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Convergence and optimality of the adaptive Morley element method

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Abstract This paper is devoted to the convergence and optimality analysis of the adaptive Morley element method for the fourth order elliptic problem. A new technique is developed to establish a quasi-orthogonality which is crucial for the convergence analysis of the adaptive nonconforming method. By introducing a new parameter-dependent error estimator and further establishing a discrete reliability property, sharp convergence and optimality estimates are then fully proved for the fourth order elliptic problem.

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1 Introduction

This paper is devoted to the study of adaptive nonconforming finite element methods for high order elliptic boundary value problems. The adaptive conforming finite element method for the second order elliptic problems has been a subject of extensive studies for many years since the pioneering work of Babuska and Rheinboldt [2], and its theory has become rather mature [1,6,14,15,19,26,27,29–31]. For the nonconforming method, the a posteriori error theory of the second order elliptic problems has been studied only very recently [9–12,17,18]; for the fourth order elliptic problem, only the a posteriori error estimate of the Morley element method can be found in the literature [3,21,32] and there have been no works on either convergence or optimality for any finite element methods for fourth order problems.

The main difficulty for the analysis of nonconforming finite element methods arises from the nonconformity of the discrete space and consequently the lack of the Galerkin-orthogonality which is a key ingredient for the convergence analysis of the adaptive conforming method of the second order elliptic problem [14,19,26,27,29]. For the nonconforming linear element of the Poisson equation, a quasi-orthogonality is established instead in [13] by using some special equivalency between the nonconforming linear element and the lowest order Raviart–Thomas element [24]. For the Morley element of the fourth order elliptic problem, however, it is unclear whether such type of equivalency still holds. We also note that the convergence (not to mention optimality) analysis of the adaptive conforming method is still missing for the fourth order elliptic problem in the literature.

This paper is devoted to the convergence and optimality analysis of the adaptive version of the Morley element [25,28,33]. Our analysis is based on an observation that a quasi-orthogonality can be obtained from a crucial local conservative property (that plays a critical role in a general study in [33]), of the Morley element method. Another ingredient is a new parameter dependent estimator which is introduced to analyze optimality of the adaptive nonconforming method. With the help of the discrete reliability which is established by introducing two interpolation operators between two nonconforming spaces, we show convergence and optimality of the adaptive algorithm.

The rest of the paper is organized as follows. In Sect. 2, we present the Kirchhoff plate problem and the Morley finite element method, and recall a posteriori error analysis due to [21]. In Sect. 3, we prove the quasi-orthogonality and then show reduction of some total error in Sect. 4 by introducing a new parameter-dependent estimator. To obtain optimality of the adaptive algorithm, we establish the discrete reliability in Sect. 5. Consequently, we show optimality of the adaptive Morley element method in Sect. 6. We give a brief comment on the extension of the theory to the Morley element method in three dimensions in Sect. 7. Also, we discuss the generalization to the nonconforming linear elements in both two and three dimensions therein. This extension gives an alternative analysis of the convergence result from [13]. The paper ends with Sect. 8 where we give the conclusion and some comments.

2 The Morley element for the Kirchhoff plate problem and an a posteriori error estimate

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $\mathbb{E}$ the Young modulus, and ν the Poisson ratio. For all 2×2 symmetric matrices, the linear operator $\mathcal{C}$ is defined by

$$\mathcal{C}\tau := \frac{\mathbb{E}}{12(1-\nu^2)}\big((1-\nu)\tau + \nu\,\mathrm{tr}(\tau)I\big).$$

The bilinear form $a(u, v)$ is defined by

$$a(u, v) = (\mathcal{C}\nabla^2 u, \nabla^2 v)_{L^2(\Omega)}, \quad \text{for any } u, v \in W := H_0^2(\Omega), \tag{2.1}$$

where $\nabla^2 u$ is the Hessian matrix of u. The corresponding energy norm is given by

$$\|u\|_{\mathcal{C}}^2 := a(u, u) \quad \text{for any } u \in W, \tag{2.2}$$

which is equivalent to the usual norm $|\cdot|_{H^2(\Omega)}$ for any $u \in W$.

We consider the Kirchhoff plate bending problem as follows: Given $f \in L^2(\Omega)$, find $u \in W$ such that

$$a(u, v) = (f, v)_{L^2(\Omega)} \quad \text{for all } v \in W. \tag{2.3}$$

We now present the Morley element. Suppose that $\overline{\Omega}$ is covered exactly by a shape-regular triangulation $\mathcal{T}_h$ consisting of triangles in $2D$, see [16]. $\mathcal{E}_h$ is the set of all edges in $\mathcal{T}_h$, $\mathcal{E}_h(\Omega)$ is the set of interior edges, and $\mathcal{E}(K)$ is the set of edges of any given element K in $\mathcal{T}_h$; $h_K = |K|^{1/2}$, the size of the element $K \in \mathcal{T}_h$. ω_K is the union of elements $K' \in \mathcal{T}_h$ that share an edge with K, and ω_e is the union of elements that share a common edge e. Given any edge $e \in \mathcal{E}_h(\Omega)$ with the length h_e we assign one fixed unit normal $\nu_e := (\nu_1, \nu_2)$ and tangential vector $\tau_e := (-\nu_2, \nu_1)$. For e on the boundary we choose $\nu_e = \nu$ the unit outward normal to Ω. Once ν_e and τ_e have been fixed on e, in relation to ν_e one defines the elements $K_- \in \mathcal{T}_h$ and $K_+ \in \mathcal{T}_h$, with $e = K_+ \cap K_-$. Given $e \in \mathcal{E}_h(\Omega)$ and some $\mathbb{R}^d$-valued function v defined in Ω, with $d = 1, 2$, we denote by $[v] := (v|_{K_+})|_e - (v|_{K_-})|_e$ the jump of v across e.

The discrete space of the Morley finite element method is defined as follows [25, 28, 33]

$$W_h := \left\{ v \in M_{2,h}, \int_e [\nabla_h v \cdot \nu_e] ds = 0 \text{ on } e \in \mathcal{E}_h(\Omega), \right.$$
$$\left. \text{and} \int_e \nabla v \cdot \nu_e ds = 0 \text{ on } e \in \mathcal{E}_h \cap \partial\Omega \right\}, \tag{2.4}$$

where $M_{2,h}$ is the space of piecewise polynomials of degree ≤ 2 over $\mathcal{T}_h$ which are continuous at all the internal nodes and vanish at all the nodes on the boundary $\partial\Omega$, and ∇_h the discrete gradient operator which is defined elementwise. We define

$$a_h(u_h, v_h) := (\mathcal{C}\nabla_h^2 u_h, \nabla_h^2 v_h)_{L^2(\Omega)} \quad \text{for any } u_h\,,\, v_h \in W + W_h,$$
$$\|u_h\|_{\mathcal{C}_h}^2 := a_h(u_h, u_h) \quad \text{for any } u_h \in W + W_h, \tag{2.5}$$

where the discrete Hessian operator ∇_h^2 is defined elementwise with respect to the triangulation $\mathcal{T}_h$.

We now consider the finite element discretization of (2.3) as follows: Find $u_h \in W_h$ such that

$$a_h(u_h, v_h) = (f, v_h)_{L^2(\Omega)} \quad \text{for all } v_h \in W_h. \tag{2.6}$$

To recall the a posteriori error estimate for the Morley element, we first define an estimator on each element $K \in \mathcal{T}_h$ as

$$\eta_K = h_K^2 \|f\|_{L^2(K)} + \left(\sum_{e \subset \partial K} h_K \|[\nabla_h^2 u_h \tau_e]\|_{L^2(e)}^2 \right)^{1/2}. \tag{2.7}$$

For any $S_h \subset \mathcal{T}_h$, we define the estimator over S_h by

$$\eta^2(u_h, S_h) := \sum_{K \in S_h} \eta_K^2. \tag{2.8}$$

In particular, for $S_h = \mathcal{T}_h$, we have

$$\eta^2(u_h, \mathcal{T}_h) := \sum_{K \in \mathcal{T}_h} \eta_K^2. \tag{2.9}$$

We further define the oscillation $\mathrm{osc}(f, \mathcal{T}_h)$ by

$$\mathrm{osc}^2(f, \mathcal{T}_h) := \sum_{K \in \mathcal{T}_h} h_K^4 \|f - f_K\|_{L^2(K)}^2, \tag{2.10}$$

where f_K is the constant projection of f over K. For the estimator (2.9), we have the following reliability and efficiency whose proof can be found in [21].

Lemma 2.1 *Let u be the solution of Problem* (2.3)*, and u_h be the solution of Problem* (2.6)*. Then,*

$$\|u - u_h\|_{\mathcal{C}_h} \lesssim \eta(u_h, \mathcal{T}_h) \lesssim \|u - u_h\|_{\mathcal{C}_h} + \mathrm{osc}(f, \mathcal{T}_h). \tag{2.11}$$

Here and throughout the paper, we shall follow [34] to use the notation $\lesssim$ and $\approxeq$. When we write

$$A_1 \lesssim B_1, \text{ and } A_2 \approxeq B_2,$$

then there exist possible constants C_1, c_2 and C_2 such that

$$A_1 \le C_1 B_1, \text{ and } c_2 B_2 \le A_2 \le C_2 B_2.$$

Given $v \in H^2(\mathcal{T}_h) := \{v \in L^2(\Omega), v|_K \in H^2(K), \text{ for any } K \in \mathcal{T}_h\}$, we define the following residual

$$\mathrm{Res}_H(v) = (f, v)_{L^2(\Omega)} - a_h(u_H, v), \quad \text{for any } v \in H^2(\mathcal{T}_h), \tag{2.12}$$

with u_H being the solution of the discrete problem (2.6) on $\mathcal{T}_H$, which is a nested and coarser mesh to $\mathcal{T}_h$; namely, $\mathcal{T}_h$ is some refinement of $\mathcal{T}_H$. It follows from the discrete problem (2.6) that

$$\mathrm{Res}_H(v) = \mathrm{Res}_H(v - v_H), \quad \text{for any } v_H \in W_H. \tag{2.13}$$

Lemma 2.2 *For any $v \in W$, it holds that*

$$|\,\mathrm{Res}_H(v)| \lesssim \left(\sum_{K \in \mathcal{T}_H} h_K^4 \|f\|_{L^2(K)}^2 \right)^{1/2} \|v\|_C \quad \textit{for any } v \in W. \tag{2.14}$$

The proof of the above lemma can be found in [3,21,32]. □

3 Quasi-orthogonality

In this section, we address one difficulty, namely the quasi-orthogonality, in the convergence analysis of the adaptive Morley element method. Our analysis is based on two interpolation operators: the canonical interpolation operator Π_h of the non-conforming space W_h, and the restriction operator I_H from the discrete space W_h on the mesh $\mathcal{T}_h$ to the discrete space W_H on the nested and coarser mesh $\mathcal{T}_H$ of $\mathcal{T}_h$.

Here and in what follows, $\mathcal{N}_h$ denotes the set of nodes of the partition $\mathcal{T}_h$. We first define the canonical interpolation operator $\Pi_h : W \to W_h$ by,

$$(\Pi_h v)(P) = v(P), \int_e \nabla_h(\Pi_h v - v) \cdot \nu_e \, ds = 0 \text{ for any } v \in W, P \in \mathcal{N}_h, e \in \mathcal{E}_h. \tag{3.1}$$

Lemma 3.1 *Let the interpolation operator Π_h be defined as in (3.1). Then,*

$$\int_e \nabla_h(v - \Pi_h v)\, ds = 0 \quad \textit{for any } e \in \mathcal{E}_h \textit{ and } v \in W, \tag{3.2}$$

$$a_h(v - \Pi_h v, v_h) = 0 \quad \textit{for any } v \in W, v_h \textit{ piecewise quadratic}, \tag{3.3}$$

$$\|v - \Pi_h v\|_{L^2(K)} \lesssim h_K^2 |v|_{H^2(K)} \quad \textit{for any } K \in \mathcal{T}_h \textit{ and } v \in W. \tag{3.4}$$

The above properties are immediate from the definition of Π_h. Now we define the restriction interpolation operator $I_H : W_h \rightarrow W_H$ by, for any $v_h \in W_h$,

$$\begin{cases} (I_H v_h)(P) = v_h(P),\ P \in \mathcal{N}_H\ , \\ \int\limits_e \frac{\partial (I_H v_h)}{\partial \nu_e} ds = \sum_{l=1}^{\ell} \int\limits_{e_l} \frac{\partial v_h}{\partial \nu_e} ds\ ,\ e \in \mathcal{E}_H \text{ with } e = e_1 \cup e_2 \cdots \cup e_\ell \text{ and } e_i \in \mathcal{E}_h. \end{cases} \tag{3.5}$$

Before analyzing the properties of this interpolation, we state the following simple result.

Lemma 3.2 *Let $K_1, K_2 \in \mathcal{T}_h$ be two elements sharing a common edge e. If $v_h \in W_h(K_1 \cup K_2)$ and $\nabla_h^2 v_h = 0$, then $v_h \in P_1(K_1 \cup K_2)$. Namely v_h is a polynomial of degree ≤ 1 over $K_1 \cup K_2$.*

Proof By the definition of W_h, v_h is continuous on $K_1 \cup K_2$. Further $\frac{\partial v_h}{\partial \nu_e}|_{K_1}$ and $\frac{\partial v_h}{\partial \nu_e}|_{K_2}$ are two constant functions that must be equal since by the definition of W_h $\int_e [\frac{\partial v_h}{\partial \nu_e}] ds = 0$. Thus v must belong to $P_1(K_1 \cup K_2)$. □

The properties of the interpolation operator I_H are summarized in the following lemma.

Lemma 3.3 *Let the interpolation operator I_H be defined as in (3.5). Then,*

$$\int\limits_e \nabla_h (v_h - I_H v_h)\, ds = 0 \quad \textit{for any } e \in \mathcal{E}_H \quad \textit{and } v_h \in W_h\ , \tag{3.6}$$

$$a_h(v_H, v_h - I_H v_h) = 0 \quad \textit{for any } v_H \in W_H, v_h \in W_h\ , \tag{3.7}$$

$$I_H v_h|_K = v_h|_K \quad \textit{for any } K \in \mathcal{T}_h \cap \mathcal{T}_H \textit{ and } v_h \in W_h\ , \tag{3.8}$$

$$\|I_H v_h - v_h\|_{L^2(K)} \lesssim h_K^2 \|\nabla_h^2 v_h\|_{L^2(K)} \quad \textit{for any } K \in \mathcal{T}_H \backslash \mathcal{T}_h \textit{ and } v_h \in W_h. \tag{3.9}$$

Proof The properties of (3.6), and (3.8) directly follow from the definition of the interpolation. We only need to prove (3.7) and the estimate (3.9).

We first define $\sigma_H = \mathcal{C} \nabla_H^2 v_H$ to assert that

$$\int\limits_e \nabla_h (I - I_H) v_h \cdot \sigma_H \nu_e\, ds = 0 \quad \text{for any } e \in \mathcal{E}_H. \tag{3.10}$$

In fact, for $e \in \mathcal{E}_H \backslash \mathcal{E}_h$, this assertion follows from the fact that σ_H is a piecewise constant matrix with respect to $\mathcal{T}_H$ and the definition of I_H in (3.5). For $e \in \mathcal{E}_h \cap \mathcal{E}_H$, the assertion follows from $(I - I_H) v_h|_e = 0$.

For the edge $e \in \mathcal{E}_h$ which lies in the interior of some $K \in \mathcal{T}_H$, we can use the continuity of $\int_e \nabla_h v_h\, ds$ over e and the fact σ_H is constant over K to show that

$$\int_e [\nabla_h (I - I_H) v_h] \cdot \sigma_H \nu_e \, ds = 0. \tag{3.11}$$

Whence, we integrate by parts and use (3.10) and (3.11) to conclude (3.7).

Now we turn to (3.9). In fact, both sides of (3.9) are semi-norms of the restriction $W_h(K)$ of W_h on K. If the right hand side vanishes for some $v_h \in W_h(K)$, then v_h is a piecewise polynomial of degree≤ 1 on K with respect to $\mathcal{T}_h$. It follows from Lemma 3.2 that v_h is a polynomial of degree≤ 1 on K. Therefore the left hand side also vanishes for the same v_h. The desired result then follows from a scaling argument. □

Lemma 3.4 (Quasi-orthogonality) *Let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$, and u_h and u_H be the solutions of* (2.6) *on $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively. Then,*

$$|a_h(u_h - u_H, u - u_h)| \lesssim \sum_{K \in \mathcal{T}_H \setminus \mathcal{T}_h} h_K^2 \|f\|_{L^2(K)} \|\nabla_h^2 (u - u_h)\|_{L^2(K)}. \tag{3.12}$$

Proof Let the interpolation operator Π_h be defined as in (3.1). Since Π_h is well-defined for any $v_h \in W_h$(in fact, $\Pi_h v_h = v_h$) and $a_h(u_h - u_H, (I - \Pi_h)(u - u_h)) = 0$ (by (3.3)), we have

$$a_h(u_h - u_H, u - u_h) = a_h(u_h - u_H, \Pi_h(u - u_h)). \tag{3.13}$$

Let $v_h = \Pi_h(u - u_h)$ and the interpolation $I_H v_h$ be defined as in (3.5). The combination of (2.6) and (2.12) leads to

$$\begin{aligned} a_h(u_h - u_H, v_h) &= (f, v_h)_{L^2(\Omega)} - a_h(u_H, v_h) \\ &= (f, (I - I_H) v_h)_{L^2(\Omega)} - a_h(u_H, (I - I_H) v_h). \end{aligned} \tag{3.14}$$

By (3.8) and (3.9), we have

$$|(f, (I - I_H) v_h)_{L^2(\Omega)}| \lesssim \sum_{K \in \mathcal{T}_H \setminus \mathcal{T}_h} h_K^2 \|f\|_{L^2(K)} \|\nabla_h^2 v_h\|_{L^2(K)}. \tag{3.15}$$

From (3.7) we have $a_h(u_H, (I - I_H) v_h) = 0$. Then, the desired result then follows from the triangle inequality and the approximation property of Π_h. □

Remark 3.5 For the nonconforming P_1 element of the Poisson equation, the quasi-orthogonality was established in [13]. The analysis therein is based on some special equivalency between the nonconforming P_1 and Raviart–Thomas elements. For the Stokes-like problem, the quasi-orthogonality of the nonconforming P_1 element has been first proved in [23] based on some special relation of the nonconforming P_1 and Raviart–Thomas elements. For the Morley element, it is unclear whether there exists similar equivalency or relation so far.

Remark 3.6 This paper is a refined version of a technical report in 2009 [22], where it was the first time in the literature to make use of the conservative property of the nonconforming finite element space to analyze the quasi-orthogonality.

4 Reduction of a properly defined total error

In the rest of the paper, we shall establish convergence and optimality of our Adaptive Nonconforming Finite Element Method(ANFEM). Our analysis is based on two main ingredients: the strict reduction of some total error between two levels and the discrete reliability of the estimator. To this end, we shall first introduce a modified estimator $\tilde{\eta}$ with a undetermined positive constant; we shall then borrow the concept of the total error of [13,14] which contains the energy norm of the error and the scaled estimator $\tilde{\eta}$; we shall finally show reduction of this total error. We shall establish the discrete reliability of the estimator in the next section.

Let us first define our adaptive algorithm. Given an initial shape regular triangulation $\mathcal{T}_0$, a right-hand side function $f \in L^2(\Omega)$, a tolerance $\varepsilon > 0$, and a parameter $\theta \in (0, 1)$. Hereafter, we shall replace the subscript h by an iteration counter called k.

Algorithm 4.1 $[\mathcal{T}_N, u_N]$=**ANFEM**$(\mathcal{T}_0, f, \varepsilon, \theta)$

$\eta = \varepsilon$, $k = 0$

WHILE $\eta \geq \varepsilon$, **DO**

(1) Solve (2.6) on $\mathcal{T}_k$, to get the solution u_k.

(2) Compute the error estimator $\eta = \eta(u_k, \mathcal{T}_k)$.

(3) Mark the minimal element set $\mathcal{M}_k$ such that

$$\eta^2(u_k, \mathcal{M}_k) \geq \theta\, \eta^2(u_k, \mathcal{T}_k). \tag{4.1}$$

(4) Refine each triangle $K \in \mathcal{M}_k$ by the newest vertex bisection and possible further refining to conformity to get $\mathcal{T}_{k+1}$.

$k = k + 1$.

END WHILE

$\mathcal{T}_N = \mathcal{T}_k$.

END ANFEM

In order to prove a strict reduction of some total error, we define the following modified estimator

$$\tilde{\eta}^2(u_H, \mathcal{T}_H) := \sum_{K \in \mathcal{T}_H} \left(\beta_1 h_K^4 \|f\|^2_{L^2(K)} + \eta_K^2\right) \text{ with } \eta_K \text{ defined in (2.7)} \tag{4.2}$$

for some positive constant β_1 to be determined later.

Remark 4.1 Note that, as we can see below, the modified error estimator $\tilde{\eta}(u_H, \mathcal{T}_H)$ is only for the analysis, the final results concerning both convergence and optimality will be proved for Algorithm 4.1.

Lemma 4.2 *Let $\mathcal{T}_h$ be some refinement of $\mathcal{T}_H$ with the bulk criterion* (4.1)*, then there exist $\rho > 0$ and a positive constant $\beta \in (1 - \rho\theta, 1)$ such that*

$$\eta^2(u_H, \mathcal{T}_h) \le \beta\eta^2(u_H, \mathcal{T}_H) + (1 - \rho\theta - \beta)\eta^2(u_H, \mathcal{T}_H). \tag{4.3}$$

Proof The result can be proved by following the idea in [14]. We give the details only for the readers' convenience. In fact, we have

$$\eta^2(u_H, \mathcal{T}_h) = \eta^2(u_H, \mathcal{T}_H \cap \mathcal{T}_h) + \eta^2(u_H, \mathcal{T}_H \backslash \mathcal{T}_h). \tag{4.4}$$

For any $K \in \mathcal{T}_H \backslash \mathcal{T}_h$, we only need to consider the case where K is subdivided into K_1 , $K_2 \in \mathcal{T}_h$ with $|K_1| = |K_2| = \frac{1}{2}|K|$. By the definitions of h_K and $\eta_K(u_H)$, we have

$$\begin{aligned}\sum_{i=1}^{2} \eta_{K_i}^2(u_H) &:= \sum_{i=1}^{2} \left(h_{K_i}^2 \|f\|_{L^2(K_i)} + \left(\sum_{\mathcal{E}_h \ni e \subset \partial K_i} h_{K_i} \|[\nabla_H^2 u_H \tau_e]\|_{L^2(e)}^2 \right)^{1/2} \right)^2 \\ &\le \frac{1}{2^{1/2}} \eta_K^2(u_H) := \frac{1}{2^{1/2}} \left(h_K^2 \|f\|_{L^2(K)} + \left(\sum_{\mathcal{E}_H \ni e \subset \partial K} h_K \|[\nabla_H^2 u_H \tau_e]\|_{L^2(e)}^2 \right)^{1/2} \right)^2,\end{aligned} \tag{4.5}$$

since $[\nabla_H^2 u_H \tau_e] = 0$ over $e = K_1 \cap K_2 \in \mathcal{E}_h$. Consequently

$$\sum_{K \in \mathcal{T}_H \backslash \mathcal{T}_h} \sum_{i=1}^{2} \eta_{K_i}^2(u_H) \le \frac{1}{2^{1/2}} \eta^2(u_H, \mathcal{T}_H \backslash \mathcal{T}_h), \tag{4.6}$$

and

$$\eta^2(u_H, \mathcal{T}_h) \le \eta^2(u_H, \mathcal{T}_H) - \rho\eta^2(u_H, \mathcal{T}_H \backslash \mathcal{T}_h), \tag{4.7}$$

with $\rho = 1 - \frac{1}{2^{1/2}}$. Taking the positive parameter β with $1 - \rho\theta < \beta < 1$, the desired result follows by combining the above inequality and the bulk criterion (4.1). □

Lemma 4.3 *Let $\mathcal{T}_h$ be some refinement of $\mathcal{T}_H$, then there exists $\rho > 0$ such that*

$$\sum_{K \in \mathcal{T}_h} h_K^4 \|f\|_{L^2(K)}^2 \le \sum_{K \in \mathcal{T}_H} h_K^4 \|f\|_{L^2(K)}^2 - \rho \sum_{K \in \mathcal{T}_H \backslash \mathcal{T}_h} h_K^4 \|f\|_{L^2(K)}^2. \tag{4.8}$$

Proof The proof immediately follows from the definition of the meshsize h_K. □

Lemma 4.4 (Continuity of the estimator) *Let u_h and u_H be the solutions to the discrete problem* (2.6) *on the meshes $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively. Given any positive constant ϵ, there exists a positive constant $\beta_2(\epsilon)$ dependent on ϵ such that*

$$\eta^2(u_h, \mathcal{T}_h) \le (1 + \epsilon)\eta^2(u_H, \mathcal{T}_h) + \frac{1}{\beta_2(\epsilon)} \|u_h - u_H\|_{\mathcal{C}_h}^2. \tag{4.9}$$

Proof Given any $K \in \mathcal{T}_h$, it follows from the definitions of $\eta_K(u_h)$ and $\eta_K(u_H)$ in (4.5) that

$$\begin{aligned}\left|\eta_K(u_h) - \eta_K(u_H)\right| &= \Bigg|\Bigg(\sum_{\mathcal{E}_h \ni e \subset \partial K} h_K \|[\nabla_h^2 u_h \tau_e]\|^2_{L^2(e)}\Bigg)^{1/2} \\ &\quad - \Bigg(\sum_{\mathcal{E}_h \ni e \subset \partial K} h_K \|[\nabla_H^2 u_H \tau_e]\|^2_{L^2(e)}\Bigg)^{1/2}\Bigg| \\ &\le \Bigg(\sum_{\mathcal{E}_h \ni e \subset \partial K} h_K \|[\nabla_h^2 (u_h - u_H)\tau_e]\|^2_{L^2(e)}\Bigg)^{1/2}. \end{aligned} \tag{4.10}$$

With $e = K_1 \cap K_2 \in \mathcal{E}_h$, we use the trace theorem and the fact that $\nabla_h^2(u_h - u_H)$ is a piecewise constant matrix to get

$$\begin{aligned}\|[\nabla_h^2(u_h - u_H)\tau_e]\|_{L^2(e)} &\le \|\nabla_h^2(u_h - u_H)\tau_e|_{K_1}\|_{L^2(e)} + \|\nabla_h^2(u_h - u_H)\tau_e|_{K_2}\|_{L^2(e)} \\ &\lesssim h_K^{-1/2} \|\nabla_h^2(u_h - u_H)\|_{L^2(\omega_e)}, \end{aligned} \tag{4.11}$$

which gives

$$\left|\eta_K(u_h) - \eta_K(u_H)\right| \lesssim \|\nabla_h^2(u_h - u_H)\|_{L^2(\omega_K)}. \tag{4.12}$$

Applying the Young inequality with any positive constant ϵ and summarizing over all elements in $\mathcal{T}_h$ completes the proof of the lemma. □

Theorem 4.5 *Let u be the solution to the problem (2.3), and u_H and u_h be the solutions to the discrete problem (2.6) on the meshes $\mathcal{T}_H$ and $\mathcal{T}_h$, respectively. Then, there exists positive constants γ_1, β_1, and $0 < \alpha < 1$ with*

$$\|u - u_h\|^2_{\mathcal{C}_h} + \gamma_1 \tilde{\eta}^2(u_h, \mathcal{T}_h) \le \alpha(\|u - u_H\|^2_{\mathcal{C}_H} + \gamma_1 \tilde{\eta}^2(u_H, \mathcal{T}_H)). \tag{4.13}$$

Proof Let δ, γ_1, and γ_2, be three positive constants to be chosen later. Applying the Young inequality to Lemma 3.4 and adding the resulting estimate to the inequality (4.9) leads to

$$\begin{aligned}&(1-\delta)\|u - u_h\|^2_{\mathcal{C}_h} + \gamma_1 \eta^2(u_h, \mathcal{T}_h) + \gamma_2 \sum_{K \in \mathcal{T}_h} h_K^4 \|f\|^2_{L^2(K)} \\ &\le \|u - u_H\|^2_{\mathcal{C}_H} + \gamma_1(1+\epsilon)\eta^2(u_H, \mathcal{T}_h) + (\frac{\gamma_1}{\beta_2(\epsilon)} - 1)\|u_h - u_H\|^2_{\mathcal{C}_h} \\ &\quad + \gamma_2 \sum_{K \in \mathcal{T}_H} h_K^4 \|f\|^2_{L^2(K)} + (C_1(\delta) - \rho\gamma_2) \sum_{K \in \mathcal{T}_H \backslash \mathcal{T}_h} h_K^4 \|f\|^2_{L^2(K)}, \end{aligned} \tag{4.14}$$

with the positive constant ρ from (4.8). We note that the bound for $\eta^2(u_H, \mathcal{T}_h)$ is given in Lemma 4.2. Hence

$$\begin{aligned}
&(1-\delta)\|u-u_h\|^2_{\mathcal{C}_h} + \gamma_1\eta^2(u_h,\mathcal{T}_h) + \gamma_2 \sum_{K\in\mathcal{T}_h} h_K^4\|f\|^2_{L^2(K)} \\
&\quad \le \|u-u_H\|^2_{\mathcal{C}_H} + \gamma_1((1-\rho\theta-\beta)(1+\epsilon)+\epsilon\beta)\eta^2(u_H,\mathcal{T}_H) + \gamma_1\beta\eta^2(u_H,\mathcal{T}_H) \\
&\qquad +\left(\frac{\gamma_1}{\beta_2(\epsilon)}-1\right)\|u_h-u_H\|^2_{\mathcal{C}_h} + (C_1(\delta)-\rho\gamma_2)\sum_{K\in\mathcal{T}_H\backslash\mathcal{T}_h} h_K^4\|f\|^2_{L^2(K)} \\
&\qquad +\gamma_2\sum_{K\in\mathcal{T}_H} h_K^4\|f\|^2_{L^2(K)},
\end{aligned} \tag{4.15}$$

with ρ and β from Lemma 4.2. In what follows we shall choose the parameters $\alpha, \beta, \gamma_1, \gamma_2$, and δ to achieve the reduction of the total error. We first set

$$\gamma_2 = \frac{C_1(\delta)}{\rho}, \gamma_1 = \beta_2(\epsilon), \text{ and } \beta = (1-\rho\theta)(1+\epsilon) \tag{4.16}$$

which leads to

$$\begin{aligned}
&(1-\delta)\|u-u_h\|^2_{\mathcal{C}_h} + \gamma_1\eta^2(u_h,\mathcal{T}_h) + \gamma_2 \sum_{K\in\mathcal{T}_h} h_K^4\|f\|^2_{L^2(K)} \\
&\quad \le \|u-u_H\|^2_{\mathcal{C}_H} + \gamma_1\beta\eta^2(u_H,\mathcal{T}_H) + \gamma_2\sum_{K\in\mathcal{T}_H} h_K^4\|f\|^2_{L^2(K)}.
\end{aligned} \tag{4.17}$$

We choose ϵ to be small enough such that $0 < \beta < 1$. Let the positive constant α with $\beta < \alpha < 1$ be determined later, this gives

$$\begin{aligned}
&(1-\delta)\|u-u_h\|^2_{\mathcal{C}_h} + \gamma_1\eta^2(u_h,\mathcal{T}_h) + \gamma_2 \sum_{K\in\mathcal{T}_h} h_K^4\|f\|^2_{L^2(K)} \\
&\quad \le \alpha\left((1-\delta)\|u-u_H\|^2_{\mathcal{C}_H} + \gamma_1\eta^2(u_H,\mathcal{T}_H) + \gamma_2\sum_{K\in\mathcal{T}_H} h_K^4\|f\|^2_{L^2(K)}\right) \\
&\qquad +(1-\alpha(1-\delta))\|u-u_H\|^2_{\mathcal{C}_H} + \gamma_1(\beta-\alpha)\eta^2(u_H,\mathcal{T}_H) \\
&\qquad +\gamma_2(1-\alpha)\sum_{K\in\mathcal{T}_H} h_K^4\|f\|^2_{L^2(K)}.
\end{aligned} \tag{4.18}$$

Recalling the reliability of $\eta(u_H, \mathcal{T}_H)$ with the reliability coefficient C_{Rel} in Lemma 2.1

$$\|u-u_H\|^2_{\mathcal{C}_H} \le C_{Rel}\eta^2(u_H,\mathcal{T}_H), \tag{4.19}$$

and the fact that

$$\sum_{K\in\mathcal{T}_H} h_K^4\|f\|^2_{L^2(K)} \le \eta^2(u_H,\mathcal{T}_H). \tag{4.20}$$

Whence we derive as

$$(1-\alpha(1-\delta))\|u-u_H\|_{\mathcal{C}_H}^2+\gamma_1(\beta-\alpha)\eta^2(u_H,\mathcal{T}_H)+\gamma_2(1-\alpha)\sum_{K\in\mathcal{T}_H}h_K^4\|f\|_{L^2(K)}^2$$
$$\leq\big((1-\alpha(1-\delta))C_{Rel}+\gamma_1(\beta-\alpha)+\gamma_2(1-\alpha)\big)\eta^2(u_H,\mathcal{T}_H), \quad (4.21)$$

provided that $0<\delta<1$. Then, the choice of $\alpha=\frac{\gamma_1\beta+\gamma_2+C_{Rel}}{\gamma_1+\gamma_2+C_{Rel}(1-\delta)}>\beta$ gives

$$(1-\delta)\|u-u_h\|_{\mathcal{C}_h}^2+\gamma_1\eta^2(u_h,\mathcal{T}_h)+\gamma_2\sum_{K\in\mathcal{T}_h}h_K^4\|f\|_{L^2(K)}^2$$
$$\leq\alpha\left((1-\delta)\|u-u_H\|_{\mathcal{C}_H}^2+\gamma_1\eta^2(u_H,\mathcal{T}_H)+\gamma_2\sum_{K\in\mathcal{T}_H}h_K^4\|f\|_{L^2(K)}^2\right). \quad (4.22)$$

We choose such that $0<\delta<\min(\frac{\gamma_1(1-\beta)}{C_{Rel}},1)$ to assure that $\alpha<1$. Finally, we take $\beta_1=\gamma_2/\gamma_1$ and redefine $\gamma_1=\gamma_1/(1-\delta)$ to end the proof. □

5 Discrete reliability

This section is devoted to the discrete reliability of the estimator $\eta(u_H,\mathcal{T}_H)$. The analysis needs the prolongation operator $I_h':W_H\to W_h$ defined as follows. Given $P\in\mathcal{N}_h$ and $e\in\mathcal{E}_h$, the nodal patch $\omega_{P,H}$ of P and the edge patch $\omega_{e,H}$ of e with respect to the mesh $\mathcal{T}_H$ are defined by, respectively,

$$\omega_{P,H}:=\{K\in\mathcal{T}_H,\ P\in\partial K \text{ or } P \text{ is in the interior of } K\},$$
$$\omega_{e,H}:=\{K\in\mathcal{T}_H,\ e\subset\partial K \text{ or } e \text{ is in the interior of } K\}. \quad (5.1)$$

Define $\xi_P=\mathrm{card}(\omega_{P,H})$ and $\xi_e=\mathrm{card}(\omega_{e,H})$. We define the prolongation interpolation $I_h'v_H\in W_h$ by, for any $v_H\in W_H$,

$$\begin{cases}(I_h'v_H)(P)=\dfrac{1}{\xi_P}\displaystyle\sum_{K\in\omega_{P,H}}v_H|_K(P) & \text{for any } P\in\mathcal{N}_h\,,\\ \displaystyle\int_e\frac{\partial(I_h'v_H)}{\partial\nu_e}ds=\frac{1}{\xi_e}\sum_{K\in\omega_{e,H}}\int_e\frac{\partial(v_H|_K)}{\partial\nu_e}ds & \text{for any } e\in\mathcal{E}_h.\end{cases} \quad (5.2)$$

Lemma 5.1 *Let $K_1,K_2\in\mathcal{T}_H$ be two elements sharing a common edge e with two endpoints P_1 and P_2. Suppose that $v_H\in W_H$ and $\nabla_H v_H$ is continuous over e. Then, v_H is continuous over e.*

Proof We can assume that the common edge e shared by K_1 and K_2 lies along the x-axis. Then, v can be expressed as

$$v_H|_{K_1} = a_0 + a_1x + a_2y + a_3xy + a_4x^2 + a_5y^2, \text{ and}$$
$$v_H|_{K_2} = b_0 + b_1x + b_2y + b_3xy + b_4x^2 + b_5y^2.$$

Since $\frac{\partial v_H}{\partial x}$ is continuous over e, we have $a_1 = b_1$ and $a_4 = b_4$. The continuity of $\frac{\partial v_H}{\partial y}$ over e gives $a_2 = b_2$ and $a_3 = b_3$. Finally, $v_H|_{K_1}(P_\ell) = v_H|_{K_2}(P_\ell), \ell = 1, 2,$ concludes $a_0 = b_0$. Therefore, v_H is continuous over e. □

Lemma 5.2 *Let the interpolation operator I'_h be defined as in (5.2). Then,*

$$\|\nabla_h^2(I'_h v_H - v_H)\|^2_{L^2(\Omega)} \lesssim \sum_{K\in\mathcal{T}_H\backslash\mathcal{T}_h} \sum_{e\subset\partial K} h_K \|[\nabla_H^2 v_H \tau_e]\|^2_{L^2(e)} \text{ for any } v_H \in W_H. \tag{5.3}$$

Proof It follows from the definition of I'_h (5.2) that $I'_h v_H|_K = v_H|_K$ for any $K \in \mathcal{T}_H \cap \mathcal{T}_h$. Therefore, we only need to estimate $\|\nabla_h^2(I'_h v_H - v_H)\|_{L^2(K)}$ for $K \in \mathcal{T}_H\backslash\mathcal{T}_h$. To prove the desired result, it is sufficient to show that

$$\|\nabla_h^2(I'_h v_H - v_H)\|_{L^2(K)} \lesssim \sum_{e\subset\partial K} h_K \|[\nabla_H^2 v_H \tau_e]\|^2_{L^2(e)} \text{ for any } v_H \in W_H \text{ and } K \in \mathcal{T}_H\backslash\mathcal{T}_h. \tag{5.4}$$

For any $e \in \mathcal{E}_H$, $\|[\nabla_H^2 v_H \tau_e]\|_{L^2(e)} = 0$ indicates that there are no jumps over e for all tangential components of $\nabla_H^2 v_H$, which in turn implies that $\nabla_H v_H$ is continuous over e since $\nabla_H v_H$ is average continuous over e. Since v_H is continuous at all the internal nodes, Lemma 5.1 proves that v_H is continuous over e. Whence, $I'_h v_H|_K = v_H|_K$ provided that $\|[\nabla_H^2 v_H \tau_e]\|_{L^2(e)} = 0$ for any $e \in \partial K \subset \mathcal{E}_H$. Finally, the local quasi-uniformity of the mesh together with a scaling argument leads to the estimate (5.4).
□

Remark 5.3 An easy observation finds that the positive constant in (5.4) depends on the following ratio

$$\mu = \max_{K\in\mathcal{T}_H\backslash\mathcal{T}_h} \max_{\mathcal{T}_h\ni T\subset K} \frac{h_K}{h_T}. \tag{5.5}$$

In the analysis of optimality of the adaptive finite element method, this dependence is not allowed since we only know that $\mathcal{T}_h$ is some refinement of $\mathcal{T}_H$ by the newest vertex bisection and the boundness of μ is not guaranteed.

To overcome the above difficulty, we introduce a modified prolongation operator J_h which preserves the local projection property $J_h v_H|_K = v_H|_K$ for $K \in \mathcal{T}_h \cap \mathcal{T}_H$. We need the prolongation operator $\Pi : W_H \to W_H^C$, where $W_H^C \subset W$ is some conforming finite element space over the mesh $\mathcal{T}_H$. Here we take W_H^C as the Hsieh–Clough–Tocher finite element space over the mesh $\mathcal{T}_H$ [7,16].

Let $\mathcal{F}$ be any (global) degree of freedom of W_H^C, i.e., $\mathcal{F}$ is either the evaluation of a shape function or its first order derivatives at an interior node of $\mathcal{T}_H$, or the evaluation of the normal derivative of a shape function at a node on an interior edge. For $v_H \in W_H$, we define [8]

$$\mathcal{F}(\Pi v_H) = \frac{1}{|\omega_{\mathcal{F}}|} \sum_{K \in \omega_{\mathcal{F}}} \mathcal{F}(v_H|_K) \tag{5.6}$$

where $\omega_{\mathcal{F}}$ is the set of triangles in $\mathcal{T}_H$ that share the degree of freedom $\mathcal{F}$, and $|\omega_{\mathcal{F}}|$ is the number of elements of $\omega_{\mathcal{F}}$. Then a similar argument of [8] proves

$$\|\nabla_H^2(v_H - \Pi v_H)\|_{L^2(\Omega)}^2 \lesssim \sum_{e \in \mathcal{E}_H} h_e \|[\nabla_H^2 v_H \tau_e]\|_{L^2(e)}^2. \tag{5.7}$$

Denote

$$\Omega_{\mathcal{R}} := \text{interior}\left(\bigcup \{K : K \in \mathcal{T}_H \backslash \mathcal{T}_h,\ \partial K \cap \partial(\mathcal{T}_H \cap \mathcal{T}_h) = \emptyset\}\right).$$

The main idea herein is to take the mixture of the prolongation operators I_h' and Π. More precisely, we use Π in the region $\Omega_{\mathcal{R}}$ where the elements of $\mathcal{T}_H$ are refined and take I_h' on $\mathcal{T}_h \cap \mathcal{T}_H$, and we define some mixture in the layer between them. This leads to the prolongation operator $J_h : W_H \to W_h$ by

$$\mathcal{J}_h v_H = \begin{cases} \Pi_h \Pi v_H & \text{on } \Omega_{\mathcal{R}}, \\ I_h' v_H & \text{on } \mathcal{T}_H \cap \mathcal{T}_h, \\ v_{h,tr} & \text{on } \Omega \backslash (\Omega_{\mathcal{R}} \cup \mathcal{T}_H \cap \mathcal{T}_h), \end{cases}$$

where $v_{h,tr}$ is defined by

$$\begin{aligned} v_{h,tr}(P) &= \begin{cases} (\Pi v_H)(P) & \text{if } P \in \partial\Omega_{\mathcal{R}}, \\ (I_h' v_H)(P) & \text{otherwise}, \end{cases} \quad \text{for } P \in \mathcal{N}_h, \\ \int_e \nabla_h v_{h,tr} \cdot \nu_e ds &= \begin{cases} \int_e \nabla_h \Pi v_H \cdot \nu_e ds & \text{if } e \subset \partial\Omega_{\mathcal{R}} \\ \int_e \nabla_h I_h' v_H \cdot \nu_e ds & \text{otherwise} \end{cases} \quad \text{for } e \in \mathcal{E}_h. \end{aligned} \tag{5.8}$$

Lemma 5.4 *It holds true that*

$$\|\nabla_h^2(J_h v_H - v_H)\|_{L^2(\Omega)}^2 \lesssim \sum_{K \in \mathcal{T}_H \backslash \mathcal{T}_h} \sum_{e \subset \partial K} h_K \|[\nabla_H^2 v_H \tau_e]\|_{L^2(e)}^2 \quad \textit{for any } v_H \in W_H. \tag{5.9}$$

Proof We only need to use the scaling argument like that in Lemma 5.2 in the layer $\Omega \backslash (\Omega_{\mathcal{R}} \cup \mathcal{T}_H \cap \mathcal{T}_h)$. The desired result follows from the estimate (5.7) and the local projection property $J_h v_H|_K = v_H|_K$ for $K \in \mathcal{T}_h \cap \mathcal{T}_H$. □

Lemma 5.5 (Discrete reliability) *It holds that*

$$\|u_h - u_H\|^2_{\mathcal{C}_h} \lesssim \eta^2(u_H, \mathcal{T}_H \backslash \mathcal{T}_h). \tag{5.10}$$

Proof For any $v_h \in W_h$, we deduce from the discrete problem (2.6) that

$$\|u_h - u_H\|^2_{\mathcal{C}_h} = a_h(u_h - u_H, u_h - v_h) + a_h(u_h - u_H, v_h - u_H) = J_1 + J_2, \tag{5.11}$$

where

$$J_1 = \mathrm{Res}_H(u_h - v_h), \text{ and } J_2 = a_h(u_h - u_H, v_h - u_H).$$

Thanks to (2.13), (3.8) and (3.9), the residual J_1 can be bounded by a similar argument for the term on the right hand-side of (3.14), which implies

$$\begin{aligned} |J_1| &= |\mathrm{Res}_H(u_h - v_h)| = |\mathrm{Res}_H((I - I_H)(u_h - v_h))| \\ &\lesssim \sum_{K \in \mathcal{T}_H \backslash \mathcal{T}_h} h_K^2 \|f\|_{L^2(K)} \|\nabla_h^2 (u_h - v_h)\|_{L^2(K)}. \end{aligned} \tag{5.12}$$

Since $v_h \in W_h$ is arbitrary, we apply the Young and Cauchy–Schwarz inequalities in (5.11) to obtain that

$$\|u_h - u_H\|^2_{\mathcal{C}_h} \lesssim \sum_{K \in \mathcal{T}_H \backslash \mathcal{T}_h} h_K^4 \|f\|^2_{L^2(K)} + \inf_{v_h \in W_h} \|v_h - u_H\|^2_{\mathcal{C}_h}. \tag{5.13}$$

Taking $v_h = J_h u_H$ and applying (5.9) we complete the proof of the Lemma. □

We end this section by applying the previous discrete reliability to show a result indicating that the bulk criterion is in some sense a necessary condition for reduction of the energy norm between two levels.

Lemma 5.6 *If $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$ such that the following reduction holds*

$$\|u - u_h\|^2_{\mathcal{C}_h} + \mathrm{osc}^2(f, \mathcal{T}_h) \le \alpha'(\|u - u_H\|^2_{\mathcal{C}_H} + \mathrm{osc}^2(f, \mathcal{T}_H)), \tag{5.14}$$

for some $0 < \alpha' < 1$, then there exists $0 < \theta_* < 1$ such that

$$\theta_* \eta^2(u_H, \mathcal{T}_H) \le \eta^2(u_H, \mathcal{T}_H \backslash \mathcal{T}_h). \tag{5.15}$$

Proof We start with the following decomposition

$$\begin{aligned} &(1 - \alpha')(\|u - u_H\|^2_{\mathcal{C}_H} + \mathrm{osc}^2(f, \mathcal{T}_H)) \\ &\quad \le \|u - u_H\|^2_{\mathcal{C}_H} + \mathrm{osc}^2(f, \mathcal{T}_H) - \|u - u_h\|^2_{\mathcal{C}_h} - \mathrm{osc}^2(f, \mathcal{T}_h) \\ &\quad = \|u_H - u_h\|^2_{\mathcal{C}_h} + 2a_h(u - u_h, u_h - u_H) + \mathrm{osc}^2(f, \mathcal{T}_H) - \mathrm{osc}^2(f, \mathcal{T}_h). \end{aligned} \tag{5.16}$$

By the discrete reliability of Lemma 5.5 with the coefficient C_{Drel}, we have

$$\|u_h - u_H\|^2_{\mathcal{C}_h} \leq C_{Drel}\eta^2(u_H, \mathcal{T}_H \backslash \mathcal{T}_h). \tag{5.17}$$

The quasi-orthogonality in Lemma 3.4 with the coefficient C_{QO} yields

$$|2a_h(u - u_h, u_h - u_H)| \leq 2C_{QO}\|u - u_h\|_{\mathcal{C}_h} \left(\sum_{K \in \mathcal{T}_H \backslash \mathcal{T}_h} h_K^4 \|f\|^2_{L^2(K)} \right)^{1/2}. \tag{5.18}$$

It follows from (5.14) that

$$\|u - u_h\|_{\mathcal{C}_h} \leq \sqrt{\alpha'}\big(\|u - u_H\|^2_{\mathcal{C}_H} + \mathrm{osc}^2(f, \mathcal{T}_H)\big)^{1/2}. \tag{5.19}$$

Therefore, we apply the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} |2a_h(u - u_h, u_h - u_H)| \leq & \frac{1}{2}(1 - \alpha') \left(\|u - u_H\|^2_{\mathcal{C}_H} + \mathrm{osc}^2(f, \mathcal{T}_H) \right) \\ & + 2(C_{QO})^2 \frac{\alpha'}{1 - \alpha'} \sum_{K \in \mathcal{T}_H \backslash \mathcal{T}_h} h_K^4 \|f\|^2_{L^2(K)}. \end{aligned} \tag{5.20}$$

Since it is obvious that

$$|\,\mathrm{osc}^2(f, \mathcal{T}_H) - \mathrm{osc}^2(f, \mathcal{T}_h)| \leq \eta^2(u_H, \mathcal{T}_H \backslash \mathcal{T}_h), \tag{5.21}$$

we combine (5.16)–(5.21) to prove the desired result by the parameter

$$\theta_* = \frac{(1 - \alpha')^2 C_{Eff}}{2(2\alpha'(C_{QO})^2 + (1 - \alpha')(C_{Drel} + 1))}$$

with the efficiency constant C_{Eff} of the estimator $\eta(u_H, \mathcal{T}_H)$ from Lemma 2.1. □

6 Optimality

To analyze the optimality, we follow an idea commonly used in the adaptive finite element literature to introduce a nonlinear approximation class [4,5]. First, we have the following quasi-optimality.

$$\|u - u_H\|^2_{\mathcal{C}_H} \approxeq \inf_{v_H \in W_H} \|u - v_H\|^2_{\mathcal{C}_H} + \kappa^2(u, \mathcal{T}_H)\,, \tag{6.1}$$

where the consistency error term is given by

$$\kappa(u, \mathcal{T}_H) = \sup_{v_H \in W_H} \frac{(f, v_H)_{L^2(\Omega)} - a_H(u, v_H)}{\|v_H\|_{\mathcal{C}_H}}. \tag{6.2}$$

It follows from [20, Section 4.1] that

$$\kappa(u, \mathcal{T}_H) \lesssim \inf_{v_H \in W_H} \|u - v_H\|_{\mathcal{C}_H} + \operatorname{osc}(f, \mathcal{T}_H).$$

Therefore, we define

$$\mathfrak{E}(N; u, f) := \inf_{\mathcal{T} \in \mathbb{T}_N} \inf_{v \in W_{\mathcal{T}}} \left(\|u - v\|^2_{\mathcal{C}_{\mathcal{T}}} + \operatorname{osc}^2(f, \mathcal{T})\right). \tag{6.3}$$

Finally, we choose the nonlinear approximation class as follows:

$$\mathbb{A}_s := \left\{(u, f), |u, f|_s := \sup_{N > N_0} (N - N_0)^s \mathfrak{E}(N; u, f) < +\infty\right\}. \tag{6.4}$$

Compared to the adaptive conforming method for the second order elliptic problem [14,29], we have not the following monotone convergence:

$$\inf_{v_h \in W_h} \|u - v_h\|^2_{\mathcal{C}_h} + \kappa^2(u, \mathcal{T}_h)^2 \le \inf_{v_H \in W_H} \|u - v_H\|^2_{\mathcal{C}_H} + \kappa^2(u, \mathcal{T}_H), \tag{6.5}$$

where T_h is some refinement of T_H. However, it follows from the quasi-orthogonality in Lemma 3.4, the efficiency of the estimator in Lemma 2.1, and the Young inequality that

$$\|u - u_h\|^2_{\mathcal{C}_h} + \operatorname{osc}^2(f, \mathcal{T}_h) \le C_2(\|u - u_H\|^2_{\mathcal{C}_H} + \operatorname{osc}^2(f, \mathcal{T}_H)). \tag{6.6}$$

Theorem 6.1 *Let $\mathcal{M}_k$ be a set of marked elements with minimal cardinality from Algorithm 4.1, u the solution of Problem (2.3), and $(\mathcal{T}_k, W_k, u_k)$ the sequence of meshes, finite element spaces, and discrete solutions produced by the adaptive finite-element methods with $0 < \theta < \frac{C_{Eff}}{2(2(C_{QO})^2 + C_{Drel} + 1)}$. Then, the following estimate holds:*

$$\#\mathcal{M}_k \lesssim (\alpha')^{-\frac{1}{s}} |u, f|_s^{\frac{1}{s}} (C_2)^{\frac{1}{s}} \left(\|u - u_k\|^2_{\mathcal{C}_k} + \operatorname{osc}^2(f, \mathcal{T}_k)\right)^{-\frac{1}{s}} \quad \text{for any } (u, f) \in \mathbb{A}_s, \tag{6.7}$$

where the parameter $0 < \alpha' < 1$ is from Lemma 5.6.

Proof We set $\epsilon = \alpha'(C_2)^{-1}\left(\|u - u_k\|^2_{\mathcal{C}_k} + \operatorname{osc}^2(f, \mathcal{T}_k)\right)$ with $0 < \alpha' < 1$. Since $(u, f) \in \mathbb{A}_s$, there exists a $\mathcal{T}_\epsilon$ of the refinement of $\mathcal{T}_0$ and $u_\epsilon \in W_{\mathcal{T}_\epsilon}$ with

$$\#\mathcal{T}_\epsilon - \#\mathcal{T}_0 \le |u, f|_s^{1/s} \epsilon^{-1/s} \text{ and } \|u - u_\epsilon\|^2_{\mathcal{C}_{\mathcal{T}_\epsilon}} + \operatorname{osc}^2(f, \mathcal{T}_\epsilon) \le \epsilon. \tag{6.8}$$

The overlay $\mathcal{T}_*$ of $\mathcal{T}_\epsilon$ and $\mathcal{T}_k$ is the smallest refinement of both $\mathcal{T}_\epsilon$ and $\mathcal{T}_k$. Let u_* be the finite element solution of (2.6) on the mesh $\mathcal{T}_*$. Since $\mathcal{T}_*$ is a refinement of $\mathcal{T}_\epsilon$, we use, (6.8), and (6.6) to obtain that

$$\begin{aligned} \|u - u_*\|^2_{\mathcal{C}_{\mathcal{T}_*}} + \operatorname{osc}^2(f, \mathcal{T}_*) &\le C_2(\|u - u_\epsilon\|^2_{\mathcal{C}_{\mathcal{T}_\epsilon}} + \operatorname{osc}^2(f, \mathcal{T}_\epsilon)) \\ &\le C_2\epsilon = \alpha'(\|u - u_k\|^2_{\mathcal{C}_k} + \operatorname{osc}^2(f, \mathcal{T}_k)). \end{aligned} \tag{6.9}$$

We deduce from Lemma 5.6 that

$$\theta_* \eta^2(u_k, \mathcal{T}_k) \le \eta^2(u_k, \mathcal{T}_k \backslash \mathcal{T}_*), \quad \text{for some } \theta_* \in (0, 1). \tag{6.10}$$

We note that the step (3) in Algorithm 4.1 with $\theta \le \theta_*$ chooses a subset of $M_k \subset \mathcal{T}_k$ with minimal cardinality with the same property. Therefore

$$\#\mathcal{M}_k \le \#\mathcal{T}_* - \#\mathcal{T}_k \le \#\mathcal{T}_\epsilon - \#\mathcal{T}_0. \tag{6.11}$$

This together with the definition of ϵ leads to

$$\#\mathcal{M}_k \lesssim (\alpha')^{-\frac{1}{s}} |u, f|_s^{\frac{1}{s}} (C_2)^{\frac{1}{s}} (\|u - u_k\|_{\mathcal{C}_k}^2 + \text{osc}^2(f, \mathcal{T}_k))^{-\frac{1}{s}}, \tag{6.12}$$

which completes the proof. □

Theorem 6.2 *Let the marking step in Algorithm 4.1 select a set $\mathcal{M}_k$ of marked elements with minimal cardinality, u the solution to Problem (2.6), and $(\mathcal{T}_k, W_k, u_k)$ the sequence of meshes, finite element spaces, and discrete solutions produced by the adaptive finite-element methods with $0 < \theta < \frac{C_{Eff}}{2(2(C_{QO})^2 + C_{Drel} + 1)}$. Then, it holds that*

$$\|u - u_N\|_{\mathcal{C}_N}^2 + \text{osc}^2(f, \mathcal{T}_N) \lesssim |u, f|_s (\#\mathcal{T}_N - \#\mathcal{T}_0)^{-s}, \quad \textit{for } (u, f) \in \mathbb{A}_s. \tag{6.13}$$

Proof Let $\mu = (\alpha')^{-\frac{1}{s}} |u, f|_s^{\frac{1}{s}} (C_2)^{\frac{1}{s}}$. We use the result that $\#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \sum_{j=0}^{k-1} \mathcal{M}_j$ from [29,30] to obtain that

$$\#\mathcal{T}_N - \#\mathcal{T}_0 \lesssim \sum_{j=0}^{N-1} \mathcal{M}_j \lesssim \mu \sum_{j=0}^{N-1} (\|u - u_j\|_{\mathcal{C}_j}^2 + \text{osc}^2(f, \mathcal{T}_j))^{-\frac{1}{s}}. \tag{6.14}$$

It follows from the efficiency of the estimator that

$$\|u - u_j\|_{\mathcal{C}_j}^2 + \text{osc}^2(f, \mathcal{T}_j) \approxeq \tilde{\eta}^2(u_j, \mathcal{T}_j), \tag{6.15}$$

which gives

$$\|u - u_j\|_{\mathcal{C}_j}^2 + \gamma_1 \tilde{\eta}^2(u_j, \mathcal{T}_j) \lesssim \|u - u_j\|_{\mathcal{C}_j}^2 + \text{osc}^2(f, \mathcal{T}_j). \tag{6.16}$$

For any $0 \le j \le N - 1$, we use the convergence result from Theorem 4.5 to derive that

$$\|u - u_N\|_{\mathcal{C}_N}^2 + \gamma_1 \tilde{\eta}^2(u_N, \mathcal{T}_N) \le \alpha^{(N-j)} (\|u - u_j\|_{\mathcal{C}_j}^2 + \gamma_1 \tilde{\eta}^2(u_j, \mathcal{T}_j)). \tag{6.17}$$

A combination of (6.14)–(6.17) yields

$$\#\mathcal{T}_N - \#\mathcal{T}_0 \lesssim \mu(\|u - u_N\|^2_{\mathcal{C}_N} + \mathrm{osc}^2(f, \mathcal{T}_N))^{-1/s} \sum_{j=1}^{N} \alpha^{j/s}. \tag{6.18}$$

Setting $C_\theta = \alpha^{1/s}(1 - \alpha^{1/s})^{-1}$, it is easy to prove that

$$\sum_{j=1}^{N} \alpha^{j/s} \leq C_\theta. \tag{6.19}$$

Inserting this bound into (6.18) leads to

$$\|u - u_N\|^2_{\mathcal{C}_N} + \mathrm{osc}^2(f, \mathcal{T}_N) \lesssim |u, f|_s(\#\mathcal{T}_N - \#\mathcal{T}_0)^{-s}, \tag{6.20}$$

which completes the proof. □

7 The extensions of the theory

This section extends the theory to the Morley element in three dimensions and the nonconforming linear elements in both two and three dimensions.

7.1 The Morley element in three dimensions

Let $\mathcal{T}_h$ be a decomposition of the domain $\Omega \subset \mathbb{R}^3$ into simplicies. Given any face F, we let ν_F denote its unit normal vector. The Morley element in three dimensions is defined and analyzed in [33], where the space reads

$$\begin{aligned} W_h := \{ & v \in L^2(\Omega), v|_K \in P_2(K), K \in \mathcal{T}_h, \int_e [v]\, ds = 0 \quad \text{for any internal edge } e, \\ & \int_e v\, ds = 0 \text{ for any boundary edge } e, \int_F [\nabla v \cdot \nu_F] dF = 0 \quad \text{for any} \\ & \text{internal face } F, \text{ and } \int_F \nabla v \cdot \nu_F dF = 0 \quad \text{for any boundary face } F\}. \end{aligned} \tag{7.1}$$

Define the estimator on each element $K \in \mathcal{T}_h$ as

$$\eta_K = h_K^2 \|f\|_{L^2(K)} + \left(\sum_{F \subset \partial K} h_F \|[\nabla_h^2 u_h \times \nu_F]\|^2_{L^2(F)} \right)^{1/2}, \tag{7.2}$$

where $\times$ denotes the usual tensor product. The estimator is defined by

$$\eta^2(u_h, \mathcal{T}_h) := \sum_{K \in \mathcal{T}_h} \eta_K^2. \tag{7.3}$$

The following reliability and efficiency of the estimator were proved in [21].

Lemma 7.1 *Let u be the solution to the fourth order elliptic problem with $u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$ in three dimensions, u_h be the finite element solution corresponding to the discrete space W_h defined in (7.1). Then,*

$$\|u - u_h\|_{\mathcal{C}_h} \approxeq \eta_h \tag{7.4}$$

up to the oscillation $\operatorname{osc}(f, \mathcal{T}_h)$, where $\|\cdot\|_{\mathcal{C}_h}$ and $\operatorname{osc}(f, \mathcal{T}_h)$ are the three dimensional counterparts of the discrete energy norm in (2.5) and the oscillation in (2.10), respectively.

Lemma 7.2 *Let $K_1, K_2 \in \mathcal{T}_h$ be two elements sharing a common face F with three edges e_ℓ and midpoints m_ℓ, $\ell = 1, 2, 3$, and v be a piecewise polynomial of degree ≤ 1 over $K_1 \cup K_2$ such that*

$$v|_{K_1}(m_\ell) = v|_{K_2}(m_\ell), \ell = 1, 2, 3, \text{ and } \int_F [\frac{\partial v}{\partial \nu_F}] dF = 0. \tag{7.5}$$

Then, v is a polynomial of degree ≤ 1 over $K_1 \cup K_2$.

With these preparations, one can generalize the theories of the quasi-orthogonality of Lemma 3.4, error reduction of Theorem 4.5, the discrete reliability of Lemma 5.5, and the optimality of Theorem 6.2 to the Morley element method in three dimensions.

7.2 The nonconforming linear elements for second order elliptic problems

In this subsection, we let $\mathcal{T}_h$ be a decomposition of the domain $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$ into simplicies in both two and three dimensions. The nonconforming linear element spaces in both two and three dimensions is defined by, respectively,

$$W_h := \left\{ v \in L^2(\Omega), v|_K \in P_1(K), K \in \mathcal{T}_h, \int_e [v]\, ds = 0 \text{ for any internal edge } e, \right.$$
$$\left. \int_e v\, ds = 0 \text{ for any boundary edge } e \right\},$$
$$W_h := \left\{ v \in L^2(\Omega), v|_K \in P_1(K), K \in \mathcal{T}_h, \int_F [v]\, dF = 0 \text{ for any internal face } F, \right.$$
$$\left. \int_F v\, dF = 0 \text{ for any boundary face } F \right\}. \tag{7.6}$$

The continuous problems read: Given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \text{for any } v \in H_0^1(\Omega). \tag{7.7}$$

The discrete problems read: Given $f \in L^2(\Omega)$, find $u_h \in W_h$ such that

$$(\nabla_h u_h, \nabla_h v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} \quad \text{for any } v_h \in W_h. \tag{7.8}$$

The convergence of the adaptive nonconforming linear element methods was first analyzed in [13]. The theory in Sects. 3–7 can be extended to this case. This extension gives another analysis of the convergence result from [13].

8 Conclusion and comments

In this paper, we carry out the convergence and optimality analysis of the Morley element for the fourth order elliptic equation. Moreover, we generalize the theory to the nonconforming linear elements. However, the analysis herein heavily depends on the conservative properties of these two classes of nonconforming elements and the fact that the discrete stress is a piecewise constant tensor. At the present time, it is unclear how to generalize these techniques to other nonconforming schemes of the fourth order elliptic problems.

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