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Analysis of a continuous finite element method for $H(\text{curl}, \text{div})$-elliptic interface problem

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Abstract In this paper, we develop a continuous finite element method for the curlcurl-grad div vector second-order elliptic problem in a three-dimensional polyhedral domain occupied by discontinuous nonhomogeneous anisotropic materials. In spite of the fact that the curlcurl-grad div interface problem is closely related to the elliptic interface problem of vector Laplace operator type, the continuous finite element discretization of the standard variational problem of the former generally fails to give a correct solution even in the case of homogeneous media whenever the physical domain has reentrant corners and edges. To discretize the curlcurl-grad div interface problem by the continuous finite element method, we apply an element-local L^2 projector to the curl operator and a pseudo-local L^2 projector to the div operator, where the continuous Lagrange linear element enriched by suitable element and face bubbles may be employed. It is shown that the finite element problem retains the same coercivity property as the continuous problem. An error estimate $\mathcal{O}(h^r)$ in an energy norm is obtained between the analytical solution and the continuous finite element solution,

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where the analytical solution is in $\prod_{l=1}^{L}(H^r(\Omega_l))^3$ for some $r \in (1/2, 1]$ due to the domain boundary reentrant corners and edges (e.g., nonconvex polyhedron) and due to the interfaces between the different material domains in $\Omega = \bigcup_{l=1}^{L} \Omega_l$.

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1 Introduction

In this paper we shall develop a continuous finite element method for the curlcurl-grad div interface problem in the physical domain $\Omega \subset \mathbb{R}^3$. To model different anisotropic nonhomogeneous materials filling Ω, we introduce two coefficient matrices $\mu, \varepsilon \in \mathbb{R}^{3\times 3}$, describing the physical properties, e.g., permeability and permittivity of the materials in electromagnetism. The curlcurl-grad div second-order elliptic system that we shall consider reads as follows:

$$\mathbf{curl}\ \mu^{-1}\,\mathbf{curl}\ \boldsymbol{u} - \varepsilon\ \nabla\ \mathrm{div}\,\varepsilon\,\boldsymbol{u} = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{1.1}$$

where $\boldsymbol{u}$ represents the unknown (e.g., electric field in electromagnetism), $\boldsymbol{f} \in (L^2(\Omega))^3$ denotes the known source function, the curl operator $\mathbf{curl}\ \boldsymbol{v} = \nabla \times \boldsymbol{v}$, and the div operator $\mathrm{div}\,\boldsymbol{v} = \nabla \cdot \boldsymbol{v}$, with ∇ denoting the gradient operator. Let ε and μ be piecewise smooth functions, i.e., there exists a partition of Ω into a finite number of subdomains Ω_l, $1 \le l \le L$, where μ and ε are smooth in all material subdomains Ω_l, but they are possibly discontinuous and differ greatly in magnitude across inter-subdomain-boundaries. We shall assume that Ω and Ω_l are simply-connected three-dimensional Lipschitz polyhedra with connected boundaries. As usual, it is assumed that μ, ε are symmetric, uniform boundedness, and piecewise smooth, and satisfy uniform ellipticity property, i.e., letting $\omega = (\omega_{ij}) \in \mathbb{R}^{3\times 3}$ denote any of μ and ε, we have

$$\omega_{ij} = \omega_{ji}, \quad \omega_{ij} \in L^\infty(\Omega), \quad \omega_{ij} \in W^{1,\infty}(\Omega_l), \quad 1 \le l \le L, \quad 1 \le i, j \le 3,$$

and there exists a constant $C_\omega > 0$ such that

$$C_\omega^{-1}\,|\xi|^2 \ge \xi^T \omega(x)\xi \ge C_\omega |\xi|^2, \quad \text{a.e. in} \quad \Omega, \forall \xi \in \mathbb{R}^3,$$

where, from the above, the inverse ω^{-1} of ω satisfies the same: $C_\omega^{-1}\,|\xi|^2 \ge \xi^T \omega^{-1}(x)\xi \ge C_\omega|\xi|^2$ a.e. in Ω and for all $\xi \in \mathbb{R}^3$, and $L^p(D)$ and $W^{s,p}(D)$ are classical Sobolev spaces over an open subset $D \subset \Omega$, see [1].

Denote by Γ the boundary of Ω and by $\Gamma_{ij} = \Omega_i \cap \Omega_j$ the material interfaces. Let $\boldsymbol{n}$ denote the outward unit normal vector to Γ or the unit normal vector to Γ_{ij} oriented from Ω_i to Ω_j. Along the interface Γ_{ij}, we define jumps $[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]|_{\Gamma_{ij}} = \boldsymbol{v}\cdot\boldsymbol{n}|_{\Omega_i} - \boldsymbol{v}\cdot\boldsymbol{n}|_{\Omega_j}$, $[\![\boldsymbol{v}\times\boldsymbol{n}]\!]|_{\Gamma_{ij}} = \boldsymbol{v}\times\boldsymbol{n}|_{\Omega_i} - \boldsymbol{v}\times\boldsymbol{n}|_{\Omega_j}$, and $[\![q]\!]|_{\Gamma_{ij}} = q|_{\Omega_i} - q|_{\Omega_j}$. We supplement Eq. (1.1) with the following boundary conditions

$$\boldsymbol{u}\times\boldsymbol{n} = 0, \quad \mathrm{div}\,\varepsilon\,\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma. \tag{1.2}$$

The unknown $\boldsymbol{u}$ additionally satisfies the following physical jump conditions on the interfaces

$$[\![\boldsymbol{u} \times \boldsymbol{n}]\!] = \mathbf{0}, \quad [\![\mu^{-1} \,\mathbf{curl}\; \boldsymbol{u} \times \boldsymbol{n}]\!] = \mathbf{0}, \quad [\![\varepsilon\, \boldsymbol{u} \cdot \boldsymbol{n}]\!] = 0, \quad [\![\mathrm{div}\, \varepsilon\, \boldsymbol{u}]\!] = 0 \quad \text{on} \quad \Gamma_{ij}. \tag{1.3}$$

We see that $\boldsymbol{u}$ belongs to $H_0(\mathbf{curl}; \Omega) \cap H(\mathrm{div}; \varepsilon; \Omega)$, with the curl and div Hilbert spaces defined as in [39],

$$\begin{aligned} H(\mathbf{curl}; \Omega) &= \{\boldsymbol{v} \in (L^2(\Omega))^3 : \mathbf{curl}\; \boldsymbol{v} \in (L^2(\Omega))^3\}, \\ H_0(\mathbf{curl}; \Omega) &= \{\boldsymbol{v} \in H(\mathbf{curl}; \Omega) : \boldsymbol{v} \times \boldsymbol{n}|_\Gamma = 0\}, \\ H(\mathrm{div}; \varepsilon; \Omega) &= \{\boldsymbol{v} \in (L^2(\Omega))^3 : \mathrm{div}\, \varepsilon\, \boldsymbol{v} \in L^2(\Omega)\}. \end{aligned}$$

Equation (1.1) is quite general, and it covers a wide range of models and forms in many physical problems involving curl and div operators. Two typical examples of (1.1) are from electromagnetism and fluid-structure interaction problems. In the case of vector potential Maxwell's equations, a model reads (See [12,20]):

$$\mathbf{curl}\;\; \mu^{-1}\, \mathbf{curl}\;\; \boldsymbol{u} = \boldsymbol{f}, \quad \mathrm{div}\, \varepsilon\, \boldsymbol{u} = 0 \quad \text{in} \quad \Omega. \tag{1.4}$$

A very detailed development and explanations of the problem (1.1)–(1.3) for time-harmonic Maxwell's equations in heterogeneous materials can be found in Section 1 of [31] and Section 7 of [29]. In fact, the 'strong' form of the variational formulation in [31] and [29] is exactly the same as (1.1)–(1.3), except that here the electromagnetic wave number/frequency is taken as zero. Another example which may be stated as the form (1.1) is the simplest model for fluid-structure interaction problem as follows (See [54]):

$$\boldsymbol{\nabla}\; \mathrm{div}\, \boldsymbol{u} = \boldsymbol{f}, \quad \mathbf{curl}\;\; \boldsymbol{u} = 0. \tag{1.5}$$

At the same time, Eq. (1.1) is widely-used for theoretical studies of many mathematical issues relating to the Maxwell's equations, including regularity-singularities (See [11,29–31,52]), solvability-uniqueness (See [2,9,12,19,20,27,36,41,53,59,60]), and numerical methods (See [15,28,42,45]). Therefore, the numerical method of (1.1) is a very important subject and may be used to motivate and justify those theoretical studies.

Also, Eq. (1.1) is closely related to the second-order elliptic problem of vector Laplace operator. As a matter of fact, if $\mu = \varepsilon = 1$, using the relationship

$$-\Delta \boldsymbol{u} = \mathbf{curl}\;\; \mathbf{curl}\;\; \boldsymbol{u} - \boldsymbol{\nabla}\, \mathrm{div}\, \boldsymbol{u}, \tag{1.6}$$

where Δ is the Laplace operator, we obtain a vector Poisson equation

$$-\Delta\, \boldsymbol{u} = \boldsymbol{f}. \tag{1.7}$$

Additionally, if Ω is smooth enough or is convex polyhedron, then the solution of problem (1.1) and (1.2) with $\mu = \varepsilon = 1$ is the same as that of the vector Poisson problem (1.7) and (1.2), and, moreover, the solution is in $(H^1(\Omega))^3$ (See [29] for more details).

Finite element discretizations of all these equations are not so simple as they look, though. As an illustration, let us consider the homogeneous case of (1.1) where $\mu = \varepsilon = 1$. Note that the analytical solution $\boldsymbol{u}$ belongs to $H(\mathbf{curl}; \Omega) \cap H(\mathrm{div}; \Omega)$, where $H(\mathrm{div}; \Omega) = H(\mathrm{div}; 1; \Omega)$. Thus, a natural choice of finite elements is the continuous Lagrange element as in the book [24], because a $H(\mathbf{curl}; \Omega) \cap H(\mathrm{div}; \Omega)$ conforming piecewise-polynomial element is necessarily in $(H^1(\Omega))^3$ and continuous. As is well known, indeed, it has been mathematically and practically justified in the literature [24,47–49,63] that the continuous finite element method is the most popular numerical method for solving (1.7), which is equivalent to (1.1) in a smooth domain or a convex polyhedron. Unfortunately, whenever the domain has reentrant corners/edges/ interface corners, the solution $\boldsymbol{u}$ is usually not in $(H^1(\Omega))^3$, instead, it is in a much larger $(H^r(\Omega))^3$ for some $r < 1$, see [9,11,29–31,52]. It is this very low regularity that causes enormous difficulties in the finite element solution of (1.1). By the continuous finite element method an incorrect solution may be obtained—the continuous finite element solution converges to a function of $(H^1(\Omega))^3$, but not to the solution which does not belong to $(H^1(\Omega))^3$, see [11,30,41–43]. It is better understood now that the reason is mainly at the standard variational formulation of (1.1) [5,9,29,31,33,36,41,49]:

$$(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}, \mathbf{curl}\;\boldsymbol{v}) + (\mathrm{div}\,\varepsilon\,\boldsymbol{u}, \mathrm{div}\,\varepsilon\,\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \tag{1.8}$$

where $\boldsymbol{u}, \boldsymbol{v} \in H_0(\mathbf{curl}; \Omega) \cap H(\mathrm{div}; \varepsilon; \Omega)$ and $(\cdot, \cdot)$ denotes the L^2 inner product.

Although this incorrect convergence has been known for quite some time in both mathematical and engineering communities, attempts have never been stopped to handle the low regularity solution with continuous elements, because there are so many well-known popular advantages of the continuous Lagrange elements. However, only over the last decade, three different continuous finite element methods have been developed which can give correct convergent finite element approximations of the low regularity solution. The first method of 'rehabilitation of nodal finite element' is proposed in [28], i.e., the weighted regularization method for Maxwell's problem (1.4) with $\mu = \varepsilon = 1$, where a weight function is applied to the div term. It uses a continuous finite element space including the gradient of some C^1 element [24]. Similar weighted methods have since been developed in [16,50]. The second method is the H^{-1} least-squares method [13] for the first-order system with a given $\boldsymbol{f}$ and g_0

$$\mathbf{curl}\;\boldsymbol{u} = \boldsymbol{f}, \quad \mathrm{div}\,\varepsilon\,\boldsymbol{u} = g_0, \tag{1.9}$$

We note that (1.9) may be also formulated into a least-squares variational problem according to the L^2 inner products, resulting in the same bilinear form as (1.8), with $\mu = 1$. In this least-squares formulation, the right-hand side of (1.8) should be accordingly replaced by $(\boldsymbol{f}, \mathbf{curl}\;\boldsymbol{v}) + (g_0, \mathrm{div}\varepsilon\boldsymbol{v})$. The H^{-1} method is to reformulate (1.9) into a least-squares variational problem based on the so-called 'minus-one' inner products, instead of the L^2 inner products. The third method is the element-local L^2 projection method in [33] for the Maxwell's problem (1.4) with $\mu = \varepsilon = 1$, where two element-local L^2 projectors are applied to both curl and div operators.

In most real applications, however, problem (1.1) is imposed in discontinuous, nonhomogeneous, anisotropic media. Most of the aforementioned continuous finite element methods either are not directly applicable or become inefficient for the general problem (1.1), due to different physical characteristics reflected by μ and ε of different media. In fact, the standard variational formulation (1.8) itself is not suitable for direct continuous finite element discretizations in the case of discontinuous ε, because $\operatorname{div} \varepsilon \boldsymbol{v}$ is not generally well-defined for continuous piecewise-polynomials $\boldsymbol{v}$. We have so far not seen how to apply the H^{-1} least-squares method of the first-order system (1.9) to the general curlcurl-grad div second-order system (1.1) even if $\mu = \varepsilon = 1$. When the element local L^2 projection method is directly applied to the general problem (1.1) with discontinuous ε, a *discontinuous* element method prevails [34]. This might be not consistent with the original intention of the use of the continuous elements.

There are a lot of finite element methods for scalar second-order elliptic interface problems (see, e.g., [18,51]) of Laplace operator type. But, the scalar solution therein at least belongs to $H^1(\Omega)$, and those finite element methods cannot be employed for problem (1.1). This is not only because the solution of (1.1) is often in $(H^r(\Omega))^3$ for some $r < 1$, but also because (1.1) is in general not a problem of vector Laplace operator type (unless $\mu = \varepsilon = 1$, cf. (1.6), (1.7)). It is also very interesting to mention a few non or nearly C^0 finite element methods for the Maxwell's interface problems in the case of piecewise constant coefficients. An edge element mixed/indefinite method is proposed in [22] for problem (1.4), where the classical edge or Nédélec element in [55,56] is used. In [23], a finite volume mixed/indefinite method is analyzed for the first-order systems of problem (1.4) in terms of electric and magnetic fields. A Singular Function method in [8] is considered for problem (1.9). See also an alternate Singular Function method in [4]. A nearly continuous mixed finite element method in the context of the weighted regularization method is also recently presented in [25] for the heterogeneous time-harmonic Maxwell's equations.

In this paper, we shall develop a continuous finite element method for the general problem (1.1) with discontinuous μ and ε. We employ the continuous Lagrange linear element in [24] enriched with element and face bubbles in order to approximate the low regularity solution. With two L^2 projections applied to both curl and div operators, plus a stabilization term $\mathcal{S}_h(\boldsymbol{u}, \boldsymbol{v})$, we propose the following finite element formulation

$$(R_h(\mu^{-1}\,\mathbf{curl}\ \boldsymbol{u}), R_h(\mu^{-1}\,\mathbf{curl}\ \boldsymbol{v}))_\mu + (\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{u}), \breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}))_{0,h} + \mathcal{S}_h(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}). \tag{1.10}$$

The operator R_h for the curl operator is an element-local L^2 projection which is defined element by element onto a discontinuous linear finite element space with respect to the μ-weighted L^2 inner product $(\cdot,\cdot)_\mu$, while $\breve{R}_h$ for the div operator is a pseudo-local L^2 projection which is defined onto a continuous linear finite element space, with respect to the mass-lumping L^2 inner product $(\cdot,\cdot)_{0,h}$ (the resulting matrix is then diagonal). The stabilization term $\mathcal{S}_h$ is defined as the element-wise analogue of the bilinear form of (1.8) weighted with local mesh sizes. Note that the computational cost of (1.10) is not much more than that of (1.8) (given that the latter can be directly discretized by the continuous finite element method), since all the additions in (1.10), compared with (1.8), are locally evaluated. We have noticed that the mesh-locally-weighted stabilization is also used in a recent method of H^{-1} type in [10].

With the stabilization role of $\mathcal{S}_h$, we show that the same coercivity property as the continuous problem (1.8) holds for (1.10)—a nice feature for using iterative methods to solve the resulting algebraic system. Just like the Poisson equation for Laplace operator, the associated condition number is $\mathcal{O}(h^{-2})$, with h being the mesh size. Both L^2 projectors R_h and $\breve{R}_h$, together with the element and face bubbles in the continuous finite element space of the solution, essentially remove the effects of curl and div partial derivatives on the analytical solution(See Theorem 4.2). With the Crouzeix-Raviart nonconforming linear element interpolations [32], an ad hoc strategy is designed to estimate the errors associated with the stabilization term which is explicitly involved with curl and div partial derivatives on the low regularity solution (See Lemma 4.5). Therefore, with respect to the regularity of the solution in material subdomains, i.e., $\boldsymbol{u} \in \prod_{l=1}^{L}(H^r(\Omega_l))^3$ for some $1/2 < r \le 1$, for a suitable regular right-hand side $\boldsymbol{f}$ (see a further explanation below), we prove the optimal error estimates $\mathcal{O}(h^r)$ in an energy norm between the analytical solution of (1.1) and the continuous finite element solution of (1.10).

We should point out that the two new key ingredients in (1.10) are the new stabilization $\mathcal{S}_h$—the element-wise analogue (weighted with local mesh sizes) of the bilinear form of (1.8) and the new pseudo-local L^2 projection $\breve{R}_h$. It is because of them that not only can we have a simple finite element method, but also can we employ the continuous elements even if ε is discontinuous. To the authors' knowledge, the continuous finite element method in this paper is the first attempt so far for the general curlcurl-grad div interface problem (1.1)–(1.3) with both μ and ε being discontinuous and nonhomogeneous and anisotropic.

Before ending this section we would like to make a remark about the regularity. The regularity of the solution may heavily depend on that of the coefficients, the domain and the right-hand side. A general dependence of the regularity of the solution on μ^{-1}, ε, Ω and $\boldsymbol{f}$ in the interface problem (1.1)–(1.3) does not seem to be available in the literature, but readers may refer to [31] for some results in the case of the heterogeneous time-harmonic Maxwell's equations. Since this issue is beyond the scope of this paper, we have made two hypotheses Hypothesis H1 and Hypothesis H2 on the regularity of the solution for technical needs when deriving the error estimates. For example, we shall use a regular-singular decomposition, which relies on the regularity of the domain boundary Γ, the material interfaces, and even the number of material subdomains, see [31]. In this paper, the regularity assumptions for the solution, the coefficients and the right-hand side can be stated as follows: $\boldsymbol{u} \in H_0(\mathbf{curl};\Omega) \cap H(\mathrm{div};\varepsilon;\Omega) \cap \prod_{l=1}^{L}(H^r(\Omega_l))^3$, $\mu^{-1}\mathbf{curl}\,\boldsymbol{u} \in H(\mathbf{curl};\Omega) \cap H_0(\mathrm{div};\mu;\Omega) \cap \prod_{l=1}^{L}(H^r(\Omega_l))^3$, $\varepsilon, \mu \in (L^\infty(\Omega))^{3\times 3} \cap \prod_{l=1}^{L}(W^{1,\infty}(\Omega_l))^{3\times 3}$, and $\boldsymbol{f} \in (L^2(\Omega))^3 \cap \prod_{l=1}^{L} H(\mathrm{div};\Omega_l) \cap H(\mathbf{curl};\varepsilon^{-1};\Omega_l) \cap (H^r(\Omega_l))^3$, where $H(\mathbf{curl};\varepsilon^{-1};\Omega_l) = \{\boldsymbol{v} \in (L^2(\Omega_l))^3 : \mathbf{curl}\ \varepsilon^{-1}\boldsymbol{v} \in (L^2(\Omega_l))^3\}$. It is assumed that the regularity index $r > 1/2$. For the solution, this is because the construction of the finite element interpolation involves the element-face integrals of $\int_{\partial T} \boldsymbol{u}$ which are well-defined if $r > 1/2$. For the right-hand side, this is because the traces of $\boldsymbol{f}$ along the domain boundary and the material interfaces are used. We will see that the additional regularity assumption on $\boldsymbol{f}$, i.e., $\boldsymbol{f} \in \prod_{l=1}^{L} H(\mathrm{div}; \Omega_l) \cap H(\mathbf{curl}; \varepsilon^{-1}; \Omega_l) \cap (H^r(\Omega_l))^3$ can ensure that $\mathrm{div}\varepsilon\boldsymbol{u}$ and $\mu^{-1}\mathbf{curl}\ \boldsymbol{u}$ have suitable regularity, so that we can derive the desired error estimates $\mathcal{O}(h^r)$ when dealing with the consistency errors caused by the L^2 projections R_h and $\breve{R}_h$ (see Sect. 4.1). On the other hand, as we have noted, the variational formulation (1.8) of the curlcurl-grad div problem may also come from a first-order system $\mathbf{curl}\ \boldsymbol{u} = \boldsymbol{f}$ and $\mathrm{div}\varepsilon\boldsymbol{u} = g_0$, like (1.9), with $\mu = 1$. In that case, the right-hand side of (1.10) becomes $(\boldsymbol{f}, R_h(\mathbf{curl}\ \boldsymbol{v})) + (g_0, \breve{R}_h(\mathrm{div}\varepsilon\boldsymbol{v}))$, up to an additional consistent right-hand side corresponding to the stabilization term $\mathcal{S}_h$ (see Remark 2.9), the just mentioned consistency errors from the L^2 projections R_h and $\breve{R}_h$ do not exist. Thus, it suffices to assume the regularity as follows: $\boldsymbol{u} \in H_0(\mathbf{curl}; \Omega) \cap H(\mathrm{div}; \varepsilon; \Omega) \cap \prod_{l=1}^{L}(H^r(\Omega_l))^3$ for some $r > 1/2$ and $\varepsilon \in (L^\infty(\Omega))^{3\times 3} \cap \prod_{l=1}^{L}(W^{1,\infty}(\Omega_l))^{3\times 3}$ and $\boldsymbol{f} \in (L^2(\Omega))^3$ and $g_0 \in L^2(\Omega)$. In other words, for first-order systems, we do not require any additional regularity of $\mathbf{curl}\ \boldsymbol{u}$ and any additional regularity on $\boldsymbol{f}$ and g_0.

The rest of the paper is organized as follows. In Sect. 2, we define the L^2 projected continuous finite element method. Section 3 is devoted to the establishment of the coercivity property and the condition number of the resulting linear system. In Sect. 4 we show the error bounds of the proposed finite element method in an energy norm. Some concluding remarks are given in the last section.

2 The finite element method

In this section, we shall describe the finite element method (1.10) for problem (1.1), (1.2), (1.3).

2.1 Finite element space

We first define the continuous finite element space U_h.

Let $\mathcal{C}_h$ denote a shape-regular conforming (also conforming with the material interfaces) triangulation [24] of $\bar{\Omega}$ into tetrahedra, with diameters h_K for all $K \in \mathcal{C}_h$ bounded by h. Let $\mathcal{E}_h$ be the collection of all element faces, and $\mathcal{E}_h^\Gamma$ the collection of all the element faces on Γ, and $\mathcal{E}_h^{\mathrm{inter}}$ the collection of all the element faces on the discontinuous interfaces of the material ε. Denote $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \mathcal{E}_h^\Gamma \cup \mathcal{E}_h^{\mathrm{inter}}$. For ease of exposition, we shall assume that ε is a piecewise polynomial with respect to the triangulation of $\mathcal{C}_h$. In Sect. 4 we remark how to deal with a general piecewise smooth ε by its finite element interpolation.

Let λ_i, $1 \le i \le 4$, denote the local basis of linear polynomial associated with the ith vertex of K, see [24]. Let F denote a face of the element boundary ∂K. If $F \notin \mathcal{E}_h^\Gamma$, we define b_F as the usual face bubble on F, satisfying

$$b_F \in H^1(K), \quad b_F \in H_0^1(F) \quad \text{on} \quad F \subset \partial K, \quad b_F = 0 \quad \text{on} \quad \partial K \setminus F. \tag{2.1}$$

For example, let a_1, a_2, a_3 be the three vertices of the face F, then

$$b_F = \lambda_1\lambda_2\lambda_3.$$

If $F \in \mathcal{E}_h^\Gamma$, we put $b_F = 0$. For any given K, let b_K denote the usual element bubble on K, i.e., $b_K \in H_0^1(K)$, for example,

$$b_K = \lambda_1\lambda_2\lambda_3\lambda_4.$$

As usual, let $\mathcal{P}_n$ be the space of polynomials of degree not greater than $n \geq 0$, see [24]. Let

$$\boldsymbol{P}_0(\varepsilon; K) := \mathrm{span}\{\varepsilon\,(\mathcal{P}_0(K))^3, (\mathcal{P}_0(K))^3\}. \tag{2.2}$$

Let $H^1(\Omega)$, $H_0^1(\Omega)$ denote the standard Hilbert spaces [1]. We introduce the following face-bubble space Φ_h and element-bubble space Θ_h:

$$\Phi_h := \left\{\boldsymbol{v} \in (H_0^1(\Omega))^3; \boldsymbol{v}|_K \in (\mathcal{P}_1(K))^3(b_{F_1}, b_{F_2}, b_{F_3}, b_{F_4}), \forall K \in \mathcal{C}_h\right\} \tag{2.3}$$

and

$$\Theta_h := \{\boldsymbol{v} \in (H_0^1(\Omega))^3; \boldsymbol{v}|_K \in \boldsymbol{P}_0(\varepsilon; K)\, b_K, \forall K \in \mathcal{C}_h\}. \tag{2.4}$$

Let

$$\mathfrak{U}_h := \left\{\boldsymbol{v} \in (H^1(\Omega))^3; \boldsymbol{v}|_K \in (\mathcal{P}_1(K))^3, \forall K \in \mathcal{C}_h\right\}. \tag{2.5}$$

We define the continuous finite element space for approximating the solution as follows:

$$U_h := \mathfrak{U}_h + \Phi_h + \Theta_h. \tag{2.6}$$

Remark 2.1 The role of the element and face bubbles help to remove the effects of the curl and div partial derivatives on the solution (See Sect. 4), so that we can obtain a correct continuous finite element solution when the analytical solution has a low regularity. Note that there is no homogeneous tangential boundary conditions imposed in U_h. Otherwise, the coexistence of several tangential vectors at the corners and edges along Γ makes it difficult to evaluate the tangential component of $\boldsymbol{u}_h \in U_h$ at those places.

Remark 2.2 We see that for any $\boldsymbol{v} \in \Phi_h$ on any given face F,

$$\boldsymbol{v}|_F \in (\mathcal{P}_1(F))^3\, b_F|_F.$$

For any face $F \in \mathcal{E}_h^\Gamma$, we have $\boldsymbol{v}|_F = 0$, since b_F has been defined as zero in advance for boundary faces. In addition, any function in Θ_h is globally continuous although ε is discontinuous, because the element bubble is zero on any face (including the material interfaces). Thus, even if ε is discontinuous, the finite element space U_h is continuous in the entire domain.

2.2 L^2 projection

In this subsection, we shall define the L^2 projections.

Let $L^2(\Omega)$ denote the Hilbert space of square integrable Lebesgue measurable functions, equipped with L^2-inner product $(u, v) = \int_\Omega u\, v$ and L^2 norm $||v||_0 = \sqrt{(v, v)}$, see [1]. Define the finite element space of discontinuous piecewise polynomials of degree not greater than n as follows:

$$P_h^n := \{q \in L^2(\Omega);\, q|_K \in \mathcal{P}_n(K), \forall K \in \mathcal{C}_h\}. \tag{2.7}$$

Let $(\cdot, \cdot)_{0,h}$ denote the mass-lumping L^2 inner product on $P_h^1 \cap H_0^1(\Omega)$, i.e.

$$(p, q)_{0,h} = \sum_{K \in \mathcal{C}_h} \frac{|K|}{4} \sum_{i=1}^{4} p(a_i)q(a_i), \tag{2.8}$$

where a_i are the four vertices of K, and $|K|$ is the volume of K. Let $(\cdot, \cdot)_\mu$ denote the μ-weighted L^2 inner product, i.e.

$$(u, v)_\mu = \int_\Omega \mu\, u\, v.$$

Given $\boldsymbol{v} \in H(\mathbf{curl}; \Omega)$, we define $\breve{R}_h(\operatorname{div} \varepsilon\, \boldsymbol{v}) \in P_h^1 \cap H_0^1(\Omega)$ and $R_h(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{v}) \in (P_h^1)^3$ as follows:

$$(\breve{R}_h(\operatorname{div} \varepsilon\, \boldsymbol{v}), q)_{0,h} := -(\varepsilon\, \boldsymbol{v}, \nabla q) \quad \forall q \in P_h^1 \cap H_0^1(\Omega), \tag{2.9}$$

$$(R_h(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{v}), \boldsymbol{q})_\mu := (\mathbf{curl}\;\, \boldsymbol{v}, \boldsymbol{q}) - \sum_{F \in \mathcal{E}_h^\Gamma} \int_F \boldsymbol{v} \times \boldsymbol{n} \cdot \boldsymbol{q}, \quad \forall \boldsymbol{q} \in (P_h^1)^3. \tag{2.10}$$

Remark 2.3 R_h is element-locally defined. $\breve{R}_h$ is essentially local, called as pseudo-local, since the mass-lumping inner product (2.8) results in a diagonal matrix for basis functions in $P_h^1 \cap H_0^1(\Omega)$. The approximation of the L^2 inner product $(\cdot, \cdot)$ by (2.8) is referred to as 'mass-lumping' [62].

Remark 2.4 Let $(\cdot, \cdot)_{0,K}$ denote the L^2 inner product on K. By Green's formula, we have from (2.9) and (2.10)

$$(\breve{R}_h(\operatorname{div} \varepsilon\, \boldsymbol{v}), q)_{0,h} = \sum_{K \in \mathcal{C}_h} (\operatorname{div} \varepsilon\, \boldsymbol{v}, q)_{0,K} - \sum_{F \in \mathcal{E}_h^{\text{inter}}} \int_F [\![\varepsilon \boldsymbol{v} \cdot \boldsymbol{n}]\!]\, q, \tag{2.11}$$

$$(R_h(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{v}), \boldsymbol{q})_\mu = \sum_{K\in\mathcal{C}_h} (\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{q})_{0,K} + \sum_{F\in\mathcal{E}_h^0\cup\mathcal{E}_h^{\text{inter}}} \int_F \boldsymbol{v}\times\boldsymbol{n}\cdot [\![\boldsymbol{q}]\!]. \quad (2.12)$$

From (2.9) and (2.12) we see that the curl and div partial derivatives no longer act on the solution. This is an essential strategy in dealing with the low regularity solution. We also see that $\breve{R}_h$ is well-defined even if ε is discontinuous. This is the key factor for which we can use the continuous finite element space in the case where ε is discontinuous.

2.3 Stabilization term

In this subsection, we shall define the stabilization term $\mathcal{S}_h$.

We define

$$\begin{aligned}\mathcal{S}_h(\boldsymbol{u},\boldsymbol{v}) = &\sum_{K\in\mathcal{C}_h} h_K^2(\operatorname{div}\varepsilon\,\boldsymbol{u}, \operatorname{div}\varepsilon\,\boldsymbol{v})_{0,K} + \sum_{K\in\mathcal{C}_h} h_K^2(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{0,K}\\ &+ \sum_{F\in\mathcal{E}_h^{\text{inter}}} h_F\,([\![\varepsilon\,\boldsymbol{u}\cdot\boldsymbol{n}]\!], [\![\varepsilon\,\boldsymbol{v}\cdot\boldsymbol{n}]\!])_{0,F}\\ &+ \sum_{F\in\mathcal{E}_h^{\Gamma}} h_F\,(\boldsymbol{u}\times\boldsymbol{n}, \boldsymbol{v}\times\boldsymbol{n})_{0,F}, \end{aligned} \quad (2.13)$$

where $(\cdot,\cdot)_{0,F}$ denotes the L^2 inner product on F, and h_F is the diameter of F.

Remark 2.5 We see that (2.13) is the elementwise analogue of the bilinear form of (1.8) weighted with local mesh sizes. The stabilization role of $\mathcal{S}_h$ is for the coercivity. Such natural and simple $\mathcal{S}_h$ greatly simplifies the design of the L^2 projection finite element method but also reduces the complexity in the corresponding coercivity analysis. Consequently, the implementation of the underlying finite element method is easier. Although the curl and div partial derivatives still exist in $\mathcal{S}_h$, we can control them reasonably well for the low regularity solution that does not belong to $(H^1(\Omega))^3$ (See Sect. 4).

Remark 2.6 For the analytical solution $\boldsymbol{u} \in H_0(\mathbf{curl};\Omega)\cap H(\operatorname{div};\varepsilon;\Omega)$ of problem (1.1)–(1.3), all the jumps, boundary terms in (2.10)–(2.13) disappear.

2.4 Finite element problem

We are now in a position to state the finite element problem: find $\boldsymbol{u}_h \in U_h$ such that for all $\boldsymbol{v}_h \in U_h$

$$\begin{aligned}&(R_h(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}_h), R_h(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{v}_h))_\mu + (\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{u}_h), \breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))_{0,h}\\ &\quad+\mathcal{S}_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = (\boldsymbol{f},\boldsymbol{v}_h).\end{aligned} \quad (2.14)$$

Remark 2.7 We should remark that all the additions in (2.14) relative to the standard formulation (1.8) are locally evaluated. So, the realization of (2.14) would not incur much more computational cost than that of (1.8).

Remark 2.8 Denote by $\mathcal{L}_h$ the bilinear form of (2.14). We see that it is symmetric and non-negative.

Remark 2.9 The finite element method stated as above is also applied to first-order systems, such as (1.9) where $\mu = 1$. In that case, as mentioned in the Introduction section, the right-hand side in (2.14) should be correspondingly replaced by the following: $(\boldsymbol{f}, R_h(\mathbf{curl}\, \boldsymbol{v}_h)) + (g_0, \breve{R}_h(\mathrm{div}\varepsilon \boldsymbol{v}_h)) + \mathcal{R}_h(\boldsymbol{f}, g_0; \boldsymbol{v}_h)$, where $\mathcal{R}_h(\boldsymbol{f}, g_0; \boldsymbol{v}_h) := \sum_{K\in\mathcal{C}_h} h_K^2(g_0, \mathrm{div}\, \varepsilon\, \boldsymbol{v})_{0,K} + \sum_{K\in\mathcal{C}_h} h_K^2(\boldsymbol{f}, \mathbf{curl}\, \boldsymbol{v})_{0,K}$. For the first-order system (1.9), problem (2.14) is consistent in the usual sense [24]. In general, problem (2.14) incurs consistency errors from the L^2 projections R_h and $\breve{R}_h$ and the stabilization term $\mathcal{S}_h$. For the rest of this paper, we shall only analyze problem (2.14), including the coercivity, the condition number, the consistency error, the construction of the finite element interpolation, and the error bound. All the analysis is completely the same when applied to the first-order system, except that the consistency error does not exist. We also point out that it is the consistency error from the L^2 projections that requires more regularity on $\boldsymbol{f}$ than $(L^2(\Omega))^3$, see Sect. 4.1. See also Remark 4.6.

Remark 2.10 We should note that if the continuous finite element space of the solution is chosen to include the gradient of some C^1 element, then the L^2 projection R_h is unnecessary and can be dropped, and, meanwhile, the sub-term associated with the curl operator in the stabilization term $\mathcal{S}_h$ can also be dropped.

3 Coercivity and condition number

In this section, we shall investigate the coercivity and the condition number of (2.14).

3.1 Coercivity

This subsection is devoted to the coercivity property of (2.14). Introduce div Hilbert spaces [39]:

$$H_0(\mathrm{div}; \Omega) = \{\boldsymbol{v} \in H(\mathrm{div}; \Omega) : \boldsymbol{v} \cdot \boldsymbol{n}|_\Gamma = 0\},$$
$$H_0(\mathrm{div}^0; \Omega) = \{\boldsymbol{v} \in H_0(\mathrm{div}; \Omega) : \mathrm{div}\, \boldsymbol{v} = 0\},$$

and the ε-weighted L^2 inner product $(\cdot, \cdot)_\varepsilon$:

$$(u, v)_\varepsilon = \int_\Omega \varepsilon\, u\, v.$$

Proposition 3.1 (See [9,11,31]). *For any $\boldsymbol{\psi} \in H(\mathbf{curl}; \Omega) \cap H_0(\mathrm{div}; \Omega)$, it can be written into the following 'regular-singular' decomposition:*

$$\boldsymbol{\psi} = \boldsymbol{\psi}^0 + \nabla p^0, \quad \boldsymbol{\psi}^0 \in (H^1(\Omega))^3,\ p^0 \in H^1(\Omega)/\mathbb{R},$$

where $\boldsymbol{\psi}^0$ *satisfies*

$$||\boldsymbol{\psi}^0||_1 \leq C\,\{||\mathbf{curl}\ \boldsymbol{\psi}||_0 + ||\mathrm{div}\ \boldsymbol{\psi}||_0\}.$$

Proposition 3.2 (See [3,35,59,60]) *For any* $\boldsymbol{v} \in (L^2(\Omega))^3$, *we have the following* L^2 *orthogonal decomposition with respect to the* ε*-weighted* L^2 *inner product* $(\cdot,\cdot)_\varepsilon$*:*

$$\boldsymbol{v} = \nabla p + \varepsilon^{-1}\mathbf{curl}\ \boldsymbol{\psi},$$

where $p \in H_0^1(\Omega)$, $\boldsymbol{\psi} \in H(\mathbf{curl};\Omega) \cap H_0(\mathrm{div}^0;\Omega)$, *satisfying*

$$||\varepsilon^{\frac{1}{2}}\boldsymbol{v}||_0^2 = ||\varepsilon^{\frac{1}{2}}\ \nabla\ p||_0^2 + ||\varepsilon^{-\frac{1}{2}}\mathbf{curl}\ \ \boldsymbol{\psi}||_0^2,$$
$$||\boldsymbol{\psi}||_0 \leq C||\mathbf{curl}\ \ \boldsymbol{\psi}||_0.$$

Proposition 3.3 (See [62]) *For the* L^2 *norm* $||\cdot||_{0,h}$ *induced from the mass-lumping* L^2 *inner product* (2.8)*, we have the following norm-equivalence property*

$$C\,||q||_{0,h} \leq ||q||_0 \leq C^{-1}\,||q||_{0,h} \quad \forall q \in P_h^1 \cap H_0^1(\Omega).$$

Theorem 3.1 *We have*

$$\mathcal{L}_h(\boldsymbol{v},\boldsymbol{v}) \geq C\,||\boldsymbol{v}||_0^2 \quad \forall \boldsymbol{v} \in U_h. \tag{3.1}$$

As a consequence, problem (2.14) *has a unique solution in* U_h*.*

Proof Noting that

$$\mathcal{L}_h(\boldsymbol{v},\boldsymbol{v}) = ||\breve{R}_h(\mathrm{div}\,\varepsilon\boldsymbol{v})||_{0,h}^2 + ||R_h(\mu^{-1}\,\mathbf{curl}\ \ \boldsymbol{v})||_\mu^2 + \mathcal{S}_h(\boldsymbol{v},\boldsymbol{v}),$$

where $||\cdot||_\mu$ is the induced norm of the μ-weighted L^2 inner product $(\cdot,\cdot)_\mu$, we need only to show

$$||\breve{R}_h(\mathrm{div}\,\varepsilon\,\boldsymbol{v})||_{0,h}^2 + ||R_h(\mu^{-1}\,\mathbf{curl}\ \ \boldsymbol{v})||_\mu^2 \geq C_1\,||\boldsymbol{v}||_0^2 - C_2\,\mathcal{S}_h(\boldsymbol{v},\boldsymbol{v}) \quad \forall \boldsymbol{v} \in U_h. \tag{3.2}$$

In fact, if (3.2) holds true, from (3.2) we can obtain

$$\begin{aligned}\mathcal{L}_h(\boldsymbol{v},\boldsymbol{v}) &\geq \frac{1}{\max(1,C_2)}\,\Big(||\breve{R}_h(\mathrm{div}\,\varepsilon\boldsymbol{v})||_{0,h}^2 + ||R_h(\mu^{-1}\,\mathbf{curl}\ \ \boldsymbol{v})||_\mu^2 + C_2\,\mathcal{S}_h(\boldsymbol{v},\boldsymbol{v})\Big)\\ &\geq \frac{C_1}{\max(1,C_2)}\,||\boldsymbol{v}||_0^2.\end{aligned} \tag{3.3}$$

To show (3.2), from Proposition 3.2 we write $\boldsymbol{v}$ as the following L^2-orthogonal decomposition with respect to the ε-weighted L^2 inner product $(\cdot,\cdot)_\varepsilon$:

$$\boldsymbol{v} = \nabla\ p + \varepsilon^{-1}\,\mathbf{curl}\ \ \boldsymbol{\psi}, \tag{3.4}$$

with $p \in H_0^1(\Omega)$, $\boldsymbol{\psi} \in H(\mathbf{curl};\Omega) \cap H_0(\mathrm{div}^0;\Omega)$, satisfying

$$||\varepsilon^{\frac{1}{2}}\,\boldsymbol{v}||_0^2 = ||\varepsilon^{\frac{1}{2}}\,\nabla\, p||_0^2 + ||\varepsilon^{-\frac{1}{2}}\,\mathbf{curl}\,\,\boldsymbol{\psi}||_0^2 \tag{3.5}$$

and

$$||\boldsymbol{\psi}||_0 \le C\,||\mathbf{curl}\,\,\boldsymbol{\psi}||_0. \tag{3.6}$$

From Proposition 3.1 we further write $\boldsymbol{\psi}$ as

$$\boldsymbol{\psi} = \boldsymbol{\psi}^0 + \boldsymbol{\psi}^1, \tag{3.7}$$

where $\boldsymbol{\psi}^0 \in (H^1(\Omega))^3$, $\mathbf{curl}\,\,\boldsymbol{\psi}^1 = \mathbf{0}$, and we have

$$||\boldsymbol{\psi}^0||_1 \le C||\mathbf{curl}\,\,\boldsymbol{\psi}||_0. \tag{3.8}$$

According to the two components $(p,\boldsymbol{\psi})$ in the decomposition of $\boldsymbol{v}$, we divide the proof into two steps. The first step on p and the second step on $\boldsymbol{\psi}$ give lower bounds for $||\breve{R}_h(\mathrm{div}\,\varepsilon\,\boldsymbol{v})||_{0,h}^2$ and $||R_h(\mu^{-1}\,\mathbf{curl}\,\,\boldsymbol{v})||_\mu^2$, respectively.

Step 1 We consider p.

We take $\widetilde{p} \in P_h^1 \cap H_0^1(\Omega)$ the interpolation of p, such that (See [7,26,61])

$$\left(\sum_{K\in\mathcal{C}_h} h_K^{-2}\,||p-\widetilde{p}||_{0,K}^2 + \sum_{F\in\mathcal{E}_h} h_F^{-1}\,||p-\widetilde{p}||_{0,F}^2\right)^{\frac{1}{2}} \le C_3\,||p||_1, \tag{3.9}$$

$$||\widetilde{p}||_0 \le C_3\,||p||_1. \tag{3.10}$$

Note that, to construct the above $\widetilde{p}$ for $p \in H_0^1(\Omega)$, we may employ the Clément interpolation in [26], or Bernardi-Girault interpolation [7], or Scott–Zhang interpolation [61].

For carrying out the analysis, we recall that the Poincaré-Friedrichs' inequality $||p||_1 \le C_{PF}||\,\nabla\, p||_0$ holds for all $p \in H_0^1(\Omega)$ with the constant C_{PF} and that the uniform ellipticity of ε: $||\boldsymbol{v}||_0 \le C_\varepsilon^{-1/2}||\varepsilon^{1/2}\boldsymbol{v}||_0$ holds for all $\boldsymbol{v} \in (L^2(\Omega))^3$ with the constant C_ε as indicated in the Introduction section.

Let $\delta > 0$ be a constant to be determined. We have

$$\begin{aligned}||\breve{R}_h(\mathrm{div}\,\varepsilon\,\boldsymbol{v})||_{0,h}^2 &= ||\breve{R}_h(\mathrm{div}\,\varepsilon\,\boldsymbol{v}) + \delta\,\widetilde{p}||_{0,h}^2 - \delta^2\,||\widetilde{p}||_{0,h}^2\\ &\quad -2\,\delta\,(\breve{R}_h(\mathrm{div}\,\varepsilon\,\boldsymbol{v}),\,\widetilde{p})_{0,h}.\end{aligned} \tag{3.11}$$

From $||\widetilde{p}||_{0,h} \le C^{-1}||\widetilde{p}||_0$ in Propositions (3.3), and (3.10), we have

$$\begin{aligned}||\widetilde{p}||_{0,h} &\le C^{-1}\,||\widetilde{p}||_0 \le C^{-1}C_3\,||p||_1 \le C^{-1}C_3C_{PF}\,||\,\nabla\, p||_0\\ &\le C^{-1}C_3C_{PF}C_\varepsilon^{-1/2}\,||\varepsilon^{\frac{1}{2}}\,\nabla\, p||_0,\end{aligned}$$

i.e., putting $C_4 := C^{-1} C_3 C_{PF} C_\varepsilon^{-1/2}$, we have

$$||\widetilde{p}||_{0,h} \leq C_4 ||\varepsilon^{\frac{1}{2}} \nabla p||_0. \tag{3.12}$$

We have

$$-2\,\delta\,(\breve{R}_h(\operatorname{div} \varepsilon\,\boldsymbol{v}), \widetilde{p})_{0,h} = 2\,\delta\,(\varepsilon\,\boldsymbol{v}, \nabla \widetilde{p}) = -2\delta \sum_{K \in \mathcal{C}_h} (\operatorname{div} \varepsilon\,\boldsymbol{v}, \widetilde{p})_{0,K} + 2\delta \sum_{F \in \mathcal{E}_h^{\text{inter}}} \int_F [\![\varepsilon\,\boldsymbol{v} \cdot \boldsymbol{n}]\!]\, \widetilde{p}, \tag{3.13}$$

$$-2\,\delta \sum_{K \in \mathcal{C}_h} (\operatorname{div} \varepsilon\,\boldsymbol{v}, \widetilde{p})_{0,K} = 2\,\delta \sum_{K \in \mathcal{C}_h} (\operatorname{div} \varepsilon\,\boldsymbol{v}, p - \widetilde{p})_{0,K} - 2\,\delta \sum_{K \in \mathcal{C}_h} (\operatorname{div} \varepsilon\,\boldsymbol{v}, p)_{0,K}, \tag{3.14}$$

$$-2\,\delta \sum_{K \in \mathcal{C}_h} (\operatorname{div} \varepsilon\,\boldsymbol{v}, p)_{0,K} = 2\,\delta\,(\varepsilon\,\boldsymbol{v}, \nabla p) - 2\,\delta \sum_{F \in \mathcal{E}_h^{\text{inter}}} \int_F [\![\varepsilon\,\boldsymbol{v} \cdot \boldsymbol{n}]\!]\, p, \tag{3.15}$$

$$2\,\delta\,(\varepsilon\,\boldsymbol{v}, \nabla p) = 2\,\delta\,(\varepsilon \nabla p, \nabla p) = 2\,\delta\,||\varepsilon^{\frac{1}{2}} \nabla p||_0^2, \tag{3.16}$$

where, from (3.9), the Poincaré-Friedrichs' inequality, and the uniform ellipticity of ε, we have

$$\begin{aligned}
&2\,\delta \sum_{K \in \mathcal{C}_h} (\operatorname{div} \varepsilon\,\boldsymbol{v}, p - \widetilde{p})_{0,K} + 2\,\delta \sum_{F \in \mathcal{E}_h^{\text{inter}}} \int_F [\![\varepsilon\,\boldsymbol{v} \cdot \boldsymbol{n}]\!]\,(\widetilde{p} - p) \\
&\geq -2\,\delta \left(\sum_{K \in \mathcal{C}_h} h_K^2\,||\operatorname{div} \varepsilon\,\boldsymbol{v}||_{0,K}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{C}_h} h_K^{-2}\,||p - \widetilde{p}||_{0,K}^2 \right)^{1/2} \\
&\quad -2\,\delta \left(\sum_{F \in \mathcal{E}_h^{\text{inter}}} h_F\,||[\![\varepsilon\,\boldsymbol{v} \cdot \boldsymbol{n}]\!]||_{0,F}^2 \right)^{1/2} \left(\sum_{F \in \mathcal{E}_h^{\text{inter}}} h_F^{-1}\,||p - \widetilde{p}||_{0,F}^2 \right)^{1/2} \\
&\geq -2\delta \left(\sum_{K \in \mathcal{C}_h} h_K^2\,||\operatorname{div} \varepsilon\,\boldsymbol{v}||_{0,K}^2 + \sum_{F \in \mathcal{E}_h^{\text{inter}}} h_F\,||[\![\varepsilon\,\boldsymbol{v} \cdot \boldsymbol{n}]\!]||_{0,F}^2 \right)^{1/2} \\
&\quad \times \left(\sum_{K \in \mathcal{C}_h} h_K^{-2}\,||p - \widetilde{p}||_{0,K}^2 + \sum_{F \in \mathcal{E}_h^{\text{inter}}} h_F^{-1}\,||p - \widetilde{p}||_{0,F}^2 \right)^{1/2}
\end{aligned}$$

$$\geq -2\,\delta\,C_3 \left(\sum_{K\in\mathcal{C}_h} h_K^2\,||\mathrm{div}\,\varepsilon\,\boldsymbol{v}||_{0,K}^2 + \sum_{F\in\mathcal{E}_h^{\mathrm{inter}}} h_F\,||[\![\varepsilon\,\boldsymbol{v}\cdot\boldsymbol{n}]\!]||_{0,F}^2 \right)^{1/2} ||p||_1$$

$$\geq -\left(\sum_{K\in\mathcal{C}_h} h_K^2\,||\mathrm{div}\,\varepsilon\,\boldsymbol{v}||_{0,K}^2 + \sum_{F\in\mathcal{E}_h^{\mathrm{inter}}} h_F\,||[\![\varepsilon\,\boldsymbol{v}\cdot\boldsymbol{n}]\!]||_{0,F}^2 \right)$$

$$-\delta^2\,(C_3 C_{PF} C_\varepsilon^{-1/2})^2 ||\varepsilon^{\frac{1}{2}}\,\nabla\,p||_0^2, \tag{3.17}$$

where $(C_3 C_{PF} C_\varepsilon^{-1/2})^2 = (CC_4)^2$. Summarizing (3.11)–(3.17) and choosing $\delta := 1/(C_4^2 + C^2 C_4^2)$, we have

$$\begin{aligned} ||\breve{R}_h(\mathrm{div}\,\varepsilon\,\boldsymbol{v})||_{0,h}^2 &\geq \delta\left[2 - \delta(C_4^2 + C^2C_4^2)\right] ||\varepsilon^{\frac{1}{2}}\,\nabla\,p||_0^2 \\ &\quad - \left(\sum_{K\in\mathcal{C}_h} h_K^2\,||\mathrm{div}\,\varepsilon\,\boldsymbol{v}||_{0,K}^2 + \sum_{F\in\mathcal{E}_h^{\mathrm{inter}}} h_F\,||[\![\varepsilon\,\boldsymbol{v}\cdot\boldsymbol{n}]\!]||_{0,F}^2 \right) \\ &= \frac{1}{C_4^2 + C^2C_4^2}\,||\varepsilon^{\frac{1}{2}}\,\nabla\,p||_0^2 \\ &\quad - \left(\sum_{K\in\mathcal{C}_h} h_K^2\,||\mathrm{div}\,\varepsilon\,\boldsymbol{v}||_{0,K}^2 + \sum_{F\in\mathcal{E}_h^{\mathrm{inter}}} h_F\,||[\![\varepsilon\,\boldsymbol{v}\cdot\boldsymbol{n}]\!]||_{0,F}^2 \right). \end{aligned} \tag{3.18}$$

Step 2 We consider $\boldsymbol{\psi}$ with $\boldsymbol{\psi}^0$ in (3.7).

We take $\widetilde{\boldsymbol{\psi}^0} \in (P_h^1)^3$ as the L^2 projection or the averaging interpolant of $\boldsymbol{\psi}^0 \in (H^1(\Omega))^3$, such that

$$\left(\sum_{K\in\mathcal{C}_h} h_K^{-2}\,||\boldsymbol{\psi}^0 - \widetilde{\boldsymbol{\psi}^0}||_{0,K}^2 + \sum_{F\in\mathcal{E}_h} h_F^{-1}\,||\boldsymbol{\psi}^0 - \widetilde{\boldsymbol{\psi}^0}||_{0,F}^2 \right)^{\frac{1}{2}} \leq C_5\,||\boldsymbol{\psi}^0||_1, \tag{3.19}$$

$$||\widetilde{\boldsymbol{\psi}^0}||_0 \leq C_5\,||\boldsymbol{\psi}^0||_1. \tag{3.20}$$

Note that the above $\widetilde{\boldsymbol{\psi}^0} \in (P_h^1)^3$ can be constructed from the L^2 projection [39,38], or from the Clément interpolation in [26], or Bernardi-Girault interpolation [7], or Scott–Zhang interpolation [61].

Let $\delta > 0$ be a constant to be determined. We have

$$\begin{aligned} ||R_h(\mu^{-1}\,\mathbf{curl}\,\ \boldsymbol{v})||_\mu^2 &= ||R_h(\mu^{-1}\,\mathbf{curl}\,\ \boldsymbol{v}) - \delta\,\widetilde{\boldsymbol{\psi}^0}||_\mu^2 - \delta^2\,||\widetilde{\boldsymbol{\psi}^0}||_\mu^2 \\ &\quad + 2\,\delta\,(R_h(\mu^{-1}\,\mathbf{curl}\,\ \boldsymbol{v}), \widetilde{\boldsymbol{\psi}^0})_\mu. \end{aligned} \tag{3.21}$$

Recall, as indicated in the Introduction section, the uniform boundedness of μ with the constant C_μ and the uniform ellipticity of ε with the constant C_ε. From (3.20) and (3.8), we have

$$
\begin{aligned}
||\widetilde{\boldsymbol{\psi}^0}||_\mu &\leq C_\mu^{-1/2}\, ||\widetilde{\boldsymbol{\psi}^0}||_0 \leq C_\mu^{-1/2} C_5\, ||\boldsymbol{\psi}^0||_1 \leq C_\mu^{-1/2} C_5 C\, ||\mathbf{curl}\ \boldsymbol{\psi}||_0 \\
&\leq C_\mu^{-1/2} C_5 C\, C_\varepsilon^{-1/2}\, ||\varepsilon^{-\frac{1}{2}}\, \mathbf{curl}\ \boldsymbol{\psi}||_0,
\end{aligned}
$$

i.e., putting $C_6 := C_\mu^{-1/2} C_5 C\, C_\varepsilon^{-1/2}$, we have

$$
||\widetilde{\boldsymbol{\psi}^0}||_\mu \leq C_6 ||\varepsilon^{-\frac{1}{2}}\, \mathbf{curl}\ \boldsymbol{\psi}||_0. \tag{3.22}
$$

We have

$$
2\,\delta\,(R_h(\mu^{-1}\,\mathbf{curl}\ \boldsymbol{v}), \widetilde{\boldsymbol{\psi}^0})_\mu = 2\,\delta\,(\mathbf{curl}\ \boldsymbol{v}, \widetilde{\boldsymbol{\psi}^0}) - 2\,\delta \sum_{F\in\mathcal{E}_h^\Gamma} \int_F \boldsymbol{v}\times\boldsymbol{n}\cdot\widetilde{\boldsymbol{\psi}^0}, \tag{3.23}
$$

$$
2\,\delta\,(\mathbf{curl}\ \boldsymbol{v}, \widetilde{\boldsymbol{\psi}^0}) = 2\,\delta\,(\mathbf{curl}\ \boldsymbol{v}, \widetilde{\boldsymbol{\psi}^0} - \boldsymbol{\psi}^0) + 2\,\delta\,(\mathbf{curl}\ \boldsymbol{v}, \boldsymbol{\psi}^0), \tag{3.24}
$$

$$
2\,\delta\,(\mathbf{curl}\ \boldsymbol{v}, \boldsymbol{\psi}^0) = 2\,\delta\,(\boldsymbol{v}, \mathbf{curl}\ \boldsymbol{\psi}^0) + 2\,\delta \sum_{F\in\mathcal{E}_h^\Gamma} \int_F \boldsymbol{v}\times\boldsymbol{n}\cdot\boldsymbol{\psi}^0, \tag{3.25}
$$

$$
\begin{aligned}
2\,\delta\,(\boldsymbol{v}, \mathbf{curl}\ \boldsymbol{\psi}^0) &= 2\,\delta\,(\boldsymbol{v}, \mathbf{curl}\ \boldsymbol{\psi}) = 2\,\delta\,(\varepsilon^{-1}\,\mathbf{curl}\ \boldsymbol{\psi}, \mathbf{curl}\ \boldsymbol{\psi}) \\
&= 2\,\delta\, ||\varepsilon^{-\frac{1}{2}}\,\mathbf{curl}\ \boldsymbol{\psi}||_0^2,
\end{aligned} \tag{3.26}
$$

where, from (3.19), (3.8), and the uniform boundedness and ellipticity of ε with the same constant C_ε (see Sect. 1), we have

$$
\begin{aligned}
&2\,\delta\,(\mathbf{curl}\ \boldsymbol{v}, \widetilde{\boldsymbol{\psi}^0} - \boldsymbol{\psi}^0) + 2\,\delta \sum_{F\in\mathcal{E}_h^\Gamma} \int_F \boldsymbol{v}\times\boldsymbol{n}\cdot(\boldsymbol{\psi}^0 - \widetilde{\boldsymbol{\psi}^0}) \\
&= 2\,\delta \sum_{K\in\mathcal{C}_h} (\mathbf{curl}\ \boldsymbol{v}, \widetilde{\boldsymbol{\psi}^0} - \boldsymbol{\psi}^0)_{0,K} + 2\,\delta \sum_{F\in\mathcal{E}_h^\Gamma} \int_F \boldsymbol{v}\times\boldsymbol{n}\cdot(\boldsymbol{\psi}^0 - \widetilde{\boldsymbol{\psi}^0}) \\
&\geq -2\,\delta \left(\sum_{K\in\mathcal{C}_h} h_K^{-2}\, ||\widetilde{\boldsymbol{\psi}^0} - \boldsymbol{\psi}^0||_{0,K}^2 \right)^{\frac{1}{2}} \left(\sum_{K\in\mathcal{C}_h} h_K^2\, ||\mathbf{curl}\ \boldsymbol{v}||_{0,K}^2 \right)^{\frac{1}{2}} \\
&\quad -2\,\delta \left(\sum_{F\in\mathcal{E}_h^\Gamma} h_F^{-1}\, ||\widetilde{\boldsymbol{\psi}^0} - \boldsymbol{\psi}^0||_{0,F}^2 \right)^{\frac{1}{2}} \left(\sum_{F\in\mathcal{E}_h^\Gamma} h_F\, ||\boldsymbol{v}\times\boldsymbol{n}||_{0,F}^2 \right)^{\frac{1}{2}} \\
&\geq -2\,\delta\, C_5\, ||\boldsymbol{\psi}^0||_1 \left(\sum_{K\in\mathcal{C}_h} h_K^2\, ||\mathbf{curl}\ \boldsymbol{v}||_{0,K}^2 + \sum_{F\in\mathcal{E}_h^\Gamma} h_F\, ||\boldsymbol{v}\times\boldsymbol{n}||_{0,F}^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\geq -2\,\delta\, C_5\, C C_\varepsilon^{-1/2} \|\varepsilon^{-\frac{1}{2}}\, \mathbf{curl}\ \boldsymbol{\psi}\|_0 \left(\sum_{K\in\mathcal{C}_h} h_K^2\, \|\mathbf{curl}\ \boldsymbol{v}\|_{0,K}^2 + \sum_{F\in\mathcal{E}_h^\Gamma} h_F\, \|\boldsymbol{v}\times\boldsymbol{n}\|_{0,F}^2 \right)^{1/2} \\
&\geq -\left(\sum_{K\in\mathcal{C}_h} h_K^2\, \|\mathbf{curl}\ \boldsymbol{v}\|_{0,K}^2 + \sum_{F\in\mathcal{E}_h^\Gamma} h_F\, \|\boldsymbol{v}\times\boldsymbol{n}\|_{0,F}^2 \right) \\
&\quad -\delta^2\, (C_5\, C C_\varepsilon^{-1/2})^2\, \|\varepsilon^{-\frac{1}{2}}\, \mathbf{curl}\ \boldsymbol{\psi}\|_0^2, \qquad (3.27)
\end{aligned}
$$

where $(C_5\, C C_\varepsilon^{-1/2})^2 = (C_6 C_\mu^{1/2})^2$. Summarizing (3.21)–(3.27) and choosing $\delta := 1/(C_6^2 + C_6^2 C_\mu)$, we have

$$
\begin{aligned}
\|R_h(\mu^{-1}\, \mathbf{curl}\ \boldsymbol{v})\|_\mu^2 \geq \delta \left[2 - \delta(C_6^2 + C_6^2 C_\mu) \right] \|\varepsilon^{-\frac{1}{2}}\, \mathbf{curl}\ \boldsymbol{\psi}\|_0^2 \\
- \left(\sum_{K\in\mathcal{C}_h} h_K^2\, \|\mathbf{curl}\ \boldsymbol{v}\|_{0,K}^2 + \sum_{F\in\mathcal{E}_h^\Gamma} h_F\, \|\boldsymbol{v}\times\boldsymbol{n}\|_{0,F}^2 \right) \\
= \frac{1}{C_6^2 + C_6^2 C_\mu} \|\varepsilon^{-\frac{1}{2}}\, \mathbf{curl}\ \boldsymbol{\psi}\|_0^2 \\
- \left(\sum_{K\in\mathcal{C}_h} h_K^2\, \|\mathbf{curl}\ \boldsymbol{v}\|_{0,K}^2 + \sum_{F\in\mathcal{E}_h^\Gamma} h_F\, \|\boldsymbol{v}\times\boldsymbol{n}\|_{0,F}^2 \right). \qquad (3.28)
\end{aligned}
$$

Finally, combining (3.5), (3.18) and (3.28), putting

$$
C_1 := C_\varepsilon \min(1/(C_4^2 + C^2 C_4^2),\, 1/(C_6^2 + C_6^2 C_\mu)), \quad C_2 := 1,
$$

we obtain

$$
\begin{aligned}
&\|\breve{R}_h(\operatorname{div} \varepsilon\, \boldsymbol{v})\|_{0,h}^2 + \|R_h(\mu^{-1}\, \mathbf{curl}\ \boldsymbol{v})\|_\mu^2 \\
&\geq C_1\, \|\boldsymbol{v}\|_0^2 - C_2 \left(\sum_{K\in\mathcal{C}_h} h_K^2\, \|\operatorname{div} \varepsilon\, \boldsymbol{v}\|_{0,K}^2 + \sum_{K\in\mathcal{C}_h} h_K^2\, \|\mathbf{curl}\ \boldsymbol{v}\|_{0,K}^2 \right. \\
&\quad \left. + \sum_{F\in\mathcal{E}_h^{\text{inter}}} h_F\, \|[\![\varepsilon\, \boldsymbol{v}\cdot\boldsymbol{n}]\!]\|_{0,F}^2 + \sum_{F\in\mathcal{E}_h^\Gamma} h_F\, \|\boldsymbol{v}\times\boldsymbol{n}\|_{0,F}^2 \right) \\
&= C_1\, \|\boldsymbol{v}\|_0^2 - C_2\, \mathcal{S}_h(\boldsymbol{v}, \boldsymbol{v}). \qquad (3.29)
\end{aligned}
$$

□

3.2 Condition number

In this subsection, we shall give the condition number of the resulting linear system.

Theorem 3.2 *Assume that the mesh is quasi-uniform as usual. Then, the condition number of the resulting linear system of problem* (2.14) *is* $\mathcal{O}(h^{-2})$.

Proof Since R_h, $\breve{R}_h$ are all L^2 projectors, we have for all $\boldsymbol{v} \in U_h$

$$||\breve{R}_h(\operatorname{div} \varepsilon \boldsymbol{v})||_{0,h}^2 + ||R_h(\mu^{-1} \mathbf{curl}\ \boldsymbol{v})||_{\mu}^2 \leq C\, h^{-2}\, ||\boldsymbol{v}||_0^2.$$

In fact, from (2.9) and Proposition 3.3, we have

$$\begin{aligned}||\breve{R}_h(\operatorname{div} \varepsilon \boldsymbol{v})||_{0,h}^2 &= -(\varepsilon \boldsymbol{v}, \nabla\, \breve{R}_h(\operatorname{div} \varepsilon \boldsymbol{v})) \leq C\, ||\boldsymbol{v}||_0\, ||\nabla\, \breve{R}_h(\operatorname{div} \varepsilon \boldsymbol{v})||_0 \\ &\leq C\, h^{-1}\, ||\boldsymbol{v}||_0\, ||\breve{R}_h(\operatorname{div} \varepsilon \boldsymbol{v})||_0 \\ &\leq C\, h^{-1}\, ||\boldsymbol{v}||_0\, ||\breve{R}_h(\operatorname{div} \varepsilon \boldsymbol{v})||_{0,h},\end{aligned}$$

where we have used the inverse estimates $||\nabla\, q||_0 \leq C\, h^{-1}\, ||q||_0$ for all $q \in P_h^1 \cap H_0^1(\Omega)$. We thus obtain

$$||\breve{R}_h(\operatorname{div} \varepsilon \boldsymbol{v})||_{0,h} \leq C\, h^{-1}\, ||\boldsymbol{v}||_0 \quad \forall \boldsymbol{v} \in U_h.$$

From (2.10) we have

$$\begin{aligned}|R_h(\mu^{-1} \mathbf{curl}\ \boldsymbol{v})||_{\mu}^2 &\leq ||\mathbf{curl}\ \boldsymbol{v}||_0\, ||R_h(\mu^{-1} \mathbf{curl}\ \boldsymbol{v})||_0 \\ &\quad + \sum_{F \in \mathcal{E}_h^{\Gamma}} ||\boldsymbol{v} \times \boldsymbol{n}||_{0,F}\, ||R_h(\mu^{-1} \mathbf{curl}\ \boldsymbol{v})||_{0,F} \\ &\leq Ch^{-1}\, ||\boldsymbol{v}||_0\, ||R_h(\mu^{-1} \mathbf{curl}\ \boldsymbol{v})||_{\mu} \\ &\quad + \left(\sum_{F \in \mathcal{E}_h^{\Gamma}} h_F^{-1}\, ||\boldsymbol{v} \times \boldsymbol{n}||_{0,F}^2\right)^{1/2} \left(\sum_{F \in \mathcal{E}_h^{\Gamma}} h_F\, ||R_h(\mu^{-1} \mathbf{curl}\ \boldsymbol{v})||_{0,F}^2\right)^{1/2} \\ &\leq C\, h^{-1}\, ||\boldsymbol{v}||_0\, ||R_h(\mu^{-1} \mathbf{curl}\ \boldsymbol{v})||_{\mu},\end{aligned}$$

where we have used the inverse estimates $||\mathbf{curl}\ \boldsymbol{v}||_0 \leq C\, h^{-1}\, ||\boldsymbol{v}||_0$ for all $\boldsymbol{v} \in U_h$ and the trace estimates $h_F^{-1/2}\, ||\boldsymbol{v} \times \boldsymbol{n}||_{0,F} \leq C\, h_K^{-1}\, ||\boldsymbol{v}||_{0,K}$ and $h_F^{1/2}\, ||\boldsymbol{q}||_{0,F} \leq C\, ||\boldsymbol{q}||_{0,K}$ for all $\boldsymbol{v} \in U_h$ and $\boldsymbol{q} \in (P_h^1)^3$ and for all K, with $F \subset \partial\, K \cap \Gamma$. Thus, we have

$$||R_h(\mu^{-1} \mathbf{curl}\ \boldsymbol{v})||_{\mu} \leq C\, h^{-1}\, ||\boldsymbol{v}||_0 \quad \forall \boldsymbol{v} \in U_h.$$

Similarly, using the inverse estimates and the trace estimates, we obtain for all $\boldsymbol{v} \in U_h$

$$\mathcal{S}_h(\boldsymbol{v}, \boldsymbol{v}) \leq C\, ||\boldsymbol{v}||_0^2.$$

We then have

$$\mathcal{L}_h(\boldsymbol{v}, \boldsymbol{v}) \leq C\, h^{-2}\, ||\boldsymbol{v}||_0^2 \quad \forall \boldsymbol{v} \in U_h,$$

which, together with the L^2 coercivity property (3.1) in Theorem 3.1 and the symmetry property of $\mathcal{L}_h$, leads to the conclusion of the theorem. $\square$

4 Error estimates

In this section, we shall estimate the errors between the exact solution and the finite element solution.

4.1 Estimation of consistency errors

Here we estimate the consistency errors caused by R_h, $\breve{R}_h$, $\mathcal{S}_h$.

Lemma 4.1 *Let* $\boldsymbol{u}$ *and* $\boldsymbol{u}_h$ *be the exact solution to problem* (1.1), (1.2), (1.3) *and the finite element solution to problem* (2.14)*, respectively. For all* $\boldsymbol{v}_h \in U_h$*, we have the following consistency formula:*

$$\begin{aligned}
\mathcal{L}_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) &= (\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u},\, R_h(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{v}_h))_\mu - (\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}, \mathbf{curl}\;\boldsymbol{v}_h)\\
&\quad + (\operatorname{div} \varepsilon\, \boldsymbol{u},\, \breve{R}_h(\operatorname{div} \varepsilon\, \boldsymbol{v}_h)) - \sum_{K\in\mathcal{C}_h} (\operatorname{div} \varepsilon\, \boldsymbol{u}, \operatorname{div} \varepsilon\, \boldsymbol{v}_h)_{0,K}\\
&\quad + \sum_{K\in\mathcal{C}_h} h_K^2 (\operatorname{div} \varepsilon\, \boldsymbol{u}, \operatorname{div} \varepsilon\, \boldsymbol{v}_h)_{0,K} + h_K^2 (\mathbf{curl}\;\boldsymbol{u}, \mathbf{curl}\;\boldsymbol{v}_h)_{0,K}\\
&\quad + \sum_{F\in\mathcal{E}_h^\Gamma} \int_F \boldsymbol{v}_h \times \boldsymbol{n} \cdot \boldsymbol{n} \times (\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u} \times \boldsymbol{n})\\
&\quad + \sum_{F\in\mathcal{E}_h^{\text{inter}}} \int_F [\![\varepsilon\, \boldsymbol{v}_h \cdot \boldsymbol{n}]\!] \operatorname{div} \varepsilon\, \boldsymbol{u}.
\end{aligned} \tag{4.1}$$

Proof For the exact solution $\boldsymbol{u}$, from (2.9), (2.11) and Remark 2.6 we have

$$\begin{aligned}
(\breve{R}_h(\operatorname{div} \varepsilon\, \boldsymbol{u}), \breve{R}_h(\operatorname{div} \varepsilon\, \boldsymbol{v}_h))_{0,h} &= -(\varepsilon\, \boldsymbol{u}, \nabla\, \breve{R}_h(\operatorname{div} \varepsilon\, \boldsymbol{v}_h))\\
&= (\operatorname{div} \varepsilon\, \boldsymbol{u}, \breve{R}_h(\operatorname{div} \varepsilon\, \boldsymbol{v}_h)) - \sum_{K\in\mathcal{C}_h} (\operatorname{div} \varepsilon\, \boldsymbol{u}, \operatorname{div} \varepsilon\, \boldsymbol{v}_h)_{0,K}\\
&\quad + \sum_{K\in\mathcal{C}_h} (\operatorname{div} \varepsilon\, \boldsymbol{u}, \operatorname{div} \varepsilon\, \boldsymbol{v}_h)_{0,K}.
\end{aligned} \tag{4.2}$$

From (2.10) and Remark 2.6 we have

$$(R_h(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}), R_h(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{v}_h))_\mu = (\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}, R_h(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{v}_h))_\mu$$
$$= (\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}, R_h(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{v}_h))_\mu - (\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}, \mathbf{curl}\;\boldsymbol{v}_h) + (\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}, \mathbf{curl}\;\boldsymbol{v}_h), \tag{4.3}$$

while we have from (1.1) for the exact solution $\boldsymbol{u}$

$$\begin{aligned}(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}, \mathbf{curl}\;\boldsymbol{v}_h) &+ \sum_{K\in\mathcal{C}_h}(\operatorname{div}\varepsilon\,\boldsymbol{u}, \operatorname{div}\varepsilon\,\boldsymbol{v}_h)_{0,K}\\ &= (\boldsymbol{f}, \boldsymbol{v}_h) + \sum_{F\in\mathcal{E}_h^\Gamma}\int_F \boldsymbol{v}_h\times\boldsymbol{n}\cdot\boldsymbol{n}\times(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}\times\boldsymbol{n})\\ &+ \sum_{F\in\mathcal{E}_h^{\mathrm{inter}}}\int_F [\![\varepsilon\,\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]\operatorname{div}\varepsilon\,\boldsymbol{u}.\end{aligned} \tag{4.4}$$

From Remark 2.6 we have

$$\mathcal{S}_h(\boldsymbol{u}, \boldsymbol{v}_h) = \sum_{K\in\mathcal{C}_h} h_K^2(\operatorname{div}\varepsilon\,\boldsymbol{u}, \operatorname{div}\varepsilon\,\boldsymbol{v}_h)_{0,K} + h_K^2(\mathbf{curl}\;\boldsymbol{u}, \mathbf{curl}\;\boldsymbol{v}_h)_{0,K}. \tag{4.5}$$

It follows that (4.1) holds. □

Lemma 4.2 *Let $\boldsymbol{u}$ be the exact solution. Then for all $\boldsymbol{v}_h \in U_h$ we have*

$$\sum_{K\in\mathcal{C}_h} h_K^2(\operatorname{div}\varepsilon\,\boldsymbol{u}, \operatorname{div}\varepsilon\,\boldsymbol{v}_h)_{0,K} + h_K^2(\mathbf{curl}\;\boldsymbol{u}, \mathbf{curl}\;\boldsymbol{v}_h)_{0,K}$$
$$\le h(\|\operatorname{div}\varepsilon\,\boldsymbol{u}\|_0 + \|\mathbf{curl}\;\boldsymbol{u}\|_0)\left(\sum_{K\in\mathcal{C}_h} h_K^2\,\|\operatorname{div}\varepsilon\,\boldsymbol{v}_h\|_{0,K}^2 + \sum_{K\in\mathcal{C}_h} h_K^2\,\|\mathbf{curl}\;\boldsymbol{v}_h\|_{0,K}^2\right)^{\frac{1}{2}}.$$

Proof This lemma can be easily proved using the Cauchy-Schwarz inequality. □

The following proposition is an intermediate step for estimating the consistency error stated in Lemma 4.1, where the involved interpolations $\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}} \in P_h^1 \cap H_0^1(\Omega)$ and $\widetilde{\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}} \in (P_h^1)^3$ to $\operatorname{div}\varepsilon\,\boldsymbol{u}$ and $\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}$ can be obtained from the Clément interpolation [26], Bernardi-Girault interpolation [7], or Scott–Zhang interpolation [61]. The latter can also be obtained from the L^2 projection [38,39].

Proposition 4.1 *Let $\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}} \in P_h^1 \cap H_0^1(\Omega)$ and $\widetilde{\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}} \in (P_h^1)^3$ denote the interpolants of $\operatorname{div}\varepsilon\,\boldsymbol{u}$ and $\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}$, respectively, where $\boldsymbol{u}$ is the exact solution. For all $\boldsymbol{v}_h \in U_h$ we have*

$$
\begin{aligned}
&(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u},\,R_h(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{v}_h))_\mu-(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u},\,\mathbf{curl}\;\boldsymbol{v}_h)\\
&\quad=(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}-\widetilde{\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}},\,R_h(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{v}_h))_\mu\\
&\qquad+(\widetilde{\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}}-\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u},\,\mathbf{curl}\;\boldsymbol{v}_h)\\
&\qquad-\sum_{F\in\mathcal{E}_h^\Gamma}\int_F\boldsymbol{v}_h\times\boldsymbol{n}\cdot\boldsymbol{n}\times(\widetilde{\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}}\times\boldsymbol{n});
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
&(\operatorname{div}\varepsilon\,\boldsymbol{u},\,\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))-\sum_{K\in\mathcal{C}_h}(\operatorname{div}\varepsilon\,\boldsymbol{u},\,\operatorname{div}\varepsilon\,\boldsymbol{v}_h)_{0,K}\\
&\quad=(\operatorname{div}\varepsilon\,\boldsymbol{u}-\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}},\,\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))+(\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}},\,\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))\\
&\qquad-(\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}},\,\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))_{0,h}\\
&\qquad+\sum_{K\in\mathcal{C}_h}(\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}}-\operatorname{div}\varepsilon\,\boldsymbol{u},\,\operatorname{div}\varepsilon\,\boldsymbol{v}_h)_{0,K}-\sum_{F\in\mathcal{E}_h^{\mathrm{inter}}}\int_F[\![\varepsilon\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]\,\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}}.
\end{aligned}
\tag{4.7}
$$

Proof Equation (4.6) holds, since from (2.10) we can obtain

$$
\begin{aligned}
&(\widetilde{\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}},\,R_h(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{v}_h))_\mu\\
&\quad=(\widetilde{\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}},\,\mathbf{curl}\;\boldsymbol{v}_h)-\sum_{F\in\mathcal{E}_h^\Gamma}\int_F\boldsymbol{v}_h\times\boldsymbol{n}\cdot\boldsymbol{n}\times(\widetilde{\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}}\times\boldsymbol{n}).
\end{aligned}
\tag{4.8}
$$

Equation (4.7) holds, since

$$
\begin{aligned}
(\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}},\,\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))&=(\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}},\,\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))-(\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}},\,\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))_{0,h}\\
&\quad+(\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}},\,\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))_{0,h},
\end{aligned}
\tag{4.9}
$$

where from (2.9) we have

$$
\begin{aligned}
(\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}},\,\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))_{0,h}&=-(\varepsilon\,\boldsymbol{v}_h,\,\nabla\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}})\\
&=\sum_{K\in\mathcal{C}_h}(\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}},\,\operatorname{div}\varepsilon\,\boldsymbol{v}_h)_{0,K}-\sum_{F\in\mathcal{E}_h^{\mathrm{inter}}}\int_F[\![\varepsilon\,\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]\,\widetilde{\operatorname{div}\varepsilon\,\boldsymbol{u}}.
\end{aligned}
\tag{4.10}
$$

$\square$

In order to estimate the consistency error in (4.1), from (4.6) and (4.7) it is obvious that $\operatorname{div}\varepsilon\,\boldsymbol{u}$ and $\mu^{-1}\mathbf{curl}\,\boldsymbol{u}$ must have suitable regularity. This issue is dealt with as follows. Introduce the div Hilbert spaces

$$
\begin{aligned}
H(\operatorname{div};\mu;\Omega)&=\{\boldsymbol{v}\in(L^2(\Omega))^3:\operatorname{div}\mu\,\boldsymbol{v}\in L^2(\Omega)\},\\
H_0(\operatorname{div}^0;\mu;\Omega)&=\{\boldsymbol{v}\in H(\operatorname{div};\mu;\Omega):\operatorname{div}\mu\,\boldsymbol{v}=0,\,(\mu\,\boldsymbol{v})\cdot\boldsymbol{n}|_\Gamma=0\}.
\end{aligned}
$$

Let $\boldsymbol{u}$ be the exact solution of (1.1), (1.2) and (1.3), and let

$$z := \mu^{-1}\,\mathbf{curl}\;\boldsymbol{u} \in H(\mathbf{curl};\Omega) \cap H_0(\mathrm{div}^0;\mu;\Omega), \quad p := \mathrm{div}\,\varepsilon\,\boldsymbol{u} \in H_0^1(\Omega).$$

Then, assuming a suitable smooth $\boldsymbol{f}$, by obvious manipulations, from (1.1)–(1.3) we can have the following two elliptic and Maxwell's interface problems:

$$\begin{gathered}
-\mathrm{div}\,\varepsilon\,\nabla\,p = \mathrm{div}\boldsymbol{f} \quad \text{in } \Omega_l \quad 1 \le l \le L, \quad p|_{\Gamma} = 0, \quad [\![p]\!] = 0,\\
-[\![\varepsilon\,\nabla\,p \cdot \boldsymbol{n}]\!] = [\![\boldsymbol{f} \cdot \boldsymbol{n}]\!] \quad \text{on } \Gamma_{ij},\\
\mathbf{curl}\;\varepsilon^{-1}\,\mathbf{curl}\;z = \mathbf{curl}\;\varepsilon^{-1}\boldsymbol{f}, \quad \mathrm{div}\,\mu\,z = 0 \quad \text{in } \Omega_l \quad 1 \le l \le L,\\
(\mu\,z) \cdot \boldsymbol{n}|_{\Gamma} = 0, \quad \varepsilon^{-1}\,\mathbf{curl}\;z \times \boldsymbol{n}|_{\Gamma} = \varepsilon^{-1}\boldsymbol{f} \times \boldsymbol{n}|_{\Gamma},\\
[\![z \times \boldsymbol{n}]\!] = 0, \quad [\![\mu\,z \cdot \boldsymbol{n}]\!] = 0, \quad [\![\varepsilon^{-1}\,\mathbf{curl}\;z \times \boldsymbol{n}]\!] = [\![\varepsilon^{-1}\boldsymbol{f} \times \boldsymbol{n}]\!] \quad \text{on } \Gamma_{ij}.
\end{gathered}$$

Hypothesis H1 For a suitable smooth $\boldsymbol{f}$, we assume that $p \in \prod_{l=1}^{L} H^{1+r}(\Omega_l)$ with $r > 1/2$, satisfying

$$\sum_{l=1}^{L} ||p||_{1+r,\Omega_l} \le C_*(\boldsymbol{f}).$$

where $C_*(\boldsymbol{f})$ is a constant depending on $\boldsymbol{f}$. In the following remark, we provide some cases where $C_*(\boldsymbol{f})$ can be explicitly given.

Remark 4.1 For a globally smooth ε (say $\varepsilon = 1$) and $\boldsymbol{f} \in H(\mathrm{div};\Omega)$, Hypothesis H1 holds (see [40]), with $C_*(\boldsymbol{f}) \le C||\mathrm{div}\boldsymbol{f}||_{r-1}$ for some $r > 1/2$. For a piecewise smooth ε as assumed, the elliptic interface problem which p solves belongs to the following general elliptic interface problem: given g and χ, to find θ such that

$$\begin{gathered}
-\mathrm{div}\,\varepsilon\,\nabla\,\theta = g \quad \text{in } \Omega_l \quad 1 \le l \le L,\\
\theta|_{\Gamma} = 0, \quad [\![\theta]\!] = 0, \quad -[\![\varepsilon\,\nabla\,\theta \cdot \boldsymbol{n}]\!] = \chi \quad \text{on } \Gamma_{ij}.
\end{gathered}$$

For $L = 2$ (i.e., $\Omega = \Omega_1 \cup \Omega_2$), if $\bar{\Omega}_1 \subset \Omega$ and $\Omega_2 = \Omega \backslash \bar{\Omega}_1$, with the material interface $\Gamma_{12} = \partial\Omega_1 \subset \Omega$, and if ε is a piecewise constant (i.e., $\varepsilon|_{\Omega_l} = c_l$, where c_l is constant), then from Remark 2.4 in [17] there exists a $r > 1/2$ such that

$$\sum_{l=1}^{L} ||\theta||_{1+r,\Omega_l} \le C(||g||_{r-1,\Omega} + ||\chi||_{r-1/2,\Gamma_{12}}),$$

where $g \in H^{r-1}(\Omega)$ and $\chi \in H^{r-1/2}(\Gamma_{12})$. In our case, we may then require that $\mathrm{div}\boldsymbol{f} \in H^{r-1}(\Omega)$ and $\boldsymbol{f}|_{\Omega_l} \in (H^r(\Omega_l))^3$ for some $r > 1/2$, and $C_*(\boldsymbol{f}) \le C(||\mathrm{div}\boldsymbol{f}||_{r-1,\Omega} + \sum_{l=1}^{L} ||\boldsymbol{f}||_{r,\Omega_l})$, where we used the trace theorem: $H^r(\Omega_l)$ is continuously embedded into $H^{r-1/2}(\partial\Omega_l)$. For piecewise smooth material interfaces, more general piecewise regularity estimates for the above elliptic interface problems may be referred to [21,46,58]. For Hypothesis H1, from [17,21,46,58] and from Theorem 4.1 in [31], a reasonable regularity on $\boldsymbol{f}$ seems to be that

$\boldsymbol{f} \in \prod_{l=1}^{L} H(\mathrm{div};\Omega_l) \cap \prod_{l=1}^{L} (H^r(\Omega_l))^3$ for some $r > 1/2$. With this $\boldsymbol{f}$ and the assumed piecewise smooth material coefficient ε (i.e., $\varepsilon|_{\Omega_l} \in (W^{1,\infty}(\Omega_l))^{3\times 3}$ for $1 \le l \le L$), since $[\![\boldsymbol{f}\cdot\boldsymbol{n}]\!]|_{\Gamma_{ij}} \in H^{r-1/2}(\Gamma_{ij})$, under the assumptions of Theorem 4.1 in [31] or under the assumptions of Remark 2.4 in [17], we could expect that Hypothesis H1 holds, with $C_*(\boldsymbol{f}) \le C\sum_{l=1}^{L}(||\mathrm{div}\boldsymbol{f}||_{0,\Omega_l} + ||\boldsymbol{f}||_{r,\Omega_l})$. This issue is however rather complicated in three-dimensions and is beyond the scope of this paper, so we do not intend to deal with it here.

Hypothesis H2 For a suitable smooth $\boldsymbol{f}$, we assume that $\boldsymbol{z} \in \prod_{l=1}^{L} (H^r(\Omega_l))^3$ with $r > 1/2$ and that $\boldsymbol{z}$ can be written into the following regular-singular decomposition

$$\boldsymbol{z} = \boldsymbol{z}_H + \nabla\varphi \quad \text{in } \Omega_l,\, 1 \le l \le L,$$

where $\boldsymbol{z}_H \in H(\mathbf{curl};\Omega) \cap \prod_{l=1}^{L} (H^{1+r}(\Omega_l))^3$ and $\varphi \in H^1(\Omega) \cap \prod_{l=1}^{L} H^{1+r}(\Omega_l)$ satisfy

$$\sum_{l=1}^{L} (||\boldsymbol{z}_H||_{1+r,\Omega_l} + ||\varphi||_{1+r,\Omega_l}) \le C_{**}(\boldsymbol{f}),$$

where $C_{**}(\boldsymbol{f})$ is a constant depending on $\boldsymbol{f}$.

Remark 4.2 Such regular-singular decompositions in Hypothesis H2 for Maxwell interface problems can be found in [29–31,57]. In the case where $\mu = \varepsilon = 1$, or μ and ε are globally in $(W^{1,\infty}(\Omega))^{3\times 3}$, from [57] we conclude that Hypothesis H2 holds, with $C_{**}(\boldsymbol{f}) \le C(||\mathbf{curl}\,\varepsilon^{-1}\boldsymbol{f}||_0 + ||\boldsymbol{f}||_r)$. In general, from Theorem 7.3 in [31] we could assume a reasonable regularity on $\boldsymbol{f}$ for Hypothesis H2 to be true, i.e., $\boldsymbol{f} \in \prod_{l=1}^{L} H(\mathbf{curl};\varepsilon^{-1};\Omega_l) \cap \prod_{l=1}^{L} (H^r(\Omega_l))^3$ for some $r > 1/2$, where $H(\mathbf{curl};\varepsilon^{-1};\Omega_l) = \{\boldsymbol{v} \in (L^2(\Omega_l))^3 : \mathbf{curl}\ \varepsilon^{-1}\boldsymbol{v} \in (L^2(\Omega_l))^3\}$. For this $\boldsymbol{f}$ and the assumed piecewise smooth material coefficients μ and ε (i.e., $\mu|_{\Omega_l}, \varepsilon|_{\Omega_l} \in (W^{1,\infty}(\Omega_l))^{3\times 3}$, $1 \le l \le L$), since $\varepsilon^{-1}\boldsymbol{f}\times\boldsymbol{n}$ belongs to $(H^{r-1/2}(\Gamma))^3$ and $[\![\varepsilon^{-1}\boldsymbol{f}\times\boldsymbol{n}]\!]$ belongs to $(H^{r-1/2}(\Gamma_{ij}))^3$, under the assumptions of Theorem 7.3 in [31], we could expect that Hypothesis H2 holds, with $C_{**}(\boldsymbol{f}) \le C\sum_{l=1}^{L}(||\mathbf{curl}\ \varepsilon^{-1}\boldsymbol{f}||_{0,\Omega_l} + ||\boldsymbol{f}||_{r,\Omega_l})$.

Proposition 4.2 (See [62]) *the mass-lumping L^2 inner product* (2.8)*, we have*

$$|(p,q)_0 - (p,q)_{0,h}| \le C\,h\,||p||_1\,||q||_0 \quad \forall p,q \in P_h^1 \cap H_0^1(\Omega).$$

Lemma 4.3 *Let $\boldsymbol{u}$ be the exact solution. Assuming Hypothesis H1, we have*

$$(\mathrm{div}\,\varepsilon\,\boldsymbol{u},\, \breve{R}_h(\mathrm{div}\,\varepsilon\,\boldsymbol{v}_h)) - \sum_{K\in\mathcal{C}_h} (\mathrm{div}\,\varepsilon\,\boldsymbol{u}, \mathrm{div}\,\varepsilon\,\boldsymbol{v}_h)_{0,K} + \sum_{F\in\mathcal{E}_h^{\mathrm{inter}}} \int_F [\![\varepsilon\,\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]\,\mathrm{div}\,\varepsilon\,\boldsymbol{u}$$
$$\le C\,h\,||\breve{R}_h(\mathrm{div}\,\varepsilon\,\boldsymbol{v}_h)||_{0,h}$$
$$+ C\,h^r \left(\sum_{K\in\mathcal{C}_h} h_K^2\,||\mathrm{div}\,\varepsilon\,\boldsymbol{v}_h||_{0,K}^2 + \sum_{F\in\mathcal{E}_h^{\mathrm{inter}}} h_F \int_F |[\![\varepsilon\,\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]|^2\right)^{\frac{1}{2}}.$$

Proof According to $p = \mathrm{div}\,\varepsilon\,\boldsymbol{u}$ in Hypothesis H1, we define $\widetilde{p} \in P_h^1 \cap H_0^1(\Omega)$ the interpolant to p, e.g., see [7,14,24,26,39,61], such that

$$\left(\sum_{K\in\mathcal{C}_h} h_K^{-2}\,||p-\widetilde{p}||_{0,K}^2\right)^{1/2} + \left(\sum_{K\in\mathcal{C}_h}\sum_{F\subset\partial K} h_F^{-1}\,||p-\widetilde{p}||_{0,F}^2\right)^{1/2}$$
$$\le C\,h^r\sum_{l=1}^{L}||p||_{1+r,\Omega_l},\ ||\widetilde{p}||_1\le C\,||p||_1,\quad ||p-\widetilde{p}||_0\le C\,h\,||p||_1.$$

Then, from Lemma 4.1 and (4.7) in Proposition 4.1 and Proposition 4.2, we have

$$\begin{aligned}&(\operatorname{div}\varepsilon\,\boldsymbol{u},\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))-\sum_{K\in\mathcal{C}_h}(\operatorname{div}\varepsilon\,\boldsymbol{u},\operatorname{div}\varepsilon\,\boldsymbol{v}_h)_{0,K}+\sum_{F\in\mathcal{E}_h^{\text{inter}}}\int_F[\![\varepsilon\,\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]\operatorname{div}\varepsilon\,\boldsymbol{u}\\
&=(\widetilde{p},\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))-(\widetilde{p},\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))_{0,h}+(p-\widetilde{p},\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))\\
&\quad+\sum_{K\in\mathcal{C}_h}(\widetilde{p}-p,\operatorname{div}\varepsilon\,\boldsymbol{v}_h)_{0,K}+\sum_{F\in\mathcal{E}_h^{\text{inter}}}\int_F[\![\varepsilon\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]\,(p-\widetilde{p}),\end{aligned}\tag{4.11}$$

where

$$\begin{aligned}(\widetilde{p},\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))-(\widetilde{p},\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))_{0,h}&\le C\,h\,||\widetilde{p}||_1\,||\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h)||_0\\
&\le C\,h\,||\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h)||_{0,h},\end{aligned}$$

$$\begin{aligned}(p-\widetilde{p},\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h))&\le||p-\widetilde{p}||_0\,||\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h)||_0\le C\,h\,||\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h)||_0\\
&\le C\,h\,||\breve{R}_h(\operatorname{div}\varepsilon\,\boldsymbol{v}_h)||_{0,h},\end{aligned}$$

$$\begin{aligned}&\sum_{K\in\mathcal{C}_h}(\widetilde{p}-p,\operatorname{div}\varepsilon\,\boldsymbol{v}_h)_{0,K}\\
&\quad\le\left(\sum_{K\in\mathcal{C}_h}h_K^{-2}\,||\widetilde{p}-p||_{0,K}^2\right)^{1/2}\left(\sum_{K\in\mathcal{C}_h}h_K^2\,||\operatorname{div}\varepsilon\,\boldsymbol{v}_h||_{0,K}^2\right)^{1/2}\\
&\quad\le Ch^r\left(\sum_{K\in\mathcal{C}_h}h_K^2\,||\operatorname{div}\varepsilon\,\boldsymbol{v}_h||_{0,K}^2\right)^{1/2}\end{aligned}$$

$$\begin{aligned}&\sum_{F\in\mathcal{E}_h^{\text{inter}}}\int_F[\![\varepsilon\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]\,(p-\widetilde{p})\\
&\le\left(\sum_{F\in\mathcal{E}_h^{\text{inter}}}h_F^{-1}\,||p-\widetilde{p}||_{0,F}^2\right)^{1/2}\left(\sum_{F\in\mathcal{E}_h^{\text{inter}}}h_F\int_F|[\![\varepsilon\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]|^2\right)^{1/2}\\
&\le C\,h^r\left(\sum_{F\in\mathcal{E}_h^{\text{inter}}}h_F\int_F|[\![\varepsilon\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]|^2\right)^{1/2}.\end{aligned}$$

□

Lemma 4.4 *Let* $\boldsymbol{u}$ *be the exact solution. Assuming Hypothesis H2, we have*

$$(\mu^{-1}\,\mathbf{curl}\ \boldsymbol{u},R_h(\mu^{-1}\,\mathbf{curl}\ \boldsymbol{v}_h))_\mu-(\mu^{-1}\,\mathbf{curl}\ \boldsymbol{u},\mathbf{curl}\ \boldsymbol{v}_h)$$

$$
\begin{aligned}
&+\sum_{F\in\mathcal{E}_h^\Gamma}\int_F \boldsymbol{v}_h\times\boldsymbol{n}\cdot\boldsymbol{n}\times(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}\times\boldsymbol{n})\\
&\le C\,h^r\,||R_h(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{v}_h)||_\mu\\
&\quad +C\,h^r\left(\sum_{K\in\mathcal{C}_h}h_K^2\,||\mathbf{curl}\;\boldsymbol{v}_h||_{0,K}^2+\sum_{F\in\mathcal{E}_h^\Gamma}h_F\int_F|\boldsymbol{v}_h\times\boldsymbol{n}|^2\right)^{\frac{1}{2}}. \qquad (4.12)
\end{aligned}
$$

Proof According to $z=\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}=z_H+\nabla\varphi$ in Hypothesis H2, we define $\widetilde{z}\in(P_h^1)^3$ as follows:

$$
\widetilde{z}=\widetilde{z}_H+\nabla\widetilde{\varphi},
$$

where $\widetilde{z}_H\in(P_h^1)^3$ is the interpolant to z_H and $\widetilde{\varphi}\in P_h^1\cap H^1(\Omega)$ is the interpolant to φ, see [7,14,24,26,39,61] for general finite element interpolation theory, satisfying

$$
\begin{aligned}
&\left(\sum_{K\in\mathcal{C}_h}h_K^{-2}\,||z_H-\widetilde{z}_H||_{0,K}^2\right)^{1/2}+\left(\sum_{K\in\mathcal{C}_h}\sum_{F\subset\partial K}h_F^{-1}\,||z_H-\widetilde{z}_H||_{0,F}^2\right)^{1/2}\\
&\qquad +||\varphi-\widetilde{\varphi}||_1\le C\,h^r,
\end{aligned}
$$

and we have

$$
||z-\widetilde{z}||_0\le C\,h^r.
$$

We also have

$$
(\nabla(\widetilde{\varphi}-\varphi),\mathbf{curl}\;\boldsymbol{v}_h)=\sum_{F\in\mathcal{E}_h^\Gamma}\int_F \boldsymbol{v}_h\times\boldsymbol{n}\cdot\boldsymbol{n}\times(\nabla(\widetilde{\varphi}-\varphi)\times\boldsymbol{n})\quad\forall\boldsymbol{v}_h\in U_h.
$$

Therefore, we have from (4.6) in Proposition 4.1 and Lemma 4.1 that

$$
\begin{aligned}
&(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u},R_h(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{v}_h))_\mu-(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u},\mathbf{curl}\;\boldsymbol{v}_h)\\
&\quad+\sum_{F\in\mathcal{E}_h^\Gamma}\int_F \boldsymbol{v}_h\times\boldsymbol{n}\cdot\boldsymbol{n}\times(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{u}\times\boldsymbol{n})\\
&=(z-\widetilde{z},R_h(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{v}_h))_\mu+(\widetilde{z}_H-z_H,\mathbf{curl}\;\boldsymbol{v}_h)+(\nabla(\widetilde{\varphi}-\varphi),\mathbf{curl}\;\boldsymbol{v}_h)\\
&\quad+\sum_{F\in\mathcal{E}_h^\Gamma}\int_F \boldsymbol{v}_h\times\boldsymbol{n}\cdot\boldsymbol{n}\times((z_H-\widetilde{z}_H)\times\boldsymbol{n})+\sum_{F\in\mathcal{E}_h^\Gamma}\int_F \boldsymbol{v}_h\times\boldsymbol{n}\cdot\boldsymbol{n}\times(\nabla(\varphi-\widetilde{\varphi})\times\boldsymbol{n})\\
&=(z-\widetilde{z},R_h(\mu^{-1}\,\mathbf{curl}\;\boldsymbol{v}_h))_\mu+(\widetilde{z}_H-z_H,\mathbf{curl}\;\boldsymbol{v}_h)\\
&\quad+\sum_{F\in\mathcal{E}_h^\Gamma}\int_F \boldsymbol{v}_h\times\boldsymbol{n}\cdot\boldsymbol{n}\times((z_H-\widetilde{z}_H)\times\boldsymbol{n}),
\end{aligned}
$$

where

$$
\begin{aligned}
&(z - \widetilde{z}, R_h(\mu^{-1} \mathbf{curl}\ \boldsymbol{v}_h))_\mu \\
&\quad \le ||z - \widetilde{z}||_\mu \, ||R_h(\mu^{-1} \mathbf{curl}\ \boldsymbol{v}_h)||_\mu \le C\, h^r \, ||R_h(\mu^{-1} \mathbf{curl}\ \boldsymbol{v}_h)||_\mu, \\
&(\widetilde{z}_H - z_H, \mathbf{curl}\ \boldsymbol{v}_h) \\
&\quad \le \left(\sum_{K \in \mathcal{C}_h} h_K^{-2} \, ||z_H - \widetilde{z}_H||_{0,K}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{C}_h} h_K^2 \, ||\mathbf{curl}\ \boldsymbol{v}_h||_{0,K}^2 \right)^{1/2} \\
&\quad \le C\, h^r \left(\sum_{K \in \mathcal{C}_h} h_K^2 \, ||\mathbf{curl}\ \boldsymbol{v}_h||_{0,K}^2 \right)^{1/2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{F \in \mathcal{E}_h^\Gamma} \int_F \boldsymbol{v}_h \times \boldsymbol{n} \cdot \boldsymbol{n} \times ((z_H - \widetilde{z}_H) \times \boldsymbol{n}) \\
&\quad \le \left(\sum_{F \in \mathcal{E}_h^\Gamma} h_F^{-1} \, ||z_H - \widetilde{z}_H||_{0,F}^2 \right)^{1/2} \left(\sum_{F \in \mathcal{E}_h^\Gamma} h_F \int_F |\boldsymbol{v}_h \times \boldsymbol{n}|^2 \right)^{1/2} \\
&\quad \le C\, h^r \left(\sum_{F \in \mathcal{E}_h^\Gamma} h_F \int_F |\boldsymbol{v}_h \times \boldsymbol{n}|^2 \right)^{1/2}.
\end{aligned}
$$

It follows that (4.12) holds. □

Simply summarizing these lemmas from Lemma 4.1 to Lemma 4.4 we obtain the estimate for the consistency errors caused by L^2 projections and the stabilization in the following theorem.

Theorem 4.1 *Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the exact solution of problem* (1.1), (1.2), (1.3) *and the finite element solution of problem* (2.14)*, respectively. Then, under Hypotheses H1 and H2,*

$$
|\mathcal{L}_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h)| \le C\, h^r \, |||\boldsymbol{v}_h|||_{\mathcal{L}_h} \quad \forall \boldsymbol{v}_h \in U_h,
$$

where

$$
|||\boldsymbol{v}_h|||_{\mathcal{L}_h}^2 := \mathcal{L}_h(\boldsymbol{v}_h, \boldsymbol{v}_h). \tag{4.13}
$$

4.2 Constructing an interpolant of the solution

In this subsection we shall construct a finite element interpolant $\widetilde{\boldsymbol{u}} \in U_h$ of the solution $\boldsymbol{u}$, which is to be used in next subsection to estimate the error of our continuous finite element method.

Theorem 4.2 *Let $\boldsymbol{u}$ be the exact solution and $\boldsymbol{u} \in \prod_{l=1}^{L} (H^r(\Omega_l))^3$ with $r > \frac{1}{2}$. Let U_h be defined as in (2.6). Then, there is a function $\widetilde{\boldsymbol{u}} \in U_h$ such that*

$$\breve{R}_h(\operatorname{div} \varepsilon\,(\boldsymbol{u} - \widetilde{\boldsymbol{u}})) = 0, \quad R_h(\mu^{-1}\,\mathbf{curl}\;(\boldsymbol{u} - \widetilde{\boldsymbol{u}})) = 0 \tag{4.14}$$

and

$$||\boldsymbol{u} - \widetilde{\boldsymbol{u}}||_0 \le C\,h^r \sum_{l=1}^{L} ||\boldsymbol{u}||_{r,\Omega_l}. \tag{4.15}$$

Proof Firstly there is a function $\boldsymbol{u}^0 \in \mathfrak{U}_h$ such that (See [6,7,14,26,61])

$$||\boldsymbol{u} - \boldsymbol{u}^0||_0 + \left(\sum_{K \in \mathcal{C}_h} \sum_{F \subset \partial K} h_F\,||\boldsymbol{u} - \boldsymbol{u}^0||_{0,F}^2\right)^{1/2} \le C\,h^r \sum_{l=1}^{L} ||\boldsymbol{u}||_{r,\Omega_l}. \tag{4.16}$$

We then define $\widetilde{\boldsymbol{u}} \in U_h$ by the following conditions:

$$\widetilde{\boldsymbol{u}}(a) = \boldsymbol{u}^0(a) \quad \text{for all vertices} \quad a, \tag{4.17}$$

$$\int_F (\widetilde{\boldsymbol{u}} - \boldsymbol{u}) \cdot \boldsymbol{p} = 0 \quad \forall \boldsymbol{p} \in (\mathcal{P}_1(F))^3, \forall F \subset \partial K, \text{with} \quad F \not\subset \Gamma, \forall K \tag{4.18}$$

and

$$\int_K (\widetilde{\boldsymbol{u}} - \boldsymbol{u}) \cdot \boldsymbol{q} = \mathbf{0} \quad \forall \boldsymbol{q} \in \boldsymbol{P}_0(\varepsilon;\, K). \tag{4.19}$$

According to (2.3)–(2.6) we can write $\widetilde{\boldsymbol{u}}$ in K as

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u}^0 + \sum_{\substack{F \subset \partial K \\ F \not\subset \Gamma}} \sum_{l=1}^{m_F} c_{F,l}\,\boldsymbol{p}_{F,l}\,b_F + \sum_{l=1}^{m_K} c_{K,l}\,\boldsymbol{q}_{K,l}\,b_K =: \hat{\boldsymbol{u}} + \sum_{l=1}^{m_K} c_{K,l}\,\boldsymbol{q}_{K,l}\,b_K, \tag{4.20}$$

where $(\mathcal{P}_1(F))^3 = \operatorname{span}\{\boldsymbol{p}_{F,l}|_F$, where $\boldsymbol{p}_{F,l}|_K \in (\mathcal{P}_1(K))^3$, $1 \le l \le m_F\}$, in which the integer m_F denotes the dimension of $(\mathcal{P}_1(F))^3$ (namely, $m_F = 9$), and $\boldsymbol{P}_0(\varepsilon;\, K) = \operatorname{span}\{\boldsymbol{q}_{K,l}, 1 \le l \le m_K\}$, and $c_{F,l}, c_{K,l} \in \mathbb{R}$ are all coefficients to be determined according to (4.18) and (4.19), i.e.

$$\sum_{l=1}^{m_F} c_{F,l} \int_F \boldsymbol{p}_{F,l} \cdot \boldsymbol{p}_{F,i} \, b_F = \int_F (\boldsymbol{u} - \boldsymbol{u}^0) \cdot \boldsymbol{p}_{F,i} \quad 1 \le i \le m_F, \forall F \subset \partial K \tag{4.21}$$

and

$$\sum_{l=1}^{m_K} c_{K,l} \int_K \boldsymbol{q}_{K,l} \cdot \boldsymbol{q}_{K,i} \, b_K = \int_K (\boldsymbol{u} - \hat{\boldsymbol{u}}) \cdot \boldsymbol{q}_{K,i} \quad 1 \le i \le m_K, \forall K. \tag{4.22}$$

Since the bubble-related terms in (4.20) are zero at any vertex and the element bubble terms are zero along any face, we can obtain that $\widetilde{\boldsymbol{u}} \in U_h$ is uniquely determined by (4.17)–(4.19). Using the standard scaling argument (See e.g. [24]), we can have

$$||\boldsymbol{u} - \hat{\boldsymbol{u}}||_{0,K} \le C \, ||\boldsymbol{u} - \boldsymbol{u}^0||_{0,K} + C \sum_{F \subset \partial K} h_F^{\frac{1}{2}} \, ||\boldsymbol{u} - \boldsymbol{u}^0||_{0,F} \tag{4.23}$$

and

$$||\boldsymbol{u} - \widetilde{\boldsymbol{u}}||_{0,K} \le C \, ||\boldsymbol{u} - \hat{\boldsymbol{u}}||_{0,K}. \tag{4.24}$$

It follows from (4.23), (4.24) and (4.16) that (4.15) holds.

Now we shall verify (4.14). By virtue of (4.19) and (2.9), we have

$$\begin{aligned} ||\breve{R}_h(\operatorname{div} \varepsilon \, (\boldsymbol{u} - \widetilde{\boldsymbol{u}}))||_{0,h}^2 &= -(\varepsilon \, (\boldsymbol{u} - \widetilde{\boldsymbol{u}}), \nabla \breve{R}_h(\operatorname{div} \varepsilon \, (\boldsymbol{u} - \widetilde{\boldsymbol{u}}))) \\ &= -(\boldsymbol{u} - \widetilde{\boldsymbol{u}}, \varepsilon \, \nabla \, \breve{R}_h(\operatorname{div} \varepsilon \, (\boldsymbol{u} - \widetilde{\boldsymbol{u}}))) \\ &= 0 \end{aligned} \tag{4.25}$$

since $\varepsilon \, \nabla \, \breve{R}_h(\operatorname{div} \varepsilon \, (\boldsymbol{u} - \widetilde{\boldsymbol{u}}))|_K \in \varepsilon(\mathcal{P}_0(K))^3 \subset \boldsymbol{P}_0(\varepsilon; K)$, while from (4.18) and (2.10), we have

$$\begin{aligned} ||R_h(\mu^{-1} \, \mathbf{curl} \, (\boldsymbol{u} - \widetilde{\boldsymbol{u}}))||_\mu^2 = &\sum_{K \in \mathcal{C}_h} (\boldsymbol{u} - \widetilde{\boldsymbol{u}}, \mathbf{curl} \, R_h(\mu^{-1} \, \mathbf{curl} \, (\boldsymbol{u} - \widetilde{\boldsymbol{u}})))_{0,K} \\ &+ \sum_{K \in \mathcal{C}_h} \sum_{\substack{F \subset \partial K \\ F \notin \mathcal{E}_h^\Gamma}} \int_F (\boldsymbol{u} - \widetilde{\boldsymbol{u}}) \times \boldsymbol{n} \cdot R_h(\mu^{-1} \, \mathbf{curl} \, (\boldsymbol{u} - \widetilde{\boldsymbol{u}})) \\ = &\, 0 \end{aligned} \tag{4.26}$$

since $\mathbf{curl} \, R_h(\mu^{-1} \, \mathbf{curl} \, (\boldsymbol{u} - \widetilde{\boldsymbol{u}}))|_K \in (\mathcal{P}_0(K))^3 \subset \boldsymbol{P}_0(\varepsilon; K)$ and $\boldsymbol{n} \times R_h(\mu^{-1}$ $\mathbf{curl} \, (\boldsymbol{u} - \widetilde{\boldsymbol{u}}))|_F \in (\mathcal{P}_1(F))^3$. So (4.14) holds. □

Remark 4.3 The assumption that the exact solution $\boldsymbol{u} \in \prod_{l=1}^{L}(H^r(\Omega_l))^3$ for some $r > 1/2$ is commonly used in the literature, see, e.g., [22], while in [44] even more regularity is assumed, i.e., $r = 1$, and in [23] a piecewise regularity $W^{1,p}(\Omega_l)$ for some $p > 2$ is assumed. Meanwhile, we have known that $r = 1$ is shown for the case where two materials occupy a convex polyhedron Ω in [46]. We should point out that, from Sobolev embedding theorem [1,39], the main reason for $r > 1/2$ is that it makes the element-face integrals of $\boldsymbol{u}$ well-defined, and the other reason is that $r > 1/2$ ensures that the traces of $\boldsymbol{f}$ in Hypotheses H1 and H2 are well-defined.

Remark 4.4 The technique for constructing the finite element interpolant from Eqs. (4.17)–(4.19) is quite classical, e.g., see Lemma A.3 in [39], and somewhat similar technique may be also found in Sect. 2, Chapter II of [39].

4.3 Error estimates in an energy norm

We introduce an energy norm:

$$
\begin{aligned}
|||\boldsymbol{v}|||_{h,\mathbf{curl}\,,\mathrm{div}}^2 := ||\boldsymbol{v}||_0^2 + |||\boldsymbol{v}|||_{\mathcal{L}_h}^2 &= ||\boldsymbol{v}||_0^2 + \mathcal{L}_h(\boldsymbol{v}, \boldsymbol{v}) \\
&= ||\boldsymbol{v}||_0^2 + ||\breve{R}_h(\mathrm{div}\, \varepsilon\, \boldsymbol{v})||_{0,h}^2 + ||R_h(\mu^{-1}\, \mathbf{curl}\ \boldsymbol{v})||_{\mu}^2 \\
&\quad + \mathcal{S}_h(\boldsymbol{v}, \boldsymbol{v}). \qquad (4.27)
\end{aligned}
$$

Recall $\Omega = \cup_{l=1}^{L}\Omega_l$, where $\{\Omega_l, 1 \le l \le L\}$ is the partition of Ω associated with discontinuities of ε. Let $\mathcal{E}_h^l$ denote the collection of element faces in $\mathcal{C}_h^l$, the subset of $\mathcal{C}_h$ corresponding to Ω_l. On Ω_l, let V_h be the Crouzeix-Raviart nonconforming linear element [32], i.e.,

$$
V_h = \left\{ z \in L^2(\Omega_l);\, z|_K \in \mathcal{P}_1(K),\, \int_F [\![z]\!] = 0, \forall F \in \mathcal{E}_h^l \right\}. \qquad (4.28)
$$

Let $J_h : H^r(\Omega_l) \to V_h$, $r > 1/2$ be the interpolation operator, for z we define $J_h\, z \in V_h$ by

$$
\int_F J_h\, z = \int_F z \quad \forall F \in \mathcal{E}_h^l, \qquad (4.29)
$$

and let $\boldsymbol{J}_h : (H^r(\Omega_l))^3 \to (V_h)^3$ be the vector interpolation operator. Let $\Pi_0 : L^2(\Omega_l) \to P_h^0$ denote the piecewise-constant L^2 projection, i.e., for z we have

$$
\Pi_0 z|_K = \frac{\int_K z}{|K|} \quad \forall K \in \mathcal{C}_h^l, \qquad (4.30)
$$

and $\boldsymbol{\Pi}_0 : (L^2(\Omega_l))^3 \to (P_h^0)^3$ the vector L^2 projection.

Proposition 4.3 (See [32]) *Given $z \in H^r(\Omega_l)$ or $\boldsymbol{z} \in (H^r(\Omega_l))^3$. We have the following relations:*

$$||J_h z - z||_{0,K} + h_F^{1/2}\, ||J_h z - z||_{0,F} \le C\, h_K^r\, ||z||_{r,K} \quad \forall K \in \mathcal{C}_h^l,$$

$$\operatorname{div} \boldsymbol{J}_h \boldsymbol{z} = \Pi_0 \operatorname{div} \boldsymbol{z}, \quad ||\operatorname{div}(\boldsymbol{J}_h \boldsymbol{z} - \boldsymbol{z})||_{0,K} \le 2\, ||\operatorname{div} \boldsymbol{z}||_{0,K} \quad \forall K \in \mathcal{C}_h^l,$$

$$\mathbf{curl}\ \boldsymbol{J}_h \boldsymbol{z} = \boldsymbol{\Pi}_0\, \mathbf{curl}\ \boldsymbol{z}, \quad ||\mathbf{curl}\ (\boldsymbol{J}_h \boldsymbol{z} - \boldsymbol{z})||_{0,K} \le 2\, ||\mathbf{curl}\ \boldsymbol{z}||_{0,K} \quad \forall K \in \mathcal{C}_h^l.$$

Lemma 4.5 *Recall that $\varepsilon = (\varepsilon_{ij})$ is piecewise smooth, i.e., $\varepsilon_{ij}|_{\Omega_l} \in W^{1,\infty}(\Omega_l)$, $1 \le l \le L$. For the exact solution $\boldsymbol{u}$ and its interpolation $\widetilde{\boldsymbol{u}} \in U_h$ as in Theorem 4.2, we have*

$$(\mathcal{S}_h(\boldsymbol{u} - \widetilde{\boldsymbol{u}}, \boldsymbol{u} - \widetilde{\boldsymbol{u}}))^{1/2} \le C h^r (||\operatorname{div} \varepsilon\, \boldsymbol{u}||_0 + ||\mathbf{curl}\ \boldsymbol{u}||_0 + \sum_{l=1}^{L} ||\boldsymbol{u}||_{r,\Omega_l}). \tag{4.31}$$

Proof Let $\widehat{\varepsilon \boldsymbol{u}} := \boldsymbol{J}_h\, \varepsilon\, \boldsymbol{u} \in (V_h)^3$ be the nonconforming linear element interpolation of $\varepsilon\, \boldsymbol{u}|_{\Omega_l}$. Under the regularity assumption on ε, we have from Proposition 4.3 that for any $K \in \mathcal{C}_h^l$

$$\begin{aligned} ||\widehat{\varepsilon \boldsymbol{u}} - \varepsilon\, \boldsymbol{u}||_{0,K} + h_F^{1/2}\, ||\widehat{\varepsilon \boldsymbol{u}} - \varepsilon\, \boldsymbol{u}||_{0,F} &\le C\, h_K^r ||\boldsymbol{u}||_{r,K}, \\ ||\operatorname{div}\, (\varepsilon\, \boldsymbol{u} - \widehat{\varepsilon \boldsymbol{u}})||_{0,K} &\le C\, ||\operatorname{div} \varepsilon\, \boldsymbol{u}||_{0,K}. \end{aligned}$$

In addition, let ε_0 be some finite element interpolation of ε, satisfying [24]

$$||\varepsilon - \varepsilon_0||_{0,\infty,K} + h_K |\varepsilon - \varepsilon_0|_{1,\infty,K} \le C h_K\, |\varepsilon|_{1,\infty,K},$$

where $||\cdot||_{0,\infty,K}, |\cdot|_{1,\infty,K}$ denote the semi-norms of Sobolev space $W^{1,\infty}(K)$, see [1].

We first consider the error term

$$\left(\sum_{K \in \mathcal{C}_h} h_K^2\, ||\operatorname{div} \varepsilon\, (\boldsymbol{u} - \widetilde{\boldsymbol{u}})||_{0,K}^2 \right)^{1/2}.$$

We have

$$\begin{aligned} h_K ||\operatorname{div} \varepsilon\, (\boldsymbol{u} - \widetilde{\boldsymbol{u}})||_{0,K} \le{}& h_K ||\operatorname{div}\, (\varepsilon\, \boldsymbol{u} - \widehat{\varepsilon \boldsymbol{u}})||_{0,K} + h_K\, ||\operatorname{div}\, (\widehat{\varepsilon \boldsymbol{u}} - \varepsilon_0\, \widetilde{\boldsymbol{u}})||_{0,K} \\ &+ h_K\, ||\operatorname{div}\, (\varepsilon_0 - \varepsilon)\, \widetilde{\boldsymbol{u}}||_{0,K}, \end{aligned}$$

where

$$
\begin{aligned}
&h_K ||\mathrm{div}\,(\varepsilon\,\boldsymbol{u} - \widehat{\varepsilon\,\boldsymbol{u}})||_{0,K} \le C\, h_K ||\mathrm{div}\,\varepsilon\,\boldsymbol{u}||_{0,K},\\
&h_K\, ||\mathrm{div}\,(\widehat{\varepsilon\,\boldsymbol{u}} - \varepsilon_0\,\widetilde{\boldsymbol{u}})||_{0,K} \le C||\widehat{\varepsilon\,\boldsymbol{u}} - \varepsilon_0\,\widetilde{\boldsymbol{u}}||_{0,K} \le C\left(||\widehat{\varepsilon\,\boldsymbol{u}} - \varepsilon\,\boldsymbol{u}||_{0,K}\right.\\
&\left.+||\varepsilon\,(\boldsymbol{u} - \widetilde{\boldsymbol{u}})||_{0,K} + ||(\varepsilon-\varepsilon_0)\,(\widetilde{\boldsymbol{u}} - \boldsymbol{u})||_{0,K} + ||(\varepsilon-\varepsilon_0)\,\boldsymbol{u}||_{0,K}\right),\\
&h_K\, ||\mathrm{div}\,(\varepsilon_0-\varepsilon)\,\widetilde{\boldsymbol{u}}||_{0,K} \le C\, h_K\, |\varepsilon_0-\varepsilon|_{1,\infty,K}\, ||\widetilde{\boldsymbol{u}}||_{0,K} + C\, h_K\, ||\varepsilon_0-\varepsilon||_{0,\infty,K}\, |\widetilde{\boldsymbol{u}}|_{1,K},\\
&h_K\, |\varepsilon_0-\varepsilon|_{1,\infty,K}\, ||\widetilde{\boldsymbol{u}}||_{0,K} \le h_K\, |\varepsilon_0-\varepsilon|_{1,\infty,K}\, ||\widetilde{\boldsymbol{u}}-\boldsymbol{u}||_{0,K} + h_K\, |\varepsilon_0-\varepsilon|_{1,\infty,K}\, ||\boldsymbol{u}||_{0,K},\\
&h_K\, ||\varepsilon_0-\varepsilon||_{0,\infty,K}\, |\widetilde{\boldsymbol{u}}|_{1,K} \le C\, ||\varepsilon_0-\varepsilon||_{0,\infty,K}\, ||\widetilde{\boldsymbol{u}}||_{0,K} \le C\, ||\varepsilon_0-\varepsilon||_{0,\infty,K}\, ||\widetilde{\boldsymbol{u}}-\boldsymbol{u}||_{0,K}\\
&+C\, ||\varepsilon_0-\varepsilon||_{0,\infty,K}\, ||\boldsymbol{u}||_{0,K}.
\end{aligned}
$$

By collecting all above estimates we obtain

$$
\begin{aligned}
&\left(\sum_{K\in\mathcal{C}_h} h_K^2\, ||\mathrm{div}\,\varepsilon\,(\boldsymbol{u} - \widetilde{\boldsymbol{u}})||_{0,K}^2\right)^{1/2}\\
&\le C\sum_{l=1}^{L}(h\, ||\mathrm{div}\,\varepsilon\,\boldsymbol{u}||_{0,\Omega_l} + ||\widehat{\varepsilon\,\boldsymbol{u}} - \varepsilon\,\boldsymbol{u}||_{0,\Omega_l} + ||\widetilde{\boldsymbol{u}} - \boldsymbol{u}||_{0,\Omega_l})\\
&\quad + C\sum_{l=1}^{L}(||\varepsilon-\varepsilon_0||_{0,\infty,\Omega_l} + h\,|\varepsilon-\varepsilon_0|_{1,\infty,\Omega_l})\, ||\boldsymbol{u}||_{0,\Omega_l}\\
&\le C\, h^r\left(||\mathrm{div}\,\varepsilon\,\boldsymbol{u}||_0 + \sum_{l=1}^{L}\, ||\boldsymbol{u}||_{r,\Omega_l}\right).
\end{aligned}
$$

Regarding the error term

$$
\left(\sum_{F\in\mathcal{E}_h^{\mathrm{inter}}} h_F \int_F\, |[\![\varepsilon(\boldsymbol{u} - \widetilde{\boldsymbol{u}})\cdot\boldsymbol{n}]\!]|^2\right)^{1/2},
$$

with two sides F^+, F^- of F, we have

$$
h_F\int_F [\![\varepsilon(\boldsymbol{u} - \widetilde{\boldsymbol{u}})\cdot\boldsymbol{n}]\!]^2 \le C\, h_F\, ||\varepsilon(\boldsymbol{u} - \widetilde{\boldsymbol{u}})||_{0,F^+}^2 + C\, h_F\, ||\varepsilon(\boldsymbol{u} - \widetilde{\boldsymbol{u}})||_{0,F^-}^2,
$$

where for either side of F with the corresponding element K we have

$$
\begin{aligned}
&h_F^{1/2}\, ||\varepsilon(\boldsymbol{u} - \widetilde{\boldsymbol{u}})||_{0,F} \le h_F^{1/2}\, ||\varepsilon\boldsymbol{u} - \widehat{\varepsilon\,\boldsymbol{u}}||_{0,F} + h_F^{1/2}\, ||\widehat{\varepsilon\,\boldsymbol{u}} - \varepsilon\,\widetilde{\boldsymbol{u}}||_{0,F},\\
&h_F^{1/2}\, ||\widehat{\varepsilon\,\boldsymbol{u}} - \varepsilon\,\widetilde{\boldsymbol{u}}||_{0,F} \le h_F^{1/2}\, ||\widehat{\varepsilon\,\boldsymbol{u}} - \varepsilon_0\,\widetilde{\boldsymbol{u}}||_{0,F} + h_F^{1/2}\, ||\varepsilon_0\,\widetilde{\boldsymbol{u}} - \varepsilon\,\widetilde{\boldsymbol{u}}||_{0,F},\\
&h_F^{1/2}\, ||\widehat{\varepsilon\,\boldsymbol{u}} - \varepsilon_0\,\widetilde{\boldsymbol{u}}||_{0,F} \le C\, ||\widehat{\varepsilon\,\boldsymbol{u}} - \varepsilon_0\,\widetilde{\boldsymbol{u}}||_{0,K},\\
&h_F^{1/2}\, ||\varepsilon_0\,\widetilde{\boldsymbol{u}} - \varepsilon\,\widetilde{\boldsymbol{u}}||_{0,F} \le C\,(||\varepsilon_0\,\widetilde{\boldsymbol{u}} - \varepsilon\,\widetilde{\boldsymbol{u}}||_{0,K} + h_K|\varepsilon_0\,\widetilde{\boldsymbol{u}} - \varepsilon\,\widetilde{\boldsymbol{u}}|_{1,K})
\end{aligned}
$$

and

$$|\varepsilon_0 \widetilde{\boldsymbol{u}} - \varepsilon \widetilde{\boldsymbol{u}}|_{1,K} \leq C(|\varepsilon_0 - \varepsilon|_{1,\infty,K} \, ||\widetilde{\boldsymbol{u}}||_{0,K} + ||\varepsilon_0 - \varepsilon||_{0,\infty,K} \, |\widetilde{\boldsymbol{u}}|_{1,K}).$$

Then, all the subsequent estimations are the same as those for the error term $\left(\sum_{K\in\mathcal{C}_h} h_K^2 \, ||\mathrm{div}\, \varepsilon\, (\boldsymbol{u} - \widetilde{\boldsymbol{u}})||_{0,K}^2\right)^{1/2}$, we thus obtain

$$\left(\sum_{F\in\mathcal{E}_h^{\text{inter}}} h_F \int_F |[\![\varepsilon(\boldsymbol{u} - \widetilde{\boldsymbol{u}}) \cdot \boldsymbol{n}]\!]|^2\right)^{1/2} \leq C\, h^r \left(||\mathrm{div}\, \varepsilon\, \boldsymbol{u}||_0 + \sum_{l=1}^{L} ||\boldsymbol{u}||_{r,\Omega_l}\right).$$

To estimate $(\sum_{K\in\mathcal{C}_h} h_K^2 \, ||\mathbf{curl}\ (\boldsymbol{u}-\widetilde{\boldsymbol{u}})||_{0,K}^2)^{1/2}$ and $(\sum_{F\in\mathcal{E}_h^\Gamma} h_F \int_F |(\boldsymbol{u}-\widetilde{\boldsymbol{u}})\times\boldsymbol{n}|^2)^{1/2}$, following similar arguments as above, with $\hat{\boldsymbol{u}} := \boldsymbol{J}_h\, \boldsymbol{u} \in (V_h)^3$ we have

$$\left(\sum_{K\in\mathcal{C}_h} h_K^2 \, ||\mathbf{curl}\ (\boldsymbol{u} - \widetilde{\boldsymbol{u}})||_{0,K}^2\right)^{1/2} + \left(\sum_{F\in\mathcal{E}_h^\Gamma} h_F \int_F |(\boldsymbol{u} - \widetilde{\boldsymbol{u}}) \times \boldsymbol{n}|^2\right)^{1/2}$$
$$\leq C\, h^r \left(||\mathbf{curl}\ \boldsymbol{u}||_0 + \sum_{l=1}^{L} ||\boldsymbol{u}||_{r,\Omega_l}\right).$$

□

Remark 4.5 If $\boldsymbol{u}$ is in $(H^1(\Omega))^3$, then from $||\widetilde{\boldsymbol{u}}||_1 \leq C||\boldsymbol{u}||_1$ we easily have

$$(\mathcal{S}_h(\boldsymbol{u} - \widetilde{\boldsymbol{u}}, \boldsymbol{u} - \widetilde{\boldsymbol{u}}))^{1/2} \leq C\, h\, ||\boldsymbol{u}||_1,$$

which is consistent with (4.31). In that case, similar error estimates have been known in the stabilization finite element methods in the literature (e.g. [37]). However, for $\boldsymbol{u}$ not in $(H^1(\Omega))^3$ but in $(H^r(\Omega))^3$ with $r < 1$, the estimate in Lemma 4.5 is new, to the best of the authors' knowledge.

Theorem 4.3 *Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solution of problem* (1.1), (1.2), (1.3) *and the finite element solution of* (2.14)*, respectively. Assume that Hypotheses H1 and H2 hold and that $\boldsymbol{u} \in \prod_{l=1}^{L} (H^r(\Omega_l))^3$ with $r > 1/2$. Then*

$$||\boldsymbol{u} - \boldsymbol{u}_h||_{h,\mathbf{curl}\,,\mathrm{div}} \leq C\, h^r. \tag{4.32}$$

Proof From the symmetry and coercivity properties of $\mathcal{L}_h$, we have the Cauchy-Schwarz inequality:

$$\mathcal{L}_h(\boldsymbol{u}, \boldsymbol{v}) \leq (\mathcal{L}_h(\boldsymbol{u}, \boldsymbol{u}))^{\frac{1}{2}} \, (\mathcal{L}_h(\boldsymbol{v}, \boldsymbol{v}))^{\frac{1}{2}}.$$

From Theorem 4.1 we have

$$\begin{aligned}|||\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}|||_{\mathcal{L}_h}^2=\mathcal{L}_h(\boldsymbol{u}_h-\widetilde{\boldsymbol{u}},\boldsymbol{u}_h-\widetilde{\boldsymbol{u}})&=\mathcal{L}_h(\boldsymbol{u}-\widetilde{\boldsymbol{u}},\boldsymbol{u}_h-\widetilde{\boldsymbol{u}})+\mathcal{L}_h(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{u}_h-\widetilde{\boldsymbol{u}})\\&\le |||\boldsymbol{u}-\widetilde{\boldsymbol{u}}|||_{\mathcal{L}_h}\,|||\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}|||_{\mathcal{L}_h}+C\,h^r\,|||\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}|||_{\mathcal{L}_h}\\&\le C\,(|||\boldsymbol{u}-\widetilde{\boldsymbol{u}}|||_{\mathcal{L}_h}+h^r)\,|||\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}|||_{\mathcal{L}_h},\end{aligned}$$

that is,

$$|||\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}|||_{\mathcal{L}_h}\le C\,(|||\boldsymbol{u}-\widetilde{\boldsymbol{u}}|||_{\mathcal{L}_h}+h^r),\tag{4.33}$$

where, from Theorem 4.2 and Lemma 4.5,

$$\begin{aligned}|||\boldsymbol{u}-\widetilde{\boldsymbol{u}}|||_{\mathcal{L}_h}^2&=\mathcal{L}_h(\boldsymbol{u}-\widetilde{\boldsymbol{u}},\boldsymbol{u}-\widetilde{\boldsymbol{u}})\\&=||\breve{R}_h(\operatorname{div}\varepsilon\,(\boldsymbol{u}-\widetilde{\boldsymbol{u}}))||_{0,h}^2+||R_h(\mu^{-1}\,\mathbf{curl}\,\,(\boldsymbol{u}-\widetilde{\boldsymbol{u}}))||_{\mu}^2+\mathcal{S}_h(\boldsymbol{u}-\widetilde{\boldsymbol{u}},\boldsymbol{u}-\widetilde{\boldsymbol{u}})\\&=\mathcal{S}_h(\boldsymbol{u}-\widetilde{\boldsymbol{u}},\boldsymbol{u}-\widetilde{\boldsymbol{u}})\\&\le C\,h^{2r}\,(||\operatorname{div}\varepsilon\,\boldsymbol{u}||_0+||\mathbf{curl}\,\,\boldsymbol{u}||_0+\sum_{l=1}^{L}||\boldsymbol{u}||_{r,\Omega_l})^2.\end{aligned}\tag{4.34}$$

Therefore, by the triangle inequality, from Theorem 3.1 and (4.34) and Theorem 4.2, we have

$$\begin{aligned}||\boldsymbol{u}-\boldsymbol{u}_h||_0&\le||\boldsymbol{u}-\widetilde{\boldsymbol{u}}||_0+||\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}||_0\le||\boldsymbol{u}-\widetilde{\boldsymbol{u}}||_0+C\,|||\boldsymbol{u}_h-\widetilde{\boldsymbol{u}}|||_{\mathcal{L}_h}\\&\le||\boldsymbol{u}-\widetilde{\boldsymbol{u}}||_0+C\,(|||\boldsymbol{u}-\widetilde{\boldsymbol{u}}|||_{\mathcal{L}_h}+h^r)\le C\,||\boldsymbol{u}-\widetilde{\boldsymbol{u}}||_0+C\,h^r\le C\,h^r,\end{aligned}\tag{4.35}$$

and by the triangle inequality again, from (4.33) and (4.34), we have

$$|||\boldsymbol{u}-\boldsymbol{u}_h|||_{\mathcal{L}_h}\le|||\boldsymbol{u}-\widetilde{\boldsymbol{u}}|||_{\mathcal{L}_h}+|||\widetilde{\boldsymbol{u}}-\boldsymbol{u}_h|||_{\mathcal{L}_h}\le C\,h^r.\tag{4.36}$$

Summarizing (4.35), (4.36) and (4.27), we have (4.32). □

Remark 4.6 From Sect. 4.1 the constant in (4.32) would take the form $||\operatorname{div}\varepsilon\,\boldsymbol{u}||_0+||\mathbf{curl}\,\,\boldsymbol{u}||_0+\sum_{l=1}^{L}||\boldsymbol{u}||_{r,\Omega_l}+\sum_{l=1}^{L}(||\operatorname{div}\boldsymbol{f}||_{0,\Omega_l}+||\mathbf{curl}\,\,\varepsilon^{-1}\boldsymbol{f}||_{0,\Omega_l}+||\boldsymbol{f}||_{r,\Omega_l})$ up to a multiplicative constant which is independent of $\boldsymbol{u},\boldsymbol{f}$ and h. On the other hand, as remarked in Remark 2.9, for the first-order system like (1.9), since no consistency errors exist, from the above argument the constant in (4.32) takes the form $||\operatorname{div}\varepsilon\,\boldsymbol{u}||_0+||\mathbf{curl}\,\,\boldsymbol{u}||_0+\sum_{l=1}^{L}||\boldsymbol{u}||_{r,\Omega_l}$, up to a multiplicative constant which is independent of $\boldsymbol{u},\boldsymbol{f}$ and h.

Remark 4.7 For piecewise non-polynomial ε, we may consider to replace ε by a suitable piecewise polynomial approximation, say ε_h (see below). With this ε_h, all the previous analysis and results are almost unchanged, except the additional consistency error terms caused by that replacement as follows:

$$(\mathrm{div}\,(\varepsilon-\varepsilon_h)\,\boldsymbol{u},\,\breve{R}_h(\mathrm{div}\,\varepsilon_h\,\boldsymbol{v}_h)), \tag{4.37}$$

$$\sum_{F\in\mathcal{E}_h^{\mathrm{inter}}} h_F \int_F [\![(\varepsilon-\varepsilon_h)\,\boldsymbol{u}\cdot\boldsymbol{n}]\!]\,[\![\varepsilon_h\,\boldsymbol{v}_h\cdot\boldsymbol{n}]\!] \tag{4.38}$$

and

$$\sum_{K\in\mathcal{C}_h} h_K^2\,(\mathrm{div}\,(\varepsilon-\varepsilon_h)\,\boldsymbol{u},\,\mathrm{div}\,\varepsilon_h\,\boldsymbol{v}_h)_{0,K}. \tag{4.39}$$

Recall $\varepsilon_{ij}\in\prod_{l=1}^{L} W^{1,\infty}(\Omega_l)$, $1\le i,j\le 3$. We choose ε_h^l in Ω_l $(1\le l\le L)$ as a continuous finite element approximation of $\varepsilon|_{\Omega_l}$ with $\varepsilon_{ij,h}^l|_K\in\mathcal{P}_1^{\square}(K)$, where $\mathcal{P}_1^{\square}(K)$ denotes the linear element enriched by three face bubbles and one element bubble. Recall that K is a tetrahedron. From [6,7,24,39] we have $||\varepsilon-\varepsilon_h||_{0,\infty,K}+h_K\,|\varepsilon-\varepsilon_h|_{1,\infty,K}\le C\,h_K\,|\varepsilon|_{1,\infty,K}$ for all $K\in\mathcal{C}_h$, where $|\cdot|_{1,\infty}$ denotes the semi-norms of $W^{1,\infty}$. Note that ε_h satisfies the same uniform ellipticity property as ε for a sufficiently small h. Also, note that the bubbles in $\mathcal{P}_1^{\square}(K)$ are introduced in order that all $\varepsilon_{ij,h}^l$, $1\le i,j\le 3$, satisfy the following interpolation properties:

$$\int_F (\varepsilon_{ij}^l-\varepsilon_{ij,h}^l)\,q=0\quad\forall q\in\mathcal{P}_1(F),\quad\forall F\subset\partial K,\forall K\in\mathcal{C}_h \tag{4.40}$$

and

$$\int_K (\varepsilon_{ij}^l-\varepsilon_{ij,h}^l)=0\quad\forall K. \tag{4.41}$$

We shall only explain how to estimate (4.37) below. Other error terms can be estimated similarly. On each face $F\subset\partial K$ for any $K\in\mathcal{C}_h$, let $\bar{\boldsymbol{u}}|_F:=\frac{1}{|F|}\int_F\boldsymbol{u}$. We have $||\boldsymbol{u}-\bar{\boldsymbol{u}}||_{0,F}\le C\,h_K^{r-\frac12}\,||\boldsymbol{u}||_{r,K}$. Letting $\hat{\boldsymbol{u}}:=\frac{\int_K\boldsymbol{u}}{|K|}$, we have $||\boldsymbol{u}-\hat{\boldsymbol{u}}||_{0,K}\le C\,h_K^r\,||\boldsymbol{u}||_{r,K}$. Then, from (4.40), (4.41) and the interpolation properties of ε_h, $\bar{\boldsymbol{u}}$ and $\hat{\boldsymbol{u}}$, we have

$$\begin{aligned}(\mathrm{div}\,(\varepsilon-\varepsilon_h)\,\boldsymbol{u},\,\breve{R}_h(\mathrm{div}\,\varepsilon_h\,\boldsymbol{v}_h)) &= -((\varepsilon-\varepsilon_h)\,\boldsymbol{u},\,\nabla\,\breve{R}_h(\mathrm{div}\,\varepsilon_h\,\boldsymbol{v}_h))\\ &+\sum_{K\in\mathcal{C}_h}\sum_{F\subset\partial K}\int_F(\varepsilon-\varepsilon_h)\,\boldsymbol{u}\cdot\boldsymbol{n}\,\breve{R}_h(\mathrm{div}\,\varepsilon_h\,\boldsymbol{v}_h)=-((\varepsilon-\varepsilon_h)\,(\boldsymbol{u}-\hat{\boldsymbol{u}}),\,\nabla\,\breve{R}_h(\mathrm{div}\,\varepsilon_h\,\boldsymbol{v}_h))\\ &+\sum_{K\in\mathcal{C}_h}\sum_{F\subset\partial K}\int_F(\varepsilon-\varepsilon_h)\,(\boldsymbol{u}-\bar{\boldsymbol{u}})\cdot\boldsymbol{n}\,\breve{R}_h(\mathrm{div}\,\varepsilon_h\,\boldsymbol{v}_h)\le C\,h^r\,||\breve{R}_h(\mathrm{div}\,\varepsilon_h\,\boldsymbol{v}_h)||_{0,h},\end{aligned} \tag{4.42}$$

where C depends on $||\boldsymbol{u}||_r$, $\sum_{l=1}^{L}||\varepsilon||_{1,\infty,\Omega_l}$, but it is independent of h.

5 Conclusion

In this paper we have proposed a continuous finite element method to solve the curlcurl-grad div interface problem in a three-dimensional polyhedron domain. Different anisotropic nonhomogeneous materials are allowed to occupy multiple subdomains of the domain, with the two characteristic coefficients being discontinuous from subdomain to subdomain. Due to spatial inhomogeneities, such interface problems are frequently encountered in many physical science and engineering applications such as electromagnetism and fluid-structure interaction. The key feature of the method proposed in this paper is the application of two suitable L^2 projections to curl and div operators. With respect to the regularity associated with the problem in material subdomains, the error bound $\mathcal{O}(h^r)$ in an energy norm between the exact solution and the continuous finite element solution is proved. From Sect. 4 of this paper, the regularity requirement of the error bound $\mathcal{O}(h^r)$ can be generally summarized as follows: $\boldsymbol{u} \in H_0(\mathbf{curl};\Omega) \cap H(\mathrm{div};\varepsilon;\Omega) \cap \prod_{l=1}^{L}(H^r(\Omega_l))^3$, $\mu^{-1}\mathbf{curl}\,\boldsymbol{u} \in H(\mathbf{curl};\Omega) \cap H_0(\mathrm{div};\mu;\Omega) \cap \prod_{l=1}^{L}(H^r(\Omega_l))^3$, $\varepsilon, \mu \in (L^\infty(\Omega))^{3\times 3} \cap \prod_{l=1}^{L}(W^{1,\infty}(\Omega_l))^{3\times 3}$, and $\boldsymbol{f} \in (L^2(\Omega))^3 \cap \prod_{l=1}^{L} H(\mathrm{div};\Omega_l) \cap H(\mathbf{curl};\varepsilon^{-1};\Omega_l) \cap (H^r(\Omega_l))^3$, where $r > 1/2$ and the regularity on $\boldsymbol{f}$ could ensure the regularity of $\mu^{-1}\mathbf{curl}\,\boldsymbol{u}$, as stated in Hypothesis H2. On the other hand, if problem (1.1) comes from a first-order system like (1.9), then $\boldsymbol{u} \in H_0(\mathbf{curl};\Omega) \cap H(\mathrm{div};\varepsilon;\Omega) \cap \prod_{l=1}^{L}(H^r(\Omega_l))^3$ for $r > 1/2$ and $\varepsilon \in (L^\infty(\Omega))^{3\times 3} \cap \prod_{l=1}^{L}(W^{1,\infty}(\Omega_l))^{3\times 3}$ are sufficient. The continuous finite element method developed in this paper is a general-purpose method and has broad applications, since it can work no matter if the general tensor coefficients are discontinuous or not, and no matter if the exact solution has a low regularity or not, and no matter if the continuous problem comes from Maxwell's second-order or first-order curl/ div interface problems or from other similar interface problems involving curl/div operators.

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