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C^0-nonconforming tetrahedral and cuboid elements for the three-dimensional fourth order elliptic problem

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Abstract In this paper, a theoretical framework is constructed on how to develop C^0-nonconforming elements for the fourth order elliptic problem. By using the bubble functions, a simple practical method is presented to construct one tetrahedral C^0-nonconforming element and two cuboid C^0-nonconforming elements for the fourth order elliptic problem in three spacial dimensions. It is also proved that one element is of first order convergence and other two are of second order convergence. From the best knowledge of us, this is the first success in constructing the second-order convergent nonconforming element for the fourth order elliptic problem.

Mathematics Subject Classification (2000) 65N15 · 65N30

1 Introduction

In this paper we consider the following three-dimensional fourth order elliptic boundary value problem:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset R^3$ is a bounded convex domain with Lipschitz continuous boundary $\partial\Omega$, $f \in L^2(\Omega)$, n is the unit vector outer normal to $\partial\Omega$ and Δ is the standard Laplacian operator. The fourth order elliptic problem is not only important from the mathematical point of view but also potentially of practical importance with many applications. It has been widely used to model the linear plates in the two-dimensional space [8] and the three-dimensional biharmonic operator has shown its importance in the study of the complex microstructure evolutions for many material processes [3]. For the detailed description, please refer to [21] and the reference therein.

There have been an enormous amount of research work, and still growing, on the numerical analysis of the finite element methods for the fourth order elliptic problems. When a conforming finite element is employed to discretize the fourth order problem (1.1), it should consist of piecewise polynomials that are globally continuously differentiable (C^1). To meet this smoothness requirement, it is forced to use polynomials of degree five or higher in the two-dimensional space. For example, the Argyris element [6] with 5-degree polynomials and 21 degrees of freedom, the Bell element [6] with incomplete 5-degree polynomials and 18 degrees of freedom are conforming triangular elements; the Bogner-Fox-Schmit (BFS) element [6] with bicubic polynomials and 16 degrees of freedom is a conforming rectangular element. A conforming rectangular element with biquadratic polynomials and 25 degrees of freedom was constructed in [5]. The convergence rate of this element is one order higher than that of the BFS element. In the three-dimensional case, the situation is more complicated. Even higher order polynomials are needed to construct a conforming finite element. A conforming tetrahedral element was constructed in [23] using 9-degree polynomials and requiring C^1 globally, C^2 on all element edges and C^4 on all element vertices. The number of degrees of freedom is 220. A three-dimensional conforming BFS element on the cuboid mesh with tri-cubic polynomials and 64 degrees of freedom was constructed in [5]. This element is of second order convergence.

Using higher order derivatives, the constructions of conforming elements for the fourth order problems are complicated and not computationally desirable. As a result, many lower degree nonconforming elements in the two-dimensional case have been constructed and used in practice. The Morley element [6,7,9] with 2-degree polynomials and 6 degrees of freedom, the Veubeke-1 element [4,18] with incomplete 3-degree polynomials and 9 degrees of freedom, and the Veubeke-2 element [4,18] with 3-degree polynomials and 10 degrees of freedom, are triangular elements and even not C^0-continuous. The Zienkiewicz element [6,9] with incomplete 3-degree polynomials and 9 degrees of freedom is a C^0-triangular element, but it is convergent only on some special meshes [12], because the mean values of normal derivatives on the boundary of the element are not continuous across the element. The Adini or ACM element [6,9] with incomplete 4-degree polynomials and 12 degrees of freedom is a C^0-rectangular element, the mean values of normal derivatives on the boundary of the element are not continuous across the element. Its convergence depends on the special geometric property of the rectangular mesh. Quasi-conforming elements [17,24], generalized-conforming elements [10,14] and double set parameter elements [4] are nonstandard elements, we do not describe them in detail here.

In [15], Stummel presented a sufficient and necessary condition for the convergence of nonconforming finite elements, named Generalized Patch-Test, but it is difficult to use in practice. In [13], Shi presented a sufficient condition, named F-E-M Test, which is easier to use in practice. For the fourth order elliptic problem, to satisfy the strong F-E-M Test, the function values and the first-order derivatives of the shape functions should be continuous in the mean across the elements. In the three-dimensional case, it makes the order of element interpolation matrix very high. As a result, it is difficult to check the nonsingularity of this matrix. So it is a nightmare to construct nonconforming elements for the fourth order elliptic problem in the three-dimensional space. Recently, great efforts have been made in successfully constructing some nonconforming elements for the three-dimensional fourth order elliptic problem, see e.g., [19–22]. On the tetrahedral meshes, the three-dimensional Morley element was presented in [22]; a 3-degree polynomial element, an incomplete 3-degree polynomial element, the three-dimensional Zienkiewicz element, and a quasi-conforming element by modifying the three-dimensional Zienkiewicz element were presented in [19,21]; in [16], a C^0-element was presented for the Darcy–Stokes flow problem. It is a modified form of three-dimensional Morley element. It was also pointed out in [16] that this element can potentially be used to the three-dimensional fourth order elliptic singular perturbation problem. On the cuboid meshes, the three-dimensional Morley-type element, the three-dimensional Adini element, and the three-dimensional BFS-type element were presented in [20]. All of the above nonconforming elements are first order convergent and are the generalizations of the corresponding two-dimensional elements. Among them, three-dimensional Zienkiewicz element and three-dimensional BFS-type element are C^0-continuous, while others are non-C^0-continuous. It should be pointed out that the above three-dimensional BFS-type element is different from that in [5]. This BFS-type element is nonconforming and only first order convergent, while the one in [5] is conforming and of second order convergence.

In this paper we present a method to construct C^0-nonconforming elements for the fourth order elliptic problem. The idea of this method is to divide the shape function space into two subspaces by using bubble functions. One subspace is responsible for the C^0-continuity of the shape functions and getting the approximation error. Another one which contains the bubble functions is responsible for the continuity in the mean of the normal derivatives of the shape functions across the elements and getting the consistence error. The resulting element interpolation matrix is a block lower triangular matrix which greatly simplifies the proof of the nonsingularity of this matrix. Using this method, we construct one tetrahedral C^0-nonconforming element and two cuboid C^0-nonconforming elements for the three-dimensional fourth order problem. The method to construct the C^0 nonconforming element with the bubble function was also used in [11] for the two-dimensional fourth order elliptic singular perturbation problem. We also prove that one element is of first order convergence and other two are of second order convergence.

The main contributions of this paper are: an abstract convergence theorem is given, which builds a theoretical frame to construct C^0-nonconforming elements for the fourth order elliptic problem; the use of the bubble function gives a simple practical method to construct C^0-nonconforming elements for the three-dimensional fourth order elliptic problem. The rest of the paper is organized as follows. Section 2 gives an abstract convergence theorem. Sections 3 and 4 give detailed descriptions of one tetrahedral element and two cuboid elements, respectively. Section 5 gives the convergent analysis and Sect. 6 contains some concluding remarks.

2 An abstract convergence theorem

The weak form of (1.1) is: find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = f(v), \quad \forall v \in H_0^2(\Omega), \tag{2.1}$$

where

$$a(u, v) = \int_\Omega \sum_{i,j=1}^{3} \partial_{ij} u \partial_{ij} v dx, \quad f(v) = \int_\Omega f v dx. \tag{2.2}$$

Here $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. We adopt the standard notation $H^m(\Omega)$ for the Sobolev space [1] on Ω with norm

$$\|v\|_{m,\Omega}^2 = \sum_{|\alpha| \le m} \|D^\alpha v\|_{0,\Omega}^2,$$

and semi-norm

$$|v|_{m,\Omega}^2 = \sum_{|\alpha| = m} \|D^\alpha v\|_{0,\Omega}^2,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is an index, $|\alpha| = \sum_{i=1}^{3} \alpha_i$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}$, $\|w\|_{0,\Omega}^2 = \int_\Omega w^2 dx$.

We set

$$H_0^m(\Omega) = \left\{ v \in H^m(\Omega);\ v = \frac{\partial^j v}{\partial n^j} = 0, \text{on } \partial\Omega,\ 1 \le j \le m-1 \right\}.$$

The energy norm of (2.1) is defined by

$$|||v||| = a(v, v)^{\frac{1}{2}} = |v|_{2,\Omega}.$$

By *Poincaré* inequality, it is well known that $|\cdot|_{2,\Omega}$ is a norm on $H_0^2(\Omega)$ and is equivalent to $\|\cdot\|_{2,\Omega}$, so (2.1) has the unique solution by the Lax–Milgram Theorem [6].

Let T_h be a triangulation of Ω into tetrahedrons or cuboids with mesh size h, $\Omega = \bigcup_{T \in T_h} T$, T be an element. The nonconforming finite element space V_h is a piecewise polynomial space such that $V_h \not\subset H_0^2(\Omega)$. The discrete problem of (2.1) is: find $u_h \in V_h$ satisfying

$$a_h(u_h, v_h) = f(v_h), \quad \forall v_h \in V_h, \tag{2.3}$$

where

$$a_h(u_h, v_h) = \sum_{T\in T_h} \int_T \sum_{i,j=1}^{3} \partial_{ij} u_h \partial_{ij} v_h dx. \tag{2.4}$$

The corresponding discrete energy norm is:

$$|||v_h|||_h = \left(\sum_{T\in T_h} |v_h|^2_{2,T} \right)^{\frac{1}{2}}. \tag{2.5}$$

Throughout this paper, we assume that T_h is regular and quasi-uniform, namely, it satisfies that:

$$h_T/\rho_T \le \sigma_1, \quad h_T/h_{T'} \le \sigma_2, \quad \forall T, T^{'} \in T_h, \quad \forall h. \tag{2.6}$$

where h_T and ρ_T are the diameters of T and the largest ball contained in T, respectively, $\sigma_1 > 0, \sigma_2 > 0$ are constants independent of h.

Let $F \subset \partial T$ be a face of T and $F_h = \{F; F \subset \partial T, T \in T_h\}$. Suppose $F = T \cap T^{'}$, define

$$[w]|_F = w|_{T\cap F} - w|_{T'\cap F}; \quad [w]|_F = w|_F, \quad \text{if } F \subset \partial\Omega.$$

The following result is the well known Strang Lemma (see [2] or [6]).

Lemma 2.1 *Assume that $|||\cdot|||_h$ is a norm of V_h. Let u and u_h be the solutions of* (2.1) *and* (2.3), *respectively, then*

$$|||u - u_h|||_h \le C \left(\inf_{v_h\in V_h} |||u - v_h|||_h + \sup_{w_h\in V_h} \frac{|a_h(u, w_h) - f(w_h)|}{|||w_h|||_h} \right), \tag{2.7}$$

where $C > 0$ is a constant independent of h.

The first term of (2.7) is the approximation error and the second term of (2.7) is the consistence error.

For any $F \in F_h$, let $n = (n_1, n_2, n_3)^T$ be the unit vector outer normal to F and τ, s be two unit vectors and orthogonal to each other on F, then we have

$$\partial_j = \beta_{\tau j}\partial_\tau + \beta_{sj}\partial_s + \beta_{nj}\partial_n, \quad \beta^2_{\tau j} + \beta^2_{sj} + \beta^2_{nj} = 1, \quad 1 \le j \le 3.$$

where

$$\partial_j = \frac{\partial}{\partial x_j}, \quad \partial_\tau = \frac{\partial}{\partial \tau}, \quad \partial_s = \frac{\partial}{\partial s}, \quad \partial_n = \frac{\partial}{\partial n}.$$

By Green's formula, for $u \in H^4(\Omega)$,

$$
\begin{aligned}
a_h(u, w_h) &= \sum_{T\in T_h} \int_T \sum_{i,j=1}^{3} \partial_{ij} u \partial_{ij} w_h dx \\
&= \sum_{T\in T_h} \sum_{i,j=1}^{3} \left\{ \int_{\partial T} (\partial_{ij} u \partial_j w_h n_i - \partial_{iij} u w_h n_j) ds + \int_T \partial_{iijj} u w_h dx \right\} \\
&= \sum_{T\in T_h} \int_{\partial T} \left\{ \sum_{i,j=1}^{3} \partial_{ij} u (\beta_{\tau j} \partial_\tau w_h + \beta_{sj} \partial_s w_h + \beta_{nj} \partial_n w_h) n_i - \partial_n \Delta u w_h \right\} ds \\
&\quad + \sum_{T\in T_h} \int_T \Delta^2 u w_h dx.
\end{aligned}
$$

Since $\Delta^2 u = f$, we have

$$
\begin{aligned}
&a_h(u, w_h) - f(w_h) \\
&= \sum_{T\in T_h} \int_{\partial T} \left\{ \sum_{i,j=1}^{3} \partial_{ij} u (\beta_{\tau j} \partial_\tau w_h + \beta_{sj} \partial_s w_h + \beta_{nj} \partial_n w_h) n_i - \partial_n \Delta u w_h \right\} ds.
\end{aligned}
$$

If $V_h \subset H_0^1(\Omega)$, then

$$
\forall F \subset \partial T, \quad \forall T \in T_h, \quad [w_h]|_F = [\partial_\tau w_h]|_F = [\partial_s w_h]|_F = 0,
$$

we get

$$
a_h(u, w_h) - f(w_h) = \sum_{T\in T_h} \sum_{i,j=1}^{3} \beta_{nj} \sum_{F\subset\partial T} \int_F \partial_{ij} u \partial_n w_h n_i ds, \quad \forall w_h \in V_h \subset H_0^1(\Omega). \tag{2.8}
$$

Since $H^4(\Omega)$ is dense in $H^3(\Omega)$ with norm $||\cdot||_{3,\Omega}$ and

$$
|a_h(u, w_h)| \le c|u|_{2,\Omega} |||w_h|||_h,
$$

$$
\left| \sum_{T\in T_h} \sum_{i,j=1}^{3} \beta_{nj} \sum_{F\subset\partial T} \int_F \partial_{ij} u \partial_n w_h n_i ds \right| \le c(h) ||u||_{3,\Omega} \left(\sum_{T\in T_h} ||w_h||_{2,T}^2 \right)^{\frac{1}{2}},
$$

we know that the formula (2.8) is also true for $u \in H^3(\Omega)$.

For the tetrahedral mesh, let I_k be the usual C^0 piecewise k-degree polynomial interpolation operator, and for the cuboid mesh, let I_k be the usual C^0 piecewise tri-k-degree polynomial interpolation operator, then it is well known that [2,6]:

$$
|v - I_k v|_{l,T} \le C h^{k+1-l} |v|_{k+1,T}, \quad 0 \le l \le k, \quad \forall v \in H^{k+1}(T). \tag{2.9}
$$

Now we give the following abstract convergence theorem for C^0-nonconforming elements for the fourth order elliptic problem.

Theorem 2.1 *Assume that* $|||\cdot|||_h$ *is a norm of* V_h. *Suppose that there is an integer* $m \geq 2$, *such that*

(H1) $V_h \subset H_0^1(\Omega)$,
(H2) $|||v - \Pi_h v|||_h \leq Ch^{m-1}|v|_{m+1,\Omega}, \ \forall v \in H^{m+1}(\Omega)$,
(H3) $\int_F p[\partial_n w_h]ds = 0, \ \forall p \in P_{m-2}(F), \ \forall F \in F_h, \ \forall w_h \in V_h$,

then

$$|||u - u_h|||_h \leq Ch^{m-1}|u|_{m+1,\Omega}. \tag{2.10}$$

Here u and u_h are the solutions of (2.1) and (2.3), respectively, $C > 0$ is a constant independent of h and Π_h is the finite element interpolation operator on V_h.

Proof By (H2),

$$\inf_{v_h \in V_h} |||u - v_h|||_h \leq |||u - \Pi_h u|||_h \leq Ch^{m-1}|u|_{m+1,\Omega}. \tag{2.11}$$

Let $\hat{T}$ be the reference element, G_T be the affine transformation from $\hat{T}$ to T and under $G_T : \hat{x} \mapsto x, \hat{T} \to T, \hat{F} \to F$, where $\hat{F}$ is the face of $\hat{T}$. Let

$$P_F v = \frac{1}{|F|}\int_F v ds,$$

then

$$P_F v = \frac{1}{|F|}\int_F v ds = \frac{1}{|\hat{F}|}\int_{\hat{F}} \hat{v} d\hat{s} = P_{\hat{F}}\hat{v}.$$

By (H3), we have

$$\begin{aligned}
&\sum_{T\in T_h}\sum_{i,j=1}^{3}\beta_{nj}\sum_{F\subset\partial T}\int_F (\partial_{ij}u)(P_F\partial_n w_h)n_i ds \\
&\quad = \sum_F\sum_{i,j=1}^{3} c_{ij}\left(\int_F \partial_{ij}u\right)\left(\int_F [\partial_n w_h]\right) = 0,
\end{aligned}$$

$$\begin{aligned}
&\sum_{T\in T_h}\sum_{i,j=1}^{3}\beta_{nj}\sum_{F\subset\partial T}\int_F (I_{m-2}\partial_{ij}u)(\partial_n w_h - P_F\partial_n w_h)n_i \\
&\quad = \sum_F\sum_{i,j=1}^{3} c_{ij}\left\{\int_F (I_{m-2}\partial_{ij}u)[\partial_n w_h] - \left(\int_F I_{m-2}\partial_{ij}u\right)\left(\int_F [\partial_n w_h]\right)/|F|\right\} = 0.
\end{aligned}$$

Then from (2.8), we get

$$
\begin{aligned}
&a_h(u, w_h) - f(w_h) \\
&= \sum_{T\in T_h} \sum_{i,j=1}^{3} \beta_{nj} \sum_{F\subset \partial T} \int_F (\partial_{ij}u - I_{m-2}\partial_{ij}u)(\partial_n w_h - P_F \partial_n w_h) n_i ds.
\end{aligned}
\tag{2.12}
$$

Put $\mu = \partial_{ij}u,\ \varphi = \partial_n w_h$, then by (2.9), trace Theorem [1] and scaling, we have

$$
\begin{aligned}
&\left| \int_F (\partial_{ij}u - I_{m-2}\partial_{ij}u)(\partial_n w_h - P_F\partial_n w_h) ds \right| \\
&= \left| \int_F (\mu - I_{m-2}\mu)(\varphi - P_F\varphi) ds \right| \\
&\leq Ch \int_{\hat F} |(\hat\mu - \hat{I_{m-2}}\hat\mu)(\hat\varphi - P_{\hat F}\hat\varphi)| d\hat s \\
&\leq Ch \|\hat\mu - \hat{I_{m-2}}\hat\mu\|_{0,\hat F} \|\hat\varphi - P_{\hat F}\hat\varphi\|_{0,\hat F} \\
&\leq Ch \|\hat\mu - \hat{I_{m-2}}\hat\mu\|_{1,\hat T} \|\hat\varphi - P_{\hat F}\hat\varphi\|_{1,\hat T} \\
&\leq Ch |\hat\mu|_{m-1,\hat T} |\hat\varphi|_{1,\hat T} \leq Ch^{m-1} |\mu|_{m-1,T} |\varphi|_{1,T} \\
&\leq Ch^{m-1} |u|_{m+1,T} |w_h|_{2,T}.
\end{aligned}
\tag{2.13}
$$

Substituting (2.13) into (2.12) we get

$$
\sup_{w_h\in V_h} \frac{|a_h(u, w_h) - f(w_h)|}{|||w_h|||_h} \leq Ch^{m-1} |u|_{m+1,\Omega}. \tag{2.14}
$$

Then (2.10) follows from (2.7), (2.11) and (2.14). □

Remark 2.1 Theorem 2.1 is also true for the C^0-triangular and rectangular nonconforming elements for the two-dimensional fourth order elliptic problem.

3 A C^0-nonconforming tetrahedral element

Let T be the tetrahedral element with nodes a_i, $1 \leq i \leq 4$. The face of T opposites to a_i is denoted by F_i, $1 \leq i \leq 4$. The volume coordinates, named λ_i, $1 \leq i \leq 4$, have the following properties [6]:

$$
\lambda_i \in P_1(T), \quad \lambda_i(a_j) = \delta_{ij}, \quad \sum_{i=1}^{4} \lambda_i = 1, \quad \lambda_i|_{F_i} = 0, \quad 1 \leq i, j \leq 4, \tag{3.1}
$$

where $P_k(T)$ is the polynomial space of degree not greater than k.

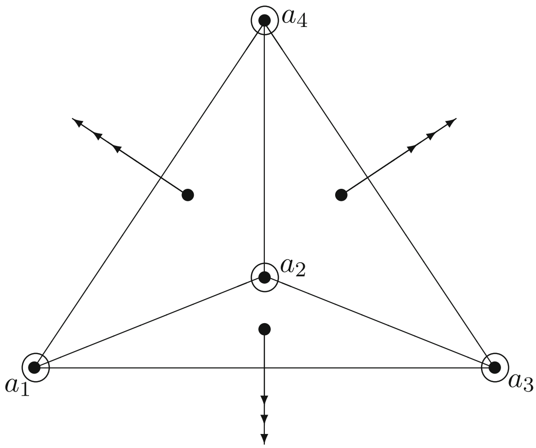

Fig. 1 Degrees of freedom of C^0T2 element

The shape function space of C^0T2 element is taken as

$$P_{T2} = P_3(T) \oplus b_T\{\lambda_i\lambda_{i+1}, \lambda_i\lambda_{i+1}\lambda_{i+2}, \lambda_i^2\lambda_{i+1}\lambda_{i+2},\ 1 \le i \le 4, mod\ 4\}, \quad (3.2)$$

It is easy to see that the dimension of P_{T2} is 32. The degrees of freedom are given as follows:

$$v_i, \quad v_{ix_j}, \quad 1 \le j \le 3, \quad v_{i0}, \quad \int_{F_i} \frac{\partial v}{\partial n} pds, \quad p \in P_1(F_i), \quad 1 \le i \le 4. \quad (3.3)$$

Here $v_i = v(a_i)$, $v_{ix_j} = \frac{\partial v}{\partial x_j}(a_i)$, $1 \le j \le 3$, $v_{i0} = v(a_{i0})$, and a_{i0} is the barycenter of F_i, $1 \le i \le 4$ (Fig. 1).

The corresponding interpolation operator $\Pi_{T2} : H^4(T) \to P_{T2}$ is defined by

$$\begin{cases} (v - \Pi_{T2}v)(a_i) = 0, \quad (v - \Pi_{T2}v)(a_{i0}) = 0, \quad \frac{\partial(v-\Pi_{T2}v)}{\partial x_j}(a_i) = 0, \ 1 \le j \le 3, \\ \int_{F_i} \frac{\partial(v-\Pi_{T2}v)}{\partial n} pds = 0, \quad p \in P_1(F_i), \quad 1 \le i \le 4. \end{cases} \quad (3.4)$$

Lemma 3.1 *The interpolation operator Π_{T2} is well posed, namely, the degrees of freedom* (3.3) *are P_{T2}-unisolvent.*

Proof It is easy to see that the number of degrees of freedom (3.3) is also 32, so it is sufficient to show that if $v \in P_{T2}$ such that all the degrees of freedom of v are zero, then $v \equiv 0$.

Let b_1, b_2, b_3 be the vertices of F_i and b_0 be the barycenter of F_i, τ and s be the unite vectors on F_i and orthogonal each other, by (3.3) we have

$$v(b_i) = \frac{\partial v}{\partial \tau}(b_i) = \frac{\partial v}{\partial s}(b_i) = v(b_0) = 0, \quad 1 \le i \le 3.$$

Since $v|_{F_i} \in P_3(F_i)$, we get

$$v|_{F_i} = 0, \quad 1 \le i \le 4. \tag{3.5}$$

By (3.2), v has the following expression

$$v = b_T q, \quad q = \sum_{i=1}^{4} (\alpha_i \lambda_i \lambda_{i+1} + \alpha_{i+4} \lambda_i \lambda_{i+1} \lambda_{i+2} + \alpha_{i+8} \lambda_i^2 \lambda_{i+1} \lambda_{i+2}).$$

By (3.3) we have

$$\int_{F_i} \frac{\partial v}{\partial n} p ds = 0, \quad p = \lambda_{i+1}, \lambda_{i+2}, \lambda_{i+3}.$$

Since $b_T|_{F_i} = \lambda_i|_{F_i} = 0$, we get

$$\int_{F_i} \frac{\partial v}{\partial n} p ds = \int_{F_i} \frac{\partial b_T}{\partial n} q p ds = \frac{\partial \lambda_i}{\partial n} \int_{F_i} \lambda_{i+1} \lambda_{i+2} \lambda_{i+3} q p|_{\lambda_i = 0} ds$$

$$= 0, \quad 1 \le i \le 4, \quad p = \lambda_{i+1}, \lambda_{i+2}, \lambda_{i+3}. \quad mod \ \ 4.$$

The above linear systems can be expressed by

$$AX = 0,$$

where $X = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{12})^T$,

$$A = \begin{pmatrix}
3 & 2 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{4}{15} & 0 & 0 & 0 \\
3 & 3 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 \\
2 & 3 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 \\
0 & 3 & 2 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{4}{15} & 0 & 0 \\
0 & 3 & 3 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 \\
0 & 2 & 3 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 \\
0 & 0 & 3 & 2 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{4}{15} & 0 \\
0 & 0 & 3 & 3 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{5} & 0 \\
0 & 0 & 2 & 3 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{5} & 0 \\
2 & 0 & 0 & 3 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{4}{15} \\
3 & 0 & 0 & 3 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{5} \\
3 & 0 & 0 & 2 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{5}
\end{pmatrix}.$$

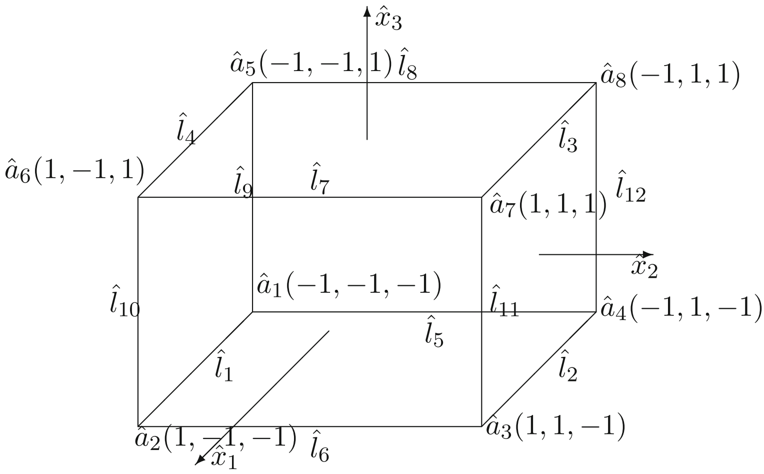

Fig. 2 Reference cube element

By simple computations, we get

$$\det A = \left(\frac{2}{45}\right)^4 \neq 0.$$

Hence $X = 0$, namely, $\alpha_i = 0, \quad 1 \leq i \leq 12$, then $v \equiv 0$. □

4 C^0-nonconforming cuboid elements

Let $\hat{T} = [-1, 1]^3$ be the reference cube element with nodes $\hat{a}_1(-1, -1, -1)$, $\hat{a}_2(1, -1, -1)$, $\hat{a}_3(1, 1, -1)$, $\hat{a}_4(-1, 1, -1)$, $\hat{a}_5(-1, -1, 1)$, $\hat{a}_6(1, -1, 1)$, $\hat{a}_7(1, 1, 1)$, $\hat{a}_8(-1, 1, 1)$. The 6 faces of $\hat{T}$ are defined by $\hat{F}_1 = \Box\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4$, $\hat{F}_2 = \Box\hat{a}_5\hat{a}_6\hat{a}_7\hat{a}_8$, $\hat{F}_3 = \Box\hat{a}_1\hat{a}_5\hat{a}_6\hat{a}_2$, $\hat{F}_4 = \Box\hat{a}_4\hat{a}_8\hat{a}_7\hat{a}_3$, $\hat{F}_5 = \Box\hat{a}_1\hat{a}_4\hat{a}_8\hat{a}_5$, $\hat{F}_6 = \Box\hat{a}_2\hat{a}_3\hat{a}_7\hat{a}_6$. The 12 edges of $\hat{T}$ are defined by $\hat{l}_1 = \hat{a}_1\hat{a}_2$, $\hat{l}_2 = \hat{a}_3\hat{a}_4$, $\hat{l}_3 = \hat{a}_7\hat{a}_8$, $\hat{l}_4 = \hat{a}_5\hat{a}_6$, $\hat{l}_5 = \hat{a}_1\hat{a}_4$, $\hat{l}_6 = \hat{a}_2\hat{a}_3$, $\hat{l}_7 = \hat{a}_6\hat{a}_7$, $\hat{l}_8 = \hat{a}_5\hat{a}_8$, $\hat{l}_9 = \hat{a}_1\hat{a}_5$, $\hat{l}_{10} = \hat{a}_2\hat{a}_6$, $\hat{l}_{11} = \hat{a}_3\hat{a}_7$, $\hat{l}_{12} = \hat{a}_4\hat{a}_8$. The middle points of $\hat{l}_i$ is denoted by $\hat{g}_i$, $1 \leq i \leq 12$. See Fig. 2.

Let

$$b_{\hat{T}} = (1 - \hat{x}_1^2)(1 - \hat{x}_2^2)(1 - \hat{x}_3^2).$$

Then $b_{\hat{T}}$ is the bubble function such that

$$b_{\hat{T}} \in Q_2(\hat{T}), \quad b_{\hat{T}}|_{\hat{F}_i} = 0, \quad 1 \leq i \leq 6,$$

where $Q_k(\hat{T})$ is the polynomial space of degree in each coordinate not greater than k.

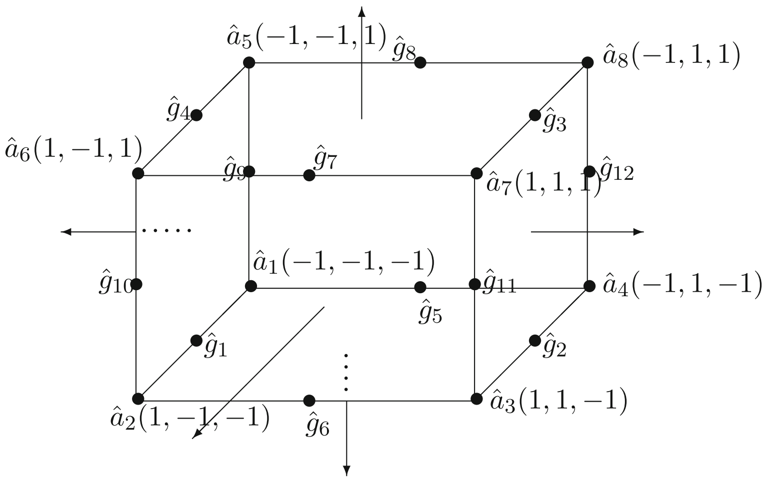

Fig. 3 Degrees of freedom of C^0C1 element

4.1 C^0C1 Element

The shape function space of C^0C1 element is taken as:

$$\hat{P}_{C1} = \hat{P}_2^* \oplus b_{\hat{T}}\{\hat{x}_i,\ \hat{x}_i^2,\ 1 \le i \le 3\}, \tag{4.1}$$

where $\hat{P}_2^* = P_2(\hat{T}) \oplus \{\hat{x}_1\hat{x}_2\hat{x}_3,\ \hat{x}_i^2\hat{x}_{i+1},\ \hat{x}_i^2\hat{x}_{i+2},\ \hat{x}_i^2\hat{x}_{i+1}\hat{x}_{i+2},\ 1 \le i \le 3,\ mod\ 3\}$. The dimension of $\hat{P}_{C1}$ is 26. The degrees of freedom are given as follows (Fig. 3):

$$\hat{v}(\hat{a}_i),\quad 1 \le i \le 8,\quad \hat{v}(\hat{g}_i),\quad 1 \le i \le 12,\quad \int_{\hat{F}_i} \frac{\partial \hat{v}}{\partial \hat{n}} d\hat{s},\quad 1 \le i \le 6. \tag{4.2}$$

The corresponding interpolation operator $\hat{\Pi}_{C1} : H^3(\hat{T}) \to \hat{P}_{C1}$ is defined by

$$\begin{cases} (\hat{v} - \hat{\Pi}_{C1}\hat{v})(\hat{a}_i) = 0, \quad 1 \le i \le 8, \quad (\hat{v} - \hat{\Pi}_{C1}\hat{v})(\hat{g}_i) = 0, \quad 1 \le i \le 12, \\ \int_{\hat{F}_i} \frac{\partial(\hat{v} - \hat{\Pi}_{C1}\hat{v})}{\partial \hat{n}} d\hat{s} = 0, \quad 1 \le i \le 6 \end{cases} \tag{4.3}$$

Lemma 4.1 *The interpolation operator $\hat{\Pi}_{C1}$ is well posed, namely, the degrees of freedom* (4.2) *are $\hat{P}_{C1}$-unisolvent.*

Proof Because the number of the degrees of freedom (4.2) are also 26, it is sufficient to show that if $\hat{v} \in \hat{P}_{C1}$ and

$$\hat{v}(\hat{a}_i) = 0, \quad 1 \le i \le 8, \quad \hat{v}(\hat{g}_i) = 0, \quad 1 \le i \le 12, \quad \int\limits_{\hat{F}_i} \frac{\partial \hat{v}}{\partial \hat{n}} d\hat{s} = 0, \quad 1 \le i \le 6.$$

Then $\hat{v} \equiv 0$.

Suppose that $\hat{F}_i$ is on $(\hat{x}_j, \hat{x}_{j+1})$ plane, it is easy to see that

$$\hat{v}|_{\hat{F}_i} \in P_2(\hat{F}_i) \oplus \{\hat{x}_j^2 \hat{x}_{j+1}, \hat{x}_j \hat{x}_{j+1}^2\},$$

and $\hat{v} = 0$, at the four vertices of $\hat{F}_i$ and the middle points of four sides of $\hat{F}_i$, hence

$$\hat{v}|_{\hat{F}_i} = 0, \quad 1 \le i \le 6. \tag{4.4}$$

By (4.1) $\hat{v}$ has the following expression

$$\hat{v} = b_{\hat{T}} \hat{q}, \quad \hat{q} = \sum_{i=1}^{3} (\alpha_i \hat{x}_i + \alpha_{i+3} \hat{x}_i^2)$$

Since $b_{\hat{T}}|_{\hat{F}_i} = 0, \ 1 \le i \le 6$, we have

$$\int\limits_{\hat{F}_i} \frac{\partial \hat{v}}{\partial \hat{n}} d\hat{s} = \int\limits_{\hat{F}_i} \frac{\partial b_{\hat{T}}}{\partial \hat{n}} \hat{q} d\hat{s} = 0, \quad 1 \le i \le 6.$$

The above linear systems can be expressed by

$$\hat{A}\hat{X} = 0,$$

where $\hat{X} = (\alpha_1, \alpha_2, \ldots, \alpha_6)^T$,

$$\hat{A} = \begin{pmatrix} 0 & 0 & -1 & \frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & 1 \\ 0 & -1 & 0 & \frac{1}{5} & 1 & \frac{1}{5} \\ 0 & 1 & 0 & \frac{1}{5} & 1 & \frac{1}{5} \\ -1 & 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} \\ 1 & 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{pmatrix}.$$

It is easy to get

$$\det \hat{A} = 14 \cdot \frac{4^3}{125} \ne 0,$$

then $\hat{X} = 0$, namely, $\alpha_i = 0, \quad 1 \le i \le 6$, and $\hat{v} \equiv 0$. □

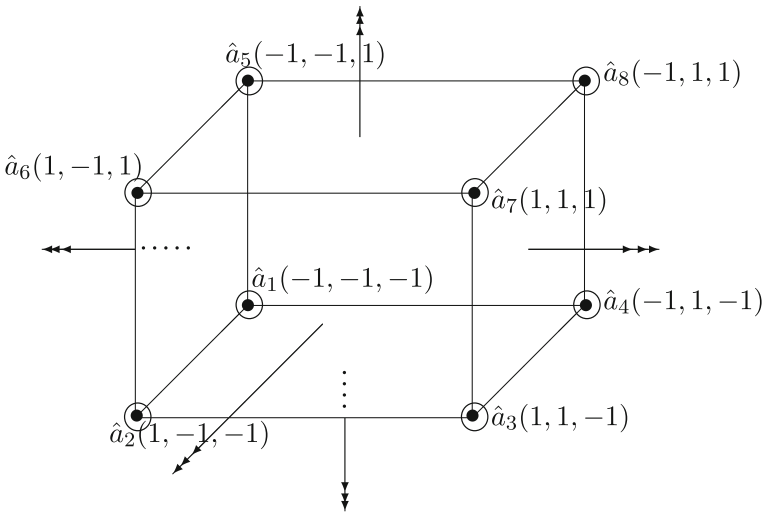

Fig. 4 Degrees of freedom of C^0C2 element

4.2 C^0C2 element

The shape function space of C^0C2 element is taken as :

$$\hat{P}_{C2} = \hat{P}_3^* \oplus b_{\hat{T}}\{\hat{x}_i, \quad \hat{x}_i^2, \quad \hat{x}_i^2\hat{x}_{i+1}, \quad \hat{x}_i^2\hat{x}_{i+2}, \quad \hat{x}_i^3\hat{x}_{i+1}, \quad \hat{x}_i^3\hat{x}_{i+2}, \quad 1 \le i \le 3 \quad mod\, 3\}, \tag{4.5}$$

where

$$\hat{P}_3^* = P_3(\hat{T}) \oplus \{\hat{x}_i^2\hat{x}_{i+1}\hat{x}_{i+2}, \quad \hat{x}_i^3\hat{x}_{i+1}, \quad \hat{x}_i^3\hat{x}_{i+2}, \quad \hat{x}_i^3\hat{x}_{i+1}\hat{x}_{i+2}, \quad 1 \le i \le 3, \quad mod\, 3\}.$$

The dimension of $\hat{P}_{C2}$ is $32 + 18 = 50$. The degrees of freedom are given as follows:

$$\hat{v}_i, \quad \hat{v}_{i\hat{x}_j}, \quad 1 \le i \le 8, \quad 1 \le j \le 3, \quad \int_{\hat{F}_i} \frac{\partial \hat{v}}{\partial \hat{n}} \hat{p} d\hat{s}, \quad \hat{p} \in P_1(\hat{F}_i) \quad 1 \le i \le 6, \tag{4.6}$$

where $\hat{v}_i = \hat{v}(\hat{a}_i),\ \hat{v}_{i\hat{x}_j} = \frac{\partial \hat{v}}{\partial \hat{x}_j}(\hat{a}_i),\ 1 \le i \le 8,\ 1 \le j \le 3$ (Fig. 4).

The corresponding interpolation operator $\hat{\Pi}_{C2} : H^3(\hat{T}) \to \hat{P}_{C2}$ is defined by

$$\begin{cases} (\hat{v} - \hat{\Pi}_{C2}\hat{v})(\hat{a}_i) = 0, \quad \frac{\partial(\hat{v} - \hat{\Pi}_{C2}\hat{v})}{\partial \hat{x}_j}(\hat{a}_i) = 0, \quad 1 \le i \le 8, \quad 1 \le j \le 3 \\ \int_{\hat{F}_i} \frac{\partial(\hat{v} - \hat{\Pi}_{C2}\hat{v})}{\partial \hat{n}} \hat{p} d\hat{s} = 0, \quad \hat{p} \in P_1(\hat{F}_i), \quad 1 \le i \le 6. \end{cases} \tag{4.7}$$

Lemma 4.2 *The interpolation operator* $\hat{\Pi}_{C2}$ *is well posed, namely, the degrees of freedom* (4.6) *are* $\hat{P}_{C2}$*-unisolvent.*

Proof Because the number of the degrees of freedom (4.6) is also 50, it is sufficient to show that if $\hat{v} \in \hat{P}_{C2}$ and

$$\hat{v}_i = 0, \quad \hat{v}_{i\hat{x}_j} = 0, \quad 1 \le i \le 8, \quad 1 \le j \le 3, \quad \int\limits_{\hat{F}_i} \frac{\partial \hat{v}}{\partial \hat{n}} \hat{p} d\hat{s} = 0, \quad \hat{p} \in P_1(\hat{F}_i), \quad 1 \le i \le 6,$$

then $\hat{v} \equiv 0$.

Suppose that $\hat{F}_i$ is on $(\hat{x}_j, \hat{x}_{j+1})$ plane, it is easy to see that

$$\hat{v}|_{\hat{F}_i} \in P_3(\hat{F}_i) \oplus \{\hat{x}_j^3 \hat{x}_{j+1}, \hat{x}_j \hat{x}_{j+1}^3\},$$

and

$$\hat{v} = \frac{\partial \hat{v}}{\partial \hat{x}_j} = \frac{\partial \hat{v}}{\partial \hat{x}_{j+1}} = 0,$$

at the four vertices of $\hat{F}_i$, it is just the construction of Adini element, then

$$\hat{v}|_{\hat{F}_i} = 0, \quad 1 \le i \le 6. \tag{4.8}$$

By (4.5), $\hat{v}$ has the following expression

$$\hat{v} = b_{\hat{T}} \hat{q}, \quad \hat{q} = \sum_{i=1}^{3} (\alpha_i \hat{x}_i + \alpha_{i+3} \hat{x}_i^2 + \alpha_{i+6} \hat{x}_i^2 \hat{x}_{i+1} + \alpha_{i+9} \hat{x}_i^2 \hat{x}_{i+2} + \alpha_{i+12} \hat{x}_i^3 \hat{x}_{i+1} + \alpha_{i+15} \hat{x}_i^3 \hat{x}_{i+2})$$

Since $b_{\hat{T}}|_{\hat{F}_i} = 0, \quad 1 \le i \le 6$, we have

$$\int\limits_{\hat{F}_i} \frac{\partial \hat{v}}{\partial \hat{n}} \hat{p} d\hat{s} = \int\limits_{\hat{F}_i} \frac{\partial b_{\hat{T}}}{\partial \hat{n}} \hat{q} \hat{p} d\hat{s} = 0, \quad \hat{p} \in P_1(\hat{F}_i), \quad 1 \le i \le 6. \tag{4.9}$$

We arrange (4.9) according to the following order:

(1) on $\hat{F}_1, \hat{x}_3 = -1, \hat{p} = 1$, (2) on $\hat{F}_2, \hat{x}_3 = 1, \hat{p} = 1$, (3) on $\hat{F}_1, \hat{x}_3 = -1, \hat{p} = \hat{x}_1$, (4) on $\hat{F}_2, \hat{x}_3 = 1, \hat{p} = \hat{x}_1$, (5) on $\hat{F}_1, \hat{x}_3 = -1, \hat{p} = \hat{x}_2$, (6) on $\hat{F}_2, \hat{x}_3 = 1, \hat{p} = \hat{x}_2$, (7) on $\hat{F}_3, \hat{x}_2 = -1, \hat{p} = 1$, (8) on $\hat{F}_4, \hat{x}_2 = 1, \hat{p} = 1$, (9) on $\hat{F}_3, \hat{x}_2 = -1, \hat{p} = \hat{x}_1$, (10) on $\hat{F}_4, \hat{x}_2 = 1, \hat{p} = \hat{x}_1$, (11) on $\hat{F}_3, \hat{x}_2 = -1, \hat{p} = \hat{x}_3$, (12) on $\hat{F}_4, \hat{x}_2 = 1$, $\hat{p} = \hat{x}_3$, (13) on $\hat{F}_5, \hat{x}_1 = -1, \hat{p} = 1$, (14) on $\hat{F}_6, \hat{x}_1 = 1, \hat{p} = 1$, (15) on $\hat{F}_5$, $\hat{x}_1 = -1, \hat{p} = \hat{x}_2$, (16) on $\hat{F}_6, \hat{x}_1 = 1, \hat{p} = \hat{x}_2$, (17) on $\hat{F}_5, \hat{x}_1 = -1, \hat{p} = \hat{x}_3$, (18) on $\hat{F}_6, \hat{x}_1 = 1, \hat{p} = \hat{x}_3$.

(4.9) can be expressed as

$$\hat{A}\hat{X} = 0,$$

where $\hat{X} = (\alpha_1, \alpha_2, \ldots, \alpha_{18})^T$, and

$$\hat{A} = \begin{pmatrix}
0 & 0 & -1 & \frac{1}{5} & \frac{1}{5} & 1 & 0 & -\frac{1}{5} & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & 1 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{5} & 0 & 0 & 0 & -1 & -\frac{3}{7} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{5} & 0 & 0 & 0 & 1 & \frac{3}{7} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{3}{7} & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 1 & 0 & \frac{3}{7} & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & \frac{1}{5} & 1 & \frac{1}{5} & -\frac{1}{5} & 0 & 0 & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & \frac{1}{5} & 1 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 1 & 0 & -\frac{3}{7} & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 1 & 0 & \frac{3}{7} & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \frac{1}{5} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -\frac{3}{7} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \frac{1}{5} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{3}{7} \\
-1 & 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & -\frac{1}{5} & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{5} & -1 & 0 & 0 & 0 & -\frac{3}{7} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{5} & 1 & 0 & 0 & 0 & \frac{3}{7} & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{5} & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{7} & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{5} & 1 & 0 & 0 & 0 & 0 & 0 & \frac{3}{7} & 1 & 0 & 0
\end{pmatrix}.$$

By some computations, we obtain

$$\det \hat{A} = \frac{2^{34}}{7^5 \cdot 5^6} \neq 0.$$

Then $\hat{X} = 0$, namely, $\alpha_i = 0, \quad 1 \leq i \leq 18$, and $\hat{v} \equiv 0$. □

5 Convergence analysis

For the tetrahedral mesh T_h, the finite element space for the C^0T2 element is defined by

$$V_{hT2} = \left\{ v_h : v_h|_T \in P_{T2}, [v_h]|_{\partial T} = 0, \int\limits_{\partial T} \left[\frac{\partial v_h}{\partial n} \right] p ds = 0, \right.$$
$$\left. \forall p \in P_1(\partial T), \quad \forall T \in T_h \right\}. \tag{5.1}$$

The finite element interpolation operator $\Pi_{hT2} : H^4(\Omega) \to V_{hT2}$ is defined by

$$\Pi_{hT2}|_T = \Pi_{T2}, \quad \forall T \in T_h.$$

For the cuboid mesh T_h, let $T \in T_h$ be an element with the center (x_{10}, x_{20}, x_{30}) and h_{T1}, h_{T2}, h_{T3} be the lengths of T along x_1, x_2, x_3 coordinates, respectively.
The affine transformation $x = F(\hat{x}) : \hat{T} \to T$ is

$$x_i = h_{Ti}\hat{x}_i + x_{i0}, \quad 1 \le i \le 3.$$

Under $x = F(\hat{x})$, let $\hat{a}_i \leftrightarrow a_i, \ 1 \le i \le 8; \ \hat{F}_i \leftrightarrow F_i, 1 \le i \le 6; \ \hat{l}_i \leftrightarrow l_i, \ \hat{g}_i \leftrightarrow g_i, \ 1 \le i \le 12; \quad \hat{P}_{C1} \leftrightarrow P_{C1}, \quad \hat{P}_{C2} \leftrightarrow P_{C2}; \quad \hat{v}(\hat{x}) = v(x)$. Then the degrees of freedom of P_{C1} on T are

$$v_i, \quad 1 \le i \le 8, \quad v(g_i), \quad 1 \le i \le 12, \quad \int\limits_{F_i} \frac{\partial v}{\partial n} ds, \quad 1 \le i \le 6. \tag{5.2}$$

The corresponding interpolation operator $\Pi_{C1} : H^3(T) \to P_{C1}$ satisfies that

$$\begin{cases} (v - \Pi_{C1}v)(a_i) = 0, \quad 1 \le i \le 8, \quad (v - \Pi_{C1}v)(g_i) = 0, \quad 1 \le i \le 12, \\ \int_{F_i} \frac{\partial (v - \Pi_{C1}v)}{\partial n} ds = 0, \quad 1 \le i \le 6. \end{cases} \tag{5.3}$$

It is easy to see that

$$(I - \Pi_{C1})v(x) = (\hat{I} - \hat{\Pi}_{C1})\hat{v}(\hat{x}), \quad x = F(\hat{x}). \tag{5.4}$$

Namely, the interpolation operate Π_{C1} is affine interpolation equivalent [2].
The finite element space for C^0C1 element is defined by

$$V_{hC1} = \left\{ v_h : v_h|_T \in P_{C1}, [v_h]|_{\partial T} = 0, \int\limits_{\partial T} \left[\frac{\partial v_h}{\partial n} \right] ds = 0, \forall T \in T_h \right\}. \tag{5.5}$$

The corresponding finite element interpolation operator $\Pi_{C1} : H^3(\Omega) \to V_{hC1}$ is defined by

$$\Pi_{hC1}|_T = \Pi_{C1}, \quad \forall T \in T_h.$$

For C^0C2 element, the degrees of freedom are :

$$v_i, \quad v_{ix_j}, \quad 1 \le i \le 8, \quad 1 \le j \le 3, \quad \int\limits_{F_i} \frac{\partial v}{\partial n} p ds, \quad p \in P_1(F_i) \quad 1 \le i \le 6. \tag{5.6}$$

where $v_i = v(a_i)$, $v_{ix_j} = \frac{\partial v}{\partial x_j}(a_i)$, $1 \le i \le 8$, $1 \le j \le 3$.

The corresponding interpolation operator $\Pi_{C2} : H^4(T) \to P_{C2}$ satisfies that

$$\begin{cases} (v - \Pi_{C2}v)(a_i) = 0, \quad \frac{\partial(v-\Pi_{C2}v)}{\partial x_j}(a_i) = 0, \quad 1 \le i \le 8, \quad 1 \le j \le 3 \\ \int_{F_i} \frac{\partial(v-\Pi_{C2}v)}{\partial n} p ds = 0, \quad p \in P_1(F_i), \quad 1 \le i \le 6. \end{cases} \tag{5.7}$$

It is easy to see that

$$(I - \Pi_{C2})v(x) = (\hat{I} - \hat{\Pi}_{C2})\hat{v}(\hat{x}), \quad x = F(\hat{x}). \tag{5.8}$$

Namely, the interpolation operate Π_{C2} is affine interpolation equivalent [2].

The finite element space for C^0C2 element is defined by

$$V_{hC2} = \Bigg\{ v_h : v_h|_T \in P_{C2}, [v_h]|_{\partial T} = 0, \\ \int\limits_{\partial T} \left[\frac{\partial v_h}{\partial n}\right] p ds = 0, \quad \forall p \in P_1(\partial T), \quad \forall T \in T_h \Bigg\}. \tag{5.9}$$

The corresponding finite element interpolation operator $\Pi_{C2} : H^4(\Omega) \to V_{hC2}$ is defined by

$$\Pi_{hC2}|_T = \Pi_{C2}, \quad \forall T \in T_h.$$

The discrete variational problems using the C^0T2, C^0C1, C^0C2 elements to solve (2.1) are:

Find $u_{hT2} \in V_{hT2}$ such that

$$a_h(u_{hT2}, v_h) = f(v_h), \quad \forall v_h \in V_{hT2}. \tag{5.10}$$

Find $u_{hC1} \in V_{hC1}$ such that

$$a_h(u_{hC1}, v_h) = f(v_h), \quad \forall v_h \in V_{hC1}. \tag{5.11}$$

Find $u_{hC2} \in V_{hC2}$ such that

$$a_h(u_{hC2}, v_h) = f(v_h), \quad \forall v_h \in V_{hC2}. \tag{5.12}$$

It is easy to check that $|||\cdot|||_h$ is a norm of V_{hT2}, V_{hC1}, V_{hC2}, respectively, so (5.10)–(5.12) are unisolvent by the Lax–Milgram Theorem [6].

To get the error estimates of the C^0T2, C^0C1 and C^0C2 elements, it is only needed to check (H1) (H2) (H3) of Theorem 2.1.

By (3.5), (4.4) and (4.8), it is easy to prove that

$$V_{hT2} \subset H_0^1(\Omega), \quad V_{hC1} \subset H_0^1(\Omega), \quad V_{hC2} \subset H_0^1(\Omega). \tag{5.13}$$

Because $P_3(T) \subset P_{T2}$, $P_2(T) \subset P_{C1}$, $P_3(T) \subset P_{C2}$, by the well-known interpolation theorem [2,6], we have

$$\begin{cases} |||u - \Pi_{hT2}u|||_h \leq Ch^2|u|_{r,\Omega}, \\ |||u - \Pi_{hC1}u|||_h \leq Ch|u|_{r,\Omega}, \\ |||u - \Pi_{hC2}u|||_h \leq Ch^2|u|_{r,\Omega}. \end{cases} \tag{5.14}$$

Here $u \in H_0^2(\Omega)$ is the solution of (2.1) with the additional regularity $u \in H^r(\Omega)$, where $r = 3$ for the C^0C1 element and $r = 4$ for the C^0T2 and C^0C2 elements. By the last sets of the degrees of freedom (3.3), (5.2) and (5.6), we obtain that

$$\begin{cases} \int_F \left[\frac{\partial w_h}{\partial n}\right] ds = 0, \quad \forall F \in F_h, \quad \forall w_h \in V_{hC1}, \\ \int_F p \left[\frac{\partial w_h}{\partial n}\right] ds = 0, \quad \forall F \in F_h, \quad \forall p \in P_1(F), \quad \forall w_h \in V_{hT2} \quad \text{or} \quad V_{hC2}. \end{cases} \tag{5.15}$$

By (5.13), (5.14), (5.15), we know that (H1) (H2) and (H3) are satisfied for C^0C1 with $m = 2$ and for C^0T2 and C^0C2 with $m = 3$. Then by Theorem 2.1, we obtain the following convergence theorem for the C^0T2, C^0C1 and C^0C2 elements.

Theorem 5.1 *Suppose that the mesh T_h, into tetrahedrons for the C^0T2 element and into cuboids for the C^0C1 and C^0C2 elements, is regular in the sense of* (2.6), *$u \in H_0^2(\Omega)$ is the solution of* (2.1) *with the additional regularity $u \in H^r(\Omega)$, and u_{hT2}, u_{hC1} and u_{hC2} are the solutions of* (5.10)–(5.12), *respectively, then*

$$\begin{cases} |||u - u_{hT2}|||_h \leq Ch^2|u|_{r,\Omega}, \\ |||u - u_{hC1}|||_h \leq Ch|u|_{r,\Omega}, \\ |||u - u_{hC2}|||_h \leq Ch^2|u|_{r,\Omega}, \end{cases} \tag{5.16}$$

where $r = 3$ for the C^0C1 element and $r = 4$ for the C^0T2 and C^0C2 elements.

6 Conclusion

In this paper, we proved an abstract convergence theorem, which builds a theoretical frame to construct C^0-nonconforming elements for the fourth order elliptic problem. It gives a direction on how to construct nonconforming elements for the fourth order elliptic problem with expected convergence order. Then we presented a method to construct the C^0-nonconforming elements for the fourth order elliptic problem by using the bubble functions which makes the element interpolation matrix being block lower triangular and easy to choose the matched shape function space and degrees of freedom. One tetrahedral C^0-nonconforming element and two cuboid C^0-nonconforming elements for the fourth-order elliptic problem in three spacial dimensions were constructed. The unisolvent of the degrees of freedom for these elements was proved clearly. And we have proved that one element are first order convergent and other two are second order convergent.

The method to construct the element with the bubble function can be used to any dimensional fourth-order elliptic problems. Furthermore, since the elements in this paper are C^0-nonconforming elements, the convergence results can be obtained easily for the second order elliptic problem. So it makes the application of these elements on the fourth order elliptic perturbation problem possible. It is our ongoing work.

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