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Optimal L^2, H^1 and L^∞ analysis of finite volume methods for the stationary Navier–Stokes equations with large data

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Abstract Previous work on the stability and convergence analysis of numerical methods for the stationary Navier–Stokes equations was carried out under the uniqueness condition of the solution, which required that the data be small enough in certain norms. In this paper an optimal analysis for the finite volume methods is performed for the stationary Navier–Stokes equations, which relaxes the solution uniqueness condition and thus the data requirement. In particular, optimal order error estimates in the H^1-norm for velocity and the L^2-norm for pressure are obtained with large data, and a new residual technique for the stationary Navier–Stokes equations is introduced for the first time to obtain a convergence rate of optimal order in the L^2-norm for the velocity. In addition, after proving a number of additional technical lemmas including weighted L^2-norm estimates for regularized Green's functions associated with the Stokes problem, optimal error estimates in the L^∞-norm are derived for the first time

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for the velocity gradient and pressure without a logarithmic factor $O(|\log h|)$ for the stationary Naiver–Stokes equations.

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1 Introduction

According to the definition of a finite volume method, volume integrals for a partial differential equation that contains a divergence term are converted into surface integrals by using the divergence theorem. These terms are then approximated by numerical fluxes at the surface of each finite volume. Because the flux entering into a volume is identical to that leaving an adjacent volume sharing a common face, this method is conservative. In addition, it can easily be formulated to allow for use of unstructured meshes to deal with complicated geometries. The method lies somewhere between the finite element and finite difference methods; it has a flexibility similar to that of the finite element method for handling complicated geometries, and its implementation is comparable to that of the finite difference method. The finite volume method is also referred to as the control volume method, the covolume method, or the first-order generalized difference method in the literature [2,6,9,13,15,20,26]. Comparatively, its theoretical analysis is the hardest among all three methods.

Although there are some results of finite volume methods for the Stokes equations [7,8,14,21,31,32], an analysis for these methods for the Navier–Stokes equations is lacking. In particular, there is a difficulty in handling the nonlinear discrete terms of the Navier–Stokes equations because these terms lack skew-symmetry in the context of a Petrov-Galerkin method which uses different trial and test functions in different finite dimensional spaces. Hence an analysis for these equations must take special care of the nonlinear discrete terms arising from the finite volume discretization. Furthermore, previous work on the stability and convergence analysis of the finite volume methods for the Navier–Stokes equations was carried out under the uniqueness condition of the solution, which required that the data be small enough in certain norms [23].

In this paper we perform a stability and convergence analysis for a finite volume method for the stationary Navier–Stokes equations without relying on the unique solution condition. Optimal order error estimates in the H^1-norm for velocity and the L^2-norm for pressure are obtained. The analysis depends on an abstract theory of Brezzi et al. [4] and Girault and Raviart [17] for a branch of nonsingular solutions for these equations, which overcomes the uniqueness condition with small data.

There is still no result available in the literature on a convergence rate of optimal order for the finite volume velocity in the L^2-norm for the stationary Navier–Stokes equations. In this paper a new duality argument by using a residual technique is introduced to establish this optimal convergence rate under the same assumptions as for the Navier–Stokes equations [23]. For the first time, the convergence analysis also shows an important superconvergence result between the conforming mixed finite element solution and the finite volume solution using the same finite element pair for these equations with large data.

Furthermore, the derivation of error estimates in the L^∞-norm is another difficult task for the analysis of the finite volume method (even the finite element method) for the Stokes equations. Estimates in this norm were obtained in the literature. However, these estimates bear a logarithmic factor $O(|\log h|)$ [10], where h is a grid size. The technique in this paper in removing this factor relies on new weighted L^2-norm estimates for regularized Green's functions for the finite element method [16] and the relationship between the finite element method and the finite volume method for the Stokes problem [21,23,24,32]. A stability and optimal analysis in the L^∞-norm is carried out for the velocity gradient and pressure for the stationary Navier–Stokes equations without relying on the solution unique condition.

This paper is organized as follows: in the next section, we introduce notation and the stationary Navier–Stokes equations. Then, in the third section, some useful results of the finite element and finite volume methods for the stationary Navier–Stokes equations are recalled. In the fourth section, stability and estimates in the L^2- and H^1-norm for velocity and the L^2-norm for pressure of a branch of nonsingular solutions for the finite volume methods are obtained. Finally, the L^∞-norm analysis for the velocity gradient and pressure is given in the fifth section.

2 Preliminaries

Let Ω be a bounded domain in $\Re^2$, assumed to have a Lipschitz continuous boundary Γ and to satisfy a further condition stated in (**A1**) below. The stationary Navier–Stokes equations are

$$-\Delta u + \lambda \nabla p = \lambda \left(f - (u \cdot \nabla) u - \frac{1}{2} (\operatorname{div} u) u \right), \quad \text{in } \Omega, \tag{2.1a}$$

$$\operatorname{div} u = 0, \quad \text{in } \Omega, \tag{2.1b}$$

$$u|_\Gamma = 0, \quad \text{on } \Gamma, \tag{2.1c}$$

where $u = (u_1(x), u_2(x))$ represents the velocity vector, $p = p(x)$ the pressure, $f = f(x)$ the prescribed body force, $\lambda = \nu^{-1}$, and $\nu > 0$ the viscosity. The consistent term $(\operatorname{div} u)u/2 = 0$ is added to ensure the dissipativity of the Navier–Stokes equations [30].

The Sobolev spaces to be used are collected:

$$X = [H_0^1(\Omega)]^2, M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \right\}, Z = [L^{3/2}(\Omega)]^2,$$

$$\bar{X} = X \times M, Y = [H^{-1}(\Omega)]^2, \quad V = \{v \in X : \operatorname{div} v = 0\},$$

$$H = \left\{ v \in [L^2(\Omega)]^2 : \operatorname{div} v = 0 \right\}, \quad D(A) = [H^2(\Omega)]^2 \cap V,$$

where the Stokes operator $A : D(A) \to H$ is defined by $A = -P\Delta$ and $P : [L^2(\Omega)]^2 \to H$ is the standard L^2-orthogonal projection. The spaces $[L^2(\Omega)]^m$, $m = 1, 2$, or 4, are endowed with the L^2-scalar product $(\cdot, \cdot)$ and the L^2-norm $\|\cdot\|_{L^2}$, as appropriate. In addition, $\|\cdot\|_{L^r}$, $1 \le r \le \infty$, denotes the norm of the space $L^r(\Omega)$. The space X is equipped with the usual scalar product $(\nabla u, \nabla v)$ and the norm $\|u\|_{H^1}$ (or equivalently $\|\nabla u\|_{L^2}$), $u, v \in X$. In particular, define the norm on $\bar{X}$:

$$|||(v, q)||| = (\|\nabla v\|_{L^2}^2 + \lambda^2 \|q\|_0^2)^{1/2}, \ (v, q) \in \bar{X}.$$

In this paper standard definitions are used for the Sobolev spaces $W^{m,r}(\Omega)$ [1], with the norm $\|\cdot\|_{W^{m,r}}$ and the seminorm $|\cdot|_{W^{m,r}}$, $m, r \ge 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_{H^m}$ for $\|\cdot\|_{W^{m,2}(\Omega)}$.

A linear operator $T : Y \to \bar{X}$ is defined as follows: Given $g \in Y$, the solution of the Stokes problem

$$\begin{aligned} -\Delta v + \lambda \nabla q &= g, \quad \text{in } \Omega, \\ \operatorname{div} v &= 0, \quad \text{in } \Omega, \\ v|_\Gamma &= 0, \quad \text{on } \Gamma, \end{aligned}$$

is denoted by $\tilde{v}(\lambda) = (v, \lambda q) = Tg \in \bar{X}$. Furthermore, a C^2-mapping $G : R^+ \times \bar{X} \to Y$ is defined by

$$G(\lambda, \tilde{v}(\lambda)) = \lambda \left((v \cdot \nabla) v + \frac{1}{2} (\operatorname{div} v) v - f \right).$$

Finally, we define

$$F(\lambda, \tilde{v}(\lambda)) = \tilde{v}(\lambda) + TG(\lambda, \tilde{v}(\lambda)), \quad \lambda \in R^+, \ \tilde{v}(\lambda) \in \bar{X}.$$

In this section, a branch of nonsingular solutions of the stationary Navier–Stokes equations, as introduced in [4, 17], are studied. Let Λ be a compact interval in R^+; $\{(\lambda, \tilde{u}(\lambda))\}$, with $\tilde{u}(\lambda) = (u, \lambda p)$, is a branch of nonsingular solutions to the equation

$$F(\lambda, \tilde{u}(\lambda)) = 0, \tag{2.2}$$

if $D_u F(\lambda, \tilde{u}(\lambda))$ is an isomorphism from $\bar{X}$ onto Y for all $\lambda \in \Lambda$.

As mentioned above, a further assumption on Ω is needed:

Assumption (A1) Assume that Ω is regular in the sense that the unique solution $\tilde{v}(\lambda) = (v, \lambda q) = Tg \in \bar{X}$ of the stationary Stokes problem for a prescribed $g \in [L^r(\Omega)]^2$ exists and satisfies

$$\|v\|_{W^{2,r}} + \lambda \|q\|_{W^{1,r}} \le C \|g\|_{L^r},$$

where $C > 0$ is a constant depending on Ω. Here and later, $C_0, \ C_1, \ldots$ are positive constants depending only on the data (λ, Ω, f).

Obviously, the validity of assumption (**A1**) is known if Γ is of C^2 or if Ω is a two-dimensional convex polygon. In addition, it is well known [1] that there holds the following inequalities

$$\|v\|_{L^4} \leq C_0\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{1/2}, \quad \|v\|_{L^2} \leq C_1\|\nabla v\|_{L^2} \quad \forall v \in X, \tag{2.3}$$

$$\|v\|_{L^\infty} \leq C_2\|v\|_{L^2}^{1/2}\|Av\|_{L^2}^{1/2} \quad \forall v \in X \cap \left[H^2(\Omega)\right]^2. \tag{2.4}$$

Using integration by parts, the weak formulation of the stationary Navier–Stokes equations (2.1) is: Find $(u, p) \in \bar{X}$ such that

$$a(u, v) - \lambda d(v, p) + \lambda d(u, q) + \lambda b(u, u, v) = \lambda(f, v) \quad \forall (v, q) \in \bar{X}, \tag{2.5}$$

where the bilinear forms a and d are defined as follows:

$$a(u, v) = (\nabla u, \nabla v) \quad \forall u, v \in X,$$
$$d(v, q) = (\operatorname{div} v, q) \quad \forall (v, q) \in \bar{X}.$$

Obviously, the bilinear form $a(\cdot, \cdot)$ is continuous and coercive on the space pair $X \times X$; the bilinear form $d(\cdot, \cdot)$ is continuous and satisfies the *inf-sup* condition: There exists a positive constant $\beta_1 > 0$ such that, for all $q \in M$,

$$\sup_{v \in X} \frac{d(v, q)}{\|\nabla v\|_{L^2}} \geq \beta_1\|q\|_{L^2}. \tag{2.6}$$

The trilinear form $b(\cdot, \cdot, \cdot)$ is continuous on the space triplet $X \times X \times X$

$$b(u, v, w) = ((u \cdot \nabla)v, w) + \frac{1}{2}((\operatorname{div} u)v, w)$$
$$= \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}((u \cdot \nabla)w, v) \quad \forall u, v, w \in X,$$

and satisfies

$$b(u, v, w) = -b(u, w, v) \quad \forall u,\ v,\ w \in X, \tag{2.7a}$$

$$|b(u, v, w)| \leq C_3\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}\|\nabla w\|_{L^2} \quad \forall u,\ v,\ w \in X, \tag{2.7b}$$

$$|b(u, v, w)| + |b(v, u, w)| + |b(w, u, v)| \leq C_3\|\nabla u\|_{L^2}\|\nabla v\|_{L^2}^{1/2}\|Av\|_{L^2}^{1/2}\|w\|_{L^2}\ \forall u \in X,\ v \in D(A),\ w \in [L^2(\Omega)]^2. \tag{2.7c}$$

Furthermore, the existence and uniqueness results of (2.5) can be referred in [17,18].

Lemma 2.1 *[17,18] If λ satisfies the following uniqueness condition:*

$$\lambda < \lambda_0 = \frac{1}{\sqrt{C_3\|f\|_{-1}}},$$

then (2.5) admits a unique solution (u, p). Moreover, the pair $(u, p) \in \bar{X}$ is a solution of the problem (2.5) if and only if $\tilde{u}(\lambda) \in \bar{X}$ is a solution of (2.2).

Similarly, we can apply the same approach as in [18] to obtain the following stability of (2.5):

Lemma 2.2 *Assume that* (**A1**) *holds, $f \in [L^2(\Omega)]^2$, and the pair $\tilde{u}(\lambda) = (u, \lambda p) \in \bar{X}$ is a solution of problem* (2.2). *Then $\tilde{u}(\lambda) \in D(A) \times [H^1(\Omega) \cap M]$ and $G(\lambda, \tilde{u}(\lambda)) \in Y$ satisfy*

$$\|u\|_{H^2} + \lambda\|p\|_{H^1} \leq C_4. \tag{2.8}$$

3 Finite element and finite volume methods

Let K_h be a regular, quasi-uniform triangulation of the polygonal domain Ω into a union of triangles [5,10]. Associated with K_h, we consider the finite element spaces for the velocity and pressure: $X_h \subset X$ and $M_h \subset M$.

Let I_h and J_h be two interpolation operators from $X \cap [C^0(\bar{\Omega})]^2$ and M into X_h and M_h, respectively, such that, for $v \in X \cap [H^2(\Omega)]^2$ and $q \in H^1(\Omega) \cap M$,

$$\|v - I_h v\|_{L^r} + h\|\nabla(v - I_h v)\|_{L^r} \leq Ch^2|v|_{W^{2,r}}, \tag{3.1}$$

$$\|q - J_h q\|_{L^r} \leq Ch|q|_{W^{1,r}}, \qquad 1 \leq r \leq \infty. \tag{3.2}$$

In particular,

$$\|\nabla I_h v_h\|_{L^2} \leq C\|\nabla v\|_{L^2}, v_h \in X_h. \tag{3.3}$$

Due to the quasi-uniformness of the triangulation K_h, the following properties hold [10,30]:

$$\|\nabla v_h\|_{L^2} \leq C_5 h^{-1}\|v_h\|_{L^2}, \quad \|v_h\|_{L^\infty} \leq C_6|\log h|^{1/2}\|v_h\|_{H^1} \quad \forall v_h \in X_h. \tag{3.4}$$

Usually, we assume that the finite element spaces satisfy the discrete *inf-sup* condition:

$$\sup_{v_h \in X_h} \frac{d(v_h, q_h)}{\|\nabla v_h\|_{L^2}} \geq \beta_2\|q_h\|_{L^2}, \tag{3.5}$$

where the constant $\beta_2 > 0$ is independent of h. However, there still are some attractive finite element pairs. Examples of the spaces that satisfy these assumptions include the following [5,10,29]:

$$X_h = \left\{v_h \in [C^0(\bar{\Omega})]^2 \cap X : v_h|_K \in [P_1(K)]^2 \quad \forall K \in K_h\right\},$$

$$M_h = \{q_h \in C^0(\bar{\Omega}) \cap M : q_h|_K \in P_i(K), i = 0, 1 \quad \forall K \in K_h\},$$

where $P_i(K), i = 0, 1$ represents piecewise linear (constant) subspace on set K. We note that neither of these methods are stable in the standard Babuska-Brezzi sense since there are more discrete incompressibility constraints than velocity degrees of freedom. A technical "macroelement condition" [29] is applied to verify the classical Babuska-Brezzi inequality. Namely, a way [27] is to approximate the P_2 velocity field defined on a macro-element mesh obtained by refining K_h uniformly to obtain the mesh $K_{h/2}$. Furthermore, the stability, and optimal order of convergence, of several known mixed finite element methods are easily valid. These two pairs are stable and this method is called iso$P_2 - P_i, i = 0, 1$ method.

Accordingly, set $\bar{X}_h \equiv X_h \times M_h$. Then, a bilinear form on $\bar{X}_h \times \bar{X}_h$ for the finite element method introduced in [27,29] is defined by

$$\begin{aligned}\mathcal{B}_h((\bar{u}_h, \lambda \bar{p}_h), (v_h, \lambda q_h)) = a(\bar{u}_h, v_h) - \lambda d(v_h, \bar{p}_h) \\ + \lambda d(\bar{u}_h, q_h) \forall (\bar{u}_h, \bar{p}_h), (v_h, q_h) \in \bar{X}_h.\end{aligned} \tag{3.6}$$

This bilinear form satisfies the continuity and weak coercivity properties [30]:

$$|\mathcal{B}_h((\bar{u}_h, \lambda \bar{p}_h), (v_h, \lambda q_h))| \le C |||(\bar{u}_h, \bar{p}_h)||| \, |||(v_h, q_h)|||, \tag{3.7a}$$

$$\sup_{(v_h, q_h) \in \bar{X}_h} \frac{|\mathcal{B}_h((u_h, \lambda p_h), (v_h, \lambda q_h))|}{|||(v_h, q_h)|||} \ge \beta_3 |||(\bar{u}_h, \bar{p}_h)|||, \tag{3.7b}$$

where the constant $\beta_3 > 0$ is independent of h.

Using the above notation, the corresponding finite element formulation of system (2.1) reads: Find $(\bar{u}_h, \bar{p}_h) \in \bar{X}_h$, such that, for all $(v_h, q_h) \in \bar{X}_h$,

$$\mathcal{B}_h((\bar{u}_h, \lambda \bar{p}_h), (v_h, \lambda q_h)) + \lambda b(\bar{u}_h, \bar{u}_h, v_h) = \lambda (f, v_h); \tag{3.8a}$$

i.e.,

$$F(\lambda, \bar{\bar{u}}_h(\lambda)) \equiv \bar{\bar{u}}_h(\lambda) + T_h G(\lambda, \bar{\bar{u}}_h(\lambda)) = 0, \tag{3.8b}$$

where T_h is the discrete counterpart of the operator T.

In the coming purpose, the finite volume methods are developed and presented. Let N_h be the set containing all the interior nodes associated with the triangulation K_h, and N be the total number of the nodes. To define the finite volume method, a dual mesh $\tilde{K_h}$ is introduced based on K_h; the elements in $\tilde{K_h}$ are called control volumes. The dual mesh can be constructed by the following rule: For each element $K \in K_h$ with vertices P_j, $j = 1, 2, \ldots N$, select its barycenter Q_j and the midpoint M_j on each of the edges of K, and construct the control volumes in $\tilde{K_h}$ by connecting Q_j to M_j as shown in Fig. 1.

The dual finite element space is defined by

$$\tilde{X_h} = \left\{ v \in [L^2(\Omega)]^2 : v|_{\tilde{K}} \in [P_0(\tilde{K})]^2 \quad \forall \tilde{K} \in \tilde{K_h};\ v|_{\partial \tilde{K}} = 0 \right\},$$

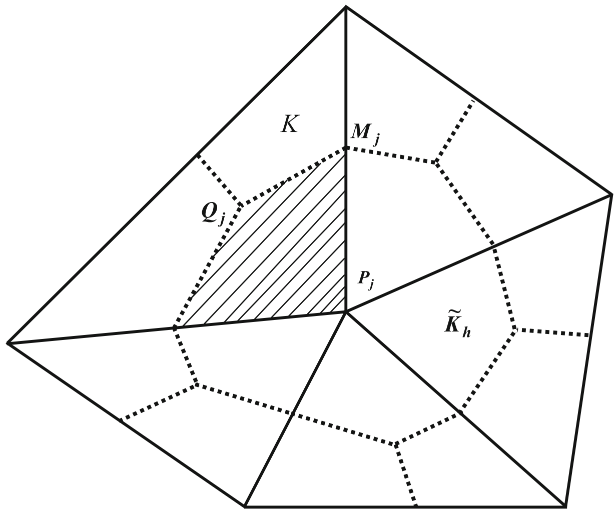

Fig. 1 Control volumes associated with triangles

which has the same dimensions as the finite element space X_h. Furthermore, there exists an invertible linear mapping $\Gamma_h : X_h \to \tilde{X}_h$ such that

$$\Gamma_h v_h(x) = \sum_{j=1}^{N} v_h(P_j)\chi_j(x), \quad x \in \Omega,\ v_h \in X_h, \tag{3.9}$$

where

$$v_h(x) = \sum_{j=1}^{N} v_h(P_j)\phi_j(x), \quad x \in \Omega,\ v_h \in X_h,$$

and $\{\phi_j\}$ and $\{\chi_j\}$ denote the bases of the finite element space X_h and finite volume space $\tilde{X}_h$. The latter are the characteristic functions associated with the dual partition $\tilde{K}_h$:

$$\chi_j(x) = \begin{cases} 1 & \text{if } x \in \tilde{K}_j \in \tilde{K}_h, \\ 0 & \text{otherwise.} \end{cases}$$

The above idea of connecting the trial and test spaces in the Petrov-Galerkin method through the mapping Γ_h was first introduced in [25] in the context of elliptic problems. Furthermore, the mapping Γ_h satisfies the following properties [25]:

Lemma 3.1 *Let $K \in K_h$. If $v_h \in X_h$ and $1 \le r \le \infty$, then*

$$\int_K (v_h - \Gamma_h v_h)dx = 0, \tag{3.10}$$

$$\|v_h - \Gamma_h v_h\|_{L^r(K)} \le C_7 h_K \|v_h\|_{W^{1,r}(K)}, \ \|\Gamma_h v_h\|_{L^2} \le C_8 \|v_h\|_{L^2}, \quad (3.11)$$

where h_K is the diameter of the element K.

To obtain the finite volume formulation of system (2.1), we multiply equation (2.1a) by $\Gamma_h v_h \in \tilde{X}_h$ and integrate over the dual elements $\tilde{K} \in \tilde{K}_h$, multiply equation (2.1b) by $q_h \in M_h$ and integrate over the primal elements $K \in K_h$, and then apply Green's formula for both equations to yield the following bilinear forms:

$$A(u_h, \Gamma_h v_h) = -\sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial \tilde{K}_j} \frac{\partial u_h}{\partial \vec{n}} ds, \quad u_h, \ v_h \in X_h,$$

$$D(\Gamma_h v_h, p_h) = -\sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial \tilde{K}_j} p_h \vec{n}\, ds, \quad p_h \in M_h,$$

$$(f, \Gamma_h v_h) = \sum_{j=1}^{N} v_h(P_j) \cdot \int_{\tilde{K}_j} f\, dx, \quad v_h \in X_h,$$

where $\vec{n}$ is the unit normal outward to $\partial \tilde{K}_j$. Using a technique similar to that for the trilinear form of the finite element method in the previous section, we define the trilinear form $b(\cdot;\cdot,\cdot):\ X_h \times X_h \times \tilde{X}_h \to \Re$ of the finite volume method:

$$b(u_h, v_h, \Gamma_h w_h) = \left((u_h \cdot \nabla) v_h + \frac{1}{2}(\mathrm{div}\, u_h) v_h, \Gamma_h w_h\right) \quad \forall u_h, v_h, w_h \in X_h.$$

Note that the definition of $b(\cdot,\cdot,\cdot)$ of the finite volume method remains consistent with the continuous case. A fundamental difference between it and that of the finite element method lies in the test and trial functions defined in two different spaces. As noted, the difficulty in the finite volume method is that the trilinear term no longer satisfies the useful skew-symmetry property in the context of the Petrov-Galerkin method. Thus the stability and error estimate analysis of this method is more difficult than that of the finite element method for the stationary Navier–Stokes equations.

Now, the finite volume variational formulation for the stationary Navier–Stokes equations (2.1) is: find $\tilde{u}_h(\lambda) = (u_h, \lambda p_h) \in \bar{X}_h \subset \bar{X}$ such that

$$F_h(\lambda, \tilde{u}_h(\lambda)) \equiv \tilde{u}_h(\lambda) + T_h G(\lambda, \tilde{u}_h(\lambda)) = 0; \quad (3.12a)$$

i.e.,

$$\mathcal{C}_h((u_h, \lambda p_h), (v_h, \lambda q_h)) + \lambda b(u_h, u_h, \Gamma_h v_h) = \lambda (f, \Gamma_h v_h) \quad \forall (v_h, q_h) \in \bar{X}_h, \quad (3.12b)$$

where the bilinear form $\mathcal{C}_h(\cdot,\cdot)$ on $\bar{X}_h \times \bar{X}_h$ is

$$\mathcal{C}_h((u_h, \lambda p_h), (v_h, \lambda q_h)) = A(u_h, \Gamma_h v_h) + \lambda D(\Gamma_h v_h, p_h) + \lambda d(u_h, q_h). \quad (3.13)$$

The following results can be found in [9,21,32]:

Lemma 3.2 *It holds that*

$$A(u_h, \Gamma_h v_h) = a(u_h, v_h) \qquad \forall u_h,\ v_h \in X_h. \quad (3.14a)$$

Moreover, the bilinear form $D(\cdot,\cdot)$ *satisfies*

$$D(\Gamma_h v_h, q_h) = -d(v_h, q_h) \qquad \forall (v_h, q_h) \in \bar{X}_h. \quad (3.14b)$$

Applying Lemma 3.2 and (3.7), the continuity and weak coercivity of the bilinear form $\mathcal{C}_h(\cdot,\cdot)$ can be easily verified:

$$|\mathcal{C}_h((u_h, \lambda p_h), (v_h, \lambda q_h))| \le C |||(u_h, p_h)||| \, |||(v_h, q_h)||| \qquad \forall (u_h, p_h),\ (v_h, q_h) \in \bar{X}_h, \quad (3.15)$$

and

$$\sup_{(v_h, q_h) \in \bar{X}_h} \frac{|\mathcal{C}_h((u_h, \lambda p_h), (v_h, \lambda q_h))|}{|||(v_h, q_h)|||} \ge \beta_4 |||(u_h, p_h)||| \qquad \forall (u_h, p_h) \in \bar{X}_h, \quad (3.16)$$

where the constant $\beta_4 > 0$ is independent of h.

4 L^2 and H^1 analysis for a branch of nonsingular solutions

In this section, the main goal is to provide the existence and optimal error analysis for a branch of nonsingular solutions of the finite volume methods for the stationary Navier–Stokes equations with large data. In particular, a new argument is introduced to obtain the L^2-norm estimate for velocity by using a residual technique in a Petrov-Galerkin system without the same symmetrical property as in the Galerkin system.

For the subsequent analysis, we now introduce a discrete analogue A_h of the Laplace operator A through the condition [19]

$$(A_h u_h, v_h) = (\nabla u_h, \nabla v_h) \qquad u_h, v_h \in X_h.$$

Define

$$V_h = \{v_h \in X_h : d(v_h, q_h) = 0 \quad \forall\, q_h \in M_h\}.$$

The restriction of A_h to V_h is invertible, with the inverse A_h^{-1}. In addition, A_h is self-adjoint and positive definite. Especially, we define the discrete Sobolev norm on V_h

for any $r \in R$ by

$$\|v_h\|_{H^r} = \|A_h^{r/2} v_h\|_{L^2}, \quad v_h \in V_h.$$

4.1 Stability of the finite volume methods

Due to the complexity of the nonlinear Navier–Stokes problem, the Brouwer fixed theory is applied in establishing the stability of the finite volume solution for this problem. The similar proof can be found in [23].

Lemma 4.1 *Assume that* **(A1)** *holds and the problem* (3.12) *has a set of solutions* $\tilde{u}_h = (u_h, \lambda p_h) \in \bar{X}_h$ *such that*

$$\|\nabla u_h\|_{L^2} + \lambda \|p_h\|_{L^2} \le C_9, \ \|A_h u_h\|_{L^2} \le C_{10}. \tag{4.1}$$

where the positive constants C_9 *and* C_{10} *depend on the previous positive constants defined above.*

4.2 Optimal error estimates of the finite volume methods

Similar to the continuous case, $\{(\lambda, \tilde{\bar{u}}_h(\lambda))\}$ with $\tilde{\bar{u}}_h(\lambda) = (\bar{u}_h, \lambda \bar{p}_h)$ is a branch of nonsingular solutions to (3.8) if

$$D_{\bar{u}_h} F(\lambda, \tilde{\bar{u}}_h(\lambda)) \text{ is an isomorphism from } \bar{X}_h \text{ onto } Y \text{ for all } \lambda \in \Lambda. \tag{4.2}$$

Recall that $T_h : Y \rightarrow \bar{X}_h$ is the solution operator of the discrete Stokes equations. This operator yields the solution $\tilde{\bar{u}}_h(\lambda) = (\bar{u}_h, \lambda \bar{p}_h)$ to problem (3.8). Apparently, this solution is also a solution of the discrete Navier–Stokes equation (3.8a) if and only if it is a solution of (3.8b). Furthermore, by (4.2) and the results in [4,17,18], we have the following Proposition.

Proposition 4.2 $\tilde{\bar{u}}_h(\lambda) \in \bar{X}_h$ *is a branch of non-singular solutions to Eq.* (3.8) *if there exist constants* $\gamma > 0$ *dependent of the data* (λ, f, Ω), *such that*

$$\sup_{(v_h, q_h) \in \bar{X}_h} \frac{\bar{B}_\lambda((\bar{w}_h, \lambda \bar{\chi}_h); (v_h, \lambda q_h))}{|||(v_h, q_h)|||} \ge \gamma |||(\bar{w}_h, \bar{\chi}_h)|||, \quad (\bar{w}_h, \bar{\chi}_h) \in \bar{X}_h. \tag{4.3}$$

where

$$\bar{B}_\lambda((\bar{w}_h, \lambda \bar{\chi}_h); (v_h, \lambda q_h)) \equiv A_\lambda(\bar{u}_h; \bar{w}_h, v_h) - \lambda d(v_h, \bar{\chi}_h) + \lambda d(\bar{w}_h, q_h),$$

and $A_\lambda(\bar{u}_h; w_h, v_h) = a(w_h, v_h) + \lambda b(\bar{u}_h, w_h, v_h) + \lambda b(w_h, \bar{u}_h, v_h)$.

Suppose that problem (2.1) has a branch of nonsingular solutions $\{(\lambda, \tilde{u}(\lambda)); \lambda \in \Lambda\}$ and that the following assumption (**A2**) holds:

Assumption (A2) There exists another Banach space Z contained in Y, with continuous imbedding, such that

$$D_u G(\lambda, \tilde{u}(\lambda)) \in \pounds(\bar{X}, Z) \quad \forall \lambda \in \Lambda, \quad \tilde{u} \in \bar{X}, \tag{4.4a}$$

$$\lim_{h\to 0} \|(T_h - T)g\|_{\bar{X}} = 0 \quad \forall g \in Y, \tag{4.4b}$$

$$\lim_{h\to 0} \|(T_h - T)\|_{\pounds(Z,\bar{X})} = 0. \tag{4.4c}$$

Then the next result holds for the finite element methods [17,18].

Theorem 4.3 *[17,18]. Assume that G is a C^2-mapping from $\Lambda \times \bar{X}$ onto Y, the mapping $D_{uu}G(\lambda, \tilde{u}(\lambda))$ is bounded on all bounded subsets of $\Lambda \times \bar{X}$, the assumptions* (**A1**) *and* (**A2**) *hold, and $\{(\lambda, \tilde{u}(\lambda)); \lambda \in \Lambda\}$ is a branch of nonsingular solutions of* (2.2)*. Then there exists a neighborhood ϑ of the origin in $\bar{X}$ and, for $0 < h \le h_0$ small enough, a unique C^2-function $\lambda \in \Lambda \to \bar{\tilde{u}}_h(\lambda) \in \bar{X}_h$ such that*

$$\{(\lambda, \bar{\tilde{u}}_h(\lambda)); \lambda \in \Lambda\} \ \textit{ is a branch of nonsingular solutions to } (3.8), \tag{4.5a}$$

$$\bar{\tilde{u}}_h(\lambda) - \tilde{u}(\lambda) \in \vartheta \ \textit{ for all } \lambda \in \Lambda. \tag{4.5b}$$

Furthermore, there exists a constant $\kappa > 0$, independent of h and λ, such that, for all $\lambda \in \Lambda$,

$$\|\bar{u}_h - u\|_{L^2} + h|||\bar{\tilde{u}}_h(\lambda) - \tilde{u}(\lambda)||| \le \kappa h(\|u\|_{H^2} + \|p\|_{H^1} + \|f\|_{L^2}). \tag{4.6}$$

Clearly, it follows from Theorem 4.3 that there is a branch of nonsingular solutions $\{(\lambda, \bar{\tilde{u}}_h(\lambda)); \lambda \in \Lambda\}$ in the neighborhood ϑ for a sufficiently small mesh scale $h > 0$ and all $\lambda \in \Lambda$. Assume that $\{(\lambda, \tilde{u}_h(\lambda)); \lambda \in \Lambda\}$ is a branch of nonsingular solutions of the finite volume methods for the stationary Navier–Stokes equations (3.12). We now need to show that these solutions are also located in the same neighborhood ϑ.

In a similar manner as for the derivation of Proposition 4.2, we give the following proposition:

Proposition 4.4 $\{(\lambda, \tilde{u}_h)\} \in \bar{X}_h$ *is a non-singular solution to Eq.* (3.12) *if there exist constants $\gamma^* > 0$, dependent of the data (λ, Ω, f), such that*

$$\sup_{(v_h,q_h)\in\bar{X}_h} \frac{B_\lambda((w_h, \lambda\chi_h); (v_h, \lambda q_h))}{|||(v_h, q_h)|||} \ge \gamma^* |||(w_h, \chi_h)|||, \tag{4.7}$$

where

$$B_\lambda((w_h, \lambda\chi_h); (v_h, \lambda q_h)) = A_\lambda(u_h; w_h, \Gamma_h v_h) + \lambda D(\Gamma_h v_h, \chi_h) + \lambda d(w_h, q_h)$$

and

$$A_\lambda(u_h; w_h, \Gamma_h v_h) = A(w_h, \Gamma_h v_h) + \lambda b(u_h, w_h, \Gamma_h v_h) + \lambda b(w_h, u_h, \Gamma_h v_h).$$

Although there holds the equivalence between the bilinear terms in Lemma 3.2. However, it is different for the trilinear term defined in the finite element method and finite volume method. Thus, the positive constant γ^* is different from γ in (4.3).

Now, we prove the stability and convergence results for the finite volume methods (3.12) for the stationary Navier–Stokes equations.

Theorem 4.5 *Under the assumptions of Theorem 4.3, then there exists a neighborhood ϑ of the origin in $\bar{X}$ and, for $h \leq h_0$ small enough, a unique C^2-function $\lambda \in \Lambda \to \tilde{u}_h(\lambda) \in \bar{X}_h$ such that*

$$\{(\lambda, \tilde{u}_h(\lambda)); \lambda \in \Lambda\} \text{ is a branch of nonsingular solutions to } (3.12), \tag{4.8a}$$

$$\tilde{u}_h(\lambda) - \tilde{u}(\lambda) \in \vartheta \ \text{ for all } \lambda \in \Lambda. \tag{4.8b}$$

Furthermore, there exists a constant $\kappa > 0$ independent of h such that, for all $\lambda \in \Lambda$,

$$|||\tilde{\tilde{u}}_h(\lambda) - \tilde{u}_h(\lambda)||| \leq \kappa |\log h|^{1/2} h^2 \|f\|_{H^1}. \tag{4.9}$$

Proof We deduce from (3.8) and (3.12) that

$$\mathcal{C}_h((\bar{u}_h - u_h, \lambda(\bar{p}_h - p_h)), (v_h, \lambda q_h)) + \lambda b(\bar{u}_h - u_h, \bar{u}_h, v_h) + \lambda b(\bar{u}_h, \bar{u}_h - u_h, v_h) \\ + \lambda b(u_h, u_h, \Gamma_h v_h - v_h) = \lambda(f, v_h - \Gamma_h v_h). \tag{4.10}$$

Taking $(v_h, q_h) = (e, \lambda\eta) = \tilde{\tilde{u}}_h - \tilde{u}_h \equiv (\bar{u}_h - u_h, \lambda(\bar{p}_h - p_h))$, noting that $\tilde{\tilde{u}}_h(\lambda) = (\bar{u}_h, \lambda\bar{p}_h)$ is a branch of nonsingular solutions of (3.8), and using Proposition 4.2 and Theorem 4.3, we see that

$$\begin{aligned} \gamma |||(e, \eta)||| &\leq \sup_{(v_h, q_h) \in \bar{X}_h} \frac{\bar{B}_\lambda((e, \lambda\eta); (v_h, \lambda q_h))}{|||(v_h, q_h)|||}, \\ &= \sup_{(v_h, q_h) \in \bar{X}_h} \frac{\lambda(f, v_h - \Gamma_h v_h) + \lambda b(u_h, u_h, v_h - \Gamma_h v_h)}{|||(v_h, q_h)|||}, \end{aligned} \tag{4.11}$$

Let $\hat{\pi}_h$ be the average interpolation operator satisfying $\hat{\pi}_h f|_K = \frac{1}{|K|}\int_K f dx$ and

$$\|f - \hat{\pi}_h f\|_{L^r(K)} \leq C h_K \|f\|_{W^{1,r}(K)}, \ 1 \leq r \leq \infty. \tag{4.12}$$

Then we obtain

$$\begin{aligned} |(f, v_h - \Gamma_h v_h)| &= (f - \hat{\pi}_h f, v_h - \Gamma_h v_h) \\ &\leq C h^{1+i} \|f\|_{H^i} \|\nabla v_h\|_{L^2}, \ i = 0, 1. \end{aligned} \tag{4.13}$$

For the last trilinear term in (4.10), it follows from Lemma 3.1 and (3.3) that

$$
\begin{aligned}
&|b(u_h, u_h, v_h - \Gamma_h v_h)| \\
&= \left| \left(((u_h - \hat{\pi}_h u_h) \cdot \nabla) u_h + \frac{1}{2} \text{div}\, u_h (u_h - \hat{\pi}_h u_h), v_h - \Gamma_h v_h \right) \right| \\
&\le \left\{ \|A_h^{1/2} u_h\|_{L^\infty} + \frac{1}{2} \|A_h^{1/2} u_h\|_{L^\infty} \right\} \|u_h - \hat{\pi}_h u_h\|_{L^2} \|e - \Gamma_h v_h\|_{L^2} \\
&\le C |\log h|^{1/2} h^2 \|A_h u_h\|_{L^2} \|\nabla u_h\|_{L^2} \|\nabla v_h\|_{L^2}.
\end{aligned} \tag{4.14}
$$

Then combining all these inequalities, Lemma 4.1, and using a straightforward computation yields

$$
|||(e, \eta)||| \le C |\log h|^{1/2} h^2 \|f\|_{H^1}. \tag{4.15}
$$

By Proposition 4.2 and Lemma 3.2, we have the following relationship between two terms $A_\lambda(u_h, w_h, \Gamma_h v_h)$ and $A_\lambda(\bar{u}_h, w_h, v_h)$

$$
\begin{aligned}
A_\lambda(u_h, w_h, \Gamma_h v_h) &= A_\lambda(\bar{u}_h, w_h, v_h) - \lambda b(e, w_h, v_h) - \lambda b(w_h, e, v_h) \\
&\quad - \lambda b(u_h, w_h, v_h - \Gamma_h v_h) - \lambda b(w_h, u_h, v_h - \Gamma_h v_h).
\end{aligned} \tag{4.16}
$$

Then, we estimate the above equality by (2.7) and (4.15) as follows

$$
\begin{aligned}
|\lambda b(e, w_h, v_h) + \lambda b(w_h, e, v_h)| &\le C \|\nabla e\|_{L^2} \|\nabla w_h\|_{L^2} \|\nabla v_h\|_{L^2} \\
&\le C |\log h|^{1/2} h^2 \|f\|_{H^1} \|\nabla w_h\|_{L^2} \|\nabla v_h\|_{L^2}.
\end{aligned}
$$

Similarly, using the same approach as (4.14) to obtain that

$$
\begin{aligned}
&|\lambda b(u_h, w_h, v_h - \Gamma_h v_h) + \lambda b(w_h, u_h, v_h - \Gamma_h v_h)| \\
&\le 4\lambda (\|u_h\|_{L^\infty} \|\nabla w_h\|_{L^2} + \|w_h\|_{L^\infty} \|\nabla u_h\|_{L^2}) \|v_h - \Gamma_h v_h\|_{L^2} \\
&\le 4\lambda (\|u_h\|_{L^2}^{1/2} \|A_h u_h\|_{L^2}^{1/2} \|\nabla w_h\|_{L^2} + \|w_h\|_{L^2}^{1/2} \|A_h w_h\|_{L^2}^{1/2} \|\nabla u_h\|_{L^2}) \|\Gamma_h v_h - v_h\|_{L^2} \\
&\le C h^{1/2} \|f\|_{L^2} \|\nabla w_h\|_{L^2} \|\nabla v_h\|_{L^2} \\
&\le C h^{1/2} \|f\|_{L^2} |||(w_h, \chi_h)||| \|\nabla v_h\|_{L^2}.
\end{aligned}
$$

Thus, by choosing $\gamma^* = \gamma - C h^{1/2} \|f\|_{L^2}$, we derive from (4.15) and (4.16) that, for sufficient small $h > 0$

$$
\begin{aligned}
\sup_{(v_h, q_h) \in \bar{X}_h} \frac{B_\lambda((w_h, \lambda \chi_h); (v_h, \lambda q_h))}{|||(v_h, q_h)|||} &\ge (\gamma - 2C \|\nabla e\|_{L^2}) |||(w_h, \chi_h)||| \\
&= \gamma^* |||(w_h, \chi_h)|||.
\end{aligned} \tag{4.17}
$$

Thus we complete the proof of (4.12) by Proposition 4.4. □

Apparently, a superconvergence result is obtained between the finite element solution and the finite volume solution. Using a result in [17] and the estimate between them in Theorem 4.5, (4.9) still holds with respect to the solution of the finite volume method around the same neighborhood ϑ of the origin in $\bar{X}$. Furthermore, we now give an optimal analysis for a branch of the finite volume solutions for the stationary Navier–Stokes equations with large data.

Theorem 4.6 *Under the assumption of Theorem 4.5, let* $\{\lambda, \tilde{u}(\lambda);\ \lambda \in \Lambda\}$ *and* $\{\lambda, \tilde{u_h}(\lambda);\ \lambda \in \Lambda\}$ *be a branch of nonsingular solutions of (2.2) (or 2.5) and (3.12), respectively. Then it holds that*

$$\|\tilde{u}_h(\lambda) - \tilde{u}(\lambda)\| \le \kappa h(\|u\|_{H^2} + \|p\|_{H^1} + \|f\|_{L^2}), \tag{4.18a}$$
$$\|u - u_h\|_{L^2} \le \kappa h^2(\|u\|_{H^2} + \|p\|_{H^1} + \|f\|_{H^1}). \tag{4.18b}$$

Proof By a triangle inequality, (4.6) and (4.9),

$$\begin{aligned}\|\tilde{u}_h(\lambda) - \tilde{u}(\lambda)\| &\le \|\tilde{u}_h(\lambda) - \tilde{\tilde{u}}_h(\lambda)\| + \|\tilde{\tilde{u}}_h(\lambda) - \tilde{u}(\lambda)\| \\ &\le \kappa h(\|u\|_{H^2} + \|p\|_{H^1} + \|f\|_{L^2}),\end{aligned} \tag{4.19}$$

which completes the proof of (4.18a).

Thanks to the Aubin-Nitsche duality technique for the general framework of mixed problems, consider the dual problem for given solution (u, p) of (2.1a), (2.1b) and any $(v, q) \in \bar{X}$ to find $(\Phi, \Psi) \in \bar{X}$

$$a(v, \Phi) + \lambda d(v, \Psi) - \lambda d(\Phi, q) + \lambda b(u, v, \Phi) + \lambda b(v, u, \Phi) = (u - \tilde{u}_h, v). \tag{4.20}$$

Because of the convexity of the domain Ω and the Lax-Milgram Theorem, this problem has a unique solution satisfying [30]

$$\|\Phi\|_{H^2} + \lambda\|\Psi\|_{H^1} \le C\|u - \tilde{u}_h\|_{L^2}. \tag{4.21}$$

For completeness, we here provide detail proof as in [23]. Below set $(\Phi_h, \Psi_h) = (I_h\Phi, J_h\Psi) \in \bar{X}_h$, which satisfies, by (3.1),

$$\|\Phi - \Phi_h\|_{L^2} + h(\|\Phi - \Phi_h\|_{H^1} + \|\Psi - \Psi_h\|_{L^2}) \le Ch^2(\|\Phi\|_{H^2} + \|\Psi\|_{H^1}). \tag{4.22}$$

Then, multiplying (2.1a) and (2.1b) by $\Gamma_h\Phi_h$ and $\lambda\Psi_h$, respectively, and adding them to find

$$A(u, \Gamma_h\Phi_h) + \lambda D(\Gamma_h\Phi_h, p) + \lambda d(u, \Psi_h) + \lambda b(u, u, \Gamma_h\Phi_h) = \lambda(f, \Gamma_h\Phi_h), \tag{4.23}$$

which, together with (3.12b), yields by setting $(e, \eta) = (u - u_h, p - p_h)$ that

$$A(e, \Gamma_h\Phi_h) + \lambda D(\Gamma_h\Phi_h, \eta) + \lambda d(e, \Psi_h) + \lambda b(e, u, \Gamma_h\Phi_h) + \lambda b(u, e, \Gamma_h\Phi_h)$$
$$-\lambda b(e, e, \Gamma_h\Phi_h) = 0. \tag{4.24}$$

Subtracting (4.24) from (4.20) with $(v, q) = (e, \eta)$ and using (2.1), we obtain

$$\begin{aligned}\|e\|_{L^2}^2 &= a(e, \Phi - \Phi_h) + \lambda d(e, \Psi - \Psi_h) - \lambda d(\Phi - \Phi_h, \eta)\\ &\quad + a(e, \Phi_h) - A(e, \Gamma_h\Phi_h) - \lambda d(\Phi_h, \eta) - \lambda D(\Gamma_h\Phi_h, \eta)\\ &\quad + \lambda b(u, e, \Phi - \Gamma_h\Phi_h) + \lambda b(e, u, \Phi - \Gamma_h\Phi_h) + \lambda b(e, e, \Gamma_h\Phi_h)\\ &= a(e, \Phi - \Phi_h) + \lambda d(e, \Psi - \Psi_h) - \lambda d(\Phi - \Phi_h, \eta)\\ &\quad + \lambda b(u, e, \Phi - \Gamma_h\Phi_h) + \lambda b(e, u, \Phi - \Gamma_h\Phi_h) + \lambda b(e, e, \Gamma_h\Phi_h)\\ &\quad + \lambda(f - (u \cdot \nabla)u, \Phi_h - \Gamma_h\Phi_h).\end{aligned} \tag{4.25}$$

Applying (4.21), (4.22), and (4.18a), we see that

$$\begin{aligned}&|a(e, \Phi - \Phi_h) + \lambda d(e, \Psi - \Psi_h) - \lambda d(\Phi - \Phi_h, \eta)|\\ &\quad \leq C\left(\|\nabla e\|_{L^2} + \|\eta\|_{L^2}\right)\left(\|\Phi - \Phi_h\|_{H^1} + \|\Psi - \Psi_h\|_{L^2}\right)\\ &\quad \leq Ch^2\left(\|u\|_{H^2} + \|p\|_{H^1}\right)\left(\|\Phi\|_{H^2} + \|\Psi\|_{H^1}\right)\\ &\quad \leq Ch^2\left(\|u\|_{H^2} + \|p\|_{H^1}\right)\|e\|_{L^2}.\end{aligned}$$

By the estimates of the trilinear terms in (2.7), Lemma 3.1, (4.18a), and (4.21), we see that

$$\begin{aligned}&|\lambda b(u, e, \Phi - \Gamma_h\Phi_h) + \lambda b(e, u, \Phi - \Gamma_h\Phi_h)|\\ &\quad \leq C\|u\|_{H^2}\|\nabla e\|_{L^2}\left(\|\Phi_h - \Gamma_h\Phi_h\|_{L^2} + \|\Phi - \Phi_h\|_{L^2}\right)\\ &\quad \leq Ch^2\left(\|u\|_{H^2} + \|p\|_{H^1} + \|f\|_{L^2}\right)\|\Phi\|_{H^1}\\ &\quad \leq Ch^2\left(\|u\|_{H^2} + \|p\|_{H^1} + \|f\|_{L^2}\right)\|e\|_{L^2}.\end{aligned}$$

Using the Hӧlder inequality, (2.3), (3.3) and (4.21), we have

$$\begin{aligned}|\lambda b(e, e, \Gamma_h\Phi_h)| &= |\lambda b(e, e, \Gamma_h\Phi_h - \Phi_h) + \lambda b(e, e, \Phi_h)|\\ &\leq C\left(\|e\|_{L^4}\|\nabla e\|_{L^2}\|\Gamma_h\Phi_h - \Phi_h\|_{L^4} + \|\nabla e\|_{L^2}^2\|\nabla\Phi_h\|_{L^2}\right)\\ &\leq Ch^2\left(\|u\|_{H^2} + \|p\|_{H^1}\right)\|e\|_{L^2}.\end{aligned}$$

Furthermore, the following estimate follows from (4.12), (4.21), and Lemma 3.1:

$$\begin{aligned}&|\lambda(f - (u \cdot \nabla)u, \Phi_h - \Gamma_h\Phi_h)|\\ &\quad = |\lambda([f - \hat{\pi}_h f] - [(u \cdot \nabla)u - \hat{\pi}_h(u \cdot \nabla)u], \Phi_h - \Gamma_h\Phi_h)|\\ &\quad \leq Ch^2(\|f\|_{H^1} + \|\nabla[(u \cdot \nabla)u]\|_{L^2})\|\Phi_h\|_{H^1}\\ &\quad \leq Ch^2(\|f\|_{H^1} + \|u\|_{L^2}^{1/2}\|Au\|_{L^2}^{3/2} + \|\nabla u\|_{L^4}^2)\|e\|_{L^2}.\end{aligned}$$

Finally, combining all these inequalities and (4.25) yields (4.18b). □

As noted earlier, for the nonlinear Navier–Stokes equations, the skew-symmetry property of the trilinear term is no longer valid, and the approximate Galerkin orthogonality relation loses its effectiveness. Moreover, the regularity of the source term may affect the convergence rate of a finite volume method. In this paper, the L^2-norm estimate for velocity is one of the major difficulties in the analysis of the finite volume method for these equations without any additional regularity on the original solution. The counterexample in [12,20] showed that the finite volume solutions approximated by the conforming linear elements cannot have an optimal L^2-norm convergence rate if the exact solution is in $H^2(\Omega)$ but the source term is only in $L^2(\Omega)$ for a saddle point problem. Hence, based on the previous analysis, the results in Theorem 4.6 should be reasonable and optimal with additional regularity on source term.

5 L^∞ analysis for a branch of nonsingular solutions

In this section, the aim is to give a stability and optimal convergence analysis in the L^∞-norm for velocity gradient and pressure, which is not available in the literature for the finite volume approximations of the stationary Stokes equations. The main difficulty of the convergence analysis is to obtain the optimal error estimates in this norm by removing the logarithmic factor $O(|\log h|)$ that appeared in the traditional estimates. The analysis in this section is based on a technique using the weighted Sobolev norms introduced in [3,11,16] for the finite element approximations of the Stokes equations. Duran et al. [11] provided a sharp L^∞-norm error estimate for the finite element approximations of the Stokes problem with the logarithmic factor. Girault et al. [16] adapted the analysis in [3] to remove the logarithmic factor by working with the weight $\sigma^{\mu/2}$ to be defined below. Here, we focus on an optimal analysis in this norm for the finite volume methods for the stationary Navier–Stokes equations. The analysis is still required to deal with the complexity of the trilinear terms and different test and trial functions in different finite dimensional spaces.

In the coming analysis, we require that the interpolation operators I_h and J_h satisfy additional properties:

Assumption (A3) • I_h is quasi-local: For all $K \in K_h$,

$$\|I_h v - v\|_{L^2(K)} + h_K\|\nabla(I_h v - v)\|_{L^2(K)} \le Ch_K^2|v|_{H^2(\Delta K)},$$
$$\|\nabla I_h v\|_{L^2(K)} \le C|v|_{H^1(\Delta K)}.$$

- I_h satisfies the discrete divergence-preserved property:

$$d(I_h v - v, q_h) = 0 \quad \forall q_h \in \overline{M}_h.$$

- J_h is also quasi-local: For all $K \in K_h$,

$$\|J_h q - q\|_{L^2(K)} \le Ch_K|q|_{H^1(\Delta K)}.$$

Here ΔK is a macro-element containing at most L elements of K_h including K, L being a fixed integer independent of h, and the functions in $\overline{M}_h$ are those in M_h without the zero mean-value constraint. The additional property of quasi-locality is fundamental here for deriving weighted estimates. For the examples considered, assumption **(A3)** holds.

5.1 Stability in the L^∞-norm

Here, we collect some basic assumption on regularity results and properties of the Green's function for the Stokes equations from the literature and use them to analyze a branch of nonsingular solutions in L^∞ norm. To analyze the stability of the finite volume methods in the L^∞-norm, following [3,28], we introduce the regularized Green's functions. Toward that end, we fix an element of the matrix ∇u_h, e.g., $\frac{\partial u_{h,i}}{\partial x_j}$, and an appropriate point x_0 located in the element $K \in K_h$ where $\left|\frac{\partial u_{h,i}}{\partial x_j}\right|$ is maximum. An approximate mollifier δ_M supported by K is defined so that

$$D\delta_M = \frac{\partial(\delta_M e_i)}{\partial x_j}, \quad \int_\Omega \delta_M dx = 1, \quad \left\|\frac{\partial u_{h,i}}{\partial x_j}\right\|_{L^\infty} = \left(\delta_M, \frac{\partial u_{h,i}}{\partial x_j}\right), \tag{5.1}$$

where e_i is the unit vector in the i-direction ($i = 1$ or 2). Now, the regularized Green's functions are defined by

$$a(G, v) - \lambda d(v, Q) + \lambda b(v, u, G) + \lambda b(u, v, G) = -(D\delta_M, v), \ \forall\, v \in X, \tag{5.2a}$$

$$\lambda d(G, q) = 0, \qquad \forall q \in M. \tag{5.2b}$$

Similarly, there holds the following estimate [16]:

$$\|\sigma^{\mu/2-1}\nabla G\|_{L^2} + \|\sigma^{\mu/2-1} Q\|_{L^2} \le C h^{\theta/2-1}, \tag{5.3a}$$

$$\|\sigma^{\mu/2}\Delta G\|_{L^2} + \|\sigma^{\mu/2}\nabla Q\|_{L^2} \le C h^{\theta/2-1}, \tag{5.3b}$$

where $\sigma(x) = [|x - x_0|^2 + (\kappa h)^2]^{1/2}$ ($|x - x_0| < R,\ R > 0$), $\mu = 2 + \theta$ with $0 < \theta < 1$, and $C > 0$ is independent of the constant $\kappa > 1$ and the mesh size h.

Also, we define the Stokes projection $(G_h, Q_h) \in \bar{X}_h$ of (G, Q):

$$a(G - G_h, v_h) - \lambda d(v_h, Q - Q_h) = 0 \ \ \forall v_h \in X_h, \tag{5.4a}$$

$$\lambda d(G - G_h, q_h) = 0 \ \ \forall q_h \in M_h. \tag{5.4b}$$

Under assumption **(A3)**, it holds that [16]

$$\|\nabla G_h\|_{L^2} + \|Q_h\|_{L^2} \le C(\|\nabla G\|_{L^2} + \|Q\|_{L^2}). \tag{5.5a}$$

Furthermore, by this assumption, the solution to problem (5.4) satisfies

$$\|\sigma^{\mu/2}\nabla(G - G_h)\|_{L^2} + \|\sigma^{\mu/2}(Q - Q_h)\|_{L^2} \leq Ch^{\theta/2}. \tag{5.5b}$$

We now analyze the solution stability in terms of $\|\nabla u_h\|_{L^\infty}$ and $\|p_h\|_{L^\infty}$ in order to obtain the optimal estimates in the same norm for the finite volume approximations of the nonsingular solutions of the stationary Navier–Stokes equations.

Lemma 5.1 *Under the assumptions of Theorem 4.6 and* **(A3)***, let* $\{\lambda, \tilde{u}(\lambda);\ \lambda \in \Lambda\}$ *and* $\{\lambda, \tilde{u}_h(\lambda);\ \lambda \in \Lambda\}$ *be a branch of nonsingular solutions of (2.2) (or 2.5) and (3.12), respectively. Then it holds that*

$$\|\nabla u_h\|_{L^\infty} \leq C(\|\nabla u\|_{L^\infty} + \|p\|_{L^\infty} + \|f\|_{L^2}). \tag{5.6}$$

Proof Taking $(v, q) = (u_h, p_h)$ in (5.2), we see that

$$\|\nabla u_h\|_{L^\infty} = a(G, u_h) - \lambda d(u_h, Q) - \lambda d(G, p_h) + \lambda b(u_h, u, G) + \lambda b(u, u_h, G),$$

which, together with the Stokes projection defined in (5.4), yields

$$\|\nabla u_h\|_{L^\infty} = a(G_h, u_h) - \lambda d(u_h, Q_h) - \lambda d(G_h, p_h) + \lambda b(u_h, u, G) + \lambda b(u, u_h, G). \tag{5.7}$$

Moreover, it follows from (2.5), (3.12) and Lemma 3.2 that

$$\begin{aligned} a(u - u_h, v_h) - \lambda d(v_h, p - p_h) + \lambda d(u - u_h, q_h) \\ +\lambda b(u, u, v_h) - \lambda b(u_h, u_h, \Gamma_h v_h) = \lambda(f, v_h - \Gamma_h v_h). \end{aligned} \tag{5.8}$$

Thus, we derive from (5.7), (5.8) with $(v_h, q_h) = (G_h, -Q_h)$ and (5.2b) with $q = p$ that

$$\begin{aligned} \|\nabla u_h\|_{L^\infty} &= a(u, G_h) + \lambda d(G_h, p) + \lambda b(u, u, G_h) - \lambda b(u_h, u_h, \Gamma_h G_h) \\ &\quad +\lambda b(u_h, u, G) + \lambda b(u, u_h, G) - \lambda(f, G_h - \Gamma_h G_h) \\ &= a(u, G_h - G) + a(u, G) - \lambda d(G_h - G, p) - \lambda(f, G_h - \Gamma_h G_h) \\ &\quad +\lambda b(u, u, G_h) - \lambda b(u_h, u_h, \Gamma_h G_h) + \lambda b(u_h, u, G) + \lambda b(u, u_h, G), \end{aligned} \tag{5.9}$$

since $d(u, Q_h) = 0$. Applying (5.2a) with $v = u$ and (2.1b) leads to that

$$a(u, G) = -2\lambda b(u, u, G) - (D\delta_M, u). \tag{5.10}$$

Then using (5.9) and (5.10) gives the main equality

$$\begin{aligned}\|\nabla u_h\|_{L^\infty} = &-(D\delta_M, u) + a(u, G_h - G) - \lambda d(G_h - G, p) - \lambda(f, G_h - \Gamma_h G_h)\\ &+\lambda b(u - u_h, u, G_h) - \lambda b(u - u_h, u - u_h, G_h) + \lambda b(u, u - u_h, G_h)\\ &+\lambda b(u_h, u_h, G_h - \Gamma_h G_h) + \lambda b(u_h, u, G) + \lambda b(u, u_h, G) - 2\lambda b(u, u, G).\end{aligned} \tag{5.11}$$

Obviously, it follows from Lemma 3.1, (5.5a) and the Hölder inequality that

$$\begin{aligned} -(D\delta_M, u) &= \|\nabla u\|_{L^\infty},\\ |(f, G_h - \Gamma_h G_h)| &\le Ch\|f\|_{L^2}(\|\nabla G\|_{L^2} + \|Q\|_{L^2}),\\ |a(u, G_h - G) - \lambda d(G_h - G, p)| &\le (\|\nabla u\|_{L^\infty} + \|p\|_{L^\infty})\|\nabla(G - G_h)\|_{L^1}.\end{aligned}$$

Similarly, we estimate the trilinear terms as follows:

$$\begin{aligned} |b(u - u_h, u, G_h) + b(u, u - u_h, G_h)| &\le C\|\nabla u\|_{L^2}(\|\nabla G\|_{L^2} + \|Q\|_{L^2})\|\nabla(u - u_h)\|_{L^2},\\ |b(u - u_h, u - u_h, G_h)| &\le C(\|\nabla G\|_{L^2} + \|Q\|_{L^2})\|\nabla(u - u_h)\|^2_{L^2},\\ |b(u_h, u, G) + b(u, u_h, G) - 2b(u, u, G)| &= |b(u_h - u, u, G) + b(u, u_h - u, G)|\\ &\le C\|\nabla u\|_{L^2}\|\nabla G\|_{L^2}\|\nabla(u - u_h)\|_{L^2}.\end{aligned}$$

By using the same approach as for (4.14) and Lemma 4.1, it follows that

$$\begin{aligned} |b(u_h, u_h, G_h - \Gamma_h G_h)| &\le C|\log h|^{1/2}h^2\|u_h\|_{L^2}\|A_h u_h\|_{L^2}(\|\nabla G\|_{L^2} + \|Q\|_{L^2})\\ &\le C|\log h|^{1/2}h^2\|f\|_{L^2}(\|\nabla G\|_{L^2} + \|Q\|_{L^2}).\end{aligned}$$

Thus it remains to estimate $\|\nabla(G_h - G)\|_{L^1}$, $\|\nabla G\|_{L^2}$ and $\|Q\|_{L^2}$. To this end, note that

$$\begin{aligned}\|\nabla(G_h - G)\|_{L^1} &= \int_\Omega \nabla(G_h - G)dx\\ &\le \left(\int_\Omega \sigma^\mu|\nabla(G_h - G)|^2dx\right)^{1/2}\left(\int_\Omega \sigma^{-\mu}dx\right)^{1/2}.\end{aligned} \tag{5.12}$$

It follows from [16] that the last term in (5.12) can be bounded as follows:

$$\int_\Omega \sigma^{-\mu}dx \le Ch^{-\theta},\ 0 < \theta < 1,\ \mu = 2 + \theta. \tag{5.13}$$

Thus we see from (5.5b), (5.12), and (5.13) that

$$\|\nabla(G_h - G)\|_{L^1} \le C. \tag{5.14}$$

As for the term $\|\nabla G\|_{L^2}$, using again the Hölder inequality and (5.3a), we see that

$$\|\nabla G\|_{L^2}^2 \le \max \sigma^{2-\mu} \int_\Omega \sigma^{\mu-2}|\nabla G|^2 dx \\ \le \kappa h^{2-\mu} \|\sigma^{\mu/2-1}\nabla G\|_{L^2(\Omega)}^2 \le Ch^{-2}. \tag{5.15}$$

Similarly,

$$\|Q\|_{L^2} \le Ch^{-1},$$

which together with these inequalities (4.18) and (5.11–5.15), we obtain the desired result. □

It is important to note that the stability of the pressure in the L^∞-norm does not directly follow from the above result on the velocity and the discrete *inf-sup* condition. The analysis for the pressure requires a different regularized Green's function [3,28]:

$$a(U,v) + \lambda d(v,V) + \lambda b(v,u,U) + \lambda b(u,v,U) = 0, \ \ v \in X, \tag{5.16a}$$
$$\lambda d(U,q) = (\delta_M - B, q), \ \ q \in M, \tag{5.16b}$$

where B is a fixed function in $C_0^\infty(\Omega)$ such that $\int_\Omega B(x)dx = 1$ and thus $\delta_M - B \in L_0^2(\Omega)$. Analogically, the solution of problem (5.16) satisfies [16]

$$\|\sigma^{\mu/2-1}\nabla U\|_{L^2} + \|\sigma^{\mu/2-1}V\|_{L^2} \le Ch^{\theta/2-1}. \tag{5.17}$$

Also, we define its Stokes projection $(U_h, V_h) \in \bar{X}_h$ as follows:

$$a(U - U_h, v_h) + \lambda d(v_h, V - V_h) - d(U - U_h, q_h) = 0 \ \ \forall (v_h, q_h) \in \bar{X}_h, \tag{5.18}$$

which has the following result [16]:

$$\|\nabla U_h\|_{L^2} + \|V_h\|_{L^2} \le C(\|\nabla U\|_{L^2} + \|V\|_{L^2}), \tag{5.19a}$$
$$\|\sigma^{\mu/2}\nabla(U - U_h)\|_{L^2} + \|\sigma^{\mu/2}(V - V_h)\|_{L^2} \le Ch^{\theta/2}. \tag{5.19b}$$

Based on the above preparation, we need to estimate $\|p_h\|_{L^\infty}$ in Lemma 5.2.

Lemma 5.2 *Under the assumptions of Theorem 4.6 and* **(A3)***, let* $\{\lambda, \tilde{u}(\lambda);\ \lambda \in \Lambda\}$ *and* $\{\lambda, \tilde{u}_h(\lambda);\ \lambda \in \Lambda\}$ *be a branch of nonsingular solutions of (2.2) (or (2.5)) and (3.12), respectively. Then it holds that*

$$\|p_h\|_{L^\infty} \le C(\|\nabla u\|_{L^\infty} + \|p\|_{L^\infty} + \|f\|_{L^2}). \tag{5.20}$$

Proof Taking $(v, q) = (u - u_h, p - p_h)$ in (5.16), we find that

$$(\delta_M - B, p - p_h) = a(U, u - u_h) + \lambda d(u - u_h, V) - \lambda d(U, p - p_h) \\ + \lambda b(u - u_h, u, U) + \lambda b(u, u - u_h, U). \tag{5.21}$$

Moreover, setting $(v_h, q_h) = (U_h, V_h)$ in (5.8) yields

$$a(u-u_h, U_h) - \lambda d(U_h, p-p_h) + \lambda d(u-u_h, V_h) + \lambda b(u,u,U_h) \\ -\lambda b(u_h, u_h, \Gamma_h U_h) = \lambda(f, U_h - \Gamma_h U_h). \quad (5.22)$$

Then, using (5.21) and (5.22) and noting that

$$d(U-U_h, p_h) = 0$$

in (5.18), we obtain

$$\|p_h\|_{L^\infty} = a(u-u_h, U-U_h) + \lambda d(u-u_h, V-V_h) - \lambda d(U-U_h, p) \\ +(B, p-p_h) - (\delta_M, p) - \lambda b(u-u_h, u, U_h) - \lambda b(u_h, u-u_h, U_h) \\ -\lambda b(u_h, u_h, U_h - \Gamma_h U_h) + \lambda b(u-u_h, u, U) + \lambda b(u, u-u_h, U) \\ +\lambda(f, U_h - \Gamma_h U_h). \quad (5.23)$$

It follows from Lemma 5.1, the Hölder inequality, and (5.19a) that

$$|a(u-u_h, U_h-U) - \lambda d(u-u_h, V_h-V) + \lambda d(U_h-U, p)| \\ \le (\|\nabla(u-u_h)\|_{L^\infty} + \|p\|_{L^\infty})(\|\nabla(U_h-U)\|_{L^1} + \|V_h-V\|_{L^1}) \\ \le (\|\nabla u\|_{L^\infty} + \|p\|_{L^\infty} + \|f\|_{L^2})(\|\nabla(U_h-U)\|_{L^1} + \|V_h-V\|_{L^1}),$$

since $d(U-U_h, p_h) = 0$. By the estimates of the trilinear terms in (2.7),

$$|\lambda b(u-u_h, u, U) + \lambda b(u, u-u_h, U)| \\ \le C\|\nabla u\|_{L^\infty}\|\nabla(u-u_h)\|_{L^2}\|\nabla U\|_{L^2}, \\ b(u-u_h, u, U_h) + b(u_h, u-u_h, U_h)| \\ \le C\|\nabla(u-u_h)\|_{L^2}(\|\nabla u\|_{L^2} + \|\nabla u_h\|_{L^2})(\|\nabla U\|_{L^2} + \|V\|_{L^2}).$$

Thanks to the same approach as for (4.14), we derive from Theorems 4.1, 5.1 and (5.19a) that

$$|\lambda b(u_h, u_h, U_h - \Gamma_h U_h)| \\ \le C\|\nabla u_h\|_{L^\infty}\|u_h - \hat{\pi}_h u_h\|_{L^2}\|U_h - \Gamma_h U_h\|_{L^2} \\ \le Ch^2\|\nabla u_h\|_{L^\infty}\|\nabla u_h\|_{L^2}\|\nabla U_h\|_{L^2} \\ \le Ch^2(\|\nabla u\|_{L^\infty} + \|p\|_{L^\infty} + \|f\|_{L^2})(\|\nabla U\|_{L^2} + \|V\|_{L^2}).$$

In addition, using Lemma 5.1, and the Hölder inequality, gives

$$|(B, p-p_h) - (\delta_M, p)| \le C(\|u\|_{L^\infty} + \|p\|_{L^\infty}), \\ |(f, U_h - \Gamma_h U_h)| \le Ch\|f\|_{L^2}(\|\nabla U\|_{L^2} + \|Q\|_{L^2}),$$

Furthermore, we apply the same procedure as in Lemma 5.1 with respect to $\|\nabla(U - U_h)\|_{L^1(\Omega)}$, $\|V - V_h\|_{L^1(\Omega)}$, $\|\nabla U\|_{L^2}$ and $\|V\|_{L^2}$ to obtain

$$\|\nabla U\|_{L^2} + \|V\|_{L^2} \le Ch^{-1}, \ \|\nabla(U - U_h)\|_{L^1(\Omega)} + \|V - V_h\|_{L^1(\Omega)} \le C. \tag{5.24}$$

Therefore, combining all these inequalities with the convergence results (4.18) of the finite volume methods yields the desired result. □

5.2 Optimal error estimates

Based on the maximum-norm stability analysis, we will show the optimal estimates in L^∞-norm for the stationary Navier–Stokes equations.

Lemma 5.3 *Under the assumptions of Theorem 4.6 and* **(A3)***, let* $\{\lambda, \tilde{u}(\lambda);\ \lambda \in \Lambda\}$ *and* $\{\lambda, \tilde{u}_h(\lambda);\ \lambda \in \Lambda\}$ *be a branch of nonsingular solution of (2.2) (or 2.5) and (3.12), respectively. Then it holds that*

$$\|\nabla(u - u_h)\|_{L^\infty} \le Ch(|u|_{W^{2,\infty}} + |p|_{W^{1,\infty}} + \|f\|_{H^1}). \tag{5.25}$$

Proof Taking $v = e = I_h u - u_h$ in (5.2), recalling the properties of δ_M in (5.2a), and using the definition of the Stokes projection (5.4), we see that

$$\begin{aligned}\|\nabla e\|_{L^\infty} &= a(G, e) - \lambda d(e, Q) + \lambda b(e, u, G) + \lambda b(u, e, G)\\ &= a(G_h, e) - \lambda d(e, Q_h) + \lambda b(e, u, G) + \lambda b(u, e, G).\end{aligned} \tag{5.26}$$

By (5.2b), (5.4b),

$$d(G_h, p - p_h) = d(G_h, p - J_h p) = d(G_h - G, p - J_h p).$$

Also, it follows from (5.2b), (5.4b) and (5.8) with $(v_h, q_h) = (G_h, 0)$ that

$$\begin{aligned}a(e, G_h) &= a(I_h u - u, G_h) + \lambda d(G_h, p - p_h) - \lambda b(u, u, G_h) + \lambda b(u_h, u_h, \Gamma_h G_h)\\ &\quad + \lambda(f, G_h - \Gamma_h G_h)\\ &= a(I_h u - u, G_h - G) + a(I_h u - u, G) + \lambda d(G_h - G, p - J_h p)\\ &\quad - \lambda b(u, u, G_h) + \lambda b(u_h, u_h, \Gamma_h G_h) + \lambda(f, G_h - \Gamma_h G_h).\end{aligned}$$

Furthermore, a consequence of (5.2a) with $v = I_h u - u$ is

$$\begin{aligned}a(I_h u - u, G) &= -(D\delta_M, I_h u - u) + \lambda d(I_h u - u, Q) - \lambda b(I_h u - u, u, G)\\ &\quad - \lambda b(u, I_h u - u, G).\end{aligned}$$

Thus, noting $Q_h \in M_h \subset \bar{M}_h$, using all these equations, Assumption (**A3**) and (5.26), we obtain

$$\begin{aligned}\|\nabla e\|_{L^\infty} = {} & a(I_h u - u, G_h - G) + \lambda d(G_h - G, p - J_h p) + \lambda d(I_h u - u, Q - Q_h)\\ & -\lambda d(e, Q_h) - (D\delta_M, I_h u - u) + \lambda(f, G_h - \Gamma_h G_h)\\ & -\lambda b(I_h u - u, u, G) - \lambda b(u, I_h u - u, G) - \lambda b(u, u, G_h) + \lambda b(u_h, u_h, G_h)\\ & -\lambda b(u_h, u_h, G_h) + \lambda b(u_h, u_h, \Gamma_h G_h) + \lambda b(e, u, G) + \lambda b(u, e, G).\end{aligned} \tag{5.27}$$

Clearly, it follows from (5.1) that

$$-(D\delta_M, I_h u - u) = \|\nabla(I_h u - u)\|_{L^\infty},$$

and

$$\begin{aligned}&|a(I_h u - u, G_h - G) + \lambda d(G_h - G, p - J_h p) + \lambda d(I_h u - u, Q - Q_h)|\\ &\quad \le (\|\nabla(I_h u - u)\|_{L^\infty} + \|p - J_h p\|_{L^\infty})(\|\nabla(G_h - G)\|_{L^1} + \|Q - Q_h\|_{L^1}),\\ &\quad |(f, G_h - \Gamma_h G_h)| = |(f - \hat{\pi}_h f, G_h - \Gamma_h G_h)| \le Ch^2 \|f\|_{H^1} \|\nabla G\|_{L^2}.\end{aligned}$$

Using the estimates of the trilinear terms in (2.7), and the property of the projection operator (5.5a), we have

$$\begin{aligned}&|b(I_h u - u, u, G) + b(u, I_h u - u, G)|\\ &\quad \le C\|I_h u - u\|_{L^2}\|Au\|_{L^2}\|\nabla G\|_{L^2},\\ &\quad\ |b(e, u, G) + b(u, e, G)|\\ &\quad \le C(\|u - u_h\|_{L^2} + \|I_h u - u\|_{L^2})\|Au\|_{L^2}\|\nabla G\|_{L^2},\\ &\quad\ |b(u, u, G_h) - b(u_h, u_h, G_h)|\\ &\quad = |b(u - u_h, u, G_h) + b(u, u - u_h, G_h) - b(u - u_h, u - u_h, G_h)|\\ &\quad \le C(\|Au\|_{L^2}\|u - u_h\|_{L^2} + \|\nabla(u - u_h)\|_{L^2}^2)(\|\nabla G\|_{L^2} + \|Q\|_{L^2}).\end{aligned} \tag{5.28}$$

In view of Lemma 3.1 and the Hölder inequality, it follows that

$$\begin{aligned}&|b(u_h, u_h, G_h - \Gamma_h G_h)|\\ &\quad = \left(((u_h - \hat{\pi}_h u_h) \cdot \nabla)u_h + \frac{1}{2}\mathrm{div}\, u_h(u_h - \hat{\pi}_h u_h), G_h - \Gamma_h G_h\right)\\ &\quad \le \left(1 + \frac{\sqrt{2}}{2}\right)\|\nabla u_h\|_{L^\infty}\|u_h - \hat{\pi}_h u_h\|_{L^2}\|G_h - \Gamma_h G_h\|_{L^2}\\ &\quad \le Ch^2\|\nabla u_h\|_{L^\infty}(\|\nabla G\|_{L^2} + \|Q\|_{L^2}).\end{aligned} \tag{5.29}$$

Using the estimates of $\|\nabla G\|_{L^2}$, $\|Q\|_{L^2}$, $\|\nabla u_h\|_{L^\infty}$, $\|\nabla(G-G_h)\|_{L^1}$, and $\|Q-Q_h\|_{L^1}$ again, we find that

$$\|\nabla(u-u_h)\|_{L^\infty} \le \|\nabla(I_h u - u)\|_{L^\infty} + \|\nabla e\|_{L^\infty},$$

which, together with (3.1)–(3.2), (5.27)–(5.29), Theorem 4.6, and Lemma 5.1, gives the desired result. □

It is worth noticing that the analysis for $\|p-p_h\|_{L^\infty}$ is still required the stability result in the L^∞-norm for the velocity and pressure, the Stokes projection (U_h, V_h), and assumption (**A3**). Then, the proof of the best approximation property will be given in the coming theorem.

Lemma 5.4 *Under the assumptions of Theorem 4.6 and* (**A3**)*, let* $\{\lambda, \tilde{u}(\lambda);\ \lambda \in \Lambda\}$ *and* $\{\lambda, \tilde{u}_h(\lambda);\ \lambda \in \Lambda\}$ *be a branch of nonsingular solutions of (2.2) (or 2.5) and (3.12), respectively. Then it holds that*

$$\|p-p_h\|_{L^\infty} \le Ch(|u|_{W^{2,\infty}} + |p|_{W^{1,\infty}} + \|f\|_{H^1}). \tag{5.30}$$

Proof Using (5.21) and (5.22) and setting $\eta = J_h p - p_h$ gives

$$\begin{aligned}\|\eta\|_{L^\infty} = {} & a(u-u_h, U-U_h) + \lambda d(u-u_h, V-V_h) - \lambda d(U-U_h, p-J_h p)\\ & +(B, p-p_h)+(\delta_M, J_h p-p)-\lambda b(u-u_h, u, U_h)+\lambda b(u-u_h, u-u_h, U_h)\\ & -\lambda b(u, u-u_h, U_h) - \lambda b(u_h, u_h, U_h - \Gamma_h U_h) + \lambda b(u-u_h, u, U)\\ & +\lambda b(u, u-u_h, U) + \lambda(f, U_h - \Gamma_h U_h).\end{aligned} \tag{5.31}$$

Thanks to Lemmas 4.1 and 5.3, and the Hölder inequality, we see that

$$\begin{aligned}&|a(u-u_h, U_h-U) - d(u-u_h, V_h-V) + d(U_h-U, p-J_h p)|\\ &\quad \le (\|\nabla(u-u_h)\|_{L^\infty} + \|p-J_h p\|_{L^\infty})(\|\nabla(U_h-U)\|_{L^1} + \|V_h-V\|_{L^1})\\ &\quad \le Ch(|u|_{W^{2,\infty}} + |p|_{W^{1,\infty}} + \|f\|_{H^1})(\|\nabla(U_h-U)\|_{L^1} + \|V_h-V\|_{L^1}).\end{aligned}$$

By (3.1), Theorem 4.6, and a simple calculation, we obtain

$$\begin{aligned}|(B, p-p_h) + (\delta_M, J_h p - p)| &\le C(\|p-p_h\|_{L^2} + \|p-J_h p\|_{L^\infty})\\ &\le Ch(\|u\|_{W^{2,\infty}} + |p|_{W^{1,\infty}} + \|f\|_{H^1}).\end{aligned}$$

Furthermore, we deduce from (2.7), (5.19a) and (5.29) that

$$\begin{aligned}|b(u-u_h, u, U) + b(u, u-u_h, U)| &\le C\|Au\|_{L^2}\|u-u_h\|_{L^2}\|\nabla U\|_{L^2},\\ |b(u-u_h, u, U_h) + b(u, u-u_h, U_h)| &\le C\|Au\|_{L^2}\|u-u_h\|_{L^2}(\|\nabla U\|_{L^2} + \|V\|_{L^2}),\\ |b(u-u_h, u-u_h, U_h)| &\le C\|\nabla(u-u_h)\|_{L^2}^2(\|\nabla U\|_{L^2} + \|V\|_{L^2}),\\ |b(u_h, u_h, U_h - \Gamma_h U_h)| &\le Ch^2\|\nabla u_h\|_{L^\infty}(\|\nabla U\|_{L^2} + \|V\|_{L^2}),\\ |(f, U_h - \Gamma_h U_h)| &\le Ch^2\|f\|_{H^1}(\|\nabla U\|_{L^2} + \|V\|_{L^2}).\end{aligned}$$

Combining all these inequalities with the estimates of $\|\nabla(U-U_h)\|_{L^1}$, $\|V-V_h\|_{L^1}$, $\|\nabla U\|_{L^2}$ and $\|V\|_{L^2}$ and using Theorem 4.6 and Lemma 5.1, we finally obtain the desired estimate. □

Now, the main result in the L^∞-norm for the velocity and pressure is summarized in the next theorem.

Theorem 5.5 *Under the assumptions of Theorem 4.6 and* **(A3)**, *let* $\{\lambda, \tilde{u}(\lambda);\ \lambda \in \Lambda\}$ *and* $\{\lambda, \tilde{u}_h(\lambda);\ \lambda \in \Lambda\}$ *be a set of nonsingular solutions of (2.2) (or 2.5) and (3.12), respectively. Then it holds that*

$$\|\nabla(u-u_h)\|_{L^\infty} + \lambda\|p-p_h\|_{L^\infty} \le Ch(|u|_{W^{2,\infty}} + |p|_{W^{1,\infty}} + \|f\|_{H^1}).$$

6 Conclusion

In this paper, we have performed an optimal L^2, H^1 and L^∞ analysis for a finite volume method for the stationary 2D Navier–Stokes equations with large data by using new techniques. Using these techniques, together with some inequalities in [19,22], we can also carry out a similar optimal L^∞ analysis for the same finite volume method for the stationary 3D Navier–Stokes equations.

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