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Stability of symmetric and nonsymmetric FEM–BEM couplings for nonlinear elasticity problems

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Abstract We consider symmetric as well as non-symmetric coupling formulations of FEM and BEM in the frame of nonlinear elasticity problems. In particular, the Johnson-Nédélec coupling is analyzed. We prove that these coupling formulations are well-posed and allow for unique Galerkin solutions if standard discretizations by piecewise polynomials are employed. Unlike prior works, our analysis does neither rely on an interior Dirichlet boundary to tackle the rigid body motions nor on any assumption on the mesh-size of the discretization used.

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1 Introduction and overview

The coupling of the finite element method (FEM) and the boundary element method (BEM) became very popular when it first appeared in the late seventies of the last century. These methods combine the advantages of FEM, which allows to resolve

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nonlinear problems in bounded domains, and BEM, which allows to solve problems with elliptic differential operators with constant coefficients in unbounded domains. The two methods are coupled via transmission conditions on the coupling boundary.

In 1979, Zienkiewicz et al. [37] introduced a non-symmetric one-equation coupling which is based on the first equation of the Calderón system and only relies on the simple-layer integral operator $\boldsymbol{V}$ as well as the double-layer integral operator $\boldsymbol{K}$. In 1980, Johnson and Nédélec [23] gave a first mathematical proof that this coupling procedure is well-posed and stable. This coupling is therefore also referred to as Johnson-Nédélec coupling. Their analysis relied on Fredholm theory and the compactness of $\boldsymbol{K}$ and was thus restricted to smooth coupling boundaries and potential problems. Moreover, we refer to [34] for an overview on asymptotic error estimates. The main idea therein, which goes back to [3], is to use a finer mesh for BEM than for FEM. Based on these works, other coupling methods such as the one-equation Bielak-MacCamy coupling and the (quasi-symmetric) Bielak-MacCamy coupling [4] have been proposed. The requirement for smooth boundaries is a severe restriction when dealing with standard FEM or BEM discretizations. Moreover, numerical experiments in [9] gave empirical evidence that this assumption and hence the compactness of $\boldsymbol{K}$ can be avoided. It took until 2009 when Sayas [28] gave a first mathematical proof for the stability of the Johnson-Nédélec coupling on polygonal boundaries.

In the meantime and because of the lack of satisfying theory, the symmetric coupling has been proposed independently by Costabel [11] and Han [21]. Relying on the symmetric formulation of the exterior Steklov-Poincaré operator, [11,21] proved stability of the symmetric coupling. Early works for linear as well as nonlinear problems [11,12,16–18,21,27,29] used interior Dirichlet boundaries to tackle constant functions for Laplace transmission problems resp. rigid body motions for elasticity problems. We also refer to the monograph [19] for further details.

To the best of the authors' knowledge, the very first work which avoided the use of an additional artificial Dirichlet boundary was [14], where a nonlinear Laplace transmission problem is considered. In the latter work the authors used the exterior Steklov-Poincaré operator to reduce the coupling equations to an operator equation with a strongly monotone operator. Although their analysis avoids an artificial Dirichlet boundary, their proof of ellipticity of the discrete exterior Steklov-Poincaré operator, and hence of unique solvability of the discrete coupling equations, involved sufficiently small mesh-sizes. Bootstrapping the original proof of [14], this assumption could recently be removed [2]. The authors of [10] then transferred the ideas of [14] to nonlinear elasticity problems in 2D. Further works in this direction include [5–8,15]. From an implementational point of view, however, the symmetric coupling seems not to be as attractive as the one-equation coupling methods, since all four integral operators of the Calderón system are involved.

While Sayas' work [28] focused on the linear Yukawa transmission problem as well as the Laplace transmission problem, Steinbach [31] proved stability for a class of linear Laplace transmission problems. He introduced an explicit stabilization for the coupling equations so that the stabilized equations turn out to be elliptic. Of & Steinbach [26] improved the results from [31], and also gave a sharp condition under which the stabilized problem is elliptic. Based on and inspired by the analysis of [28,31], Aurada et al. [1] introduced the idea of *implicit stabilization*. They proved that all (continuous and discrete) coupling equations are equivalent to associated stabilized formulations, even with the same solution. Since the stabilized formulations appear to be strongly monotone, this proves well-posedness and stability of the original coupling formulations, i.e., no explicit stabilization is needed or has to be implemented in practice. For the Johnson-Nédélec and Bielak-MacCamy coupling, their analysis covers the same problem class as [31] and moreover extends it to handle certain nonlinearities. For the symmetric coupling, the analysis of [1] provides an alternate proof for the results of [14], but avoids any restriction on the mesh-size.

In the very recent work [32], Steinbach extended the results from [26,31] to linear elasticity problems. We also refer to [20], where stability of the Johnson-Nédélec, the one-equation Bielak-MacCamy, and the (quasi-) symmetric Bielak-MacCamy coupling for a Yukawa transmission problem is proven. Moreover, they also show that the Johnson-Nédélec coupling applied to elasticity problems with interior Dirichlet boundary is stable for certain specific material parameters.

In our work, we consider (possibly) nonlinear transmission problems in elasticity. As a novelty, we introduce a general framework to handle both, the symmetric and non-symmetric couplings. We transfer and extend the idea of implicit theoretical stabilization from [1] to the present setting. This allows us to prove well-posedness of the non-stabilized coupling equations, although they seem to lack ellipticity. The basic idea is the following: We add appropriate terms to the right-hand side and left-hand side of the equations and prove that this modified (continuous or discrete) problem is equivalent to the original problem, even with the same solution. This means that a solution of the modified problem also solves the original problem and vice versa. Then, we prove existence and uniqueness of the solution of the modified problem and, due to equivalence, we infer that the original problem is well-posed. As in [10,20,32], our analysis applies to polygonal resp. polyhedral coupling boundaries. From our point of view, the advances over the state of art are fourfold:

- Unlike [10], we do not have to impose any assumption on the mesh-size h in case of the symmetric coupling.
- Unlike [12,19–21,29], we avoid the use of an artificial Dirichlet boundary to tackle the rigid body motions.
- Unlike [32], we prove well-posedness and stability of the original coupling equations and thus avoid any explicit stabilization.
- Unlike [20,32], our analysis for the one-equation couplings also covers certain nonlinear material laws, e.g., nonlinear elastic Hencky material laws.

The remainder of this work is organized as follows: In Sect. 2, we state the nonlinear elasticity transmission problem as well as the precise assumptions on the nonlinearity. Furthermore, we fix some notation and collect some important properties of linear elasticity problems and boundary integral operators, which are used throughout the work.

Section 3 deals with the symmetric coupling. Here, we introduce the concept of implicit stabilization, and prove unique solvability of the coupling equations (Theorem 1). We prove that the necessary assumption on the BEM discretization is satisfied, if the BEM ansatz space contains the piecewise constants (Theorem 2).

In Sect. 4, we apply the ideas worked out in Sect. 3 to the Johnson-Nédélec coupling. Moreover, we incorporate analytical techniques from [26,32] to our method and prove unique solvability under an additional assumption on the material parameters.

Finally, the short Sect. 5 analyzes the one-equation Bielak-MacCamy coupling which seems not to be as present as the symmetric resp. Johnson-Nédélec coupling in the literature.

2 Model problem

Throughout this work, $\Omega \subseteq \mathbb{R}^d$ ($d = 2, 3$) denotes a connected Lipschitz domain with polyhedral boundary $\Gamma = \partial\Omega$ and complement $\Omega^{\text{ext}} = \mathbb{R}^d \backslash \overline{\Omega}$.

2.1 Notation

We use bold symbols for d-dimensional vectors, e.g. $\boldsymbol{x}$, and vector valued functions $\boldsymbol{u} : \mathbb{R}^d \to \mathbb{R}^d$. The components of such objects will be indexed, e.g. $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2)^T$. For a finite set or a sequence of vector-valued objects we use upper indices for each element of the set resp. sequence, e.g., $\{\boldsymbol{u}^j\}_{j=1}^n$ resp. $\{\boldsymbol{u}^j\}_{j=1}^\infty$.

Let $X \subseteq \mathbb{R}^d$ be a nonempty, measurable set and let $L^2(X)$ resp. $H^1(X)$, $H^{1/2}(X) = (H^{-1/2}(X))^*$ denote the usual Lebesgue resp. Sobolev spaces. We define $\langle u, v\rangle_X := \int_X uv\,dx$ for $u, v \in L^2(X)$. For $u \in H^{-1/2}(\Gamma)$ and $v \in H^{1/2}(\Gamma)$, the brackets $\langle u, v\rangle_\Gamma$ denote the continuously extended L^2-scalar product.

For vector-valued Lebesgue resp. Sobolev spaces we use bold symbols, e.g., $\boldsymbol{L}^2(X) := [L^2(X)]^d$ resp. $\boldsymbol{H}^1(X) := [H^1(X)]^d$ and so on. Then, we define $\langle \boldsymbol{u}, \boldsymbol{v}\rangle_X := \int_X \boldsymbol{u} \cdot \boldsymbol{v}\,dx$ for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{L}^2(X)$. The product space $\mathcal{H} := \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{-1/2}(\Gamma)$, equipped with the norm $\|(\boldsymbol{u}, \boldsymbol{\phi})\|_{\mathcal{H}} := \big(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{\phi}\|^2_{\boldsymbol{H}^{-1/2}(\Gamma)}\big)^{1/2}$ for $(\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H}$, will be used throughout the work. Moreover, let $\boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\sigma}(\boldsymbol{v}) = \sum_{j,k=1}^d \boldsymbol{\epsilon}_{jk}(\boldsymbol{u})\boldsymbol{\sigma}_{jk}(\boldsymbol{v})$ denote the Frobenius inner product for arbitrary tensors $\boldsymbol{\epsilon}, \boldsymbol{\sigma}$, and define $\langle \boldsymbol{\sigma}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{v})\rangle_\Omega := \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})\,dx$. The divergence $\operatorname{div}(\boldsymbol{\epsilon}(\boldsymbol{u}))$ of a tensor is understood row-wise $(\operatorname{div}(\boldsymbol{\epsilon}(\boldsymbol{u})))_j = \sum_{k=1}^d \partial\boldsymbol{\epsilon}_{jk}(\boldsymbol{u})/\partial x_k$ for $j = 1, \ldots, d$. Finally, we write $\|\boldsymbol{\epsilon}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} := \langle \boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{u})\rangle_\Omega$.

2.2 Linear elasticity

As usual, the linear and symmetric strain tensor $\boldsymbol{\epsilon}$ is defined component-wise by

$$\boldsymbol{\epsilon}_{jk}(\boldsymbol{u}) = \frac{1}{2}\Big(\frac{\partial \boldsymbol{u}_j}{\partial \boldsymbol{x}_k} + \frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{x}_j}\Big) \tag{1}$$

for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $j, k = 1, \ldots, d$. Together with the Young modulus $E > 0$ and the Poisson ratio $\nu \in (0, \frac{1}{2})$, the linear stress tensor $\boldsymbol{\sigma}$ is defined by

$$\boldsymbol{\sigma}_{jk}(\boldsymbol{u}) = \delta_{jk}\frac{E\nu}{(1+\nu)(1-2\nu)}\operatorname{div}\boldsymbol{u} + \frac{E}{1+\nu}\boldsymbol{\epsilon}_{jk}(\boldsymbol{u}) \tag{2}$$

for all $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $j, k = 1, \dots, d$. To simplify notation, one usually introduces the so-called Lamé constants

$$\lambda := \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu := \frac{E}{2(1+\nu)}. \tag{3a}$$

With the identity matrix $\boldsymbol{I} \in \mathbb{R}^{d \times d}$, the stress tensor $\boldsymbol{\sigma}$ then satisfies

$$\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{u}) &= \lambda \mathrm{div}(\boldsymbol{u})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}(\boldsymbol{u}) \quad \text{as well as} \\ \mathrm{div}\, \boldsymbol{\sigma}(\boldsymbol{u}) &= \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\mathrm{div}(\boldsymbol{u}). \end{aligned} \tag{3b}$$

The kernel of the strain tensor $\boldsymbol{\epsilon}$ is given by the space of rigid body motions $\mathcal{R}_d := \ker(\boldsymbol{\epsilon}) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \,:\, \boldsymbol{\epsilon}(\boldsymbol{v}) = \boldsymbol{0}\}$ which reads

$$\mathcal{R}_2 := \mathrm{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-\boldsymbol{x}_2\\ \boldsymbol{x}_1\end{pmatrix}\right\} \quad \text{for } d = 2 \tag{4}$$

and

$$\mathcal{R}_3 := \mathrm{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}-\boldsymbol{x}_2\\ \boldsymbol{x}_1\\0\end{pmatrix}, \begin{pmatrix}0\\-\boldsymbol{x}_3\\ \boldsymbol{x}_2\end{pmatrix}, \begin{pmatrix}\boldsymbol{x}_3\\0\\-\boldsymbol{x}_1\end{pmatrix}\right\} \quad \text{for } d = 3. \tag{5}$$

Therefore, it holds $\boldsymbol{\sigma}(\boldsymbol{v}) = \boldsymbol{0}$ for all $\boldsymbol{v} \in \mathcal{R}_d$ as well. Note that (3) defines a plain strain problem in 2D. For a plain stress problem, the Lamé constants (3a) are replaced by $\lambda = E\nu/(1+\nu)(1-\nu)$, $\mu = E/(2(1+\nu))$. The analysis in this paper holds true for both cases.

2.3 Nonlinear transmission problem

As model problem, we consider the following nonlinear transmission problem in free space

$$-\mathrm{div}\, \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{6a}$$

$$-\mathrm{div}\, \boldsymbol{\sigma}^{\mathrm{ext}}(\boldsymbol{u}^{\mathrm{ext}}) = \boldsymbol{0} \quad \text{in } \Omega^{\mathrm{ext}}, \tag{6b}$$

$$\boldsymbol{u} - \boldsymbol{u}^{\mathrm{ext}} = \boldsymbol{u}_0, \quad \text{on } \Gamma \tag{6c}$$

$$\big(\mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{\sigma}^{\mathrm{ext}}(\boldsymbol{u}^{\mathrm{ext}})\big)\boldsymbol{n} = \boldsymbol{\phi}_0, \quad \text{on } \Gamma, \tag{6d}$$

$$|\boldsymbol{u}^{\mathrm{ext}}(\boldsymbol{x})| = \mathcal{O}(1/|\boldsymbol{x}|) \quad \text{for } |\boldsymbol{x}| \to \infty, \tag{6e}$$

where $\boldsymbol{n}$ denotes the exterior unit normal vector on Γ pointing from Ω to Ω^{ext}. The nonlinear operator $\mathfrak{A} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is used to describe a (possibly) nonlinear material law in Ω. Our assumptions on the operator $\mathfrak{A}$ and a more detailed description will be given later on in Sect. 2.5. The stress tensor $\boldsymbol{\sigma}^{\mathrm{ext}}$, which corresponds to the linear elasticity problem in the exterior domain, is defined as in (2)–(3) with Lamé constants λ^{ext}, μ^{ext}.

2.4 Boundary integral operators

The fundamental solution for linear elastostatics is given by the Kelvin tensor $\boldsymbol{G}(z) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ with

$$\boldsymbol{G}_{jk}(z) = \frac{\lambda+\mu}{2\mu(\lambda+2\mu)}\left(\frac{\lambda+3\mu}{\lambda+\mu}G(z)\delta_{jk} + \frac{1}{2(d-1)\pi}\frac{z_j z_k}{|z|^d}\right) \tag{7}$$

for all $z \in \mathbb{R}^d\backslash\{\mathbf{0}\}$ and $j, k = 1, \ldots, d$, where G denotes the fundamental solution of the Laplacian, i.e.,

$$G(z) = \begin{cases} -\frac{1}{2\pi}\log|z| & \text{for } d = 2, \\ \frac{1}{4\pi}\frac{1}{|z|} & \text{for } d = 3. \end{cases} \tag{8}$$

We stress that the natural conormal derivative γ_1^{int} is

$$\gamma_1^{\mathrm{int}}\boldsymbol{u} := \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} \quad \text{on } \Gamma. \tag{9}$$

There holds Betti's first formula, cf. e.g. [25, Theorem 4.4],

$$\langle\boldsymbol{\sigma}(\boldsymbol{u}),\, \boldsymbol{\epsilon}(\boldsymbol{v})\rangle_\Omega = \langle L\boldsymbol{u},\, \boldsymbol{v}\rangle_\Omega + \langle\gamma_1^{\mathrm{int}}(\boldsymbol{u}),\, \boldsymbol{v}\rangle_\Gamma, \tag{10}$$

with the linear differential operator $L\boldsymbol{u} = -\mathrm{div}\,\boldsymbol{\sigma}(\boldsymbol{u})$. Throughout this work, $\boldsymbol{V}$ denotes the simple-layer integral operator, $\boldsymbol{K}$ the double-layer integral operator with adjoint $\boldsymbol{K}'$, and $\boldsymbol{W}$ denotes the hypersingular integral operator. These boundary integral operators are induced by the two layer potentials

$$\widetilde{\boldsymbol{V}}\boldsymbol{\phi}(\boldsymbol{x}) := \int_\Gamma \boldsymbol{G}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\phi}(\boldsymbol{y})\,d\Gamma_{\boldsymbol{y}}, \tag{11}$$

$$\widetilde{\boldsymbol{K}}\boldsymbol{v}(\boldsymbol{x}) := \int_\Gamma \gamma_{1,\boldsymbol{y}}^{\mathrm{int}}\boldsymbol{G}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{v}(\boldsymbol{y})\,d\Gamma_{\boldsymbol{y}}, \tag{12}$$

It holds

$$\boldsymbol{V} := \gamma_0^{\mathrm{int}}\widetilde{\boldsymbol{V}} \in L(\boldsymbol{H}^{-1/2}(\Gamma);\, \boldsymbol{H}^{1/2}(\Gamma)), \tag{13}$$

$$\boldsymbol{K} := \gamma_0^{\mathrm{int}}\widetilde{\boldsymbol{K}} + 1 - \mathfrak{S} \in L(\boldsymbol{H}^{1/2}(\Gamma);\, \boldsymbol{H}^{1/2}(\Gamma)), \tag{14}$$

$$\boldsymbol{K}' := \gamma_1^{\mathrm{int}}\widetilde{\boldsymbol{V}} - \mathfrak{S} \in L(\boldsymbol{H}^{-1/2}(\Gamma);\, \boldsymbol{H}^{-1/2}(\Gamma)), \tag{15}$$

$$\boldsymbol{W} := -\gamma_1^{\mathrm{int}}\widetilde{\boldsymbol{K}} \in L(\boldsymbol{H}^{1/2}(\Gamma);\, \boldsymbol{H}^{-1/2}(\Gamma)), \tag{16}$$

where γ_0^{int} denotes the trace operator and $\mathfrak{S}(\boldsymbol{x}) = \frac{1}{2}$ for almost all $\boldsymbol{x} \in \Gamma$. We summarize some important properties of these operators. In 3D, the simple-layer integral operator is symmetric and elliptic, i.e., with some constant $c_{\mathrm{ell}} > 0$, which depends only on Ω, it holds

$$\langle \boldsymbol{\phi},\ \boldsymbol{V}\boldsymbol{\psi}\rangle_\Gamma = \langle \boldsymbol{\psi},\ \boldsymbol{V}\boldsymbol{\phi}\rangle_\Gamma \quad \text{and} \quad \|\boldsymbol{\phi}\|^2_{\boldsymbol{H}^{-1/2}(\Gamma)} \le c_{\text{ell}} \langle \boldsymbol{\phi},\ \boldsymbol{V}\boldsymbol{\phi}\rangle_\Gamma \quad \text{for all } \boldsymbol{\phi}, \boldsymbol{\psi} \in \boldsymbol{H}^{-1/2}(\Gamma). \tag{17}$$

Thus, $\|\boldsymbol{\phi}\|_V := \langle \boldsymbol{\phi},\ \boldsymbol{V}\boldsymbol{\phi}\rangle_\Gamma^{1/2}$ defines an equivalent Hilbert norm on $\boldsymbol{H}^{-1/2}(\Gamma)$. In 2D, ellipticity can be achieved by an appropriate scaling of the domain Ω, see e.g. [30, Section 6.7] for further details, and we may thus assume that $\boldsymbol{V}$ is elliptic. The hypersingular operator is symmetric positive semidefinite, i.e.,

$$\langle \boldsymbol{W}\boldsymbol{v},\ \boldsymbol{w}\rangle_\Gamma = \langle \boldsymbol{W}\boldsymbol{w},\ \boldsymbol{v}\rangle_\Gamma \quad \text{and} \quad \langle \boldsymbol{W}\boldsymbol{v},\ \boldsymbol{v}\rangle_\Gamma \ge 0 \quad \text{for all } \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^{1/2}(\Gamma). \tag{18}$$

There holds $\ker(\boldsymbol{W}) = \ker(\frac{1}{2} + \boldsymbol{K}) = \mathcal{R}_d$, see e.g. [30, Section 6.7], [25, Chapter 10]. Throughout this work, the boundary integral operators $\boldsymbol{V}$, $\boldsymbol{K}$, $\boldsymbol{K}'$, and $\boldsymbol{W}$ are always understood with respect to the exterior Lamé constants λ^{ext}, μ^{ext}.

2.5 Nonlinear material law and strongly monotone operators

We assume $\mathfrak{A}$ to be strongly monotone (19) and Lipschitz continuous (20), i.e., there exist constants $c_{\text{mon}} > 0$ and $c_{\text{lip}} > 0$ such that

$$c_{\text{mon}} \|\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{\epsilon}(\boldsymbol{v})\|^2_{\boldsymbol{L}^2(\Omega)} \le \langle \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}) - \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{v}),\ \boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{\epsilon}(\boldsymbol{v})\rangle_\Omega \quad \text{and} \tag{19}$$

$$\|\mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}) - \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{v})\|_{\boldsymbol{L}^2(\Omega)} \le c_{\text{lip}} \|\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{\epsilon}(\boldsymbol{v})\|_{\boldsymbol{L}^2(\Omega)} \tag{20}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$. In the case $\mathfrak{A}\boldsymbol{\epsilon}(\cdot) = \boldsymbol{\sigma}(\cdot)$, simple calculations show

$$|\langle \boldsymbol{\sigma}(\boldsymbol{u}),\ \boldsymbol{\epsilon}(\boldsymbol{v})\rangle_\Omega| \le C_1 \|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{\boldsymbol{L}^2(\Omega)}, \tag{21}$$

and

$$\langle \boldsymbol{\sigma}(\boldsymbol{u}),\ \boldsymbol{\epsilon}(\boldsymbol{u})\rangle_\Omega \ge C_2 \|\boldsymbol{\epsilon}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} \tag{22}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, with constants $C_1 = 6\lambda + 4\mu$ and $C_2 = 2\mu$.

An example for a nonlinear material law is the nonlinear elastic Hencky material, obeying the Hencky-Von Mises stress–strain relation

$$\mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}) := (K - \tfrac{2}{d}\widetilde{\mu}(\gamma(\boldsymbol{\epsilon}(u))))\text{div}(\boldsymbol{u})\boldsymbol{I} + 2\widetilde{\mu}(\gamma(\boldsymbol{\epsilon}(\boldsymbol{u})))\boldsymbol{\epsilon}(\boldsymbol{u}) \tag{23}$$

with $K > 0$ being the constant bulk modulus and Lamé function $\gamma(\boldsymbol{\epsilon}(\boldsymbol{u})) := (\boldsymbol{\epsilon}(\boldsymbol{u}) - \frac{1}{d}\text{div}(\boldsymbol{u})\boldsymbol{I}) : (\boldsymbol{\epsilon}(\boldsymbol{u}) - \frac{1}{d}\text{div}(\boldsymbol{u})\boldsymbol{I})$. Here, $\widetilde{\mu} : \mathbb{R}_{\ge 0} \to \mathbb{R}_+$ denotes a function such that the operator from (23) satisfies (19)–(20). Further information on the Hencky material law can be found in, e.g., [10,13,29,35] and the references therein.

2.6 Existence of solutions

For strongly monotone (19) and Lipschitz-continuous (20) operators $\mathfrak{A}$ and given data $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{u}_0 \in \boldsymbol{H}^{1/2}(\Gamma)$, and $\boldsymbol{\phi}_0 \in \boldsymbol{H}^{-1/2}(\Gamma)$, problem (6) admits unique solutions $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $\boldsymbol{u}^{\text{ext}} \in \boldsymbol{H}^1_{\text{loc}}(\Omega^{\text{ext}})$ in 3D. This follows from the equivalence to the symmetric coupling and its well-posedness, see Sect. 3. For the two-dimensional case, the two-dimensional compatibility condition

$$\langle \boldsymbol{f}, \boldsymbol{e}^j \rangle_\Omega + \langle \boldsymbol{\phi}_0, \boldsymbol{e}^j \rangle_\Gamma = 0 \quad j = 1, 2 \tag{24}$$

ensures unique solvability. Here, $\boldsymbol{e}^j$ are the standard unit normal vectors in $\mathbb{R}^2$. We refer to [22] for further details.

Remark The radiation condition (6e) can be generalized to

$$\boldsymbol{u}^{\text{ext}}(\boldsymbol{x}) = -\boldsymbol{G}(\boldsymbol{x})\boldsymbol{a} + \boldsymbol{r} + \mathcal{O}\Big(|\boldsymbol{x}|^{1-d}\Big) \quad \text{for } |\boldsymbol{x}| \to \infty, \tag{25}$$

with $\boldsymbol{r} \in \mathcal{R}_d$, $\boldsymbol{a} \in \mathbb{R}^d$, and $\boldsymbol{G}(\cdot)$ being the Kelvin tensor defined in (7). Moreover, $\boldsymbol{a} = \int_\Gamma \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{u}^{\text{ext}})\boldsymbol{n}\, d\Gamma$. A solution of (6a)–(6d) with (25) is unique. To see this, we stress that the pair $(\boldsymbol{u}, \boldsymbol{u}^{\text{ext}})$ solves (6a)–(6d) with (25) if and only if the pair $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}^{\text{ext}}) = (\boldsymbol{u} - \boldsymbol{r}, \boldsymbol{u}^{\text{ext}} - \boldsymbol{r})$ solves (6a)–(6d) with

$$\widetilde{\boldsymbol{u}}^{\text{ext}}(\boldsymbol{x}) = -\boldsymbol{G}(\boldsymbol{x})\boldsymbol{a} + \mathcal{O}(|\boldsymbol{x}|^{1-d}) \quad \text{for } |\boldsymbol{x}| \to \infty \tag{26}$$

and vice versa. Our analysis presented in this work still holds true if we replace $(\boldsymbol{u}, \boldsymbol{u}^{\text{ext}})$ by $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}^{\text{ext}})$ in (6a)–(6d) and the radiation condition (6e) by (26). Note that $\boldsymbol{a} = \boldsymbol{0}$ implies the compatibility condition (24) in 2D. Therefore, the compatibility condition can be dropped in 2D for $\boldsymbol{a} \neq \boldsymbol{0}$. In general, the constant $\boldsymbol{a}$ is determined by $\boldsymbol{a} = \int_\Omega \boldsymbol{f}\, d\boldsymbol{x} + \int_\Gamma \boldsymbol{\phi}_0\, d\Gamma$, which follows from (6a) and (6d). Furthermore, note that $|\boldsymbol{G}(\boldsymbol{x})| = \mathcal{O}(1/|\boldsymbol{x}|)$ for $|\boldsymbol{x}| \to \infty$ and $d = 3$. Hence, (26) coincides with (6e) in 3D.

2.7 Discretization

Let $\mathcal{T}_h$ denote a regular triangulation of Ω and let $\mathcal{E}_h^\Gamma$ denote a regular triangulation of Γ. Here, regularity is understood in the sense of Ciarlet. We define the local mesh-width function h by $h|_X := \text{diam}(X)$ for $X \in \mathcal{T}_h$ resp. $X \in \mathcal{E}_h^\Gamma$. Moreover, let $\mathcal{K}_h^\Omega$ denote the set of nodes of $\mathcal{T}_h$ and let $\mathcal{K}_h^\Gamma$ denote the set of nodes of $\mathcal{E}_h^\Gamma$. We stress that the triangulation $\mathcal{E}_h^\Gamma$ of the boundary Γ is, in general, independent of the triangulation $\mathcal{T}_h$.

Usually, one uses the space $\mathcal{P}^p(\mathcal{E}_h^\Gamma) := \{v \in L^2(\Gamma) : v|_E \text{ is a polynomial of degree } \leq p \text{ for all } E \in \mathcal{E}_h^\Gamma\}$ to approximate functions $\phi \in H^{-1/2}(\Gamma)$ and the space $\mathcal{S}^q(\mathcal{T}_h) := \mathcal{P}^q(\mathcal{T}_h) \cap C(\overline{\Omega})$ to approximate functions $u \in H^1(\Omega)$, with $q = p+1$. Here, $\mathcal{P}^q(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_T \text{ is a polynomial of degree } \leq q\}$. In Sects. 3–5, we may therefore use the space $\mathcal{H}_h := \mathcal{X}_h \times \mathcal{Y}_h = \big(\mathcal{S}^q(\mathcal{T}_h)\big)^d \times \big(\mathcal{P}^p(\mathcal{E}_h^\Gamma)\big)^d$ to approximate functions $(\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H} := \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{-1/2}(\Gamma)$.

3 Symmetric FEM–BEM coupling

The symmetric coupling of FEM and BEM has independently been introduced by Costabel and Han, see [11,21] for example. It relies on the use of all boundary integral operators from the Caldéron projector. For the derivation of the variational formulation of the symmetric coupling, cf. (27), we refer to, e.g., [10,13,19] for nonlinear elasticity problems and to, e.g., [1,14,19] for nonlinear Laplace problems. It is also shown in [13] resp. in [10] for the two-dimensional case that the symmetric coupling (27) is equivalent to the model problem (6).

3.1 Variational formulation

The symmetric coupling reads as follows: Find $(\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H} := \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{-1/2}(\Gamma)$, such that

$$\langle \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}),\, \boldsymbol{\epsilon}(\boldsymbol{v})\rangle_\Omega + \langle \boldsymbol{W}\boldsymbol{u},\, \boldsymbol{v}\rangle_\Gamma + \langle \big(\boldsymbol{K}' - \tfrac{1}{2}\big)\,\boldsymbol{\phi},\, \boldsymbol{v}\rangle_\Gamma = \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{\phi}_0 + \boldsymbol{W}\boldsymbol{u}_0,\, \boldsymbol{v}\rangle_\Gamma, \tag{27a}$$

$$\langle \boldsymbol{\psi},\, \big(\tfrac{1}{2} - \boldsymbol{K}\big)\,\boldsymbol{u} + \boldsymbol{V}\boldsymbol{\phi}\rangle_\Gamma = \langle \boldsymbol{\psi},\, \big(\tfrac{1}{2} - \boldsymbol{K}\big)\,\boldsymbol{u}_0\rangle_\Gamma \tag{27b}$$

holds for all $(\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}$.

To abbreviate notation, we define the mapping $b : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and the continuous linear functional $F \in \mathcal{H}^*$ by

$$\begin{aligned} b((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) &:= \langle \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}),\, \boldsymbol{\epsilon}(\boldsymbol{v})\rangle_\Omega \\ &\quad + \langle \boldsymbol{W}\boldsymbol{u},\, \boldsymbol{v}\rangle_\Gamma + \langle \big(\boldsymbol{K}' - \tfrac{1}{2}\big)\,\boldsymbol{\phi},\, \boldsymbol{v}\rangle_\Gamma + \langle \boldsymbol{\psi},\, \big(\tfrac{1}{2} - \boldsymbol{K}\big)\,\boldsymbol{u} + \boldsymbol{V}\boldsymbol{\phi}\rangle_\Gamma \end{aligned} \tag{28}$$

and

$$F(\boldsymbol{v}, \boldsymbol{\psi}) := \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{\phi}_0 + \boldsymbol{W}\boldsymbol{u}_0,\, \boldsymbol{v}\rangle_\Gamma + \langle \boldsymbol{\psi},\, \big(\tfrac{1}{2} - \boldsymbol{K}\big)\,\boldsymbol{u}_0\rangle_\Gamma \tag{29}$$

for all $(\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}$. Then, the symmetric coupling (27) can also be written as follows: Find $(\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H}$ such that

$$b((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) = F(\boldsymbol{v}, \boldsymbol{\psi}) \quad \text{holds for all } (\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}. \tag{30}$$

Note that $b(\cdot, \cdot)$ is nonlinear in $\boldsymbol{u}$ only, but linear in $\boldsymbol{v}$, $\boldsymbol{\psi}$, and $\boldsymbol{\phi}$. If we plug in the functions $(\boldsymbol{u}, \boldsymbol{\phi}) = (\boldsymbol{v}, \boldsymbol{\psi}) = (\boldsymbol{r}, 0)$ with $\boldsymbol{r} \in \mathcal{R}_d$ into (28), we observe

$$b((\boldsymbol{r}, \boldsymbol{0}), (\boldsymbol{r}, \boldsymbol{0})) - b((\boldsymbol{0}, \boldsymbol{0}), (\boldsymbol{r}, \boldsymbol{0})) = 0 \quad \text{for all } \boldsymbol{r} \in \mathcal{R}_d. \tag{31}$$

Therefore, $b(\cdot, \cdot)$ is not strongly monotone and unique solvability of (30) cannot be shown directly. In the following sections, we introduce an equivalent formulation of (30) which even has the same solution. Since this equivalent formulation turns out to be uniquely solvable, also (30) admits a unique solution.

The following two theorems are the main results of this section. The validity of the following Theorem 1 requires that the considered FEM–BEM space $\mathcal{H}_h = \mathcal{X}_h \times \mathcal{Y}_h \subseteq \mathcal{H}$ is sufficiently rich, i.e., $\mathcal{Y}_h$ contains a subspace $\mathcal{Y}_0$ which tackles the rigid body motions. Based on $\mathcal{Y}_0$, we shall introduce an appropriate stabilization which prevents (31) and guarantees strong monotonicity of the stabilized form $\widetilde{b}(\cdot,\cdot)$. Theorem 2 then shows that the piecewise constants $\mathcal{Y}_0 = \big(\mathcal{P}^0(\mathcal{T}_h^\Gamma)\big)^d$ are an appropriate choice. With an additional assumption on the model parameters c_{mon}, λ^{ext}, μ^{ext} these results also hold true for other coupling methods, namely the Johnson-Nédélec coupling, cf. Sect. 4, and the Bielak-MacCamy coupling, cf. Sect. 5.

Theorem 1 *Let $\mathcal{H}_h := \mathcal{X}_h \times \mathcal{Y}_h$ be a closed subspace of $\mathcal{H}$ and assume that $\mathcal{Y}_0 \subseteq \mathcal{Y}_h \cap \boldsymbol{L}^2(\Gamma)$ satisfies*

$$\forall \boldsymbol{r} \in \mathcal{R}_d \backslash \{\boldsymbol{0}\} \exists \boldsymbol{\xi} \in \mathcal{Y}_0 \quad \langle \boldsymbol{\xi}, \boldsymbol{r} \rangle_\Gamma \neq 0. \tag{32}$$

Then, the symmetric coupling

$$b((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) = F(\boldsymbol{v}, \boldsymbol{\psi}) \quad \textit{for all } (\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H} \tag{33}$$

as well as its Galerkin formulation

$$b((\boldsymbol{u}_h, \boldsymbol{\phi}_h), (\boldsymbol{v}_h, \boldsymbol{\psi}_h)) = F(\boldsymbol{v}_h, \boldsymbol{\psi}_h) \quad \textit{for all } (\boldsymbol{v}_h, \boldsymbol{\psi}_h) \in \mathcal{H}_h \tag{34}$$

admit unique solutions $(\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H}$ resp. $(\boldsymbol{u}_h, \boldsymbol{\phi}_h) \in \mathcal{H}_h$. Moreover, there holds the Céa-type quasi-optimality

$$\|(\boldsymbol{u}, \boldsymbol{\phi}) - (\boldsymbol{u}_h, \boldsymbol{\phi}_h)\|_{\mathcal{H}} \le C_{\text{Cèa}} \min_{(\boldsymbol{v}_h, \boldsymbol{\psi}_h) \in \mathcal{H}_h} \|(\boldsymbol{u}, \boldsymbol{\phi}) - (\boldsymbol{v}_h, \boldsymbol{\psi}_h)\|_{\mathcal{H}}. \tag{35}$$

The constant $C_{\text{Cèa}} > 0$ depends only on Ω, $\mathfrak{A}$, $\mathcal{Y}_0$, and on the Lamé constants λ^{ext}, μ^{ext}.

Assumption (32) is clearly satisfied if $\mathcal{Y}_0 := \big(\mathcal{P}^1(\mathcal{E}_h^\Gamma)\big)^d$ denotes the space of affine functions restricted to $\mathcal{E}_h^\Gamma$, since $\mathcal{R}_d \subseteq \big(\mathcal{P}^1(\mathcal{E}_h^\Gamma)\big)^d$ and one may thus choose $\boldsymbol{\xi} = \boldsymbol{r}$ in (32). However, we shall also show that the space $\mathcal{Y}_0 := \big(\mathcal{P}^0(\mathcal{E}_h^\Gamma)\big)^d$ is sufficiently rich to ensure (32). This is precisely the second theorem, we aim to emphasize and prove. Note that the constant $C_{\text{Cèa}}$ does not depend on the mesh-size h if $\mathcal{Y}_0 \subseteq \mathcal{Y}_h$ for all h.

Theorem 2 *For $d = 2, 3$, the space $\mathcal{Y}_0 := \big(\mathcal{P}^0(\mathcal{E}_h^\Gamma)\big)^d$ satisfies assumption (32).*

The proof of Theorem 1 resp. Theorem 2 is carried out in Sect. 3.4 resp. Sect. 3.5.

3.2 Implicit theoretical stabilization

To prove Theorem 1, we shall add appropriate terms to $b(\cdot,\cdot)$, which tackle the rigid body motions in the interior domain Ω. These (purely theoretical) linear stabilization terms are chosen in such a way that they vanish when inserting a (continuous resp. discrete) solution of (33). To be more precise, we will use (27b) to stabilize $b(\cdot,\cdot)$.

Proposition 3 *Let $\mathcal{H}_h = \mathcal{X}_h \times \mathcal{Y}_h$ be a subspace of $\mathcal{H}$. Let $\{(\boldsymbol{\xi}^j)_{j=1}^D\} \subseteq \mathcal{Y}_h$, $D \in \mathbb{N}$, be a set of functions. Define*

$$\widetilde{b}((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) := b((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) + \sum_{j=1}^{D} \langle \boldsymbol{\xi}^j, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u} + \boldsymbol{V}\boldsymbol{\phi} \rangle_\Gamma \langle \boldsymbol{\xi}^j, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{v} + \boldsymbol{V}\boldsymbol{\psi} \rangle_\Gamma \tag{36}$$

for all $(\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}_h$ and

$$\widetilde{F}(\boldsymbol{v}, \boldsymbol{\psi}) := F(\boldsymbol{v}, \boldsymbol{\psi}) + \sum_{j=1}^{D} \langle \boldsymbol{\xi}^j, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u}_0 \rangle \langle \boldsymbol{\xi}^j, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{v} + \boldsymbol{V}\boldsymbol{\psi} \rangle_\Gamma \tag{37}$$

for all $(\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}_h$. Then, there holds the following equivalence: A function $(\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H}_h$ solves

$$b((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) = F(\boldsymbol{v}, \boldsymbol{\psi}) \quad \text{for all } (\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}_h \tag{38}$$

if and only if it also solves

$$\widetilde{b}((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) = \widetilde{F}(\boldsymbol{v}, \boldsymbol{\psi}) \quad \text{for all } (\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}_h. \tag{39}$$

Proof **Step 1.** Assume that $(\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H}_h$ solves (38), then $b((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{0}, \boldsymbol{\xi}^j)) = F(\boldsymbol{0}, \boldsymbol{\xi}^j)$ and thus (39) follows directly.
Step 2. Assume that $(\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H}_h$ solves (39). By choosing $(\boldsymbol{v}, \boldsymbol{\psi}) = (\boldsymbol{0}, \boldsymbol{\xi}^\ell)$ as a test-function in (39), we infer

$$\begin{aligned}
&\langle \boldsymbol{\xi}^\ell, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u} + \boldsymbol{V}\boldsymbol{\phi} \rangle_\Gamma + \sum_{j=1}^{D} \langle \boldsymbol{\xi}^j, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u} + \boldsymbol{V}\boldsymbol{\phi} \rangle_\Gamma \langle \boldsymbol{\xi}^j, \boldsymbol{V}\boldsymbol{\xi}^\ell \rangle_\Gamma \\
&\quad = \widetilde{b}((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{0}, \boldsymbol{\xi}^\ell)) \\
&\quad = \widetilde{F}(\boldsymbol{0}, \boldsymbol{\xi}^\ell) = \langle \boldsymbol{\xi}^\ell, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u}_0 \rangle_\Gamma + \sum_{j=1}^{D} \langle \boldsymbol{\xi}^j, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u}_0 \rangle_\Gamma \langle \boldsymbol{\xi}^j, \boldsymbol{V}\boldsymbol{\xi}^\ell \rangle_\Gamma.
\end{aligned}$$

This is equivalent to

$$\sum_{j=1}^{D} \langle \boldsymbol{\xi}^j, \left(\tfrac{1}{2} - \boldsymbol{K}\right) (\boldsymbol{u} - \boldsymbol{u}_0) + \boldsymbol{V}\boldsymbol{\phi} \rangle_\Gamma \langle \boldsymbol{\xi}^j, \boldsymbol{V}\boldsymbol{\xi}^\ell \rangle_\Gamma = -\langle \boldsymbol{\xi}^\ell, \left(\tfrac{1}{2} - \boldsymbol{K}\right) (\boldsymbol{u} - \boldsymbol{u}_0) + \boldsymbol{V}\boldsymbol{\phi} \rangle_\Gamma \tag{40}$$

for all $\ell = 1, \ldots, D$. Next, we define a matrix $\boldsymbol{A} \in \mathbb{R}^{D\times D}_{\mathrm{sym}}$ with entries $\boldsymbol{A}_{jk} := \langle \boldsymbol{\xi}^k, \boldsymbol{V}\boldsymbol{\xi}^j \rangle_\Gamma$ and a vector $\boldsymbol{x} \in \mathbb{R}^D$ with entries $\boldsymbol{x}_k := \langle \boldsymbol{\xi}^k, \left(\tfrac{1}{2} - \boldsymbol{K}\right) (\boldsymbol{u} - \boldsymbol{u}_0) +$

$\boldsymbol{V}\boldsymbol{\phi}\rangle_\Gamma$ for all $j, k = 1, \ldots, d$. With $\boldsymbol{I} \in \mathbb{R}^{d\times d}$ the unit matrix, we can rewrite (40) for all $\ell = 1, \ldots, D$ simultaneously as

$$(\boldsymbol{I} + \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}. \tag{41}$$

Since $\boldsymbol{V}$ is elliptic, the matrix $\boldsymbol{A}$ is positive semi-definite and thus has only nonnegative eigenvalues. Therefore, $\boldsymbol{I} + \boldsymbol{A}$ is positive definite, and (41) is equivalent to $\boldsymbol{x} = \boldsymbol{0}$. This gives

$$\langle \boldsymbol{\xi}^j,\ \big(\tfrac{1}{2} - \boldsymbol{K}\big)\,\boldsymbol{u} + \boldsymbol{V}\boldsymbol{\phi}\rangle_\Gamma = \langle \boldsymbol{\xi}^j,\ \big(\tfrac{1}{2} - \boldsymbol{K}\big)\,\boldsymbol{u}_0\rangle_\Gamma$$

for all $j = 1, \ldots, D$. With these equalities and the definitions of $b(\cdot, \cdot)$ and $\widetilde{b}(\cdot, \cdot)$, we get

$$\begin{aligned}
&\widetilde{b}((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) - b((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi}))\\
&\quad= \sum_{j=1}^{D} \langle \boldsymbol{\xi}^j,\ \big(\tfrac{1}{2} - \boldsymbol{K}\big)\,\boldsymbol{u} + \boldsymbol{V}\boldsymbol{\phi}\rangle_\Gamma \langle \boldsymbol{\xi}^j,\ \big(\tfrac{1}{2} - \boldsymbol{K}\big)\,\boldsymbol{v} + \boldsymbol{V}\boldsymbol{\psi}\rangle_\Gamma\\
&\quad= \sum_{j=1}^{D} \langle \boldsymbol{\xi}^j,\ \big(\tfrac{1}{2} - \boldsymbol{K}\big)\,\boldsymbol{u}_0\rangle_\Gamma \langle \boldsymbol{\xi}^j,\ \big(\tfrac{1}{2} - \boldsymbol{K}\big)\,\boldsymbol{v} + \boldsymbol{V}\boldsymbol{\psi}\rangle_\Gamma\\
&\quad= \widetilde{F}(\boldsymbol{v}, \boldsymbol{\psi}) - F(\boldsymbol{v}, \boldsymbol{\psi}).
\end{aligned}$$

In particular, (39) thus implies (38). This concludes the proof. □

3.3 Equivalent norm

To show that $\widetilde{b}(\cdot, \cdot)$ from Proposition 3 yields a strongly monotone formulation, we show that the employed stabilization term provides an equivalent norm on the energy space $\mathcal{H}$. The following lemma, also known as Deny-Lions lemma, follows from a compactness argument and Korn's second inequality [25, Section 10]; see also [1, Lemma 10] for the corresponding result for Laplace-type transmission problems. It can also be understood as a consequence of a generalized Poincaré inequality (see, e.g., [24, Theorem 5.11.2]).

Lemma 4 *Let $g_j : \mathcal{H} \to \mathbb{R}$ with $j = 1, \ldots, D$ denote linear and continuous functionals such that*

$$|g(\boldsymbol{r}, \boldsymbol{0})|^2 := \sum_{j=1}^{D} g_j(\boldsymbol{r}, \boldsymbol{0})^2 \neq 0 \quad \textit{holds for all } \boldsymbol{r} \in \mathcal{R}_d \backslash \{\boldsymbol{0}\}. \tag{42}$$

Then, the definition

$$|||(\boldsymbol{u}, \boldsymbol{\phi})|||^2 := \|\boldsymbol{\epsilon}(\boldsymbol{u})\|^2_{L^2(\Omega)} + \langle \boldsymbol{\phi},\ \boldsymbol{V}\boldsymbol{\phi}\rangle_\Gamma + |g(\boldsymbol{u}, \boldsymbol{\phi})|^2 \quad \textit{for all } (\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H} \tag{43}$$

yields an equivalent norm on $\mathcal{H}$, and the norm equivalence constant $C_{\text{norm}} > 0$ in

$$C_{\text{norm}}^{-1} \|(\boldsymbol{u}, \boldsymbol{\phi})\|_{\mathcal{H}} \le |||(\boldsymbol{u}, \boldsymbol{\phi})||| \le C_{\text{norm}} \|(\boldsymbol{u}, \boldsymbol{\phi})\|_{\mathcal{H}} \quad \text{for all } (\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H} \tag{44}$$

depends only on Ω, λ^{ext}, μ^{ext}, and g. □

The following proposition provides the equivalent norm used to analyze the symmetric coupling as well as the Johnson-Nédélec coupling (see Sect. 4 below).

Proposition 5 *Let $\mathcal{Y}_0 \subseteq \mathcal{Y} \cap \boldsymbol{L}^2(\Gamma)$ be a subspace which satisfies assumption (32) of Theorem 1. Let $\boldsymbol{r}^1, \ldots, \boldsymbol{r}^D$ with $D = \dim(\mathcal{R}_d)$ denote a basis of the rigid body motions and let $\Pi_0 : \boldsymbol{L}^2(\Gamma) \to \mathcal{Y}_0$ be the $\boldsymbol{L}^2$-orthogonal projection. Then, $\boldsymbol{\xi}^j := \Pi_0(\boldsymbol{r}^j)$ for $j = 1, \ldots, D$ are linearly independent. Moreover, the functionals $g_j \in \mathcal{H}^*$ defined by*

$$g_j(\boldsymbol{u}, \boldsymbol{\phi}) := \langle \boldsymbol{\xi}^j, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u} + \boldsymbol{V}\boldsymbol{\phi} \rangle_\Gamma \quad \text{for } (\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H} \tag{45}$$

fulfill assumption (42) of Lemma 4. In particular,

$$|||(\boldsymbol{u}, \boldsymbol{\phi})|||^2 := \|\boldsymbol{\epsilon}(\boldsymbol{u})\|^2_{\boldsymbol{L}^2(\Omega)} + \langle \boldsymbol{\phi}, \boldsymbol{V}\boldsymbol{\phi} \rangle_\Gamma + \sum_{j=1}^{D} |\langle \boldsymbol{\xi}^j, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u} + \boldsymbol{V}\boldsymbol{\phi} \rangle_\Gamma|^2 \tag{46}$$

is an equivalent norm on $\mathcal{H}$, and the norm equivalence constant $C_{\text{norm}} > 0$ in

$$C_{\text{norm}}^{-1} \|(\boldsymbol{u}, \boldsymbol{\phi})\|_{\mathcal{H}} \le |||(\boldsymbol{u}, \boldsymbol{\phi})||| \le C_{\text{norm}} \|(\boldsymbol{u}, \boldsymbol{\phi})\|_{\mathcal{H}} \quad \text{for all } (\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H} \tag{47}$$

depends only on Ω, $\mathcal{Y}_0$, λ^{ext}, and μ^{ext}.

Proof We stress that condition (32) is equivalent to the fact that $\Pi_0 : \mathcal{R}_d \to \mathcal{Y}_0$ is injective. In particular, the $\boldsymbol{\xi}^j := \Pi_0(\boldsymbol{r}^j)$, for $j = 1, \ldots, D$, are linearly independent. Therefore, we can reformulate condition (32) as

$$\forall \boldsymbol{r} \in \mathcal{R}_d \backslash \{\boldsymbol{0}\} \exists j \in \{1, \ldots, D\} \quad \langle \boldsymbol{\xi}^j, \boldsymbol{r} \rangle_\Gamma \neq 0. \tag{48}$$

The functionals g_j are well-defined, linear, and bounded. To see (42), we stress that due to $\ker\left(\tfrac{1}{2} + \boldsymbol{K}\right) = \mathcal{R}_d$,

$$g_j(\boldsymbol{r}, \boldsymbol{0}) = \langle \boldsymbol{\xi}^j, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{r} \rangle_\Gamma = \langle \boldsymbol{\xi}^j, \boldsymbol{r} \rangle_\Gamma \quad \text{for } j = 1, \ldots, D \text{ and } \boldsymbol{r} \in \mathcal{R}_d.$$

From (48) we infer that there exists $j \in \{1, \ldots, D\}$ such that $g_j(\boldsymbol{r}, \boldsymbol{0}) \neq 0$. Therefore, (42) holds for

$$|g(\boldsymbol{u}, \boldsymbol{\phi})|^2 = \sum_{j=1}^{D} g_j(\boldsymbol{u}, \boldsymbol{\phi})^2 = \sum_{j=1}^{D} |\langle \boldsymbol{\xi}^j, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u} + \boldsymbol{V}\boldsymbol{\phi} \rangle_\Gamma|^2.$$

This concludes the proof. □

3.4 Proof of Theorem 1

As far as existence and uniqueness of solutions is concerned, it suffices to consider the Galerkin formulation (34), since this covers the case $\mathcal{H}_h = \mathcal{H}$ as well. With assumption (32) and $\mathcal{Y}_0 \subseteq \mathcal{Y}_h \cap \boldsymbol{L}^2(\Gamma)$, Proposition 5 allows to apply Proposition 3. Hence, we may equivalently ask for the unique solvability of (39) instead of (38) resp. (34). To this end, we define the nonlinear operator $\widetilde{\mathfrak{B}} : \mathcal{H}_h \to \mathcal{H}_h^*$ by

$$\widetilde{\mathfrak{B}}(\boldsymbol{u}_h, \boldsymbol{\phi}_h) := \widetilde{b}((\boldsymbol{u}_h, \boldsymbol{\phi}_h), \cdot).$$

First, we rewrite equation (39) as an equivalent operator equation: Find $(\boldsymbol{u}_h, \boldsymbol{\phi}_h) \in \mathcal{H}_h$ such that

$$\widetilde{\mathfrak{B}}(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = F \quad \text{in } \mathcal{H}_h^*. \tag{49}$$

Step 1 (*Lipschitz continuity of* $\widetilde{\mathfrak{B}}$). Due to the Lipschitz continuity (20) of $\mathfrak{A}$ and the boundedness of the boundary integral operators, it clearly follows that $\widetilde{\mathfrak{B}}$ is also Lipschitz continuous. The Lipschitz constant $C_{\rm lip} > 0$ in

$$\|\widetilde{\mathfrak{B}}(\boldsymbol{u}_h, \boldsymbol{\phi}_h) - \widetilde{\mathfrak{B}}(\boldsymbol{v}_h, \boldsymbol{\psi}_h)\|_{\mathcal{H}^*} \le C_{\rm lip}\|(\boldsymbol{u}_h, \boldsymbol{\phi}_h) - (\boldsymbol{v}_h, \boldsymbol{\psi}_h)\|_{\mathcal{H}}, \tag{50}$$

for all $(\boldsymbol{u}_h, \boldsymbol{\phi}_h), (\boldsymbol{v}_h, \boldsymbol{\psi}_h) \in \mathcal{H}$, thus depends only on $\mathfrak{A}$, Ω, $\lambda^{\rm ext}$, and $\mu^{\rm ext}$.
Step 2 (*Strong monotonicity of* $\widetilde{\mathfrak{B}}$). We have to prove that, for all $(\boldsymbol{u}_h, \boldsymbol{\phi}_h), (\boldsymbol{v}_h, \boldsymbol{\psi}_h) \in \mathcal{H}$,

$$\langle \widetilde{\mathfrak{B}}(\boldsymbol{u}_h, \boldsymbol{\phi}_h) - \widetilde{\mathfrak{B}}(\boldsymbol{v}_h, \boldsymbol{\psi}_h),\ (\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{\phi}_h - \boldsymbol{\psi}_h)\rangle \ge C_{\rm mon}\|(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{\phi}_h - \boldsymbol{\psi}_h)\|_{\mathcal{H}}^2. \tag{51}$$

To abbreviate notation, let $(\boldsymbol{w}_h, \boldsymbol{\chi}_h) := (\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{\phi}_h - \boldsymbol{\psi}_h)$. Then, we get

$$\begin{aligned}
&\langle \widetilde{\mathfrak{B}}(\boldsymbol{u}_h, \boldsymbol{\phi}_h) - \widetilde{\mathfrak{B}}(\boldsymbol{v}_h, \boldsymbol{\psi}_h),\ (\boldsymbol{w}_h, \boldsymbol{\chi}_h)\rangle \\
&\quad = \langle \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}_h) - \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{v}_h),\ \boldsymbol{\epsilon}(\boldsymbol{w}_h)\rangle_\Omega + \langle \boldsymbol{W}\boldsymbol{w}_h,\ \boldsymbol{w}_h\rangle_\Gamma + \langle \left(\boldsymbol{K}' - \tfrac{1}{2}\right)\boldsymbol{\chi}_h,\ \boldsymbol{w}_h\rangle_\Gamma \\
&\qquad + \langle \boldsymbol{\chi}_h,\ \left(\tfrac{1}{2} - \boldsymbol{K}\right)\boldsymbol{w}_h + \boldsymbol{V}\boldsymbol{\chi}_h\rangle_\Gamma + \sum_{j=1}^{D} |\langle \boldsymbol{\xi}^j,\ \left(\tfrac{1}{2} - \boldsymbol{K}\right)\boldsymbol{w}_h + \boldsymbol{V}\boldsymbol{\chi}_h\rangle_\Gamma|^2 =: I
\end{aligned}$$

Next, we use strong monotonicity (19) of $\mathfrak{A}$ and positive semi-definiteness (18) of $\boldsymbol{W}$ to estimate

$$\begin{aligned}
I &\ge c_{\rm mon}\|\boldsymbol{\epsilon}(\boldsymbol{w}_h)\|_{\boldsymbol{L}^2(\Omega)}^2 + \langle \boldsymbol{\chi}_h,\ \boldsymbol{V}\boldsymbol{\chi}_h\rangle_\Gamma + \sum_{j=1}^{D} |\langle \boldsymbol{\xi}^j,\ \left(\tfrac{1}{2} - \boldsymbol{K}\right)\boldsymbol{w}_h + \boldsymbol{V}\boldsymbol{\chi}_h\rangle_\Gamma|^2 \\
&\ge \min\{c_{\rm mon}, 1\}\left(\|\boldsymbol{\epsilon}(\boldsymbol{w}_h)\|_{\boldsymbol{L}^2(\Omega)}^2 + \langle \boldsymbol{\chi}_h,\ \boldsymbol{V}\boldsymbol{\chi}_h\rangle_\Gamma + \sum_{j=1}^{D} |\langle \boldsymbol{\xi}^j,\ \left(\tfrac{1}{2} - \boldsymbol{K}\right)\boldsymbol{w}_h + \boldsymbol{V}\boldsymbol{\chi}_h\rangle_\Gamma|^2\right) \\
&= \min\{c_{\rm mon}, 1\}\,|||(\boldsymbol{w}_h, \boldsymbol{\chi}_h)|||^2.
\end{aligned}$$

Finally, the norm equivalence of Proposition 5 yields strong monotonicity, where $C_{\text{mon}} = \min\{c_{\text{mon}}, 1\}C_{\text{norm}}^{-1} > 0$ depends only on $\mathfrak{A}$, Ω, λ^{ext}, μ^{ext} and $\mathcal{Y}_0$.

Step 3 (*Unique solvability and Céa lemma*). The main theorem on strongly monotone operators, see e.g. [36, Section 25], states that the operator formulation (49) and thus the Galerkin formulation (34) admits a unique solution $(\boldsymbol{u}_h, \boldsymbol{\phi}_h) \in \mathcal{H}_h$. For $\mathcal{H}_h = \mathcal{H}$, we see that also the symmetric formulation (33) admits a unique solution $(\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H}$. Finally, standard theory [36, Section 25] also proves the validity of Céa's lemma (35), where $C_{\text{Cèa}} = C_{\text{lip}}/C_{\text{mon}} > 0$ depends only on Ω, $\mathfrak{A}$, λ^{ext}, μ^{ext}, and $\mathcal{Y}_0$. □

Remark Our analysis unveils that (27b) tackles the rigid body motions in the interior domain. We have seen in (31) that this information is lost when trying to prove strong monotonicity of $b(\cdot,\cdot)$, but can be reconstructed by adding appropriate terms to $b(\cdot,\cdot)$. We stress that the radiation condition (6e) fixes the rigid body motion in the exterior Ω^{ext}, see also Sect. 2.3. Since the interior and exterior solution are coupled via equation (27b), this information is transferred by (27b) from the exterior to the interior. Thus, adding terms to $b(\cdot,\cdot)$ that satisfy (27b) for fixed test-functions seems to be a natural approach.

3.5 Proof of Theorem 2

Let $\boldsymbol{r}^1, \ldots, \boldsymbol{r}^D$ be a basis of the rigid body motions $\mathcal{R}_d$ and let $\Pi_0 : \boldsymbol{L}^2(\Gamma) \to \mathcal{P}^0(\mathcal{E}_h^\Gamma)$ denote the $\boldsymbol{L}^2$-projection. We shall use the observation from the proof of Proposition 5 that assumption (32) is equivalent to the fact that the $\Pi_0(\boldsymbol{r}^j)$, for $j = 1, \ldots, D$, are linearly independent.

Proof of Theorem 2 for $d = 2$ Let

$$\boldsymbol{r}^1 := \begin{pmatrix}1\\0\end{pmatrix}, \quad \boldsymbol{r}^2 := \begin{pmatrix}0\\1\end{pmatrix}, \quad \boldsymbol{r}^3 := \begin{pmatrix}-\boldsymbol{x}_2\\ \boldsymbol{x}_1\end{pmatrix}$$

denote the canonical basis of $\mathcal{R}_2$, and let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ fulfill

$$\alpha_1 \Pi_0(\boldsymbol{r}^1) + \alpha_2 \Pi_0(\boldsymbol{r}^2) + \alpha_3 \Pi_0(\boldsymbol{r}^3) = \boldsymbol{0}. \tag{52}$$

We stress that $\Pi_0(\boldsymbol{r}^1) = \boldsymbol{r}^1$ and $\Pi_0(\boldsymbol{r}^2) = \boldsymbol{r}^2$. For $E \in \mathcal{E}_h^\Gamma$, we get

$$\Pi_0(\boldsymbol{r}^3)|_E = \frac{1}{|E|}\begin{pmatrix}-\int_E \boldsymbol{x}_2\, d\Gamma_{\boldsymbol{x}}\\ \int_E \boldsymbol{x}_1\, d\Gamma_{\boldsymbol{x}}\end{pmatrix} = \begin{pmatrix}-\boldsymbol{s}_2^E\\ \boldsymbol{s}_1^E\end{pmatrix},$$

where $\boldsymbol{s}^E = (\boldsymbol{s}_1^E, \boldsymbol{s}_2^E)^T$ denotes the midpoint of a boundary element E. Therefore, (52) can be written as

$$\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} + \alpha_3 \begin{pmatrix}-\boldsymbol{s}_2^E\\ \boldsymbol{s}_1^E\end{pmatrix} = \boldsymbol{0} \quad \text{for all } E \in \mathcal{E}_h^\Gamma. \tag{53}$$

Altogether, we thus obtain $\alpha_3 \boldsymbol{s}^E = \alpha_3 \boldsymbol{s}^{E'}$ for all $E, E' \in \mathcal{E}_h^\Gamma$, which can only hold for $\alpha_3 = 0$. This implies $\alpha_1 \boldsymbol{r}^1 + \alpha_2 \boldsymbol{r}^2 = 0$ and hence $\alpha_1 = 0 = \alpha_2$. Therefore, $\Pi_0(\boldsymbol{r}^j)$, $j = 1, \ldots, 3 = D$, are linearly independent which is equivalent to (32). □

For the proof of the 3D case, we require the following elementary observation, whose proof is left to the reader.

Lemma 6 *Let $d = 3$ and $\mathcal{E}_h^\Gamma$ be a regular triangulation of the closed boundary $\Gamma = \partial\Omega$ into flat surface triangles. Then, there are at least three different triangles $A, B, C \in \mathcal{E}_h^\Gamma$ such that the centers of mass $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ corresponding to these elements do not lie on one line, i.e., $\boldsymbol{c} - \boldsymbol{a} \notin \{t(\boldsymbol{b} - \boldsymbol{a}) : t \in \mathbb{R}\}$.* □

Proof of Theorem 2 for $d = 3$ Let

$$\boldsymbol{r}^1 := \begin{pmatrix}1\\0\\0\end{pmatrix}, \boldsymbol{r}^2 := \begin{pmatrix}0\\1\\0\end{pmatrix}, \boldsymbol{r}^3 := \begin{pmatrix}0\\0\\1\end{pmatrix}, \boldsymbol{r}^4 := \begin{pmatrix}-\boldsymbol{x}_2\\\boldsymbol{x}_1\\0\end{pmatrix}, \boldsymbol{r}^5 := \begin{pmatrix}0\\-\boldsymbol{x}_3\\\boldsymbol{x}_2\end{pmatrix},$$
$$\boldsymbol{r}^6 := \begin{pmatrix}\boldsymbol{x}_3\\0\\-\boldsymbol{x}_1\end{pmatrix}$$

denote the canonical basis of $\mathcal{R}_3$. We stress that $\Pi_0(\boldsymbol{r}^j) = \boldsymbol{r}^j$ for $j = 1, 2, 3$, and

$$\Pi_0(\boldsymbol{r}^4)|_E = \begin{pmatrix}-\boldsymbol{s}_2^E\\\boldsymbol{s}_1^E\\0\end{pmatrix}, \quad \Pi_0(\boldsymbol{r}^5)|_E = \begin{pmatrix}0\\-\boldsymbol{s}_3^E\\\boldsymbol{s}_2^E\end{pmatrix}, \quad \Pi_0(\boldsymbol{r}^6)|_E = \begin{pmatrix}\boldsymbol{s}_3^E\\0\\-\boldsymbol{s}_1^E\end{pmatrix},$$

where $\boldsymbol{s}^E = (\boldsymbol{s}_1^E, \boldsymbol{s}_2^E, \boldsymbol{s}_3^E)^T \in \mathbb{R}^3$ denotes the center of mass of an element $E \in \mathcal{E}_h^\Gamma$. The main ingredient for the proof is the geometric observation of Lemma 6: There are at least three elements $A, B, C \in \mathcal{E}_h^\Gamma$ such that the corresponding centers of mass $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ do not lie on one line. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \mathbb{R}$ fulfill

$$\alpha_1\Pi_0(\boldsymbol{r}^1) + \alpha_2\Pi_0(\boldsymbol{r}^2) + \alpha_3\Pi_0(\boldsymbol{r}^3) + \alpha_4\Pi_0(\boldsymbol{r}^4) + \alpha_5\Pi_0(\boldsymbol{r}^5) + \alpha_6\Pi_0(\boldsymbol{r}^6) = \boldsymbol{0},$$

which is equivalent to

$$\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} + \begin{pmatrix}-\boldsymbol{s}_2^E & 0 & \boldsymbol{s}_3^E\\\boldsymbol{s}_1^E & -\boldsymbol{s}_3^E & 0\\0 & \boldsymbol{s}_2^E & -\boldsymbol{s}_1^E\end{pmatrix}\begin{pmatrix}\alpha_4\\\alpha_5\\\alpha_6\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{54}$$

for arbitrary faces E. The space of rigid body motions is invariant under translations and rotations. We translate and rotate the domain Ω and therefore also Γ such that A is transformed into the triangle A' with center of mass $\boldsymbol{a}' = \boldsymbol{0}$, B is transformed into the triangle B' with center of mass $\boldsymbol{b}' = (\boldsymbol{b}_1', 0, 0)$ and C is transformed into the triangle

C' with center of mass $\boldsymbol{c}' = (\boldsymbol{c}'_1, \boldsymbol{c}'_2, \boldsymbol{c}'_3)$. Note that $\boldsymbol{b}'_1 \neq 0$ as well as $(\boldsymbol{c}'_2, \boldsymbol{c}'_3) \neq \mathbf{0}$. Considering A' in (54) proves $(\alpha_1, \alpha_2, \alpha_3) = \mathbf{0}$. With B' in (54), we get

$$\begin{pmatrix} 0 & 0 & 0 \\ \boldsymbol{b}'_1 & 0 & 0 \\ 0 & 0 & -\boldsymbol{b}'_1 \end{pmatrix} \begin{pmatrix} \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} = \mathbf{0},$$

and thus $\alpha_4 = 0 = \alpha_6$. Finally, taking C' in (54), we are led to

$$\begin{pmatrix} -\boldsymbol{c}'_2 & 0 & \boldsymbol{c}'_3 \\ \boldsymbol{c}'_1 & -\boldsymbol{c}'_3 & 0 \\ 0 & \boldsymbol{c}'_2 & -\boldsymbol{c}'_1 \end{pmatrix} \begin{pmatrix} 0 \\ \alpha_5 \\ 0 \end{pmatrix} = \mathbf{0}$$

and therefore $\alpha_5 = 0$, since $(\boldsymbol{c}'_2, \boldsymbol{c}'_3) \neq \mathbf{0}$. Altogether, we have shown $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$ in (54). Therefore, the orthogonal projections $\Pi_0(\boldsymbol{r}^j)$, for $j = 1, \ldots, 6 = D$, are linearly independent. Since this is equivalent to (32), we conclude the proof. □

4 Johnson-Nédélec coupling

This section deals with the Johnson-Nédélec coupling, see, e.g., [23,37] for linear Laplace problems and [20,32] for linear elasticity problems. In contrast to [20], we avoid the use of interior Dirichlet boundaries. Moreover, our approach of implicit stabilization avoids an explicit stabilization of the coupling equations as is used in [32]. The derivation of the variational formulation (55) of the Johnson-Nédélec coupling and the proof of equivalence to the model problem (6) are done as for the Laplace problem, see, e.g., [1,19] for the derivation.

4.1 Variational formulation

The Johnson-Nédélec coupling reads as follows: Find $(\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H} = \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{-1/2}(\Gamma)$ such that

$$\langle \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}),\, \boldsymbol{\epsilon}(\boldsymbol{v})\rangle_\Omega - \langle \boldsymbol{\phi},\, \boldsymbol{v}\rangle_\Gamma = \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{\phi}_0,\, \boldsymbol{v}\rangle_\Gamma \tag{55a}$$

$$\langle \boldsymbol{\psi},\, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u} + \boldsymbol{V}\boldsymbol{\phi}\rangle_\Gamma = \langle \boldsymbol{\psi},\, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u}_0\rangle_\Gamma \tag{55b}$$

holds for all $(\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}$. Note that the second equation of the Johnson-Nédélec equations (55) is the same as for the symmetric coupling (27). We define a mapping $b : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and a continuous linear functional $F \in \mathcal{H}^*$ by

$$b((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) := \langle \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}),\, \boldsymbol{\epsilon}(\boldsymbol{v})\rangle_\Omega - \langle \boldsymbol{\phi},\, \boldsymbol{v}\rangle_\Gamma + \langle \boldsymbol{\psi},\, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u} + \boldsymbol{V}\boldsymbol{\phi}\rangle_\Gamma \tag{56}$$

as well as

$$F(\boldsymbol{v}, \boldsymbol{\psi}) := \langle \boldsymbol{f},\ \boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{\phi}_0,\ \boldsymbol{v}\rangle_\Gamma + \langle \boldsymbol{\psi},\ \left(\tfrac{1}{2} - \boldsymbol{K}\right)\boldsymbol{u}_0\rangle_\Gamma \tag{57}$$

for all $(\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}$. Problem (55) can equivalently be stated as follows: Find $(\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H}$ such that

$$b((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) = F(\boldsymbol{v}, \boldsymbol{\psi}) \quad \text{holds for all } (\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}. \tag{58}$$

We infer from (56) that

$$b((\boldsymbol{r}, \boldsymbol{0}), (\boldsymbol{r}, \boldsymbol{0})) - b((\boldsymbol{0}, \boldsymbol{0}), (\boldsymbol{r}, \boldsymbol{0})) = 0 \quad \text{for all } \boldsymbol{r} \in \mathcal{R}_d. \tag{59}$$

Therefore, $b(\cdot, \cdot)$ cannot be strongly monotone, and we proceed as in Sect. 3 to prove well-posedness of (55) and its Galerkin discretization.

4.2 Main result

According to [33], there exists a constant $1/2 \le c_{\boldsymbol{K}} < 1$ such that

$$\| \left(\tfrac{1}{2} + \boldsymbol{K}\right) \boldsymbol{v}\|_{\boldsymbol{V}^{-1}} \le c_{\boldsymbol{K}} \|\boldsymbol{v}\|_{\boldsymbol{V}^{-1}} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^{1/2}(\Gamma), \tag{60}$$

where $\|\boldsymbol{v}\|^2_{\boldsymbol{V}^{-1}} = \langle \boldsymbol{V}^{-1}\boldsymbol{v},\ \boldsymbol{v}\rangle$ denotes an equivalent norm on $\boldsymbol{H}^{1/2}(\Gamma)$ induced by the inverse of the simple-layer integral operator. The following theorem is the main result of this section.

Theorem 7 *Let $c_{\boldsymbol{K}} < 1$ denote the contraction constant (60) of the double-layer integral operator and assume that $2c_{\text{mon}} > c_{\boldsymbol{K}}(3\lambda^{\text{ext}} + 2\mu^{\text{ext}})$. Then, the assertions of Theorem 1 hold for the Johnson-Nédélec coupling accordingly.*

Remark (i) In the linear case $\mathfrak{A} = \boldsymbol{\sigma}^{\text{int}}$, we may also use an estimate from [32] in Step 2 of the proof of Theorem 7 and replace the assumption $2c_{\text{mon}} > c_{\boldsymbol{K}}(3\lambda^{\text{ext}} + 2\mu^{\text{ext}})$ from Theorem 7 with

$$\eta := \min\{\lambda^{\text{int}}/\lambda^{\text{ext}}, \mu^{\text{int}}/\mu^{\text{ext}}\} > \frac{c_{\boldsymbol{K}}}{4}.$$

As can be seen, solvability is guaranteed, if the Lamé constants in the exterior are not too large compared to the Lamé constants in the interior. However, for the symmetric coupling such a restriction is not needed.

(ii) The assumption $2c_{\text{mon}} > c_{\boldsymbol{K}}(3\lambda^{\text{ext}}+2\mu^{\text{ext}})$, is an assumption on the monotonicity constant c_{mon} and the Lamé constants λ^{ext}, μ^{ext} in the exterior domain. As we have seen for the symmetric coupling the assumption $c_{\text{mon}} > 0$ suffices to prove unique solvability. Since the Johnson-Nédélec coupling is equivalent to the model problem, we stress that at least the continuous formulation of the Johnson-Nédélec coupling equations is uniquely solvable. In [26], Of and Steinbach have shown that the discrete Johnson-Nédélec coupling equations may become indefinite (and hence non-elliptic) for special choices of the model parameters. However, the numerical experiments from [1] show at least numerically that the Laplace transmission problem also allows for unique Galerkin solutions in the indefinite regime.

(iii) Assume a nonlinear Hencky-Von Mises stress–strain relation, i.e., the operator from (23), with $\widetilde{\mu}(\cdot) \geq \alpha > 0$ and $\widetilde{\mu}(\cdot) \leq Kd/2 - \beta$ for some $\alpha, \beta > 0$. Then we may replace the assumption $2c_{\text{mon}} > c_{\boldsymbol{K}}(3\lambda^{\text{ext}} + 2\mu^{\text{ext}})$ from Theorem 7 with

$$\eta > \frac{c_{\boldsymbol{K}}}{4},$$

where $\eta := \min\{(K - 2/d \inf_{x\in\mathbb{R}_+} \widetilde{\mu}(x))/\lambda^{\text{ext}}, \inf_{x\in\mathbb{R}_+} \widetilde{\mu}(x)/\mu^{\text{ext}}\}$.

4.3 Auxiliary results

We stress that the results of Sects. 3.2 and 3.3 also apply to the Johnson-Nédélec coupling without further modifications. Additionally, the proof needs some properties of the boundary integral operators and some results from the works [26,32], which are stated in the following. First, we introduce the interior Steklov-Poincaré operator $\boldsymbol{S} : \boldsymbol{H}^{1/2}(\Gamma) \to \boldsymbol{H}^{-1/2}(\Gamma)$ defined by

$$\boldsymbol{S} := \boldsymbol{V}^{-1}\left(\tfrac{1}{2} + \boldsymbol{K}\right),$$

see e.g. [22]. Note that $\boldsymbol{V}$ and $\boldsymbol{K}$ are still defined with respect to the exterior Lamé constants λ^{ext}, μ^{ext}. We use the estimate

$$\|\left(\tfrac{1}{2} + \boldsymbol{K}\right)\boldsymbol{w}\|^2_{\boldsymbol{V}^{-1}} \leq c_{\boldsymbol{K}}\langle \boldsymbol{S}\boldsymbol{w},\, \boldsymbol{w}\rangle_\Gamma \quad \text{for all } \boldsymbol{w} \in \boldsymbol{H}^{1/2}(\Gamma)$$

from [26,32], which involves the contraction constant (60) of the double-layer integral operator $\boldsymbol{K}$. The last estimate yields

$$\begin{aligned}\langle \boldsymbol{\chi},\, \left(\tfrac{1}{2} + \boldsymbol{K}\right)\boldsymbol{w}\rangle_\Gamma &\leq \|\left(\tfrac{1}{2} + \boldsymbol{K}\right)\boldsymbol{w}\|_{\boldsymbol{V}^{-1}}\|\boldsymbol{\chi}\|_{\boldsymbol{V}} \\ &\leq \sqrt{c_{\boldsymbol{K}}\langle \boldsymbol{S}\boldsymbol{w},\, \boldsymbol{w}\rangle_\Gamma}\,\|\boldsymbol{\chi}\|_{\boldsymbol{V}} \quad \text{for all } (\boldsymbol{w}, \boldsymbol{\chi}) \in \mathcal{H}. \end{aligned} \tag{61}$$

For $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$, we next introduce the splitting

$$\boldsymbol{w}^0 := \boldsymbol{w} - \boldsymbol{w}^{\text{D}}, \tag{62}$$

where $\boldsymbol{w}^{\text{D}} \in \boldsymbol{H}^1(\Omega)$ is the unique weak solution of

$$\begin{aligned}\operatorname{div} \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}^{\text{D}}) &= 0 \quad \text{in } \Omega,\\ \boldsymbol{w}^{\text{D}} &= \boldsymbol{w} \quad \text{on } \Gamma.\end{aligned}$$

Then, there holds $\boldsymbol{w}^0|_\Gamma = \boldsymbol{0}$ as well as the orthogonality relation $\langle \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}^{\text{D}}),\ \boldsymbol{\epsilon}(\boldsymbol{w}^0)\rangle_\Omega = 0 = \langle \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}^0),\ \boldsymbol{\epsilon}(\boldsymbol{w}^{\text{D}})\rangle_\Omega$. Consequently, we see

$$\langle \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}),\ \boldsymbol{\epsilon}(\boldsymbol{w})\rangle_\Omega = \langle \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}^{\text{D}}),\ \boldsymbol{\epsilon}(\boldsymbol{w}^{\text{D}})\rangle_\Omega + \langle \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}^0),\ \boldsymbol{\epsilon}(\boldsymbol{w}^0)\rangle_\Omega. \tag{63}$$

Moreover, $\boldsymbol{w}^{\text{D}}$ fulfills $\gamma_1^{\text{int}}\boldsymbol{w}^{\text{D}} = \boldsymbol{S}\boldsymbol{w}^{\text{D}}$. Together with Betti's first formula (10), we infer

$$\langle \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}^{\text{D}}),\ \boldsymbol{\epsilon}(\boldsymbol{w}^{\text{D}})\rangle_\Omega = \langle \gamma_1^{\text{int}}\boldsymbol{w}^{\text{D}},\ \boldsymbol{w}^{\text{D}}\rangle_\Gamma = \langle \boldsymbol{S}\boldsymbol{w}^{\text{D}},\ \boldsymbol{w}^{\text{D}}\rangle_\Gamma. \tag{64}$$

4.4 Proof of Theorem 7

Note that Proposition 3 holds true with $b(\cdot,\cdot)$ resp. $F(\cdot)$ replaced by definition (56) resp. (57). We define the nonlinear operator $\widetilde{\mathfrak{B}} : \mathcal{H} \to \mathcal{H}^*$ by

$$\langle \widetilde{\mathfrak{B}}(\boldsymbol{u}_h, \boldsymbol{\phi}_h),\ (\cdot,\cdot)\rangle := \widetilde{b}((\boldsymbol{u}_h, \boldsymbol{\phi}_h), (\cdot,\cdot)).$$

Step 1 (*Lipschitz continuity of* $\widetilde{\mathfrak{B}}$). Arguing as in (50) in the proof of Theorem 1, we prove Lipschitz continuity of $\widetilde{\mathfrak{B}}$, where the Lipschitz constant $C_{\text{lip}} > 0$ depends only on $\mathfrak{A}$, λ^{ext}, μ^{ext}, and Ω.
Step 2 (*Strong monotonicity of* $\widetilde{\mathfrak{B}}$). We have to prove that, for all $(\boldsymbol{u}_h, \boldsymbol{\phi}_h), (\boldsymbol{v}_h, \boldsymbol{\psi}_h) \in \mathcal{H}$,

$$\langle \widetilde{\mathfrak{B}}(\boldsymbol{u}_h, \boldsymbol{\phi}_h) - \widetilde{\mathfrak{B}}(\boldsymbol{v}_h, \boldsymbol{\psi}_h),\ (\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{\phi}_h - \boldsymbol{\psi}_h)\rangle \geq C_{\text{mon}} \|(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{\phi}_h - \boldsymbol{\psi}_h)\|_{\mathcal{H}}^2. \tag{65}$$

To abbreviate notation, let $(\boldsymbol{w}_h, \boldsymbol{\chi}_h) := (\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{\phi}_h - \boldsymbol{\psi}_h)$. By use of monotonicity (19) of $\mathfrak{A}$, we see

$$\begin{aligned}
&\langle \widetilde{\mathfrak{B}}(\boldsymbol{u}_h, \boldsymbol{\phi}_h) - \widetilde{\mathfrak{B}}(\boldsymbol{v}_h, \boldsymbol{\psi}_h),\ \boldsymbol{w}_h\rangle_\Gamma \\
&\quad = \langle \mathfrak{A}\boldsymbol{u}_h - \mathfrak{A}\boldsymbol{v}_h,\ \boldsymbol{w}_h\rangle_\Omega - \langle \boldsymbol{\chi}_h,\ \boldsymbol{w}_h\rangle_\Gamma + \langle \boldsymbol{\chi}_h,\ \left(\tfrac{1}{2} - \boldsymbol{K}\right)\boldsymbol{w}_h + \boldsymbol{V}\boldsymbol{\chi}_h\rangle_\Gamma \\
&\qquad + \sum_{j=1}^{D} |\langle \boldsymbol{\xi}^j,\ \left(\tfrac{1}{2} - \boldsymbol{K}\right)\boldsymbol{w}_h + \boldsymbol{V}\boldsymbol{\chi}_h\rangle_\Gamma|^2 \\
&\quad \geq c_{\text{mon}} \|\boldsymbol{\epsilon}(\boldsymbol{w}_h)\|_{\boldsymbol{L}^2(\Omega)}^2 - \langle \boldsymbol{\chi}_h,\ \left(\tfrac{1}{2} + \boldsymbol{K}\right)\boldsymbol{w}_h\rangle_\Gamma + \langle \boldsymbol{\chi}_h,\ \boldsymbol{V}\boldsymbol{\chi}_h\rangle_\Gamma \\
&\qquad + \sum_{j=1}^{D} |\langle \boldsymbol{\xi}^j,\ \left(\tfrac{1}{2} - \boldsymbol{K}\right)\boldsymbol{w}_h + \boldsymbol{V}\boldsymbol{\chi}_h\rangle_\Gamma|^2 \\
&\quad =: I_1 - I_2 + I_3 + I_4.
\end{aligned}$$

Next, we use the splitting (62) for $\boldsymbol{w}_h = \boldsymbol{w}^0 + \boldsymbol{w}^{\text{D}}$. Together with (63) and (21), where $C_1 = 6\lambda^{\text{ext}} + 4\mu^{\text{ext}}$, we get

$$
\begin{aligned}
I_1 &\geq \frac{c_{\text{mon}}}{C_1} \langle \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}_h),\ \boldsymbol{\epsilon}(\boldsymbol{w}_h) \rangle_\Omega \\
&= \frac{c_{\text{mon}}}{C_1} \langle \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}^0),\ \boldsymbol{\epsilon}(\boldsymbol{w}^0) \rangle_\Omega + \frac{c_{\text{mon}}}{C_1} \langle \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}^{\text{D}}),\ \boldsymbol{\epsilon}(\boldsymbol{w}^{\text{D}}) \rangle_\Omega \\
&=: I_{11} + I_{12}.
\end{aligned}
$$

Estimate (61) and Young's inequality yield for $\delta > 0$

$$
\begin{aligned}
I_2 = \langle \boldsymbol{\chi}_h,\ \left(\tfrac{1}{2} + \boldsymbol{K}\right) \boldsymbol{w}^{\text{D}} \rangle_\Gamma &\leq \sqrt{c_{\boldsymbol{K}} \langle \boldsymbol{S} \boldsymbol{w}^{\text{D}},\ \boldsymbol{w}^{\text{D}} \rangle_\Gamma}\, \| \boldsymbol{\chi}_h \|_{\boldsymbol{V}} \\
&\leq \frac{\delta}{2} c_{\boldsymbol{K}} \langle \boldsymbol{S} \boldsymbol{w}^{\text{D}},\ \boldsymbol{w}^{\text{D}} \rangle_\Gamma + \frac{\delta^{-1}}{2} \| \boldsymbol{\chi}_h \|_{\boldsymbol{V}}^2.
\end{aligned}
$$

With the last inequality and (64), we get

$$
I_2 \leq \frac{\delta}{2} c_{\boldsymbol{K}} \langle \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}^{\text{D}}),\ \boldsymbol{\epsilon}(\boldsymbol{w}^{\text{D}}) \rangle_\Omega + \frac{\delta^{-1}}{2} \| \boldsymbol{\chi}_h \|_{\boldsymbol{V}}^2.
$$

Now, we can further estimate the terms $I_1 - I_2 + I_3$ by

$$
\begin{aligned}
I_1 - I_2 + I_3 &\geq I_{11} + \left(\frac{c_{\text{mon}}}{C_1} - \frac{\delta}{2} c_{\boldsymbol{K}} \right) \langle \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}^{\text{D}}),\ \boldsymbol{\epsilon}(\boldsymbol{w}^{\text{D}}) \rangle_\Omega + \left(1 - \frac{\delta^{-1}}{2} \right) \| \boldsymbol{\chi}_h \|_{\boldsymbol{V}}^2 \\
&\geq \left(\frac{c_{\text{mon}}}{C_1} - \frac{\delta}{2} c_{\boldsymbol{K}} \right) \langle \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}_h),\ \boldsymbol{\epsilon}(\boldsymbol{w}_h) \rangle_\Omega + \left(1 - \frac{\delta^{-1}}{2} \right) \langle \boldsymbol{\chi}_h,\ \boldsymbol{V} \boldsymbol{\chi}_h \rangle_\Gamma,
\end{aligned}
$$

where we used (63) again. The assumption $2c_{\text{mon}} > c_{\boldsymbol{K}}(3\lambda^{\text{ext}} + 2\mu^{\text{ext}})$ is equivalent to $c_{\text{mon}}/C_1 > c_{\boldsymbol{K}}/4$ with $C_1 = 6\lambda^{\text{ext}} + 4\mu^{\text{ext}}$. Therefore, there exists $\delta > 0$ such that $C := \min\{c_{\text{mon}}/C_1 - c_{\boldsymbol{K}}\delta/2,\, 1 - \delta^{-1}/2\} > 0$. Together with (63) and (22), we infer

$$
\begin{aligned}
I_1 - I_2 + I_3 + I_4 &\geq C \Big(\langle \boldsymbol{\sigma}^{\text{ext}}(\boldsymbol{w}_h),\ \boldsymbol{\epsilon}(\boldsymbol{w}_h) \rangle_\Omega + \langle \boldsymbol{\chi}_h,\ \boldsymbol{V} \boldsymbol{\chi}_h \rangle_\Gamma \\
&\qquad + \sum_{j=1}^{D} |\langle \boldsymbol{\xi}^j,\ \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{w}_h + \boldsymbol{V} \boldsymbol{\chi}_h \rangle_\Gamma|^2 \Big) \\
&\geq \widetilde{C} \Big(\| \boldsymbol{\epsilon}(\boldsymbol{w}_h) \|_{\boldsymbol{L}^2(\Omega)}^2 + \langle \boldsymbol{\chi}_h,\ \boldsymbol{V} \boldsymbol{\chi}_h \rangle_\Gamma \\
&\qquad + \sum_{j=1}^{D} |\langle \boldsymbol{\xi}^j,\ \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{w}_h + \boldsymbol{V} \boldsymbol{\chi}_h \rangle_\Gamma|^2 \Big) \\
&= \widetilde{C} |||(\boldsymbol{w}_h, \boldsymbol{\chi}_h)|||^2 \geq \widetilde{C} C_{\text{norm}}^{-1} \| (\boldsymbol{w}_h, \boldsymbol{\chi}_h) \|_{\mathcal{H}}^2,
\end{aligned}
$$

where $\widetilde{C} = C \min\{1, C_2\}$. The constant $C_{\text{mon}} := \widetilde{C} C_{\text{norm}}^{-1} > 0$ depends only on $\Omega, \mathfrak{A}, \mathcal{Y}_0$, and on the Lamé constants $\lambda^{\text{ext}}, \mu^{\text{ext}}$.

Step 3 (*Unique solvability and Céa lemma*). This step is essentially the same as Step 3 in the proof of Theorem 1. We thus omit the details. □

5 One-equation and symmetric Bielak-MacCamy coupling

In this section, we investigate the non-symmetric Bielak-MacCamy one-equation coupling, see e.g. [1,4,9] for the Laplace problem, as well as its symmetric variant. The derivation of the variational formulation (66) as well as the proof of equivalence to the model problem (6) essentially follow as for the Johnson-Nédélec coupling resp. symmetric coupling, cf. e.g. [1,10,19]. For brevity, we only sketch the results and proofs and leave the details to the reader.

5.1 One-equation coupling

The variational formulation of the Bielak-MacCamy coupling reads as follows: Find $(\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H} = \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{-1/2}(\Gamma)$ such that

$$\langle \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}),\, \boldsymbol{\epsilon}(\boldsymbol{v})\rangle_\Omega + \langle \big(\tfrac{1}{2} - \boldsymbol{K}'\big)\,\boldsymbol{\phi},\, \boldsymbol{v}\rangle_\Gamma = \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{\phi}_0,\, \boldsymbol{v}\rangle_\Gamma \tag{66a}$$

$$\langle \boldsymbol{\psi},\, \boldsymbol{V}\boldsymbol{\phi} - \boldsymbol{u}\rangle_\Gamma = -\langle \boldsymbol{\psi},\, \boldsymbol{u}_0\rangle_\Gamma \tag{66b}$$

holds for all $(\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}$. We sum up the left-hand side and the right-hand side of (66) and define the mapping $b : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ as well as the linear functional $F \in \mathcal{H}^*$ by

$$b((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) := \langle \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}),\, \boldsymbol{\epsilon}(\boldsymbol{v})\rangle_\Omega + \langle \big(\tfrac{1}{2} - \boldsymbol{K}'\big)\,\boldsymbol{\phi},\, \boldsymbol{v}\rangle_\Gamma + \langle \boldsymbol{\psi},\, \boldsymbol{V}\boldsymbol{\phi} - \boldsymbol{u}\rangle_\Gamma \tag{67}$$

as well as

$$F(\boldsymbol{v}, \boldsymbol{\psi}) := \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega + \langle \boldsymbol{\phi}_0,\, \boldsymbol{v}\rangle_\Gamma - \langle \boldsymbol{\psi},\, \boldsymbol{u}_0\rangle_\Gamma \tag{68}$$

for all $(\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}$. Then, problem (66) can equivalently be stated as follows: Find $(\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H}$ such that

$$b((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) = F(\boldsymbol{v}, \boldsymbol{\psi}) \quad \text{holds for all } (\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}. \tag{69}$$

As for the other coupling formulations $b(\cdot, \cdot)$ is not strongly monotone, and unique solvability cannot be shown directly. We follow the ideas of Sect. 3 resp. Sect. 4 to overcome these difficulties. Moreover, with $b_{\mathrm{JN}}(\cdot, \cdot)$ denoting the mapping defined in (56), we stress that

$$b_{\mathrm{JN}}((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{u}, \boldsymbol{\phi})) = b((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{u}, \boldsymbol{\phi})) \quad \text{for all } (\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H}. \tag{70}$$

Thus, there is a strong relation between the one-equation Bielak-MacCamy and Johnson-Nédélec coupling.

As in Sect. 3.2, we use (66b) to stabilize $b(\cdot, \cdot)$ and to tackle the rigid body motions in the interior domain Ω. We note that Proposition 3 holds with $\widetilde{b}(\cdot, \cdot)$ resp. $\widetilde{F}(\cdot)$ replaced by

$$\widetilde{b}((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) := b((\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi})) + \sum_{j=1}^{D} \langle \boldsymbol{\xi}^j, \boldsymbol{V\phi} - \boldsymbol{u} \rangle_\Gamma \langle \boldsymbol{\xi}, \boldsymbol{V v} - \boldsymbol{\psi} \rangle_\Gamma, \quad (71)$$

$$\widetilde{F}(\boldsymbol{v}, \boldsymbol{\psi}) := F(\boldsymbol{v}, \boldsymbol{\psi}) - \sum_{j=1}^{D} \langle \boldsymbol{\xi}^j, \boldsymbol{u}_0 \rangle_\Gamma \langle \boldsymbol{\xi}, \boldsymbol{V v} - \boldsymbol{\psi} \rangle_\Gamma \quad (72)$$

for all $(\boldsymbol{u}, \boldsymbol{\phi}), (\boldsymbol{v}, \boldsymbol{\psi}) \in \mathcal{H}$. Furthermore, the assertions of Proposition 5 also hold true if (45) is replaced by

$$g_j(\boldsymbol{u}, \boldsymbol{\phi}) := \langle \boldsymbol{\xi}, \boldsymbol{V\phi} - \boldsymbol{u} \rangle_\Gamma \quad \text{for } (\boldsymbol{u}, \boldsymbol{\phi}) \in \mathcal{H}. \quad (73)$$

With these observations, Theorem 7 transfers to the Bielak-MacCamy coupling. Details are left to the reader.

5.2 Combination of the Johnson-Nédélec and Bielak-MacCamy coupling

For a fixed parameter $\delta \in (0, 1)$ we consider the convex combination of (55) and (66), see also [4,20], which yields to the variational formulation: Find $(\boldsymbol{u}, \boldsymbol{\phi}, \boldsymbol{\chi}) \in \mathcal{H} := \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^{-1/2}(\Gamma) \times \boldsymbol{H}^{-1/2}(\Gamma)$ such that

$$\langle \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{v}) \rangle_\Omega - \delta \langle \boldsymbol{\phi}, \boldsymbol{v} \rangle_\Gamma - (1-\delta) \langle \left(\tfrac{1}{2} - \boldsymbol{K}'\right) \boldsymbol{\chi}, \boldsymbol{v} \rangle_\Gamma = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega + \langle \boldsymbol{\phi}_0, \boldsymbol{v} \rangle_\Gamma, \quad (74a)$$

$$\delta \langle \boldsymbol{\psi}, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u} + \boldsymbol{V\phi} \rangle_\Gamma = \delta \langle \boldsymbol{\psi}, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u}_0 \rangle_\Gamma, \quad (74b)$$

$$(1-\delta) \langle \boldsymbol{\omega}, \boldsymbol{V\chi} - \boldsymbol{u} \rangle_\Gamma = -(1-\delta) \langle \boldsymbol{\omega}, \boldsymbol{u}_0 \rangle_\Gamma \quad (74c)$$

for all $(\boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\omega}) \in \mathcal{H}$. For $\delta = \frac{1}{2}$ the coupling scheme (74) is also called (quasi-) symmetric Bielak-MacCamy coupling method. We define the mapping $b : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ for all $(\boldsymbol{u}, \boldsymbol{\phi}, \boldsymbol{\chi}), (\boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\omega}) \in \mathcal{H}$ by

$$\begin{aligned} b((\boldsymbol{u}, \boldsymbol{\phi}, \boldsymbol{\chi}), (\boldsymbol{v}, \boldsymbol{\psi}, \boldsymbol{\omega})) = {} & \langle \mathfrak{A}\boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{v}) \rangle_\Omega - \delta \langle \boldsymbol{\phi}, \boldsymbol{v} \rangle_\Gamma - (1-\delta) \langle \left(\tfrac{1}{2} - \boldsymbol{K}'\right) \boldsymbol{\chi}, \boldsymbol{v} \rangle_\Gamma \\ & + \delta \langle \boldsymbol{\psi}, \left(\tfrac{1}{2} - \boldsymbol{K}\right) \boldsymbol{u} + \boldsymbol{V\phi} \rangle_\Gamma + (1-\delta) \langle \boldsymbol{\omega}, \boldsymbol{V\chi} - \boldsymbol{u} \rangle_\Gamma. \end{aligned} \quad (75)$$

Due to the rigid body motions, the mapping $b(\cdot, \cdot)$ is not strongly monotone. As for the symmetric coupling, the Johnson-Nédélec coupling and the one-equation Bielak-MacCamy coupling, the mapping $b(\cdot, \cdot)$ can be stabilized—either by using equation (74b), equation (74c) or both of them. This yields to a modified mapping $\widetilde{b}(\cdot, \cdot)$ which is strongly monotone. The same techniques as above provide an equivalent formulation and unique solvability thereof. Thus, Theorem 7 holds true for the combined coupling (74) with any fixed $\delta \in (0, 1)$.

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