tau - 2C_1 h^\gamma |v_h|_{1,\tau}^2.$$

When h is sufficiently small, the estimate (4.11) is valid. □

Finally, we are ready to state and show the coercivity of $a_h(\cdot, \Pi\cdot)$.

Theorem 4.5 *Let $\mathcal{T}_h$ be a shape regular $h^{1+\gamma}$ parallelogram, then*

$$a_h(v_h, \Pi v_h) \gtrsim |v_h|_1^2, \quad \forall v_h \in \mathcal{U}_h^r. \tag{4.12}$$

Proof The facts that v_h and $V_{E,P}, \forall E \in \mathcal{E}_h^{b,1}$, $P \in E$ are continuous across each internal edge $E \in \mathcal{E}_h$ and that $v_h = 0$ on the boundary $\partial\Omega$ implies that

$$\sum_{\tau \in \mathcal{T}_h} \mathrm{bdy}_\tau = 0.$$

Then (4.12) is a direct consequence of local estimates (4.11). □

4.4 Inf–sup property

Summarizing the above two lemmas, we obtain the following inf–sup property.

Theorem 4.6 *Let $\mathcal{T}_h$ be a shape regular $h^{1+\gamma}$, $\gamma > 0$ quadrilateral mesh and suppose the coefficient α is piecewise continuous with respect to $\mathcal{T}_h$. Then (4.1) holds when the meshsize h is sufficiently small.*

Proof When α is piecewise constant, by (4.12) and (4.7), for any $v_h \in \mathcal{U}_h^r$,

$$\sup_{w_h \in \mathcal{V}_h} \frac{a_h(v_h, w_h)}{|w_h|_h'} \geq \frac{a_h(v_h, \Pi v_h)}{|\Pi v_h|_h'} \gtrsim \frac{|v_h|_1^2}{|\Pi v_h|_h'} \gtrsim |v_h|_1.$$

The inf–sup condition (4.1) is proved. Furthermore, an argument in the beginning of this section guarantee that (4.1) holds also for general piecewise continuous coefficient α with respect to the underlying mesh. □

Remark 4.7 Let us reiterate the essential idea in establishing the inf–sup condition, the construction of the special mapping from the trail space to the test space is the key. Recall that the trial space is the same as the standard finite element method, which contains globally continuous piecewise polynomials. On the other hand, the test space contains globally discontinuous piecewise constants with much more "pieces" on the dual mesh. The feasibility of the mapping between the two spaces can be seen from the counting of total degrees of freedom. We examine the simplest case, the Poisson equation ($\alpha = 0$) with zero Dirichlet boundary condition and $\mathcal{T}_h$ is the $n \times n$ square partition of the unit square $[0, 1]^2$. For bilinear element, the dimension of the trial space is $(n-1)^2$, while the piecewise constants in the test space has exactly $(n-1)^2$ non-zero pieces on the dual mesh. For biquadratic element, the dimension of the trial space is $(2n-1)^2$. By our construction, non-zero constant pieces are those between four adjacent Gaussian points that form a square. In each of the horizontal and vertical directions, there are $2n$ Gaussian points and $2n-1$ intervals between them, and hence there are $(2n-1)^2$ squares between Gaussian points so that the test space has exactly $(2n-1)^2$ non-zero pieces on the dual mesh. The counting can be easily extended to r-degree tensor product space with both trial and test space have the same dimension $(rn-1)^2$. As for a general domain which can be partitioned by general quadrilaterals, a careful examination of the topological structure is needed. Nevertheless, the degrees of freedom for trial and test spaces are still the same, which guarantees the the existence of the aforementioned mapping even though the construction of such a mapping requires more delicate analysis as we have done in this work.

5 Numerical results

In this section, we present two numerical examples to validate our theoretical findings.

Example 1 We consider the problem (3.1), (3.2) with $\alpha = 1$ and $\Omega = [0, 1]^2$. We choose the right-hand-side function

$$\begin{aligned} f(x, y) = [(5\pi^2 - 4x^2 - 3)\sin(2\pi x)\sin(\pi y) - 8\pi x \sin(\pi y)\cos(2\pi x) \\ -2\pi \cos(\pi y)\sin(2\pi x)]e^{x^2+y}, \ (x, y) \in [0, 1]^2 \end{aligned}$$

which allows the exact solution

$$u(x, y) = \sin(2\pi x)\sin(\pi y)e^{x^2+y}, \ (x, y) \in [0, 1]^2.$$

We use FV schemes (3.5) with $r = 1, 2, 3, 4$ to compute FVM approximate solutions of u. The partition $\mathcal{T}_k = \mathcal{T}_{h_k}$, $k = 1, \ldots, 6$, are obtained by first uniformly refining the unite square $[0, 1]^2$ and then adding some lines to obtain associated trapezoidal meshes, see Fig. 5 for an example of trapezoidal mesh.

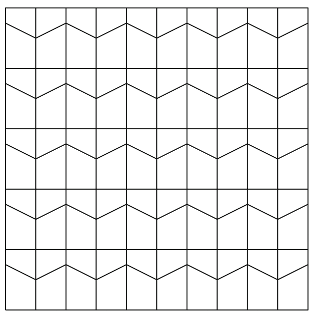


Fig. 5 A trapezoidal mesh

Table 1 Error and convergence order for the problem with a sufficiently smooth function

N	$r=1$		$r=2$		$r=3$		$r=4$	
	Error	C.O.	Error	C. O.	Error	C. O.	Error	C. O.
2	5.0e−01	–	5.1e−2	–	3.4e−03	–	1.7e−04	–
4	2.5e−01	1.0015	1.3e−2	2.0004	4.2e−04	2.9970	1.0e−05	3.9960
8	1.3e−01	1.0004	3.2e−3	2.0002	5.3e−05	2.9993	6.5e−07	3.9990
16	6.3e−02	1.0001	8.0e−4	2.0001	6.6e−06	2.9999	4.1e−08	3.9997
32	3.1e−02	1.0000	2.0e−4	2.0000	8.3e−07	3.0000	2.6e−09	3.9986
64	1.6e−02	1.0000	5.0e−5	2.0000	1.0e−07	3.0000	1.9e−10	3.9972

We present our numerical results in Table 1. In this table, r indicates the polynomial order, $N = h_k^{-1} = 2^k$, $k = 1, \ldots, 6$ indicates the number of partition along the $x-$ and $y-$ directions. The "Error" indicates the computed H^1 semi-norm error $|u - u_{h_k}|_{H^1}$ where u_{h_j} is the finite volume solution in the space $U_{h_k}^r$ and "C.O." indicates the computed convergence order of the H^1 semi-norm error. We observe that $|u - u_{h_k}|_{H^1}$ decays with the optimal convergence order h_k^r which supports our theory (3.11).

Example 2 We consider the problem (3.1), (3.2) with $\Omega = [0, 1]^2$ which admits a unique solution

$$\begin{aligned} u(x,y) = {} & \frac{x(1-x)+y(1-y)}{4} - \frac{2}{\pi^3}\sum_{i=0}^{\infty}\frac{1}{(2i+1)^3(1+e^{-(2i+1)\pi})} \\ & \cdot[(e^{-(2i+1)\pi y} + e^{-(2i+1)\pi(1-y)})\sin(2i+1)\pi x \\ & +(e^{-(2i+1)\pi x} + e^{-(2i+1)\pi(1-x)})\sin(2i+1)\pi y]. \end{aligned}$$

One can verify that $-\Delta u = 1$ in Ω. However, in this numerical experiment, we choose the non-constant coefficient as

$$\alpha(x,y) = \begin{cases} 1 & x \le y \\ 1 + x - y & x > y \end{cases}$$

which is continuous but not globally smooth in Ω. Correspondingly, the right-hand-side function is given by

$$f(x,y) = \begin{cases} 1, & x \le y \\ 1 + x - y + \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}, & x > y. \end{cases}$$

We use FV schemes (3.5) with $r = 1, 2, 3, 4$ to compute FVM approximate solutions of u. The partition $\mathcal{T}_k = \mathcal{T}_{h_k}$, $k = 1, \ldots, 7$, are quadrilateral meshes which are obtained by giving the corresponding rectangular meshes a small random perturbation. Specifically, the node coordinates of a point (x_{ij}, y_{ij}) of the quadrilateral mesh $\mathcal{T}_k$ are

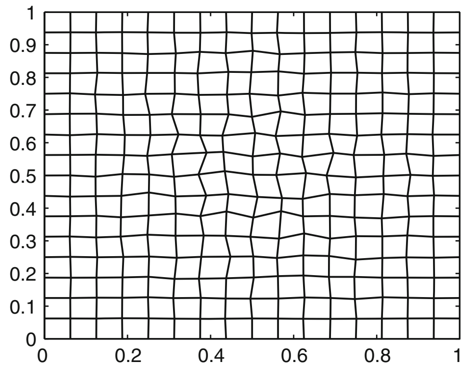


Fig. 6 A rectangular mesh with small random perturbation

Table 2 Error and convergence for the problem with a singular solution

N	$r = 1$		$r = 2$		$r = 3$		$r = 4$	
	Error	C.O.	Error	C. O.	Error	C. O.	Error	C. O.
2	5.6e−02	–	5.1e−02	–	9.1e−04	–	2.2e−04	–
4	2.8e−02	0.9871	1.5e−03	1.7733	2.2e−04	2.0223	5.4e−05	1.9981
8	1.4e−02	0.9961	4.2e−04	1.8298	5.6e−05	2.0053	1.4e−05	1.9999
16	7.2e−03	0.9989	1.2e−04	1.8628	1.4e−05	2.0013	3.4e−06	2.0000
32	3.6e−03	0.9997	3.1e−05	1.8848	3.5e−06	2.0003	8.5e−07	2.0000
64	1.8e−03	0.9999	8.4e−06	1.9007	8.7e−07	2.0001	2.1e−07	2.0000
128	8.9e−04	1.0000	2.2e−06	1.9128	2.2e−07	2.0000	5.3e−08	2.0000

given by

$$\begin{aligned} x_{ij} &= \tfrac{i}{N} + 0.1\tfrac{1}{N}\sin\left(\tfrac{i\pi}{N}\right)\sin\left(\tfrac{j\pi}{N}\right)\text{randn}(),\\ y_{ij} &= \tfrac{j}{M} + 0.1\tfrac{1}{M}\sin\left(\tfrac{i\pi}{N}\right)\sin\left(\tfrac{j\pi}{N}\right)\text{randn}(),\\ 0 &\le i, j \le N, \end{aligned}$$

where $N = 2^k, k = 1, \ldots, 7$ and `randn()` is a built-in random number generator that usually produces a uniformly distributed random number in (0, 1). Notice that the random quadrilateral meshes used in $\mathcal{T}_k$ are not nested as k increases from 1 to 8 since for each run, the random numbers are different. What we know is that the maximal distortion in the meshes is about 20 % of the uniform mesh size $h = 1/M$. Depicted in Fig. 6 is the grid $\mathcal{T}_4$.

The numerical results are demonstrated in Table 2. We observe that when $r = 1$, the error $|u - u_{h_k}|_{H^1}$ decays with the optimal convergence order r, even here the coefficient α is nonsmooth and the underlying mesh has been randomly perturbed. However, when $r = 2, 3, 4$, the error decay order is of about 2. Note that this does Not violate (3.11), since here the exact solution u has singularities in the four corners of Ω and thus it only belongs to $H^{3-\epsilon}$ with an arbitrary small ϵ. Moreover, we find that similar phenomena happen when we use corresponding FEM methods to compute the same problem. How to construct optimal order FV schemes based on adaptive (non-uniform) mesh for singular problems is our next on-going project.

6 Conclusions and future works

The design and analysis of high-order FV schemes are challenging tasks. In this paper, we constructed a family of any order FV schemes based on the Gauss points. Using a novel mapping from the trial to test space, we provide a unified proof for the *optimal-convergence-order* property of our FV schemes. Moreover, to prove this optimal-convergence-order property, we only require a very relaxed mesh condition which is similar to that in finite element methods.

It is obvious that we can use the same idea to construct FV schemes for 3D elliptic problems, and some other problems such as nonlinear problems. In fact, we have constructed and tested numerically some 3D FV schemes for some simple examples. From our numerical experiments, the 3D FV schemes also have optimal convergence order. However, since the traditional tensor-product argument is difficult to be applied to the arguments in the analysis of the current paper, the analysis of corresponding 3D FV schemes will be a challenging task and it is one of our on-going project.

References

1. Bank, R.E., Rose, D.J.: Some error estimates for the box scheme. SIAM J. Numer. Anal. **24**, 777–787 (1987)
2. Barth, T., Ohlberger, M.: Finite Volume Methods: Foundation and Analysis. In: Encyclopedia of Computational Mechanics, vol. 1, chapter 15. Wiley, New York (2004)
3. Cai, Z.: On the finite volume element method. Numer. Math. **58**, 713–735 (1991)
4. Cai, Z., Douglas, J., Park, M.: Development and analysis of higher order finite volume methods over rectangles for elliptic equations. Adv. Comput. Math. **19**, 3–33 (2003)
5. Chen, L.: A new class of high order finite volume methods for second order elliptic equations. SIAM J. Numer. Anal. **47**, 4021–4043 (2010)
6. Chen, Z., Wu, J., Xu,Y.: Higher-order finite volume methods for elliptic boundary value problems. Adv. Comput. Math. **37**, 191–253 (2012)
7. Chou, S.-H., Kwak, D.Y., Li, Q.: L^p error estimates and superconvergence for covolume or finite volume element methods. Numer. Methods Partial Differ. Equ. **19**, 463–486 (2003)
8. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North Holland, Amsterdam (1978)
9. Cui, M., Ye, X.: Unified analysis of finite volume methods for the Stokes equations. SIAM J. Numer. Anal. **48**, 824–839 (2010)
10. Davis, P.J., Rabinowitz, P.: Methods of Numerical Integration, 2nd edn. Academic Press, Boston (1984)
11. Douglas, J., Dupont, T.: Galerkin approximations for the two point boundary problem using continuous, piecewise polynomial spaces. Numer. Math. **22**, 99–109 (1974)
12. Emonot, P.: Methods de volums elements finis: applications aux equations de Navier–Stokes et resultats de convergence, Lyon (1992)
13. Ewing, R., Lin, T., Lin, Y.: On the accuracy of the finite volume element based on piecewise linear polynomials. SIAM J. Numer. Anal. **39**, 1865–1888 (2002)
14. Eymard, R., Gallouet, T., Herbin, R.: Finite volume methods. In: Ciarlet, P.G., Lions, J.L. (eds.) Handbook of Numerical Analysis, vol. VII, pp. 713–1020. North-Holland, Amsterdam (2000)
15. Hackbusch, W.: On first and second order box methods. Computing **41**, 277–296 (1989)
16. Hyman, J.M., Knapp, R., Scovel, J.C.: High order finite volume approximations of differential operators on nonuniform grids. Phys. D **60**, 112–138 (1992)

17. Lazarov, R., Michev, I., Vassilevski, P.: Finite volume methods for convection–diffusion problems. SIAM J. Numer. Anal. **33**, 31–55 (1996)
18. LeVeque, R.J.: Finite Volume Methods for Hyperbolic Problems. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge (2002)
19. Li, R., Chen, Z., Wu, W.: The Generalized Difference Methods for Partial differential Equations. Marcel Dikker, New Youk (2000)
20. Liebau, F.: The finite volume element method with quadratic basis function. Computing **57**, 281–299 (1996)
21. Nicolaides, R.A., Porsching, T.A., Hall, C.A.: Covolume methods in computational fluid dynamics. In: Hafez, M., Oshima, K. (eds.) Computational Fluid Dynamics Review, pp. 279–299. Wiley, New York (1995)
22. Ollivier-Gooch, C., Altena, M.: A high-order-accurate unconstructed mesh finite-volume scheme for the advection–diffusion equation. J. Comput. Phys. **181**, 729–752 (2002)
23. Patanker, S.V.: Numerical Heat Transfer and Fluid Flow. Ser. Comput. Methods Mech. Thermal Sci. McGraw Hill, New York (1980)
24. Plexousakis, M., Zouraris, G.: On the construction and analysis of high order locally conservative finite volume type methods for one dimensional elliptic problems. SIAM J. Numer. Anal. **42**, 1226–1260 (2004)
25. Schmidt, T.: Box schemes on quadrilateral meshes. Computing **51**, 271–292 (1993)
26. Shi, Z.: A convergence condition for the quadrilateral Wilson element. Numer. Math. **44**, 349–361 (1984)
27. Shu, C.-W.: High order finite difference and finite volume weno schemes and discontinous galerkin methods for cfd. J. Comput. Fluid Dyn. **17**, 107–118 (2003)
28. Xu, J., Zou, Q.: Analysis of linear and quadratic simplitical finite volume methods for elliptic equations. Numer. Math. **111**, 469–492 (2009)
29. Zhang, Z.: Polynomial preserving gradient recovery and a posteriori estimate for bilinear element on irregular quadrilaterals. Int. J. Numer. Anal. Model. **1**(1), 1–24 (2004)
30. Zhang, Z.: Analysis of some quadrilateral nonconforming elements for incompressible elasticity. SIAM J. Numer. Anal. **34**, 640–663 (1997)