Numer. Math. (2011) 119:517–556
DOI 10.1007/s00211-011-0390-3

Numerische
Mathematik

How to build all Chebyshevian spline spaces good for geometric design?

Marie-Laurence Mazure

Received: 2 January 2011 / Revised: 23 May 2011 / Published online: 29 June 2011

Abstract In the present work we determine all Chebyshevian spline spaces good for geometric design. By Chebyshevian spline space we mean a space of splines with sections in different Extended Chebyshev spaces and with connection matrices at the knots. We say that such a spline space is good for design when it possesses blossoms. To justify the terminology, let us recall that, in this general framework, existence of blossoms (defined on a restricted set of tuples) makes it possible to develop all the classical geometric design algorithms for splines. Furthermore, existence of blossoms is equivalent to existence of a B-spline bases both in the spline space itself and in all other spline spaces derived from it by insertion of knots. We show that Chebyshevian spline spaces good for design can be described by linear piecewise differential operators associated with systems of piecewise weight functions, with respect to which the connection matrices are identity matrices. Many interesting consequences can be drawn from the latter characterisation: as an example, all Chebsyhevian spline spaces good for design can be built by means of integral recurrence relations.

Mathematics Subject Classification (2000) 65D07 · 65D17

1 Introduction

Extended Chebyshev spaces can be considered as generalised versions of polynomial spaces, in so far as they share with them the same bounds of zeros for their non-zero elements [11,13,33]. Unlike polynomial spaces, it is well known that the class of all Extended Chebyshev spaces provides us with a great variety of shape parameters which

M.-L. Mazure (✉)
Laboratoire Jean Kuntzmann, Université Joseph Fourier, BP53,
38041 Grenoble Cedex 9, France
e-mail: mazure@imag.fr

can usefully be exploited for geometric design purposes, see for instance [29]. Some of them have also been usefully exploited for spline interpolation (e.g. tension splines [12]). To obtain a maximum benefit of this great variety, it is interesting to consider splines with sections in different Extended Chebyshev spaces. On the other hand, the requirement that left and right derivatives coincide at a given knot up to some order (fixed by the multiplicity of the knot) has no special meaning for parametric spline curves. It is thus preferable to replace it by the presence of a connection matrix, assumed to be lower triangular and to have a positive diagonal. Such matrices offer shape parameters too, but they also increase the chance of obtaining "good" spline spaces. Indeed, on a given knot-vector, connecting an arbitrary family of Extended Chebyshev spaces (the section-spaces) via an arbitrary family of connection matrices does not necessarily lead to a "good" spline space, that is, a space usable for geometric design.

The splines we consider here are thus *geometrically continuous* (due to the connection matrices) *piecewise Chebyshevian splines* (the sections belong to different Extended Chebyshev spaces). In the remainder of the introduction we will use the word *spline* in this specific sense. What, therefore, are the exact properties required of a spline space for it to be considered "good for design"? We first want it to be able to control the shape of the curves, in the sense that it must possess a basis with respect to which the shape properties of the control polygons will be transmitted to the curves. Furthermore, we want this control to be local, in the sense that modifying one of the control points should only affect a part of the curve, as small as possible. Mathematically speaking these requirements mean the existence of a normalised totally positive basis whose elements have minimal supports, in other words, existence of a totally positive B-spline basis. On the other hand, we need to be able to develop in this spline space all the classical geometric design algorithms: evaluation, knot insertion, subdivision, … A necessary and sufficient condition for all these requirements is the existence of *blossoms* in the spline space [18,24]. Let us recall that, if we are dealing with $(n+1)$-dimensional section-spaces, spline blossoms are functions of n variables defined on a restricted set of n-tuples said to be admissible (with respect to the knot-vector).

This very general framework has been considered by several authors (see, for instance [4,18,20,26–28]), among whom the very first was Barry [1]. In his approach, the section-spaces were defined by means of given weight functions and associated differential operators, as is classical for Extended Chebyshev spaces. Moreover, the connection matrices he considered did not concern the ordinary derivatives but the differential operators in question. Via de Boor-Fix type dual functionals, he proved that the total positivity of all such connection matrices (i.e., all their minors are nonnegative), was sufficient to ensure existence of a B-spline basis and of a de Boor-type evaluation algorithm. Later on, under the same total positivity assumption, a further proof of the existence of a B-spline basis was given by Mühlbach via generalised Chebyshevian divided differences [4,27,28]. In the meantime we had shown that, in any such spline space, existence of blossoms was equivalent to existence of a B-spline basis in the space itself and in all spline spaces deduced from it by knot insertion [18]. In the proof of the latter equivalence the main part consists in showing that, as soon as blossoms exist, they are *pseudoaffine* in each variable. The latter property and the obvious symmetry and diagonal properties of blossoms are the three fundamental properties on which all the classical geometric design algorithms are based.

Concerning piecewise Chebyshevian splines we would like to mention the interesting work by Prautzsch [31], although the framework is not exactly the same there since the splines are no longer geometrically continuous. Indeed, the connection matrices are not assumed to be lower triangular, but only regular. Under the latter more general assumption, Prautzsch proved that almost all such matrices produced bases of minimally supported splines. However, they are not necessarily B-splines bases in so far as they may fail to be non-negative. If so, they fail to ensure shape preservation. On the other hand it is also important to briefly recall a few essential results prior to [1]. Indeed, existence of B-spline bases had been obtained by Dyn and Micchelli [7] in the framework of geometrically continuous polynomial splines with totally positive regular lower triangular connection matrices (see the work by Goodman [8] for the case of one-banded matrices, and also [6]). Later on, the latter existence enabled Seidel [34] to build blossoms for the splines in question and to use them for knot insertion and evaluation algorithms. He was the first to exploit the beautiful idea of blossoms defined by means of intersections of osculating flats. This has since become the most natural way to introduce blossoms outside the strict polynomial context presented by Ramshaw [32], in particular after the powerful work by Pottmann [29].

Let us now comment on some of the results mentioned above. Barry's result was a crucial step in the study of Chebyshevian splines. Nevertheless, in [15] we proved the limits of his total positivity sufficient condition which may be very restrictive. For $n = 3$ and for simple knots, we established practical necessary and sufficient conditions for existence of blossoms. For given weight functions associated with given section-spaces, they enabled us to "measure how far beyond total positivity" we could go, and we showed how useful this was to increase the possibilities of shape effects. As for existence of blossoms, it is indeed an elegant necessary and sufficient condition for a spline space to be good for design. Unfortunately, in practice, it is not easy to check whether or not blossoms do exist in a given spline space, especially in high dimensions. In the present article we thus investigate two main questions:

1. can we give a simple practical description of all Chebyshevian spline spaces good for design which would be valid for any dimension?
2. the knot-vector being given, and the sequence of section-spaces being given too, can we indicate all connection matrices (linking left/right ordinary derivatives at the knots) which yield spline spaces good for design?

We will give an affirmative answer to each of these two questions. To achieve our results, the paper is organised as follows. In Sect. 2, we give a precise description of the spline spaces we shall deal with and of the main tools: blossoms, B-spline bases, knot insertion. We also remind the reader of the strong connections existing between these tools, for our proofs strongly rely on them. Out of necessity, Sect. 2 is thus rather long, the present results being the culmination of a long series of previous ones. As already mentioned, classically, differential operators associated with positive weight functions produce Extended Chebyshev spaces. By analogy, in Sect. 3, we show that piecewise differential operators associated with positive piecewise weight functions produce *Extended Chebyshev piecewise spaces* and we explain why all spline spaces based on them are automatically good for design. Assuming the knot-vector to be bi-infinite, the converse property is shown in Sect. 4: a spline space good for design is always based on an Extended Chebyshev piecewise space defined by means of positive piecewise weight functions (Theorem 4.2(i) $\Leftrightarrow$ (ii)). We even find all Extended Chebyshev piecewise spaces suitable for that property (Proposition 4.7 and Theorem 4.8).

Expressed differently, Theorems 4.2 and 4.8 establish that, with each spline space good for design involving $(n+1)$-dimensional section-spaces, one can associate infinitely many nth order piecewise differential operators which are to this spline space that which the nth order ordinary differentiation is to ordinary polynomial spline spaces with sections of degree at most n. The key step, which is the core of this paper, consists in proving the following result: a spline space good for design with $(n+1)$-dimensional section-spaces, can always be transformed into another spline space good for design, but with n-dimensional section-spaces, under an appropriate piecewise generalised differentiation (Theorem 4.1). As a matter of fact, Theorem 4.2 (i) $\Leftrightarrow$ (iii), contains the answer to Question 1. In order to build all spline spaces good for design, on each interval take any positive weight functions to define the corresponding section-space. There is no need for any additional ingredients. At each knot, simply require the left and right corresponding differential operators to coincide up to some order in accordance with the multiplicity of the knot. In other words, by comparison with [1], there is no need to introduce connection matrices. Stated differently, we can limit ourselves to the simplest possible totally positive connection matrices: identity matrices. We would like to draw the reader's attention on the following point: when varying the weight functions, we do produce the global class of good spline spaces obtained by Barry, even though our requirement on the connections is much stronger than his total positivity condition (see details in Remark 4.3).

Sections 5 and 6 deal with some implications of the latter results. Instead of B-spline bases we consider the problem of existence of B-spline-like bases (i.e. no normalisation is required). We also consider the case of splines on a closed bounded interval. As a special instance, we recover the results of [25] where we determined all weight functions which can be used to define a given Extended Chebyshev space on a closed bounded interval. This leads us to the answer to Question 2: all possible connection matrices linking left/right derivatives at a knot are defined in terms of arbitrary positive shape parameters of a new type, which themselves define all possible weight functions which can be associated with two consecutive section-spaces. We would like to bring a second important point to the reader's attention: for fixed section-spaces defined by given weight functions, though we use identity matrices instead of any possible totally positive matrices, the resulting class of all possible connection matrices involved in Question 2 is much larger than the one we could directly deduce from [1]. This is due to the fact that we now have at our disposal all other weight functions leading to the same section-spaces.

Finally, in Sect. 7 we briefly address the following unexpected outcome of our results: all spline spaces good for design can also be built by means of integral recurrence relations which are the exact piecewise version of the integral approach developed by Bister and Prautzsch applied to Chebsyhevian splines based on a single Extended Chebyshev space [2].

To conclude this introduction we would like to mention that our results also have important implications in approximation theory where the presence of B-spline bases is as crucial as in geometric design.

2 The framework

We start by presenting the spline spaces we shall be dealing with. Three main tools will be essential in the present article, strongly related to each other: blossoms, B-spline type bases, and knot insertion. It is why we cannot avoid recalling their precise definitions and major connections between them.

2.1 Spline spaces based on PEC-spaces

Throughout the article we consider a fixed bi-infinite sequence of *knots* $\mathbb{T} := (t_k)_{k\in\mathbb{Z}}$, with $t_k < t_{k+1}$ for all $k \in \mathbb{Z}$, and the associated interval $I :=]\mathrm{Inf}_k t_k, \mathrm{Sup}_k t_k[$,. We shall say that F is a *piecewise function on* $(I, \mathbb{T})$ if F is defined separately on each interval $[t_k^+, t_{k+1}^-]$, implying in particular that, for any $k \in \mathbb{Z}$, both $F(t_k{}^-)$ and $F(t_k{}^+)$ are defined, with possibly $F(t_k{}^-) \neq F(t_k{}^+)$. In such a case, unless explicitly mentioned, F is not a function on I. We shall deliberately use the somewhat abusive notation $F : \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-] \to \mathbb{R}$ to stress this fact. Given two piecewise functions F and G on $(I, \mathbb{T})$, the equality $F = G$ thus means that $F(x) = G(x)$ for all $x \in I \setminus \{t_k,\ k \in \mathbb{Z}\}$ and both $F(t_k^-) = G(t_k^-)$ and $F(t_k^+) = G(t_k^+)$ for all $k \in \mathbb{Z}$, which we shall summarise as $F(x) = G(x)$ for all $x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-]$. Similarly F is positive if $F(x) > 0$ for all $x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-]$. We denote by $PC^n(I, \mathbb{T})$ the set of *piecewise C^n functions on I*, that is, all piecewise functions on $(I, \mathbb{T})$ which are C^n on each interval $[t_k^+, t_{k+1}^-]$.

From now on we assume that $n \geq 0$ is a given integer. Before introducing our spline spaces, we need to recall a few definitions and classical related properties. Let $\mathbb{E}_k \subset C^n([t_k^+, t_{k+1}^-])$ be $(n+1)$-dimensional. Then, $\mathbb{E}_k$ is said to be a *W-space on* $[t_k^+, t_{k+1}^-]$ if the Wronskian of any of its bases never vanishes on $[t_k^+, t_{k+1}^-]$, or, equivalently, if no non-zero element of $\mathbb{E}_k$ can have a zero of multiplicity $(n+1)$ in $[t_k^+, t_{k+1}^-]$. It is said to be an *Extended Chebyshev space* (in short EC-space) *on* $[t_k^+, t_{k+1}^-]$ if no non-zero element of $\mathbb{E}_k$ can vanish more than n times on $[t_k^+, t_{k+1}^-]$, counting multiplicities up to $(n+1)$, see [11,33]. Clearly an EC-space on $[t_k^+, t_{k+1}^-]$ is a W-space on $[t_k^+, t_{k+1}^-]$, but the converse is not true. Assume that $\mathbb{E}_k$ contains constants and consider the n-dimensional space $D\mathbb{E}_k := \{DF := F' \mid F \in \mathbb{E}_k\}$. Then, $\mathbb{E}_k$ is W-space on $[t_k^+, t_{k+1}^-]$ if and only if so is $D\mathbb{E}_k$. In contrast, the fact that $\mathbb{E}_k$ is an EC-space on $[t_k^+, t_{k+1}^-]$ does not guarantee that $D\mathbb{E}_k$ will be an EC-space on $[t_k^+, t_{k+1}^-]$ in turn. From Rolle's theorem it is easy to deduce that the converse implication does hold.

Definition 2.1 Let us consider

– a bi-infinite sequence $\mathcal{R} := (R_k)_{k\in\mathbb{Z}}$ of *connection matrices*: for each $k \in \mathbb{Z}$, R_k is a lower triangular matrix of order $(n+1)$ with positive diagonal elements,
– a bi-infinite sequence $\mathcal{E} := (\mathbb{E}_k)_{k\in\mathbb{Z}}$ of *section-spaces*: for each $k \in \mathbb{Z}$, $\mathbb{E}_k \subset C^n([t_k^+, t_{k+1}^-])$ is an $(n+1)$-dimensional W-space on $[t_k^+, t_{k+1}^-]$.

Then, the $(n+1)$-dimensional linear subspace $\mathbb{E}$ of $PC^n(I, \mathbb{T})$ composed of all piecewise functions such that

(1) for each $k \in \mathbb{Z}$, the restriction of F to $[t_k^+, t_{k+1}^-]$ belongs to $\mathbb{E}_k$,
(2) F satisfies the connection conditions:

$$\Big(F(t_k{}^+), F'(t_k^+), \ldots, F^{(n)}(t_k{}^+)\Big)^T = R_k . \Big(F(t_k{}^-), F'(t_k^-), \ldots, F^{(n)}(t_k{}^-)\Big)^T, \quad k \in \mathbb{Z}, \tag{1}$$

is said to be *a piecewise W-space* (in short PW-space) *on* $(I, \mathbb{T})$. If the space $\mathbb{E}_k$ is an $(n+1)$-dimensional EC-space on $[t_k^+, t_{k+1}^-]$ for any $k \in \mathbb{Z}$, we say that $\mathbb{E}$ is *a piecewise Extended Chebyshev space* (in short PEC-space) *on* $(I, \mathbb{T})$.

Remark 2.2 For design purposes it is interesting to consider the case where the PW- or PEC-space $\mathbb{E}$ contains constants. This occurs if and only the following two properties hold : firstly, each space $\mathbb{E}_k$ contains constants, and secondly the first column of each connection matrix R_k is equal to $(1, \underbrace{0, \ldots, 0}_{n \text{ times}})^T$. In such a case, all elements of $\mathbb{E}$ are continuous functions on I. They are *geometrically continuous* in the weak sense of continuity of the Frenet frames of order n (see [14]).

Remark 2.3 Assume that the PW-space $\mathbb{E}$ contains constants. Then, from the reminder preceding Definition 2.1, one can say that the space $D\mathbb{E}$ obtained by (possibly left/right differentiation) is in turn a PW-space on $(I, \mathbb{T})$, the connection matrix at a knot t_k being obtained by deleting the first row and column of R_k. Moreover, if $D\mathbb{E}$ is a PEC-space on $(I, \mathbb{T})$, so is $\mathbb{E}$, but the converse implication does not hold.

When each knot t_k is allocated a multiplicity $m_k \geq 0$, we obtain a knot-vector based on the sequence $\mathbb{T}$, defined as $\mathbb{K} := \big(t_k{}^{[m_k]}\big)_{k \in \mathbb{Z}}$. In the latter equality as well as throughout the paper, for any $x \in I$ and any non-negative integer μ, the notation $x^{[\mu]}$ will stand for x repeated μ times. With the latter knot-vector we can associate splines as follows.

Definition 2.4 Let $\mathbb{E}$ be the $(n+1)$-dimensional PW- or PEC-space on $(I, \mathbb{T})$ introduced in Definition 2.1. Based on $\mathbb{E}$ and on the knot-vector $\mathbb{K}$ we denote by $\mathbb{S}(\mathbb{E}, \mathbb{K})$ the linear space composed of all piecewise functions S on $(I, \mathbb{T})$ which satisfy

(1) on each $[t_k^+, t_{k+1}^-]$, $k \in \mathbb{Z}$, S coincides with an element of $\mathbb{E}$;
(2) S satisfies the connection conditions

$$\Big(S(t_k^+), S'(t_k^+), \ldots, S^{(n-m_k)}(t_k^+)\Big)^T = M_k . \Big(S(t_k^-), S'(t_k^-), \ldots, S^{(n-m_k)}(t_k^-)\Big)^T, \quad k \in \mathbb{Z}, \tag{2}$$

where M_k is obtained from R_k by deleting its last m_k rows and columns.

In short, depending on the case, we shall say that $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is the PW- or PEC-spline space based on $\mathbb{E}$ and on the knot-vector $\mathbb{K}$.

At each knot t_k of multiplicity $m_k \geq n + 1$, there is no connection condition. In general it is convenient to work with multiplicities at most n. However, later on, we shall "differentiate" our spline spaces, and we shall thereby have to deal with section-spaces of lower and lower dimensions, for which the latter condition will not hold. This is why we directly allow multiplicities $m_k > n$.

Remark 2.5 As soon as at least one multiplicity is positive, a given PW-spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is based on infinitely many different PW-spaces. When the PW-spline space contains constants (*i.e.*, each section-space $\mathbb{E}_k$ contains constants and, for any $k \in \mathbb{Z}$, the first row of the connection matrix M_k is equal to $(1, \underbrace{0, \ldots, 0}_{(n-m_k)\text{times}})^T$, without loss of generality we shall systematically assume that the PW-space $\mathbb{E}$ itself contains constants.

Remark 2.6 Assume that the PW-spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ contains constants. All splines in $\mathbb{S}(\mathbb{E}, \mathbb{K})$ are continuous at each knot t_k such that $m_k \leq n$. Then, the space $D\mathbb{S}(\mathbb{E}, \mathbb{K})$ is in turn a PW-spline space. In contrast (see Remark 2.3) in case $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is a PEC-space on $(I, \mathbb{T})$ there is a priori no guarantee that the space $D\mathbb{S}(\mathbb{E}, \mathbb{K})$ should be a PEC-spline space on $(I, \mathbb{T})$.

2.2 Splines spaces good for design

For design purposes it is convenient to introduce the set $\mathbb{A}_n(\mathbb{K})$ of all admissible n-tuples (relative to the knot-vector $\mathbb{K}$). Suppose for a while that $m_k \leq n$ for all $k \in \mathbb{Z}$. Then, an n-tuple $(x_1, \ldots, x_n)$ is said to be admissible when, for any integer $k \in \mathbb{Z}$ such that

$$\text{Min}(x_1, \ldots, x_n) < t_k < \text{Max}(x_1, \ldots, x_n)$$

the knot t_k appears in the sequence $(x_1, \ldots, x_n)$ a number of times at least equal to its multiplicity m_k. Let us now drop the assumption $m_k \leq n$ for all $k \in \mathbb{Z}$. Then, we can always write the interval I as a union of subintervals with pairwise disjoints interiors: $I = \cup_{q \in \mathcal{Q}_n(\mathbb{K})} I_{n,q}$, where $\mathcal{Q}_n(\mathbb{K})$ is a set of consecutive integers such that:

- for each $q \in \mathcal{Q}_n(\mathbb{K})$, each knot t_k in the interior of $I_{n,q}$ is of multiplicity $m_k \leq n$;
- if $q, q + 1 \in \mathcal{Q}_n(\mathbb{K})$, then $I_{n,q}$ has a right endpoint, $I_{n,q+1}$ has a left one, and both are equal to some knot t_k of multiplicity $m_k \geq n + 1$.

Let us split the knot-vector in subsequences $\mathbb{K}_{n,q} := \left(t_k{}^{[m_k]}\right)_{t_k \in I_{n,q}}, q \in \mathcal{Q}_n(\mathbb{K})$. For each $q \in \mathcal{Q}_n(\mathbb{K})$, we then define the set $\mathbb{A}_{n,q}(\mathbb{K}_{n,q})$ exactly as we did previously, simply replacing $\mathbb{K}$ by $\mathbb{K}_{n,q}$. The set $\mathbb{A}_n(\mathbb{K})$ is then defined as the union of all $\mathbb{A}_{n,q}(\mathbb{K}_{n,q})$. It is essential to note that if t_k is the right-hand point of $I_{n,q}$ and the left-hand point of $I_{n,q+1}$ (which implies that $m_k \geq n + 1$), then as an element of $I_{n,q}$, it always has the meaning of t_k^-, while as an element of $I_{n,q+1}$ it always has the meaning of t_k^+. One can similarly define the set $\mathbb{A}_p(\mathbb{K})$ of all admissible p-tuples, for $p \leq n + 1$.

Assume that the PW-spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ contains constants. We denote by $\mathbb{1}$ the constant function $\mathbb{1}(x) = 1$ for all $x \in \cup_{k \in \mathbb{Z}}[t_k^+, t_{k+1}^-]$, and, for each $k \in \mathbb{Z}$, by $\mathbb{1}_k$

its restriction to $[t_k^+, t_{k+1}^-]$. Let $\Sigma \in \mathbb{S}(\mathbb{E}, \mathbb{K})^n$ be a fixed non-degenerate spline. This means that, for each $k \in \mathbb{Z}$, the affine flat spanned by $\{\Sigma(x),\ x \in [t_k^+, t_{k+1}^-]\}$ is of dimension n, or, equivalently, the restrictions of its components to $[t_k^+, t_{k+1}^-]$ and the constant function $\mathbb{1}_k$ span the whole space $\mathbb{E}_k$.

Suppose that i is a given integer, $0 \le i \le n$. When $x \in I$ is not a knot, we can consider the ith order osculating flat of Σ at x, that is, the affine flat passing through $\Sigma(x)$ with direction spanned by the ith first derivatives of Σ at x, *i.e.*,

$$\mathrm{Osc}_i\, \Sigma(x) := \{\Sigma(x) + \sum_{j=1}^{i} \lambda_j \Sigma^{(j)}(x),\ \lambda_1, \ldots, \lambda_i \in \mathbb{R}\}. \tag{3}$$

It is i-dimensional. At a knot t_k, a priori we have to consider two osculating flats, $\mathrm{Osc}_i\, \Sigma(t_k^-)$ and $\mathrm{Osc}_i\, \Sigma(t_k^+)$ defined using either the left derivatives or the right ones. However, whenever $i \le n - m_k$, both osculating flats coincide due to the structure of all connection matrices. In that case we simply write $\mathrm{Osc}_i\, \Sigma(t_k)$. This is obviously no longer valid when $i > n - m_k$. In that case, we are not allowed to write $\mathrm{Osc}_i\, \Sigma(t_k)$ unless the context makes it clear whether this notation has the meaning of $\mathrm{Osc}_i\, \Sigma(t_k^-)$ or of $\mathrm{Osc}_i\, \Sigma(t_k^+)$. In any case, it is also i-dimensional.

Definition 2.7 Assume the PW-spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ to contain constants. We say that blossoms exist in $\mathbb{S}(\mathbb{E}, \mathbb{K})$, if, for any admissible $(x_1, \ldots, x_n) \in \mathbb{A}_n(\mathbb{K})$ equal to $(a_1{}^{[\mu_1]}, \ldots, a_p{}^{[\mu_p]})$, up to permutation (with positive $\mu_1, \ldots, \mu_p$ and pairwise distinct $a_1, \ldots, a_p$), all osculating flats $\mathrm{Osc}_{n-\mu_i}\, \Sigma(a_i)$ have in common a unique point, labelled as $\sigma(x_1, \ldots, x_n)$. If blossoms exist, the function σ so defined on $\mathbb{A}_n(\mathbb{K})$ is called the blossom of Σ.

Remark 2.8 1. With the notations introduced in the previous definition, suppose that $p \ge 2$. Then, the admissibility of $(x_1, \ldots, x_n)$ guarantees that all osculating flats in question are well-defined, except possibly the first and last ones which are naturally meant as $\mathrm{Osc}_{n-\mu_1}\, \Sigma(a_1^+)$ and $\mathrm{Osc}_{n-\mu_p}\, \Sigma(a_p^-)$ in case a_1 (resp. a_p) is a knot t_k with $\mu_1 < m_k$ (resp. $\mu_p < m_k$).

2. If blossoms exist in $\mathbb{S}(\mathbb{E}, \mathbb{K})$, the blossom s of any $S \in \mathbb{S}(\mathbb{E}, \mathbb{K})^d$ is then defined on $\mathbb{A}_n(\mathbb{K})$ from the blossom σ of Σ via affine maps. This does not depend of the chosen non-degenerate spline (see [18]).

Definition 2.9 A PW-spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ will be said to be *good for design* if it contains constants and if blossoms exist in $\mathbb{S}(\mathbb{E}, \mathbb{K})$.

Our terminology is justified by the fact that, in any PW-spline space good for design it is possible to develop all the classical geometric design algorithms. This is made possible by the three main properties of blossoms derived from their geometrical definition. We recall them in Theorem 2.10 below.

Theorem 2.10 ([18,24]) *Let $\mathbb{S}(\mathbb{E}, \mathbb{K})$ be a PW-spline space $(I, \mathbb{T})$ with $(n+1)$-dimensional section-spaces, $n \ge 1$. Suppose that $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for design. Then, the blossom s of any spline $S \in \mathcal{S}^d$ satisfies the following properties:*

(B)$_1$ *symmetry:* $s(x_{\varrho(1)}, \dots, x_{\varrho(n)}) = s(x_1, \dots, x_n)$ *for any* $(x_1, \dots, x_n) \in \mathbb{A}_n(\mathbb{K})$ *and any permutation* ϱ *of* $\{1, \dots, n\}$*;*

(B)$_2$ *diagonal property: for all* $x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-]$, $s(x^{[n]}) = S(x)$*;*

(B)$_3$ *pseudoaffinity property: given any admissible* $(n-1)$*-tuple* $(x_1, \dots, x_{n-1})$*, any subinterval* $J \subset I$ *such that, for any* $y, z \in J$, $(x_1, \dots, x_{n-1}, y, z)$ *is admissible, any* $a, b \in J$*, with* $a < b$*, there exists a continuous strictly increasing function* $\beta(x_1, \dots, x_{n-1}; a, b; .) : J \to \mathbb{R}$ *(independent of S) such that:*

$$s(x_1, \dots, x_{n-1}, x) = \big[1 - \beta(x_1, \dots, x_{n-1}; a, b; x)\big] s(x_1, \dots, x_{n-1}, a) + \beta(x_1, \dots, x_{n-1}; a, b; x) s(x_1, \dots, x_{n-1}, b), \quad x \in J. \tag{4}$$

With the same data as in (B)$_3$, the pseudoaffinity function β satisfies

$$\beta(x_1, \dots, x_{n-1}; a, b; a) = 0, \quad \beta(x_1, \dots, x_{n-1}; a, b; b) = 1,$$
$$0 < \beta(x_1, \dots, x_{n-1}; a, b; t) < 1 \quad \text{for } t \in]a, b[. \tag{5}$$

Let us conclude the present subsection with the following important observation.

Proposition 2.11 *If a PW-spline space* $\mathbb{S}(\mathbb{E}, \mathbb{K})$ *is good for design, then* $\mathbb{S}(\mathbb{E}, \mathbb{K})$ *and* $D\mathbb{S}(\mathbb{E}, \mathbb{K})$ *are PEC-spline spaces on* $(I, \mathbb{T})$.

Proof For each $k \in \mathbb{Z}$, blossoms do exist in the W-space $\mathbb{E}_k$. This means that the space $D\mathbb{E}_k$ is an n-dimensional space EC-space on $[t_k^+, t_{k+1}^-]$, which implies that $\mathbb{E}_k$ is an $(n+1)$-dimensional space EC-space on $[t_k^+, t_{k+1}^-]$ [19]. □

2.3 B-spline bases and knot insertion

Insertion of knots is a classical tool in geometric design, and it plays a prominent rôle in the present work. Let $\mathbb{E}^*$ be another PW-space of dimension $(n+1)$ as $\mathbb{E}$, but based on another sequence of knots $\mathbb{T}^* := (t_k^*)_{k\in\mathbb{Z}}$. Given any knot-vector $\mathbb{K}^*$ based on $\mathbb{T}^*$, the spline space $\mathbb{S}(\mathbb{E}^*, \mathbb{K}^*)$ is said to be obtained from $\mathbb{S}(\mathbb{E}, \mathbb{K})$ by knot insertion whenever $\mathbb{S}(\mathbb{E}^*, \mathbb{K}^*) \supset \mathbb{S}(\mathbb{E}, \mathbb{K})$. Equivalently, this means that:

– any knot t_k in $\mathbb{T}$ is a knot $t_{k'}^*$ in $\mathbb{T}^*$, with $m_{k'}^* \geq m_k$ (we equivalently say that $\mathbb{K}^*$ is obtained from $\mathbb{K}$ by knot insertion) and in the spline space $\mathbb{S}(\mathbb{E}^*, \mathbb{K}^*)$, the connection matrix at $t_{k'}^*$ is obtained by deleting the last $(m_{k'}^* - m_k)$ rows and columns in M_k;
– the section-spaces in $\mathbb{E}^*$ are obtained by restricting those of $\mathbb{E}$ to all intervals of the form $[t_k^*, t_{k+1}^*]$;
– at any "new" knot t_k^* in $\mathbb{T}^*$ which is not in $\mathbb{T}$, the connection matrix in $\mathbb{S}(\mathbb{E}^*, \mathbb{K}^*)$ is the identity matrix of order $(n - m_k^* + 1)$.
  If needed, one can thus assume that $\mathbb{E}^* = \mathbb{E}$ without loss of generality. When $\mathbb{K}^*$ is obtained from $\mathbb{K}$ by knot insertion, the corresponding sets of admissible n-tuples satisfy $\mathbb{A}_n(\mathbb{K}) \supset \mathbb{A}_n(\mathbb{K}^*)$. Accordingly, if the PW-spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for

design, then $\mathbb{S}(\mathbb{E}, \mathbb{K}^*)$ is good for design in turn. Moreover, blossoms in $\mathbb{S}(\mathbb{E}, \mathbb{K}^*)$ are obtained by restricting blossoms in $\mathbb{S}(\mathbb{E}, \mathbb{K})$ to $\mathbb{A}_n(\mathbb{K}^*)$.

The presence of B-spline bases being essential for design, Theorem 2.12 below gives another justification of the terminology introduced in our Definition 2.9.

Theorem 2.12 [18,24] *Let $\mathbb{E}$ be a PW-space on $(I, \mathbb{T})$, supposed to contain constants, and let $\mathbb{K}$ be a knot-vector based on $\mathbb{T}$. Then, the following two properties are equivalent:*

- (i) *the spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for design;*
- (ii) *the spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ possesses a B-spline basis, and so does any spline space obtained from $\mathbb{S}(\mathbb{E}, \mathbb{K})$ by knot insertion.*

Theorem 2.12 was first proved in [18] under the assumption that $\mathbb{E}_k \subset C^\infty([t_k^+, t_k^-])$ for all $k \in \mathbb{Z}$, then in [24] under the weaker assumption $\mathbb{E}_k \subset C^n([t_k^+, t_k^-])$ for all $k \in \mathbb{Z}$, via completely different techniques. Due to their importance in the present article, we cannot avoid recalling the precise definition of B-spline(-like) bases. For the sake of simplicity, unless explicitly stated differently, we assume the knot-vector $K := \left(t_k{}^{[m_k]}\right)_{k\in\mathbb{Z}}$ to be bi-infinite. This enables us to also write it as

$$\mathbb{K} = (\xi_\ell)_{\ell\in\mathbb{Z}}, \quad \text{with } \xi_\ell \leq \xi_{\ell+1} \text{ for all } \ell \in \mathbb{Z}.$$

Subsequently, by $\Lambda_n(\mathbb{K})$ we denote the set of all integers $\ell \in \mathbb{Z}$ such that $\xi_\ell < \xi_{\ell+n+1}$.

Definition 2.13 Given a bi-infinite knot-vector $\mathbb{K}$, let $\mathbb{S}(\mathbb{E}, \mathbb{K})$ be a PW-spline space based on $\mathbb{K}$, with $(n + 1)$-dimensional section-spaces. A sequence $N_\ell, \ell \in \Lambda_n(\mathbb{K})$, of elements of $\mathbb{S}(\mathbb{E}, \mathbb{K})$, is said to be a *B-spline-like basis of* $\mathbb{S}(\mathbb{E}, \mathbb{K})$ if it meets the following requirements:

- $(\mathrm{BSB})_1$ *support property*: for each $\ell \in \Lambda_n(\mathbb{K})$, $N_\ell(x) = 0$ for $x \notin [\xi_\ell^+, \xi_{\ell+n+1}^-]$;
- $(\mathrm{BSB})_2$ *positivity property*: for each $\ell \in \Lambda_n(\mathbb{K})$, $N_\ell(x) > 0$ for $x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-]$ such that $\xi_\ell < x < \xi_{\ell+n+1}$;
- $(\mathrm{BSB})_3$ *endpoint property*: for each $\ell \in \Lambda_n(\mathbb{K})$, N_ℓ vanishes exactly $(n - s + 1)$ times at ξ_ℓ and exactly $(n - s' + 1)$ at $\xi_{\ell+n+1}$, where $s := \sharp\{j \geq \ell \mid \xi_j = \xi_\ell\}$ and $s' := \sharp\{j \leq \ell + n + 1 \mid \xi_j = \xi_{\ell+n+1}\}$.

It is said to be a *B-spline basis of* $\mathbb{S}(\mathbb{E}, \mathbb{K})$ when it is a B-spline-like basis of $\mathbb{S}(\mathbb{E}, \mathbb{K})$ which is *normalised*, *i.e.*, which satisfies the additional requirement

- $(\mathrm{BSB})_4$ *normalisation property*: $\sum_{\ell\in\Lambda_n(\mathbb{K})} N_\ell(x) = 1$ for all $x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-]$.

The three properties $(\mathrm{BSB})_\mathrm{i}$, $i = 1, 3, 4$, guarantee the uniqueness of a possible B-spline basis. Assume that the spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for design. Then, the B-spline basis of $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is a natural product of blossoms via the so-called *de Boor algorithm*. Indeed, thanks to the three properties of blossoms recalled in Theorem 2.10, the latter algorithm computes all values of any spline $S \in \mathbb{S}(\mathbb{E}, \mathbb{K})^d$ as convex combinations of it *poles* $P_\ell, \ell \in \Lambda_n(\mathbb{K})$, with coefficients independent of S, *i.e.*,

$$S(x) = \sum_{\ell\in\Lambda_n(\mathbb{K})} N_\ell(x) P_\ell, \quad \text{with } P_\ell := s(\xi_{\ell+1}, \ldots, \xi_{\ell+n}), \quad \ell \in \Lambda_n(\mathbb{K}). \tag{6}$$

The latter coefficients form the B-spline basis of $\mathbb{S}(\mathbb{E}, \mathbb{K})$. Due to (5), the de Boor algorithm is a *corner-cutting algorithm*. Because the B-spline basis emerges from a corner-cutting algorithm, it is automatically *totally positive*, *i.e.*, for given any $x_1 < \ldots < x_q$ in $\cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-]$, the matrix with entries $N_\ell(x_j)$, $\ell \in \Lambda_n(\mathbb{K})$, $1 \le j \le q$, is totally positive (see the introduction). However our purpose in the present article is not to insist on this property. For further acquaintance with the subject, we thus refer the reader to [17] and also to [9,10,30].

Let us now assume that $\mathbb{S}(\mathbb{E}^*, \mathbb{K}^*)$ is obtained from $\mathbb{S}(\mathbb{E}, \mathbb{K})$ by insertion of knots. Then, the corresponding *knot insertion algorithm* computes the new poles of a spline $S \in \mathbb{S}(\mathbb{E}, \mathbb{K})^d$ in terms of its initial poles. To understand how this works it is sufficient to consider the case where we insert only one additional knot. For any $x \in I$, let $m(x)$ denote the multiplicity of x in $\mathbb{K}$, that is, $m(x) := 0$ if x is not a knot, and $m(t_k) := m_k$ for all $k \in \mathbb{Z}$. If the new knot-vector $\mathbb{K}^*$ is obtained from $\mathbb{K}$ by inserting one knot x such that $m(x) \ge n + 1$, then clearly $\mathbb{S}(\mathbb{E}^*, \mathbb{K}^*) = \mathbb{S}(\mathbb{E}, \mathbb{K})$, the poles remain unchanged, up to shifts in the indices. Accordingly, we can limit ourselves to considering insertion of a knot x such that $m(x) \le n$. In the new knot-vector $\mathbb{K} = (\xi_\ell^*)_{\ell\in\mathbb{Z}}$, the numbering is supposed to be as follows:

$$\xi_\ell^* := \xi_\ell \text{ if } \xi_\ell \le x, \quad \xi_\ell^* := \xi_{\ell-1} \text{ if } \xi_\ell > x.$$

As a result of the de Boor evaluation algorithm we then get the following [22]:

Proposition 2.14 *Assume that the PW-space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for design and that $\mathbb{S}(\mathbb{E}^*, \mathbb{K}^*)$ is obtained from $\mathbb{S}(\mathbb{E}, \mathbb{K})$ by insertion of one knot x such that $m(x) \le n$. Let P_ℓ, $\ell \in \Lambda_n(\mathbb{K})$ be the poles of a given $S \in \mathbb{S}(\mathbb{E}, \mathbb{K})^d$. Then, the poles of S considered as an element of $\mathbb{S}(\mathbb{E}^*, \mathbb{K}^*)^d$ can be computed as follows:*

$$P_\ell^* = (1 - \alpha_\ell)P_{\ell-1} + \alpha_\ell P_\ell, \quad \ell \in \Lambda_n(\mathbb{K}), \tag{7}$$

with

$$\alpha_\ell := \begin{cases} 1 & \textit{if } \xi_{\ell+n} \le x, \\ 0 & \textit{for } \xi_\ell \ge x, \\ \beta(\xi_{\ell+1}, \ldots, \xi_{\ell+n-1}; \xi_\ell, \xi_{\ell+n}; x) \in]0, 1[& \textit{otherwise}. \end{cases} \tag{8}$$

The dual version of Proposition 2.14 can be stated as follows:

Corollary 2.15 *The assumptions are the same as in Proposition 2.14. Then the B-spline basis N_ℓ, $\ell \in \Lambda_n(\mathbb{K})$, of $\mathbb{S}(\mathbb{E}, \mathbb{K})$ can be decomposed in the B-spline basis N_ℓ^*, $\ell \in \Lambda_n(\mathbb{K}^*) = \Lambda_n(\mathbb{K})$, of $\mathbb{S}(\mathbb{E}^*, \mathbb{K}^*)$ as follows:*

$$N_\ell = \alpha_\ell N_\ell^* + (1 - \alpha_{\ell+1})N_{\ell+1}^*, \quad \ell \in \Lambda_n(\mathbb{K}), \tag{9}$$

the coefficients α_ℓ being the same as in (8).

3 A simple process to construct good spline spaces

Weight functions and associated linear differential operators are classical tools for EC-spaces (see [11,33]). In this section we show how to build good spline spaces by analogy with the non-piecewise case: they will be based on linear piecewise differential operators associated with piecewise weight functions.

3.1 Extended Chebyshev piecewise spaces

Given a bi-infinite sequence $\mathcal{R} := (R_k)_{k\in\mathbb{Z}}$ of connection matrices as in Definition 2.1, we denote by $PC^n(I, \mathbb{T}, \mathcal{R})$ the set of all piecewise functions $F \in PC^n(I, \mathbb{T})$ which satisfy the connection conditions (1). The regular lower triangular structure of each matrix R_k enables us to count zeros in $PC^n(I, \mathbb{T}, \mathcal{R})$. Indeed, for any $k \in \mathbb{Z}$, t_k^+ is a zero of multiplicity $p \le n + 1$ (resp., of exact multiplicity $p \le n$) of some $F \in PC^n(I, \mathbb{T}, \mathcal{R})$, if and only if so is t_k^-. Hence, for any $F \in PC^n(I, \mathbb{T}, \mathcal{R})$, we can introduce the total number $Z_{n+1}(F)$ of all zeros of F in I, including their multiplicities up to $(n + 1)$.

Definition 3.1 An $(n + 1)$-dimensional linear subspace $\mathbb{E}$ of $PC^n(I, \mathbb{T}, \mathcal{R})$ is said to be *an Extended Chebyshev Piecewise space* (in short, ECP-space) *on* $(I, \mathbb{T})$ if any non-zero element $F \in \mathbb{E}$ satisfies $Z_{n+1}(F) \le n$.

We already considered such ECP-spaces in [20] where we proved that they behaved exactly as EC-spaces, as recalled in Theorem 3.2 below. Before stating it, observe that a PW-space $\mathbb{E} \subset PC^n(I, \mathbb{T}, \mathcal{R})$ coincides with the spline space $\mathbb{S}(\mathbb{E}, \mathbb{K}_0)$ based on $\mathbb{E}$ corresponding to the knot-vector $\mathbb{K}_0 := \big({t_k}^{[0]}\big)_{k\in\mathbb{Z}}$ obtained when all multiplicities are 0. When $\mathbb{E}$ contains constants, the question of existence of blossoms, defined on $\mathbb{A}_n(\mathbb{K}_0) = I^n$, thus arises (see [20]), and Definition 2.9 applies to PW-spaces as well.

Theorem 3.2 *Let $\mathbb{E}$ be an $(n + 1)$-dimensional PW-space on $(I, \mathbb{T})$ containing constants. Then the following five properties are equivalent*:

(i) *any spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ based on $\mathbb{E}$ is good for design;*
(ii) *$\mathbb{E}$ itself is good for design;*
(iii) *the PW-space $D\mathbb{E}$ obtained by (possibly left/right) differentiation is an ECP-space on $(I, \mathbb{T})$;*
(iv) *given any $(a, b) \in I^2$, $a < b$, $D\mathbb{E}$ possesses a Bernstein-like basis relative to (a, b), that is, a basis $(V_0, \dots, V_{n-1})$ such that, for $0 \le i \le n - 1$, V_i is positive on $]a, b[$ and it vanishes exactly i times at a and exactly $(n - 1 - i)$ times at b.*
(v) *given any $(a, b) \in I^2$, $a < b$, $\mathbb{E}$ possesses a Bernstein basis relative to (a, b), that is, a Bernstein-like basis $(B_0, \dots, B_n)$ which is normalised, in the sense that $\sum_{i=0}^{n} B_i = \mathbb{1}$;*

The main reason why we are interested in ECP-spaces is the equivalence (i) ⇔ (iii) of Theorem 3.2. We mentioned the other ones because they will be useful later on.

3.2 A procedure to build ECP-spaces

The equivalence between the two properties (iii) and (i) of Theorem 3.2 is a good motivation to try and build as many ECP-spaces as possible. In the present subsection we shall see that it is possible to adapt to the piecewise framework a classical procedure to build EC-spaces by means of linear differential operators associated with weight functions. Let us briefly recall the procedure in question here. Given a non-trivial real interval J, let $w_0, \dots, w_n$ be any positive functions on J such that $w_i \in C^{n-i}(J)$ for $0 \le i \le n$. We then say that $(w_0, \dots, w_n)$ is a system of weight functions on J. For any $F \in C^n(J)$, set:

$$L_0 F := \frac{F}{U_0}, \quad L_i F := \frac{1}{w_i} D L_{i-1} F, \quad 1 \le i \le n. \tag{10}$$

Then, each L_i is a linear differential operator of order i on $C^n(J)$ and the set of all $F \in C^n(J)$ for which $L_n F$ is constant on J is an $(n+1)$-dimensional EC-space on J. We say that it is the EC-space associated with $(w_0, \dots, w_n)$ and we denote it by $EC(w_0, \dots, w_n)$.

Let us introduce the following terminology by analogy with the non-piecewise case.

Definition 3.3 A given sequence $(w_0, \dots, w_n)$ of piecewise functions on $(I, \mathbb{T})$ will be said to be a *system of piecewise weight functions on* $(I, \mathbb{T})$ if it meets the two requirements below:

(1) for $i = 0, \dots, n$, $w_i \in PC^{n-i}(I, \mathbb{T})$;
(2) each w_i is positive on $\cup_{k \in \mathbb{Z}} [t_k^+, t_{k+1}^-]$.

We shall now see that systems of piecewise weight functions produce ECP-spaces on $(I, \mathbb{T})$ just as systems of weight functions produce EC-spaces. This will be due to some stability properties of the class of all ECP-spaces on $(I, \mathbb{T})$ stated in the proposition below.

Proposition 3.4 *The class of all ECP-spaces on* $(I, \mathbb{T})$ *is stable under multiplication by positive piecewise functions and integration, in the following sense. Assuming that* $\mathbb{E} \subset PC^n(\mathbb{T}, \mathcal{R})$ *is an* $(n+1)$*-dimensional ECP-space on* $(I, \mathbb{T})$*, then*:

– *for any positive piecewise function* $\omega \in PC^n(I, \mathbb{T})$*, the set* $\overline{\mathbb{E}} : \{\omega F \mid F \in \mathbb{E}\}$ *is an* $(n+1)$*-dimensional ECP-space on* $(I, \mathbb{T})$*, contained in* $PC^n(\mathbb{T}, \overline{\mathcal{R}})$*, each matrix* $\overline{R}_k$ *of the sequence* $\overline{\mathcal{R}}$ *being defined by*

$$\overline{R}_k := \mathcal{C}_n(\omega, t_k^+).\, R_k.\, \mathcal{C}_n(\omega, t_k^-)^{-1}, \quad k \in \mathbb{Z},$$

where for any $x \in \cup_{k \in \mathbb{Z}} [t_k^+, t_{k+1}^-]$, $\mathcal{C}_n(w, x) = \big(\mathcal{C}_n(w, x)_{p,q}\big)_{0 \le p,q \le n}$ *stands for the lower triangular square matrix of order* $(n+1)$ *defined by*

$$\mathcal{C}_n(w, x)_{p,q} := \binom{p}{q} w^{(p-q)}(x), \quad 0 \le q \le p \le n;$$

– *for any sequence of positive numbers a_k, $k \in \mathbb{Z}$, the set $\widehat{\mathbb{E}}$ of all piecewise functions $\widehat{F} \in PC^{n+1}(I, \mathbb{T})$ such that*

$$D\widehat{F} \in \mathbb{E} \quad \text{and } \widehat{F}(t_k^+) = a_k \widehat{F}(t_k^-) \quad \text{for each } k \in \mathbb{Z},$$

is an $(n+2)$-dimensional ECP-space on $(I, \mathbb{T})$ contained in $PC^{n+1}(\mathbb{T}, \widehat{\mathcal{R}})$, where each matrix $\widehat{R}_k$ of the sequence $\widehat{\mathcal{R}}$ is the block diagonal matrix (a_k, R_k).

Proof The fact that $\overline{\mathbb{E}} \subset PC^n(\mathbb{T}, \overline{\mathcal{R}})$ comes from Leibnitz's rule. It should be observed that each $\overline{R}_k$ has positive diagonal elements like R_k due to the positivity of the piecewise function ω. On the other hand, we clearly have $Z_{n+1}(\omega F) = Z_{n+1}(F)$ for any $F \in \mathbb{E}$. This proves the first part of the proposition. The second one follows from the piecewise version of Rolle's theorem stated below. We leave its proof to the reader. □

Lemma 3.5 *Given any positive numbers a_k, b_k, $k \in \mathbb{Z}$, assume that a given piecewise function $F \in PC^1(I, \mathbb{T})$ satisfies:*

$$F(t_k^+) = a_k F(t_k^-), \quad F'(t_k^+) = b_k F'(t_k^-), \quad k \in \mathbb{Z}.$$

Then, for any $a, b \in I, a < b$, such that $F(a) = F(b) = 0$, there exists $\xi \in]a, b[$ such that $F'(\xi) = 0$.

Remark 3.6 As a straightforward consequence of Proposition 3.4 and of (ii) ⇔ (iii) in Theorem 3.2, we can state that any PW-space on $(I, \mathbb{T})$ supposed to be good for design is an ECP-space on $(I, \mathbb{T})$.

With any system of piecewise weight functions on I, it is clearly possible to associate *linear piecewise differential operators* on the set $PC^n(I, \mathbb{T})$ defined exactly by the same formulæ (10) as in the non-piecewise case.

Theorem 3.7 *Let $L_0, \ldots, L_n$ be the piecewise differential operators on $PC^n(I, \mathbb{T})$ associated with a given system $(w_0, \ldots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$ via (10), and for $i = 0, \ldots, n$, let a_k^i, $k \in \mathbb{Z}$, be a bi-infinite sequence of positive numbers. Then, the set $\mathbb{E}$ of all piecewise functions $F \in PC^n(I, \mathbb{T})$ such that*:

(1) *$L_n F$ is piecewise constant on $(I, \mathbb{T})$;*
(2) *F satisfies the connection conditions*

$$L_i F(t_k^+) = a_k^i L_i F(t_k^-), \quad k \in \mathbb{Z}, \ 0 \le i \le n,$$

is an $(n+1)$-dimensional ECP-space on $(I, \mathbb{T})$.

Proof Let $\mathcal{R}^{\{n\}}$ denote the bi-infinite sequence a_k^n, $k \in \mathbb{Z}$. The set $\mathbb{E}^{\{n\}}$ composed of all $F^{\{n\}} \in PC^0(I, \mathbb{T}, \mathcal{R}^{\{n\}})$ which are piecewise constants on $(I, \mathbb{T})$ is clearly a one-dimensional ECP-space on $(I, \mathbb{T})$. By application of the first part of Proposition 3.4, we can thus deduce that the set $w_n \mathbb{E}^{\{n\}} := \{w_n F^{\{n\}} \mid F^{\{n\}} \in \mathbb{E}^{\{n\}}\}$ is a one-dimensional ECP-space contained in $PC^0(I, \mathbb{T}, \overline{\mathcal{R}}^{\{n\}})$, where each $\overline{R}_k^{\{n\}}$ is obtained by multiplying

a_k^n by the positive number $w_n(t_k^+)/w_n(t_k^-)$. The second part of Proposition 3.4 ensures that the set $\mathbb{E}^{\{n-1\}}$ of all $F^{\{n-1\}} \in PC^1(I, \mathbb{T})$ such that

$$DF^{\{n-1\}} \in w_n \mathbb{E}^{\{n\}} \text{ and } F^{\{n-1\}}(t_k^+) = a_k^{n-1} F^{\{n-1\}}(t_k^-) \text{ for all } k \in \mathbb{Z}, \tag{11}$$

is a two-dimensional ECP-space on $(I, \mathbb{T})$, contained in $PC^1(I, \mathbb{T}, \mathcal{R}^{\{n-1\}})$, where each $R_k^{\{n-1\}}$ is the diagonal matrix $(a_k^{n-1}, \overline{R}_k^{\{n\}})$. Continuing the same way by repeated application of the two parts of Proposition 3.4 we build a sequence $\mathbb{E}^{\{j\}}$, $j = n, n-1, \dots, 1, 0$, where each $\mathbb{E}^{\{j\}}$ is an $(n-j+1)$-dimensional ECP-space on $(I, \mathbb{T})$ contained in $PC^{n-j}(I, \mathbb{T}, \mathcal{R}^{\{j\}})$, where the matrices $R_k^{\{j\}}$ are block diagonal lower triangular matrices of order $(n-j+1)$ defined by

$$R_k^{\{j\}} := (a_k^j, \mathcal{C}_{n-j-1}(\omega, t_k^+).\, R_k^{\{j+1\}}.\mathcal{C}_{n-j-1}(\omega, t_k^-)^{-1}, \quad k \in \mathbb{Z}.$$

We clearly have $\mathbb{E} = \mathbb{E}^{\{0\}}$. □

Remark 3.8 Let us recall that a matrix is said to be *totally positive* if all its minors are non-negative. Under the same assumptions as in Theorem 3.7, all elements F in $\mathbb{E}$ satisfy the connection conditions

$$\big(L_0F(t_k^+), L_1F(t_k^+), \dots, L_nF(t_k^+)\big) = N_k.\big(L_0F(t_k^-), L_1F(t_k^-) \dots, L_nF(t_k^-)\big), \quad k \in \mathbb{Z},$$

where, for each k, N_k is the diagonal matrix $(a_k^0, \dots, a_k^n)$. Since the real numbers $a_k^0, \dots, a_k^n$ are positive, obviously each matrix N_k is totally positive. Accordingly, the fact that $\mathbb{E}$ is an ECP-space on $(I, \mathbb{T})$ also follows from arguments similar to those used by Barry [1], Corollary 1, based on Budan–Fourier's theorem for EC-spaces (see proof of Theorem 6.10 in [14]). However it was interesting to obtain Theorem 3.7 via different techniques based in particular on a generalised version Rolle's theorem, thus copying the classical procedure of the non-piecewise case. Moreover, the recursive computation of the connection matrices will be useful later on.

Choose an integer $k \in \mathbb{Z}$, consider the system $(\widetilde{w}_0, \dots, \widetilde{w}_n)$ of piecewise weight functions on $(I, \mathbb{T})$ defined by

$$\widetilde{w}_i(x^\varepsilon) := w_i(x^\varepsilon) \text{ for } x^\varepsilon \le t_k^-, \quad \widetilde{w}_i(x^\varepsilon) := \frac{1}{a_k^i} w_i(x^\varepsilon) \text{ for } x^\varepsilon \ge t_k^+, \quad 0 \le i \le n.$$

Then, the ECP-space $\widetilde{\mathbb{E}}$ built according to Theorem 3.7 is equal to $\mathbb{E}$ provided that we simultaneously replace all real numbers $a_k^0, a_k^1, \dots, a_k^n$ by 1. Accordingly, in respect of building ECP-spaces, without loss of generality we can assume that $a_k^i = 1$ for $0 \le i \le n$ and for all $k \in \mathbb{Z}$, as we are doing in the following definition.

Definition 3.9 Let $L_0, \dots, L_n$ be the piecewise differential operators associated with a given system $(w_0, \dots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$. Then, the set of all piecewise functions $F \in PC^n(I, \mathbb{T})$ such that $L_n F$ is piecewise constant on $(I, \mathbb{T})$ and for which

$$\big(L_0 F(t_k^+), L_1 F(t_k^+), \dots, L_n F(t_k^+)\big) = \big(L_0 F(t_k^-), L_1 F(t_k^-) \dots, L_n F(t_k^-)\big), \quad k \in \mathbb{Z},$$

will be called *the ECP-space associated with* $(w_0, \dots, w_n)$. We denote it by $ECP(w_0, \dots, w_n)$.

Note that one can as well define it as the set of all $F \in PC^n(I, \mathbb{T})$ such that $L_n F$ is constant on the whole of I and for which $L_i F(t_k^+) = L_i F(t_k^-)$ for $i = 0, \dots, n-1$.

Remark 3.10 As in the non-piecewise case, a given system $(w_0, \dots, w_n)$ piecewise weight functions on $(I, \mathbb{T})$ provides us with a nested sequence of ECP-spaces:

$$\begin{aligned} ECP(w_0) \subset ECP(w_0, w_1) \subset \cdots \subset ECP(w_0, \dots, w_{n-1}) \\ \subset ECP(w_0, \dots, w_{n-1}, w_n) \subset PC^n(I, \mathbb{T}), \end{aligned} \tag{12}$$

along with a non-nested one

$$\mathbb{E}^{\{i\}} := ECP(\mathbb{1}, w_{i+1}, \dots, w_n) = L_i\,(ECP(w_0, \dots, w_n)) \subset PC^{n-i}(I, \mathbb{T}), \quad 0 \le i \le n,$$

which is actually the space $\mathbb{E}^{\{i\}}$ built in the proof of Theorem 3.7. The latter sequence also satisfies

$$D\mathbb{E}^{\{i\}} = ECP(w_{i+1}, \dots, w_n), \quad 0 \le i \le n-1.$$

3.3 Splines based on ECP-spaces

Definition 3.11 Given any system $(w_0, w_1, \dots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$ and any knot-vector $K := \big(t_k{}^{[m_k]}\big)_{k \in \mathbb{Z}}$, we denote by $ECPS(w_0, \dots, w_n; \mathbb{K})$ the spline space based on the ECP-space $ECP(w_0, \dots, w_n)$ and on the knot-vector $\mathbb{K}$. If $L_0, \dots, L_n$ are the associated piecewise differential operators, it can be defined as the set of all piecewise functions $S \in PC^n(I, \mathbb{T})$ such that

(1) $L_n S$ is piecewise constant on $(I, \mathbb{T})$;
(2) S satisfies the connection conditions

$$\big(L_0 S(t_k^+), L_1(t_k^+), \dots, L_{n-m_k}(t_k^+)\big)^T = \big(L_0 S(t_k^-), L_1(t_k^-), \dots, L_{n-m_k}(t_k^-)\big)^T, \quad k \in \mathbb{Z}, \tag{13}$$

Observe that, for any $S \in ECPS(w_0, \dots, w_n; \mathbb{K})$, the piecewise function $L_n S$ is constant on any subinterval of I the interior of which contains no knot t_k of multiplicity $m_k \ge n+1$, that is, on each $I_{n,q}, q \in \mathcal{Q}_n(\mathbb{K})$ (with the notations introduced in Subsect. 2.2).

As an application of Theorem 3.2 one can then state:

Theorem 3.12 *Given any system $(w_1, \ldots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$, and any knot-vector $\mathbb{K}$, the spline space $ECPS(\mathbb{1}, w_1, \ldots, w_n; \mathbb{K})$ is good for design. Moreover, if $L_0 = Id, L_1, \ldots, L_n$ are the piecewise differential operators associated with $(\mathbb{1}, w_1, \ldots, w_n)$, then, each PW-spline space $L_p\,(ECPS(\mathbb{1}, w_1, \ldots, w_n; \mathbb{K}))$, $1 \le p \le n$, is good for design too.*

Proof We know that $\mathbb{E} := ECP(\mathbb{1}, w_1, \ldots, w_n)$ is an ECP-space on $(I, \mathbb{T})$ and that it contains constants. According to Theorem 3.2, in order to prove that the spline space $ECPS(\mathbb{1}, w_1, \ldots, w_n; \mathbb{K})$ is good for design, it thus suffices to check that $D\mathbb{E}$ is an ECP-space on $(I, \mathbb{T})$. This is due to the equality $D\mathbb{E} = ECP(w_1, \ldots, w_n)$ (see Remark 3.10). Finally, from Definition 3.11 and (10) it is easy to check that

$$L_p\,(ECPS(\mathbb{1}, w_1, \ldots, w_n; \mathbb{K})) = ECPS(\mathbb{1}, w_{p+1}, \ldots, w_n; \mathbb{K}), \quad 1 \le p \le n.$$

□

Equivalently, Theorem 3.12 can be stated as follows.

Corollary 3.13 *For any $k \in \mathbb{Z}$, let $(w_1^k, \ldots, w_n^k)$ be a system of weight functions on $[t_k, t_{k+1}]$ and let $L_0^k, L_1^k, \ldots, L_n^k$ be the differential operators on $C^n([t_k, t_{k+1}])$ associated with $(\mathbb{1}_k, w_1^k, \ldots, w_n^k)$. Given any knot-vector $\mathbb{K} = \left(t_k{}^{[m_k]}\right)_{k\in\mathbb{Z}}$, the spline space composed of all piecewise functions $S \in PC^n(I, \mathbb{T})$ meeting the following requirements:*

(1) *for any $k \in \mathbb{Z}$, the restriction of S to $[t_k^+, t_{k+1}^-]$ belongs to $\mathbb{E}_k := EC(\mathbb{1}_k, w_1^k, \ldots, w_n^k)$,*
(2) *for any $k \in \mathbb{Z}$, S satisfies the connection condition:*

$$\left(S(t_k^+), L_1^k S(t_k^+), \ldots, L_{n-m_k}^k S(t_k^+)\right)^T = \left(S(t_k^-), L_1^{k-1} S(t_k^-), \ldots, L_{n-m_k}^{k-1} S(t_k^-)\right)^T, \tag{14}$$

is good for design.

Proof It suffices to apply Theorem 3.12 to the system $(w_1, \ldots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$ defined by

$$w_i(x) := w_i^k(x) \text{ for all } x \in [t_k^+, t_{k+1}^-], \quad k \in \mathbb{Z}.$$

The connections in $ECPS(\mathbb{1}, w_1, \ldots, w_n; \mathbb{K})$ being given by (13), they can as well be written as (14). □

Remark 3.14 Given a system $(w_0, w_1, \ldots, w_n)$ of piecewise weight fuctions on $(I, \mathbb{T})$, and the associated piecewise differential operators (10), it is easy to deduce that, for any piecewise function $F \in PC^n(I, \mathbb{T})$, we have

$$F^{(p)} = \sum_{q=0}^{p} \Gamma_{p,q}(w_0, \ldots, w_q; .) L_q F, \quad 0 \le p \le n, \tag{15}$$

where each piecewise function $\Gamma_{p,q}(w_0, \dots, w_q; .)$ depends only on $w_0, \dots, w_q$, not on F. The latter piecewise functions can be calculated by induction as follows:

$$\Gamma_{0,0}(w_0; .) := w_0 \text{ and, for } p = 0, \dots, n-1 \text{ and } 0 \le q \le p :$$
$$\Gamma_{p+1,q}(w_0, \dots, w_q; .) := D\left(\Gamma_{p,q}(w_0, \dots, w_q; .)\right) + w_q \Gamma_{p,q-1}(w_0, \dots, w_{q-1}; .), \tag{16}$$

with the convention that $\Gamma_{p,-1} = 0$. For instance we thus have

$$\Gamma_{p,0}(w_0; .) = w_0{}^{(p)}, \quad \Gamma_{p,p}(w_0, \dots, w_p; .) = \prod_{i=0}^{p} w_i, \quad 0 \le p \le n.$$

Setting $\Gamma_{p,q}(w_0, \dots, w_q; .) = 0$ for $p < q \le n$, let us introduce the following lower triangular matrices of order $(n+1)$:

$$\Gamma_n(w_0, \dots, w_n; x) := \left(\Gamma_{p,q}(w_0, \dots, w_q); x\right)_{0 \le p,q \le n}, \quad x \in \cup_{k \in \mathbb{Z}} [t_k^+, t_{k+1}^-].$$

As a consequence of (15), for any knot-vector $\mathbb{K}$, in the spline space $ECPS(w_0, w_1, \dots, w_n; \mathbb{K})$, in terms of the ordinary derivatives the connections are given by (2), with the matrices

$$M_k := \Gamma_{n-m_k}(w_0, \dots, w_{n-m_k}; t_k^+).\, \Gamma_{n-m_k}(w_0, \dots, w_{n-m_k}; t_k^-)^{-1}, \quad k \in \mathbb{Z}. \tag{17}$$

4 All spline spaces good for design

Given any system $(w_1, \dots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$ and any knot-vector $\mathbb{K} = \left(t_k{}^{[m_k]}\right)_{k \in \mathbb{Z}}$, we know that the spline space $ECPS(\mathbb{1}, w_1, \dots, w_n; \mathbb{K})$ is good for design (see Theorem 3.12). The question we address here is the converse one: given any PW-spline space which is good for design, based on $\mathbb{K}$, is it of the form $ECPS(\mathbb{1}, w_1, \dots, w_n; \mathbb{K})$? We shall actually be able to give a positive answer to the latter question provided that the knot-vector $\mathbb{K}$ is bi-infinite. As a consequence, in the latter case we shall also be able to give a simple description of all PW-spline spaces which are good for design. Of course our results can easily be adapted to PW-spline spaces over a closed bounded interval (see Sect. 6).

Throughout the present section we thus assume that $\mathbb{K} = \left(t_k{}^{[m_k]}\right)_{k \in \mathbb{Z}}$ *is a given bi-infinite knot-vector.*

4.1 The main results

On account of Proposition 2.11, looking for all PW-spline spaces which are good for design amounts to looking for all PEC-spline spaces which are good for design. The most important result of the paper is the following one, in which we use the notations introduced in Subsect. 2.2.

Theorem 4.1 *Assume that $n \geq 1$. Given any PEC-space $\mathbb{E}$ on $(I, \mathbb{T})$, with $(n+1)$-dimensional section-spaces, and any bi-infinite knot-vector $\mathbb{K}$ based on $\mathbb{T}$, assume that the spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for design. Let a_ℓ, $\ell \in \Lambda_n(\mathbb{K})$, denote the poles of a given element $U \in \mathbb{S}(\mathbb{E}, \mathbb{K})$. The following properties are then equivalent*:

(i) *for any $q \in \mathcal{Q}_n(\mathbb{K})$, the poles a_ℓ, $\ell \in \Lambda_n(\mathbb{K}_{n,q})$, form a strictly increasing sequence;*

(ii) *the piecewise function $w_1 := DU$ is strictly positive on $\cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-]$ and the set $L_1\mathbb{S}(\mathbb{E}, \mathbb{K}) := \{L_1 S := \dfrac{DS}{w_1} \mid S \in \mathbb{S}\}$ is a PW-spline space (with n-dimensional section-spaces) which is good for design.*

Theorem 4.1 will be proved in the next subsection. The reason why it is essential is that it enables to determine all good spline spaces, as stated below:

Theorem 4.2 *Let $\mathbb{E}$ be an $(n+1)$-dimensional PEC-space on $(I, \mathbb{T})$ and let $\mathbb{K} = \left(t_k{}^{[m_k]}\right)_{k\in\mathbb{Z}}$ be any bi-infinite knot-vector. The following three properties are then equivalent:*

(i) *the space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for design;*

(ii) *there exists a system $(w_1, \ldots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$, such that*

$$\mathbb{S}(\mathbb{E}, \mathbb{K}) = ECPS(\mathbb{1}, w_1, \ldots, w_n; \mathbb{K}); \tag{18}$$

(iii) *for each $k \in \mathbb{Z}$, there exists a system $(w_1^k, \ldots, w_n^k)$ of weight functions on $[t_k^+, t_{k+1}^-]$ such that $\mathbb{S}(\mathbb{E}, \mathbb{K})$ can be described as the set of all piecewise functions $S \in PC^n(I, \mathbb{T})$ such that*
   - *for any $k \in \mathbb{Z}$, the restriction of S to $[t_k^+, t_{k+1}^-]$ belongs to $EC(\mathbb{1}_k, w_1^k, \ldots, w_n^k)$;*
   - *for any $k \in \mathbb{Z}$, S satisfies the connection condition*

$$\left(S(t_k^+), L_1^k S(t_k^+) \ldots, L_{n-m_k}^k S(t_k^+)\right)^T = \left(S(t_k^-), L_1^{k-1} S(t_k^-) \ldots, L_{n-m_k}^{k-1} S(t_k^-)\right)^T, \tag{19}$$

   where, for each k, $L_0^k = Id$, $L_1^k, \ldots, L_n^k$ are the differential operators on $C^n([t_k^+, t_{k+1}^-])$ associated with $(\mathbb{1}_k, w_1^k, \ldots, w_n^k)$.

Proof The equivalence between (ii) and (iii) is clear. The implication (ii) $\Rightarrow$ (i) was proved in Theorem 3.12. Assuming that (i) holds, let us prove (ii). It is actually sufficient to prove the existence of a system $(w_1, \ldots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$ such that

$$ECP(\mathbb{1}, w_1, \ldots, w_n) \subset \mathbb{S}(\mathbb{E}, \mathbb{K}). \tag{20}$$

This will be done by induction on $n \geq 0$. For $n = 0$, there is nothing to prove. Suppose that $n \geq 1$, and that the result holds for $(n-1)$. Theorem 4.1 ensures the existence of a positive piecewise function w_1 such that the corresponding PW-space $L_1\mathbb{S}(\mathbb{E}, \mathbb{K})$, with section-spaces of dimension n, is good for design. From the recursive assumption we can attest the existence of a system $(w_2, \ldots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$ such that $ECP(\mathbb{1}, w_2, \ldots, w_n) \subset L_1\mathbb{S}(\mathbb{E}, \mathbb{K})$. For any $k \in \mathbb{Z}$ such that $m_k \leq n$, each spline $S \in \mathbb{S}(\mathbb{E}, \mathbb{K})$ is continuous at t_k. The inclusion (20) readily follows. □

Remark 4.3 The bi-infinite knot-vector $\mathbb{K}$ being given, let us now consider the class $\mathcal{C}$ of all spline spaces on $(I, \mathbb{T})$ described by all requirements in (iii) of Theorem 4.2, obtained when involving all possible bi-infinite sequences of systems of weight functions $(w_1^k, \ldots, w_n^k)$, $k \in \mathbb{Z}$. The latter theorem tells us that it is the class of all PEC-spline spaces on $(I, \mathbb{T})$ based on $\mathbb{K}$ which are good for design. Let us now compare it with the class $\widehat{\mathcal{C}}$ of all spaces obtained in a similar way, but after replacing the connection conditions (19) by

$$\left(S(t_k^+), L_1^k S(t_k^+) \ldots, L_{n-m_k}^k S(t_k^+)\right)^T = N_k. \left(S(t_k^-), L_1^{k-1} S(t_k^-) \ldots, L_{n-m_k}^{k-1} S(t_k^-)\right)^T, \tag{21}$$

where, for each k, N_k denotes any totally positive regular lower triangular matrix. Note that the class $\widehat{\mathcal{C}}$ is somewhat larger than the class of spline spaces which was considered by Barry [1]. Indeed, in [1], each w_i^k, $1 \leq i \leq n$, $k \in \mathbb{Z}$, was supposed to belong to $C^{\max(n-i,i-1)}([t_k^+, t_{k+1}^-])$, each multiplicity to be at least 1, and the last colum of each N_k to be $(0, \ldots, 0, 1)^T$, due in particular to the proofs strongly involving dual EC-spaces, and dual connection matrices.

A priori, the class $\widehat{\mathcal{C}}$ seems much larger than $\mathcal{C}$, since we have a choice for each N_k. Nevertheless, let us choose a spline space $\mathbb{S}$ in the class $\widehat{\mathcal{C}}$. For each k, let us add $\min(m_k, n+1)$ rows and columns to the corresponding matrix N_k so as to obtain a totally positive regular lower triangular matrix $\widetilde{N}_k$, of order $(n+1)$, with $(1, 0, \ldots, 0)^T$ as its first column. Each such choice leads to an $(n+1)$-dimensional PEC-space $\mathbb{E}$ on $(I, \mathbb{T})$ which is contained in $\mathbb{S}$. A slight adaptation of Corollary 1 of [1] (see [14]) ensures that this is actually an ECP-space on $(I, \mathbb{T})$. Accordingly, Theorem 3.2 guarantees that the space $\mathbb{S}$ is good for design, *i.e.*, it belongs to the class $\mathcal{C}$. The two classes $\mathcal{C}$ and $\widehat{\mathcal{C}}$ are thus identical.

4.2 Generalised differentiation within existence of blossoms: proof of Theorem 4.1

Throughout the present subsection we work under the same global assumption as in Theorem 4.1. The PEC-spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ being good for design, it possesses a B-spline basis, which we denote by N_ℓ, $\ell \in \Lambda_n(\mathbb{K})$. Let us start with the following two lemmas.

Lemma 4.4 *Consider the following piecewise functions in $PC^{n-1}(I, \mathbb{T})$:*

$$B_\ell(x) := \sum_{i \geq \ell,\ i \in \Lambda_n(\mathbb{K})} DN_i(x), \quad x \in \cup_{k \in \mathbb{Z}}[t_k^+, t_{k+1}^-], \quad \ell \in \mathbb{Z}. \tag{22}$$

If $\ell \notin \Lambda_{n-1}(\mathbb{K})$*, then* $B_\ell \equiv 0$*. The sequence* B_ℓ*,* $\ell \in \Lambda_{n-1}(\mathbb{K})$*, is a B-spline-like basis in the space* $D\mathbb{S}(\mathbb{E}, \mathbb{K})$*.*

Proof The B-spline being normalised, we can as well define each piecewise function B_ℓ as

$$B_\ell(x) = - \sum_{i \le \ell-1,\ i \in \Lambda_n(\mathbb{K})} DN_i(x), \quad x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-], \quad \ell \in \Lambda_{n-1}(\mathbb{K}). \tag{23}$$

Using the two expressions (22) and (23) along with the support property $(\mathrm{BSLB})_1$ of the B-spline basis N_ℓ, $\ell \in \Lambda_n(\mathbb{K})$, it is easy to derive that, for any $\ell \in \mathbb{Z}$,

$$B_\ell(x) = 0 \quad \text{for any } x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-] \text{ such that either } x \le \xi_\ell^- \text{ or } x \ge \xi_{\ell+n}^+. \tag{24}$$

This guarantees that, whenever $\ell \notin \Lambda_{n-1}(\mathbb{K})$, the piecewise $B_\ell \in PC^{n-1}(I, \mathbb{T})$ is identically 0. Let us thus only consider integers $\ell \in \Lambda_{n-1}(\mathbb{K})$. From (22) and (23) and the endpoint property$(\mathrm{BSLB})_1$ of the B-spline basis one can derive that, for any $\ell \in \Lambda_{n-1}(\mathbb{K})$,

$$B_\ell \text{ vanishes exactly } (n-s) \text{ times at } \xi_\ell^+ \text{ and exactly } (n-s') \text{ times at } \xi_{\ell+n}^-, \tag{25}$$

where $s := \sharp\{j \ge \ell \mid \xi_j = \xi_\ell\}$ and $s' := \sharp\{j \le \ell+n+1 \mid \xi_j = \xi_{\ell+n+1}\}$. Taking account of the positivity property $(\mathrm{BSB})_2$ of the B-spline basis we can additionally state that

$$\begin{aligned} B_\ell{}^{(n-s)}(\xi_\ell^+) &= N_\ell{}^{(n-s+1)}(\xi_\ell^+) > 0, \\ (-1)^{n-s'} B_\ell{}^{(n-s')}(\xi_{\ell+n}^-) &= (-1)^{n-s'+1} N_{\ell-1}{}^{(n-s+1)}(\xi_{\ell+n}^-) > 0. \end{aligned} \tag{26}$$

Let us now prove that

$$B_\ell(x) > 0 \quad \text{for any } x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-] \text{ such that } \xi_\ell^+ < x < \xi_{\ell+n}^-, \quad \ell \in \Lambda_{n-1}(\mathbb{K}). \tag{27}$$

Without loss of generality we can assume that $m_k \le n+1$ for all $k \in \mathbb{Z}$, in which case $\Lambda_n(\mathbb{K}) = \mathbb{Z}$. Consider a knot-vector $\mathbb{K}^*$ obtained by inserting one knot x in $\mathbb{K}$, such that $m(x) \le n$. We also have $\Lambda_n(\mathbb{K}^*) = \mathbb{Z}$. From the B-spline basis N_ℓ^*, $\ell \in \mathbb{Z}$, of the new spline space $\mathbb{S}(\mathbb{E}, \mathbb{K}^*)$, we can similarly build the piecewise functions

$$B_\ell^*(x) := \sum_{i \ge \ell} DN_i^*(x) = - \sum_{i \le \ell-1} DN_i^*(x), \quad x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-], \quad \ell \in \mathbb{Z}.$$

From relations (9) we can derive, for any $x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-]$:

$$\sum_{j \ge \ell} N_\ell(x) = \alpha_\ell N_\ell^*(x) + \sum_{k \ge \ell+1}^{m} N_\ell^*(x) = \alpha_\ell \sum_{k \ge \ell} N_k^*(x) + \alpha_\ell \sum_{k \ge \ell+1} N_k^*(x), \quad \ell \in \mathbb{Z}.$$

By differentiation, this yields

$$B_\ell = \alpha_\ell\, B^*_\ell + (1-\alpha_\ell)B^*_{\ell+1}, \quad \ell \in \mathbb{Z}. \tag{28}$$

If ever the new sequence B^*_ℓ, $\ell \in \mathbb{Z}$, is known to satisfy the positivity property (27), then the decomposition relations (28) ensure that so does the initial sequence B_ℓ, $\ell \in \mathbb{Z}$. It is thus sufficient to check that the positivity property (27) is satisfied by the sequence B_ℓ, $\ell \in \mathbb{Z}$, when each knot t_k, $k \in \mathbb{Z}$, has multiplicity $m_k = n+1$. In the latter case, the support of each B-spline is then composed of a single interval. Choose any integer $k \in \mathbb{Z}$. Denote by j_k the unique integer which satisfies

$$\xi_{j_k} \le t_k, \quad \xi_{j_k+1} \ge t_{k+1}. \tag{29}$$

Then, only the piecewise functions $B_{j_k-n+1}, \dots, B_{j_k}$, are not identically 0 on the interval $[t_k^+, t_{k+1}^-]$. Let us denote by $\beta_{k,0}, \dots, \beta_{k,n-1}$ their restrictions to $[t_k^+, t_{k+1}^-]$. On account of $(\mathrm{BSLB})_3$ and (26), each function β_i belongs to $D\mathbb{E}_k$, it vanishes exactly i times at t_k^+ and exactly $(n-1-i)$ times at t_{k+1}^-, and it satisfies $\beta_i{}^{(i)}(t_k^+) > 0$. Since the space $D\mathbb{E}_k$ is an n-dimensional EC-space on $[t_k^+, t_{k+1}]^-$, each function β_i is positive on $[t_k^+, t_{k+1}^-]$. As a matter of fact, $(\beta_{k,0}, \dots, \beta_{k,n-1})$ is a Bernstein-like basis relative to (t_k, t_{k+1}) in the space $D\mathbb{E}_k$. □

Lemma 4.5 *The spline U in $\mathbb{S}(\mathbb{E}, \mathbb{K})$ satisfies (i) of Theorem 4.1 if and only if the decomposition of $w_1 := DU$ in any given B-spline-like basis of the spline space $D\mathbb{S}(\mathbb{E}, \mathbb{K})$ involves only positive coefficients.*

Proof In Lemma 4.4 we have exhibited one special B-spline-like basis in the spline space $D\mathbb{S}(\mathbb{E}, \mathbb{K})$. There exists many other ones, each of their elements being completely determined by its three properties $(\mathrm{BSB})_\mathrm{i}$, $i = 1, 2, 3$, up to multiplication by a positive constant (see, for instance, the proof of Proposition 2.7 in [16]). Accordingly, it is sufficient to check that the claimed result holds when using the special B-spline-like basis defined in Lemma 4.4. Now, from

$$U(x) = \sum_{\ell \in \Lambda_n(\mathbb{K})} a_\ell N_\ell(x), \quad x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-],$$

and from (22), one can easily deduce that w_1 can be decomposed as

$$w_1(x) = \sum_{\ell \in \Lambda_{n-1}(\mathbb{K})} b_\ell B_\ell(x), \quad x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-], \tag{30}$$

with

$$b_\ell = a_\ell - a_{\ell-1} \quad \text{for any } \ell \in \Lambda_{n-1}(\mathbb{K}).$$

One can readily check that (i) is satisfied if and only if $b_\ell > 0$ for all $\ell \in \Lambda_{n-1}(\mathbb{K})$. □

Lemma 4.6 *Suppose that the spline $U \in \mathbb{S}(\mathbb{E}, \mathbb{K})$ satisfies (i) of Theorem 4.1. Then U satisfies the same property when considered as an element of any PW-spline space $\mathbb{S}(\mathbb{E}, \mathbb{K}^*)$ obtained from $\mathbb{S}(\mathbb{E}, \mathbb{K})$ by insertion of knots.*

Proof It suffices to prove this when $\mathbb{K}^*$ is obtained from $\mathbb{K}$ by insertion of one additional knot $x \in I$ of multiplicity $m(x) \leq n$. Let $a_\ell^*, \ell \in \Lambda_n(\mathbb{K}^*)$, denote the poles of U considered as an element of the new spline space $\mathbb{S}(\mathbb{E}, \mathbb{K}^*)$. Let us write $I = \cup_{q \in \mathcal{Q}_n(\mathbb{K}^*)} I_{n,q}^*, q \in \mathcal{Q}_n(\mathbb{K}^*)$. We have to prove that, for any $q \in \mathcal{Q}_n(\mathbb{K}^*)$, the poles $a_\ell^*, \ell \in \Lambda_n(\mathbb{K}_{n,q}^*)$, form a strictly increasing sequence. Without loss of generality we can assume that $\Lambda_n(\mathbb{K}) = \mathbb{Z}$, in which case we shall also have $\Lambda_n(\mathbb{K}^*) = \mathbb{Z}$. Then, x is located in the interior of some $I_{n,q}$. The new poles of U are given by (see (7)):

$$a_\ell^* = (1 - \alpha_\ell) a_{\ell-1} + \alpha_\ell a_\ell, \quad \ell \in \mathbb{Z},$$

where the α's are given in (8). Two cases must be considered:

1. $m(x) < n$: then, x is interior to some $I_{n,q} = I_{n,q}^*$, and we only have to consider the sequence $a_\ell^*, \ell \in \Lambda_n(\mathbb{K}_{n,q}^*)$. Choose any ℓ such that both ℓ and $(\ell + 1)$ belong to $\Lambda_n(\mathbb{K}_{n,q}^*)$. Then, $\ell - 1, \ell, \ell + 1 \in \Lambda_n(\mathbb{K}_{n,q})$. From (5) and (8) one can say that

$$a_\ell^* \leq a_\ell \leq a_{\ell+1}^*.$$

   The equality $a_\ell^* = a_{\ell+1}^*$ could be satisfied if and only if we had both $\alpha_\ell = 1$ and $\alpha_{\ell+1} = 0$, which would occur if and only if we had both $x \geq \xi_{\ell+n}$ and $x \leq \xi_{\ell+1}$, that is, if and only if $x = \xi_{\ell+1} = \cdots = \xi_{\ell+n}$. The assumption $m(x) < n$ makes it impossible. Accordingly, the sequence $a_\ell, \ell \in \Lambda_n(\mathbb{K}_{n,q}^*)$, is strictly increasing.
2. $m(x) = n$: this means that x is a knot t_k interior to $I_{n,q}$, with multiplicity $m_k = n$. The insertion of t_k splits $I_{n,q}$ in two parts $I_{n,q}^*$ and $I_{n,q+1}^*$, and the sequence $a_\ell, \ell \in \Lambda_n(\mathbb{K}_{n,q})$, in two parts too, each of which is strictly increasing: firstly $a_\ell = a_\ell^*, \ell \in \Lambda_n(\mathbb{K}_{n,q}), \ell \leq j_k - m_k = j_k - n$, and secondly $a_\ell = a_{\ell+1}^*, \ell \in \Lambda_n(\mathbb{K}_{n,q}), \ell \geq j_k - n$, the integer j_k being defined in (29). □

After the preliminary results above we are in a position to prove Theorem 4.1.

Theorem 4.1: Proof of (ii)⇒(i).
Let us assume that (ii) holds. Then, according to Theorem 2.12, the space $L_1\mathbb{S}(\mathbb{E}, \mathbb{K})$ possesses a B-spline basis, which we denote by $N_\ell^{\{1\}}, \ell \in \Lambda_{n-1}(\mathbb{K})$. Multiplying both hand-sides of its normalisation property

$$\sum_{\ell \in \Lambda_{n-1}(\mathbb{K})} N_\ell^{\{1\}}(x) = 1, \quad x \in \cup_{k \in \mathbb{Z}} [t_k^+, t_{k+1}^-],$$

by w_1 leads to

$$w_1(x) = \sum_{\ell \in \Lambda_{n-1}(\mathbb{K})} w_1(x) N_\ell^{\{1\}}(x), \quad x \in \cup_{k \in \mathbb{Z}} [t_k^+, t_{k+1}^-].$$

The sequence $w_1 N_\ell^{\{1\}}$, $\ell \in \Lambda_{n-1}(\mathbb{K})$, satisfies the same support, positivity and endpoint properties as the B-spline basis $N_\ell^{\{1\}}$, $\ell \in \Lambda_{n-1}(\mathbb{K})$. It is thus a B-spline-like basis of the piecewise space $D\mathbb{S}(\mathbb{E}, \mathbb{K})$. All coefficients of the expansion of w_1 in the latter basis are thus positive, since they all are equal to 1. Lemma 4.5 ensures that (i) holds.

Theorem 4.1: Proof of (i)⇒(ii).

According to Lemmas 4.4 and 4.5 we can use the special B-spline-like basis B_ℓ, $\ell \in \Lambda_{n-1}(\mathbb{K})$, defined in (22) and the corresponding decomposition (30) in which we assume all coefficients b_ℓ, $\ell \in \Lambda_{n-1}(\mathbb{K})$, to be positive. Select any $x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-]$. Each $B_\ell(x)$, $\ell \in \Lambda_{n-1}(\mathbb{K})$, is non-negative and there is at least one such integer ℓ for which $B_\ell(x) > 0$. We thus have $w_1(x) > 0$. The positivity of w_1 on $\cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-]$ is proved.

We can thus introduce the piecewise differential operator L_1 defined on $PC^n(I, \mathbb{T})$ by:

$$L_1 F(x) := \frac{DF(x)}{w_1(x)}, \quad x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-].$$

The first two things to observe is that the space $L_1\mathbb{S}(\mathbb{E}, \mathbb{K})$ contains constants (since $U_1 \in \mathbb{S}(\mathbb{E}, \mathbb{K})$) and that, restricted to any $[t_k^+, t_{k+1}^-]$, it is a W-space. In order to make sure that $L_1\mathbb{S}(\mathbb{E}, \mathbb{K})$ is indeed a PW-space on $(I, \mathbb{T})$, we must consider the connections at any knot t_k. Obviously, if $m_k \geq n$, there is no connection condition at t_k in $L_1\mathbb{S}(\mathbb{E}, \mathbb{K})$. Let us assume that $m_k \leq n - 1$. Select any $S \in \mathbb{S}(\mathbb{E}, \mathbb{K})$. Since the space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ contains constants, we know that

$$\big(DS(t_k^+), D^2S(t_k^+), \ldots, D^{n-m_k}S(t_k^+)\big)^T = \overline{M}_k . \big(DS(t_k^-), D^2S(t_k^-), \ldots, D^{n-m_k}S(t_k^-)\big)^T,$$

where the matrix $\overline{M}_k$, of order $(n - m_k)$, is obtained by deleting the first row and the first column in M_k. Exactly as in the proof of Proposition 3.4, this leads to

$$\big(L_1S(t_k^+), \ldots, D^{n-1-m_k}L_1S(t_k^+)\big)^T = M_k^{\{1\}} . \big(L_1S(t_k^-), \ldots, D^{n-1-m_k}L_1S(t_k^-)\big)^T, \tag{31}$$

with

$$M_k^{\{1\}} := \mathcal{C}_{n-m_k-1}(w_1, t_k^+)^{-1} . \overline{M}_k . \mathcal{C}_{n-m_k-1}(w_1, t_k^-).$$

This is a lower triangular matrix the diagonal of which is positive since it is obtained by multiplying the diagonal of $\overline{M}_k$ by $w_1(t_k^-)/w_1(t_k^+)$. Accordingly, $L_1\mathbb{S}(\mathbb{E}, \mathbb{K})$ is a PW-spline space on $(I, \mathbb{T})$.

Division of both sides of (30) by $w_1(x)$ yields

$$1 = \sum_{\ell\in\Lambda_{n-1}(\mathbb{K})} \frac{b_\ell B_\ell(x)}{w_1(x)}, \quad x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-]. \tag{32}$$

The sequence $b_\ell B_\ell/w_1$, $\ell \in \Lambda_{n-1}(\mathbb{K})$, satisfies the same support, positivity and endpoint properties as the B-spline-like basis B_ℓ, $\ell \in \Lambda_{n-1}(\mathbb{K})$. Relation (32) proves that it is normalised. It follows that the space $L_1\mathbb{S}(\mathbb{E}, \mathbb{K})$ does possess a B-spline basis, namely, the sequence $b_\ell B_\ell/w_1$, $\ell \in \Lambda_{n-1}(\mathbb{K})$. In order to prove that the space $L_1\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for design, we have to also prove the existence of a B-spline basis in $L_1\mathbb{S}(\mathbb{E}, \mathbb{K}^*)$ for any knot-vector $\mathbb{K}^*$ obtained from $\mathbb{K}$ by insertion of knots (see Theorem 2.12).

Now, clearly, existence of a B-spline basis in the new space $L_1\mathbb{S}(\mathbb{E}, \mathbb{K}^*)$ in question will be obtained via the same arguments as those used for the initial space $L_1\mathbb{S}(\mathbb{E}, \mathbb{K})$ provided that we can make sure that the poles a_ℓ^*, $\ell \in \Lambda_n(\mathbb{K}^*)$, of U considered as an element of the new spline space $\mathbb{S}(\mathbb{E}, \mathbb{K}^*)$ satisfy the corresponding property (i). This was proved in Lemma 4.6.

4.3 Piecewise weight functions for spline spaces

In Theorem 4.2 we have established the existence of a system $(w_1, \ldots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$ ensuring the equality (18). In the present subsection we shall complement it by determining all such possibilities. Let us start with the following proposition.

Proposition 4.7 *The global assumptions are the same as in Theorem 4.2. For a given integer i, $0 \le p \le n-1$, assume that there exists a system $(w_1, \ldots, w_p)$ of piecewise weight functions on $(I, \mathbb{T})$, such that $ECP(\mathbb{1}, w_1, \ldots, w_p) \subset \mathbb{S}(\mathbb{E}, \mathbb{K})$, and let L_p be the linear piecewise differential operator of order p associated with $(\mathbb{1}, w_1, \ldots, w_p)$. Then, the following two properties are equivalent:*

- (i) *the PW-spline space $L_p\mathbb{S}(\mathbb{E}, \mathbb{K})$, with $(n-p+1)$ dimensional section-spaces, is good for design.*
- (ii) *there exists a positive piecewise function $w_{p+1} \in PC^{n-p-1}(I, \mathbb{T})$ such that the space $L_{p+1}\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for design, where the piecewise differential operator L_{p+1} is defined on $PC^n(I, \mathbb{T})$ by $L_{p+1} := (DL_pF)/w_{p+1}$.*
- (iii) *there exists a system $(w_{p+1}, \ldots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$, such that*

$$\mathbb{S}(\mathbb{E}, \mathbb{K}) = ECPS(\mathbb{1}, w_1, \ldots, w_p, w_{p+1}, \ldots, w_n; \mathbb{K});$$

Moreover, when (i) holds, the piecewise functions w_{p+1} leading to (ii) are all splines w_{p+1} in $DL_p\mathbb{S}(\mathbb{E}, \mathbb{K})$ of which the decompositions in a B-spline-like basis of $DL_p\mathbb{S}(\mathbb{E}, \mathbb{K})$ involve only positive coefficients.

Proof Assume that condition (i) is satisfied. We can now apply Theorems 4.1 and 4.2 in the spline space $L_p\mathbb{S}(\mathbb{E}, \mathbb{K})$. The former implies both (ii) and the last sentence of the theorem. The latter ensures the existence of a system $(w_{p+1}, \ldots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$, such that that $ECP(\mathbb{1}, w_{p+1}, \ldots, w_n) \subset L_p\mathbb{S}(\mathbb{E}, \mathbb{K})$. The assumption $ECP(\mathbb{1}, w_1, \ldots, w_p) \subset \mathbb{S}(\mathbb{E}, \mathbb{K})$ implies that any $S \in \mathbb{S}(\mathbb{E}, \mathbb{K})$ satisfies we have

$$L_iS(t_k^+) = L_iS(t_k^-), \quad 0 \le i \le \min(n - m_k, p), \quad k \in \mathbb{Z}.$$

Accordingly, from $ECP(\mathbb{1}, w_{p+1}) \subset L_p\mathbb{S}(\mathbb{E}, \mathbb{K})$ one can deduce that $ECP(\mathbb{1}, w_1, \dots, w_p, w_{p+1}) \subset \mathbb{S}(\mathbb{E}, \mathbb{K})$. We thus have proved that (i) implies (iii), but at the same time that (ii) implies (iii).

Finally the fact that (iii) implies (i) follows from Theorem 3.12. □

Theorem 4.8 *Given an $(n+1)$-dimensional PEC-space $\mathbb{E}$ on $(I, \mathbb{T})$, with $n \geq 1$, and given any bi-infinite knot-vector $\mathbb{K} = \left(t_k{}^{[m_k]}\right)_{k \in \mathbb{Z}}$, assume that the spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for design. Then there exist infinitely many different nested sequences*

$$ECP(\mathbb{1}, w_1) \subset ECP(\mathbb{1}, w_1 w_2) \subset \cdots \subset ECP(\mathbb{1}, w_1, \dots, w_n) \subset \mathbb{S}(\mathbb{E}, \mathbb{K}). \tag{33}$$

More precisely, at each stage $0 \leq p \leq n-1$, there exist infinitely many different ECP-spaces $ECP(\mathbb{1}, w_1, \dots, w_p)$ to start or continue the nested sequence (33).

Proof Consider first the case $p = 0$. Let B_ℓ, $\ell \in \Lambda_{n-1}(\mathbb{K})$, be a B-spline-like basis in $D\mathbb{S}(\mathbb{E}, \mathbb{K})$. Given two families $b_\ell, \overline{b}_\ell$, $\ell \in \Lambda_{n-1}(\mathbb{K})$, of positive real numbers, consider the splines

$$w_1(x) = \sum_{\ell \in \Lambda_{n-1}(\mathbb{K})} b_\ell B_\ell(x), \quad \overline{w}_1(x) = \sum_{\ell \in \Lambda_{n-1}(\mathbb{K})} \overline{b}_\ell B_\ell(x), \quad x \in \cup_{k \in \mathbb{Z}} \left[t_k^+, t_{k+1}^-\right],$$

The two splines satisfy $ECP(\mathbb{1}, w_1) = ECP(\mathbb{1}, \overline{w}_1)$ if and only there exist a positive real number α such that $\overline{w}_1 = \alpha w_1$, that is, if and only if $\overline{b}_\ell = \alpha b_\ell$ for each $\ell \in \Lambda_{n-1}(\mathbb{K})$. There thus exist infinitely many different spaces $ECP(\mathbb{1}, w_1) \subset \mathbb{S}(\mathbb{E}, \mathbb{K})$ such that $L_1\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for design. Given an integer p, with $0 \leq p \leq n-1$, assume that we have chosen $w_1, \dots, w_p$. With our usual notations, we now know the existence of infinitely many different spaces $ECP(\mathbb{1}, w_{p+1}) \subset L_p\mathbb{S}(\mathbb{E}, \mathbb{K})$ such that that $L_{p+1}\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for design. This proves the expected result. □

Remark 4.9 Let the PEC-spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ be good for design. Then, there exist infinitely many sequences of piecewise differential operators $L_0 = \mathrm{Id}, L_1, \dots, L_n$ so that the $\mathbb{S}(\mathbb{E}, \mathbb{K})$ can be described as in Definition 3.11. Each of them plays for $\mathbb{S}(\mathbb{E}, \mathbb{K})$ exactly the same rôle as the sequence D^j, $0 \leq j \leq n$, for ordinary polynomial spline spaces. On the other hand, each nested sequence of ECP-spaces (33) also leads to the following nested sequence of spline spaces good for design.

$$\begin{aligned} &ECPS(\mathbb{1}, w_1; \mathbb{K}_1) \subset ECPS(\mathbb{1}, w_1 w_2; \mathbb{K}_2) \\ &\qquad \subset \cdots \subset ECPS(\mathbb{1}, w_1, \dots, w_n; \mathbb{K}_n) \subset \mathbb{S}(\mathbb{E}, \mathbb{K}), \end{aligned} \tag{34}$$

where, for $p = 1, \dots, n$, the multiplicity of the knot t_k in the knot-vector $\mathbb{K}_p$ is equal to $\max(0, m_k + p - n)$.

Remark 4.10 We would like to conclude the present section with an important observation concerning blossoms. Let the PEC-spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ be good for design.

Then, the blossom f of any element F of the PEC-space $\mathbb{E}$ is not necessarily defined on I^n. We can only say that f is defined on the restricted set $\mathbb{A}_n(\mathbb{K})$. Nevertheless Theorem 4.8 ensures that one can find infinitely many $(n+1)$-dimensional ECP-space on $(I, \mathbb{T})$, $\widetilde{\mathbb{E}}$, such that:

- $\mathbb{S}(\mathbb{E}, \mathbb{K}) = \mathbb{S}(\widetilde{\mathbb{E}}, \mathbb{K})$;
- the blossom $\widetilde{f}$ of any element $\widetilde{F}$ in $\widetilde{E}$ is defined on the whole of I^n.

5 Spaces possessing B-spline-like bases

In geometric design we generally assume the spaces used to create parametric curves to contain constants. Nevertheless there exist lots of examples of spline spaces which do not meet the latter requirement and which are interesting to consider for other purposes, for instance for spline interpolation. The presence of minimally supported bases remains essential, and this is the question the present section is devoted to.

5.1 $\mathbb{S}$ versus $D\mathbb{S}$

Let us have a look at the equivalence "(ii) ⇔ (iv)" of Theorem 3.2 concerning PW-spaces. It characterises the property "being good for design" by the existence of a certain type of bases in the space obtained by differentiation. The result below can be considered the spline version of the latter characterisation.

Theorem 5.1 *Let $\mathbb{E}$ be an $(n+1)$-dimensional PW-space on $(I, \mathbb{T})$ containing constants. Then, for any bi-infinite knot-vector $\mathbb{K}$, the following two properties are equivalent:*

(i) *the spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for design;*
(ii) *the spline space $D\mathbb{S}(\mathbb{E}, \mathbb{K})$ possesses a B-spline-like basis and so does any spline space obtained from $D\mathbb{S}(\mathbb{E}, \mathbb{K})$ by knot insertion.*

Proof • $(ii) \Rightarrow (i)$: In Lemma 4.4 we have built a B-spline-like basis in the space $D\mathbb{S}(\mathbb{E}, \mathbb{K})$. In its proof we mentioned why the same holds too in any spline space $D\mathbb{S}(\mathbb{E}, \mathbb{K}^*)$ such that $\mathbb{K}^*$ is obtained from $\mathbb{K}$ by knot insertion.

- $(ii) \Rightarrow (i)$: Let B_ℓ, $\ell \in \Lambda_{n-1}(\mathbb{K})$, be a given B-spline-like basis in $D\mathbb{S}(\mathbb{E}, \mathbb{K})$. For each $\ell \in \Lambda_n(\mathbb{K})$, we consider the piecewise function $N_\ell \in PC^n(I, \mathbb{T})$ defined by

$$N_\ell(x) = 0 \quad \text{if } x \notin [\xi_\ell^+, \xi_{\ell+n+1}^-],$$
$$N_\ell(x) = \frac{\int_{\xi_\ell}^x B_\ell(t)\,dt}{\int_{\xi_\ell}^{\xi_{\ell+n}} B_\ell(t)\,dt} - \frac{\int_{\xi_{\ell+1}}^x B_{\ell+1}(t)\,dt}{\int_{\xi_{\ell+1}}^{\xi_{\ell+n+1}} B_{\ell+1}(t)\,dt} \quad \text{if } x \in [\xi_\ell^+, \xi_{\ell+n+1}^-]. \tag{35}$$

Due to the positivity property $(\mathrm{BSB})_2$ satisfied by the B-spline-like basis, the piecewise function N_ℓ is well defined on each interval $[t_k^+, t_{k+1}^-]$, at least with the usual convention, namely:

$$\begin{aligned} \text{if } \xi_\ell = \xi_{\ell+n}, &\text{ then replace the first quotient of integrals by } 1, \\ \text{if } \xi_{\ell+1} = \xi_{\ell+n+1}, &\text{ then replace the second one by } 0. \end{aligned} \tag{36}$$

The piecewise function N_ℓ so defined clearly belongs to $\mathbb{S}(\mathbb{E}, \mathbb{K})$. It satisfies the support and end point properties $(\mathrm{BSB})_1$ and $(\mathrm{BSB})_4$. Select any integer $k \in \mathbb{Z}$. Consider the integer j_k introduced in (29). On the interval $[t_k^+, t_{k+1}^-]$, all piecewise functions N_ℓ, such that $\ell \in \Lambda_n(\mathbb{K}) \setminus \{j_k - n, \dots, j_k\}$ are identically 0. Therefore, taking (35) into account, we obtain

$$\begin{aligned} \sum_{\ell \in \Lambda_n(\mathbb{K})} N_\ell(x) &= \sum_{\ell = j_k - n}^{j_k} N_\ell(x), \qquad x \in [t_k^+, t_{k+1}^-] \subset [\xi_{j_k}^+, \xi_{j_k+1}^-], \\ &= \frac{\int_{\xi_{j_k-n}}^{x} B_{j_k-n}(t)\,dt}{\int_{\xi_{j_k-n}}^{\xi_{j_k}} B_{j_k-n}(t)\,dt} - \frac{\int_{\xi_{j_k+1}}^{x} B_{j_k+1}(t)\,dt}{\int_{\xi_{j_k+1}}^{\xi_{j_k+n+1}} B_{j_k+1}(t)\,dt} = 1, \end{aligned} \tag{37}$$

which proves the normalisation property $(\mathrm{BSB})_3$.

Choose any points $P_\ell \in \mathbb{R}^n$, $\ell \in \Lambda_n(\mathbb{K})$, so as to ensure the affine independence of the points $P_{j_k-n}, \dots, P_{j_k}$ for each $k \in \mathbb{Z}$, where j_k is defined in (29). Such a choice is always possible. We then consider the piecewise spline $S \in \mathbb{S}(\mathbb{E}, \mathbb{K})^n$ defined by

$$S(x) := \sum_{\ell \in \Lambda_n(\mathbb{K})} N_\ell(x) P_\ell, \quad x \in \cup_{k \in \mathbb{Z}} [t_k^+, t_{k+1}^-].$$

This spline S is non-degenerate. On the other hand, choose any integer $j \in \Lambda_n(\mathbb{K})$. One can always write the n-tuple $(\xi_{j+1}, \dots, \xi_{j+n})$ as follows

$$(\xi_{j+1}, \dots, \xi_{j+n+1}) = \left({t_k}^{[\alpha]}, {t_{k+1}}^{[m_{k+1}]}, \dots, {t_{k+s}}^{[m_{k+s}]}, {t_{k+s+1}}^{[\beta]} \right),$$

with $0 \le \alpha < m_k$ and $0 \le \beta < m_{k+s+1}$. Then, because the piecewise functions N_ℓ, $\ell \in \Lambda_n(\mathbb{K})$, satisfy $(\mathrm{BSB})_1$, $(\mathrm{BSB})_3$, and $(\mathrm{BSB})_4$, one can prove that all osculating flats $\mathrm{Osc}_{n-\alpha}\, S(t_k^+)$, $\mathrm{Osc}_{n-m_i}\, S(t_i)$, $k+1 \le i \le k+s$, and $\mathrm{Osc}_{n-\beta}\, S(t_{k+s+1}^-)$ have in common the only point $P_{j_k-\alpha}$. The arguments are exactly those used in the proofs of Proposition 2.3 and Theorem 3.1 of [16] which we refer the reader to (also see the comment after the latter theorem). Similar arguments can also be applied starting with any knot-vector $\mathbb{K}^*$ obtained from $\mathbb{K}$ by insertion of knots, and any B-spline-like basis B_ℓ^*, $\ell \in \Lambda_{n-1}(\mathbb{K}^*)$, of the corresponding spline space $\mathbb{S}(\mathbb{E}, \mathbb{K}^*)$. Accordingly, existence of blossoms in $\mathbb{S}(\mathbb{E}, \mathbb{K})$ is proved. □

Remark 5.2 From (35) and from the positivity of the B_ℓ's, $\ell \in \Lambda_{n-1}(\mathbb{K})$, we could easily have deduced the positivity of each N_ℓ, $\ell \in \Lambda_n(\mathbb{K})$, close to ξ_ℓ^+ or to $\xi_{\ell+n+1}^-$.

However, this would not be sufficient to directly conclude that N_ℓ is positive on $]\xi_\ell, \xi_{\ell+n+1}[$. Once the existence of blossoms proved, we can assert that the piecewise functions $N_\ell, \ell \in \Lambda_n(\mathbb{K})$, built via (35), form the B-spline basis in $\mathbb{S}(\mathbb{E}, \mathbb{K})$. Hence they also satisfy the positivity property $(\mathrm{BSB})_2$.

5.2 Consequences

In consequence of Theorem 5.1, the results of Sect. 4 lead to the following one:

Theorem 5.3 *Let $\mathbb{E}$ be an $(n+1)$-dimensional PW-space on $(I, \mathbb{T})$ and let the knot-vector $\mathbb{K}$ be bi-infinite. The following three properties are then equivalent:*

(i) *the space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ possesses a B-spline-like basis and so does the space $\mathbb{S}(\mathbb{E}, \mathbb{K}^*)$ for any knot-vector $\mathbb{K}^*$ obtained from $\mathbb{K}$ by insertion of knots;*
(ii) *there exists a positive spline $w_0 \in \mathbb{S}(\mathbb{E}, \mathbb{K})$ such that $L_0\mathbb{S}(\mathbb{E}, \mathbb{K}) := \{S/w_0 \mid S \in \mathbb{S}(\mathbb{E}, \mathbb{K})\}$ is good for design.*
(iii) *there exists a system $(w_0, \ldots, w_n)$ of piecewise weight functions such that*

$$\mathbb{S}(\mathbb{E}, \mathbb{K}) = ECPS(w_0, \ldots, w_n; \mathbb{K}). \tag{38}$$

Suppose that (i) is satisfied, and select one B-spline-like basis $B_\ell, \ell \in \Lambda_n(\mathbb{K})$, in $\mathbb{S}(\mathbb{E}, \mathbb{K})$. Then, in order to obtain a positive spline $w_0 \in \mathbb{S}(\mathbb{E}, \mathbb{K})$ so that the corresponding spline space $L_0\mathbb{S}(\mathbb{E}, \mathbb{K})$ is good for design, it is necessary and sufficient to take

$$w_0(x) := \sum_{\ell \in \Lambda_n(\mathbb{K})} b_\ell B_\ell(x), \quad x \in \cup_{k \in \mathbb{Z}}[t_k^+, t_{k+1}^-],$$

where $b_\ell, \ell \in \Lambda_n(\mathbb{K})$, are any positive real numbers.

Proof Let $\widehat{\mathbb{E}}$ be the $(n+2)$-dimensional space composed of all piecewise functions $\widehat{F} \in PC^{n+1}(I, \mathbb{T})$ which are continuous at each knot t_k of multiplicity $m_k \le n+1$ and for which $DF \in \mathbb{E}$. We then have $\mathbb{S}(\mathbb{E}, \mathbb{K}) = D\mathbb{S}(\widehat{\mathbb{E}}, \mathbb{K})$. Theorem 5.1 says that (i) is satisfied if and only if $\mathbb{S}(\widehat{\mathbb{E}}, \mathbb{K})$ is good for design. Accordingly, the various equivalences are obtained by applying Theorem 4.2 in $\mathbb{S}(\widehat{\mathbb{E}}, \mathbb{K})$. The remaining assertion concerning how to choose w_0 follows from applying Theorem 4.1 and Lemma 4.5 in $\mathbb{S}(\widehat{\mathbb{E}}, \mathbb{K})$. □

Any "interesting" spline space is thus good for design, at least up to multiplication by a positive piecewise function. This fully justifies that we can limit investigations on Chebyshevian spline spaces to those which are good for design, see, for instance, Sect. 7.

Let us conclude the present section with the following special case of Theorem 5.3.

Corollary 5.4 *Assume that (i) of Theorem 5.3 is satisfied. Then, the following two statements are equivalent:*

(i) *the space* $\mathbb{S}(\mathbb{E}, \mathbb{K})$ *possesses a B-spline basis;*
(ii) *the space* $\mathbb{S}(\mathbb{E}, \mathbb{K})$ *is good for design.*

Proof The implication (ii)⇒(i) being contained in Theorem 2.12, the interesting part is the converse implication. Assuming that (i) holds, and let N_ℓ, $\ell \in \Lambda_n(\mathbb{K})$, denote the B-spline basis in $\mathbb{S}(\mathbb{E}, \mathbb{K})$. Then,

$$1 = \sum_{\ell \in \Lambda_n(\mathbb{K})} b_\ell N_\ell(x), \quad x \in \cup_{k \in \mathbb{Z}}[t_k^+, t_{k+1}^-], \quad \text{with } b_\ell := 1 \quad \text{for all } \ell \in \Lambda_n(\mathbb{K}).$$

Accordingly (ii) follows by application of Theorem 5.3 with $w_0 := \mathbb{1}$. □

6 Comments. Consequences

We shall now comment on several points: what about a non-bi-infinite knot-vector? what about splines on a closed bounded interval? what are the implications of our results?

6.1 Non bi-infinite knot-vectors

Throughout Sects. 4 and 5 we have assumed the knot-vector $\mathbb{K}$ to be bi-infinite. Let us now consider the case where $\mathbb{K}$ is finite, *i.e.*, only a finite number of knots have positive multiplicities, and let us assume the spline space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ to be good for design. Do all results previously obtained remain valid? As a matter of fact, the definition of B-splines bases must be slightly modified (see [18]) and in a given spline space good for design, uniqueness is no longer guaranteed. In particular we know that the B-splines are no longer necessarily positive on the interior of their supports. An analogous remark holds in the space $D\mathbb{S}(\mathbb{E}, \mathbb{K})$. For this reason, if we choose U as in (i) of Theorem 4.1 we cannot guarantee that the piecewise function $w_1 := DU$ will be positive everywhere on $\cup_{k \in \mathbb{Z}}[t_k^+, t_{k+1}^-]$. We therefore cannot ensure that we can find a system $(w_1, \ldots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$ such that $\mathbb{S}(\mathbb{E}, \mathbb{K}) = ECPS(\mathbb{1}, w_1, \ldots, w_n; \mathbb{K})$. Of course the same situation occurs when the knot-vector is infinite only on one side.

Let us now consider the case of PEC-splines on a given closed bounded interval $[a, b]$, $a < b$. To distinguish between it and the previous case of a bi-infinite knot-vector we shall use calligraphic letters. We start with a finite sequence $\mathcal{T}$ of interior knots, *i.e.*,

$$t_0 := a < t_1 < \cdots < t_q < t_{q+1} := b,$$

and with interior multiplicities $0 \le m_k \le n$ for $k = 1, \ldots, q$. The data are the same as previously apart from the fact that now, the indices for the section-spaces are limited to $0 \le k \le q$ and those for the connection matrices to $1 \le k \le q$. The knot-vector $\mathcal{K}$

is now

$$\mathcal{K} := \left(t_k{}^{[m_k]}\right)_{0\le k\le q+1} = (\xi_{-n}, \ldots, \xi_{n+m+1}),$$

with $m_0 := m_{q+1} := n+1$ and $m := \sum_{k=1}^{q} m_k$. All definitions previously introduced on $(I, \mathbb{T})$ can easily be adapted to $([a,b], \mathcal{T})$: PEC- or ECP-spaces on $([a,b], \mathcal{T})$, piecewise weight functions on $([a,b], \mathcal{T})$, and so forth. In particular, an $(n+1)$-dimensional PEC-space $\mathcal{E}$ on $([a,b], \mathcal{T})$ being given, with section-spaces $\mathbb{E}_k$, $0 \le k \le q$, the spline space $\mathcal{S}(\mathcal{E}, \mathcal{K})$ is then defined as the set of all continuous functions S on $[a,b]$ with kth sections in $\mathbb{E}_k$, $0 \le k \le q$, and with connection condition (2) at t_k, $1 \le k \le q$. It is well known that $\mathcal{S}(\mathcal{E}, \mathcal{K})$ is then $(n+m+1)$-dimensional. A possible B-spline(-like) basis is then defined as in Definition 2.13, but only for indices $\ell = -n, \ldots, m$. The space $\mathcal{S}(\mathcal{E}, \mathcal{K})$ can always be considered the restriction to $[a,b] = [t_0^+, t_{q+1}^-]$ of a space $\mathbb{S}(\mathbb{E}, \mathbb{K})$ of PEC-splines with a bi-infinite knot-vector $\mathbb{K} = \left(t_k{}^{[m_k]}\right)_{k\in\mathbb{Z}}$, chosen so that $\mathcal{S}(\mathcal{E}, \mathcal{K})$ is good for design (resp., possesses a B-spline(-like) basis) if and only if the same property holds for $\mathbb{S}(\mathbb{E}, \mathbb{K})$. For instance, one can assume that outside $[a,b]$, the elements of $\mathbb{S}(\mathbb{E}, \mathbb{K})$ are C^{n-1} functions, piecewise in the degree n polynomial space.

By restriction to $[a,b]$, we can then state:

Theorem 6.1 *All results of Sections 4 and 5 remain valid for PW-/PEC-spline spaces on a closed bounded interval as described above.*

For the rest of the subsection we consider the special case where $m_k = 0$ for $1 \le k \le q$. Then $\mathcal{S}(\mathcal{E}, \mathcal{K}) = \mathcal{E}$ is an $(n+1)$-dimensional PEC-space on $([a,b], \mathcal{T})$, and the B-spline basis in $\mathcal{S}$ (if it exists) is just its Bernstein basis relative to (a,b). Let us observe that in spite of the general statement of Theorem 6.1, there exists one precise point where we should change our statement: in Theorem 4.8, for $p = n-1$, there is obviously only one way to continue the nested sequence (33).

Applying Theorem 5.3 in $\mathbb{S}(\mathbb{E}, \mathbb{K})$ provides us with all ECP-spaces on $([a,b], \mathcal{T})$:

Theorem 6.2 *Let $\mathcal{E}$ be an $(n+1)$-dimensional PEC-space on $([a,b], \mathcal{T})$, with section-spaces $\mathbb{E}_k$, $k = 0, \ldots, q$. Then the following three properties are equivalent:*

(i) *$\mathcal{E}$ is an ECP-space on $([a,b], \mathcal{T})$;*
(ii) *there exists a system $(w_0, \ldots, w_n)$ of piecewise weight functions on $([a,b], \mathcal{T})$, such that $\mathcal{E} = ECP(w_0, \ldots, w_n)$;*
(iii) *for each $k \in \mathbb{Z}$, there exists a system $(w_0^k, \ldots, w_n^k)$ of weight functions on $[t_k^+, t_{k+1}^-]$ with associated differential operators $L_0^k, \ldots, L_n^k$ on $C^n([t_k^+, t_{k+1}^-])$ such that $\mathcal{E}$ can be described as the set of all piecewise functions $F \in PC^n(I, \mathcal{T})$ such that*
   - *for any $k = 0, \ldots q$, the restriction of F to $[t_k^+, t_{k+1}^-]$ belongs to $EC(w_0^k, \ldots, w_n^k)$;*
   - *for any $k = 1, \ldots, q$, F satisfies the connection conditions*

$$\left(L_0^k F(t_k^+), L_1^k F(t_k^+) \ldots, L_n^k F(t_k^+)\right)^T = \left(L_0^{k-1} F(t_k^-), L_1^{k-1} F(t_k^-) \ldots, L_n^{k-1} F(t_k^-)\right)^T. \quad (39)$$

We first achieved the latter result in [21], via completely different arguments which we shall now briefly recall. We first proved that an $(n+1)$-dimensional ECP-space $\mathcal{E}$ on $([a,b],\mathcal{T})$ can be viewed as the restriction to $[a,b]$ of an $(n+1)$-dimensional ECP-space $\widehat{\mathcal{E}}$ on $([\widehat{a},\widehat{b}],\mathcal{T})$, for some $\widehat{a},\widehat{b}$ such that $\widehat{a}<a$, $\widehat{b}>b$. Any $c\in[\widehat{a},\widehat{b}]\backslash[a,b]$ then yields a nested sequence of ECP-spaces on $([a,b],\mathcal{T}):\mathcal{E}_0\subset\mathcal{E}_1\subset\cdots\subset\mathcal{E}_n=\mathcal{E}$, where, for $0\le i\le n$, $\mathcal{E}_i$ is obtained by restricting to $[a,b]$ the set of all $\widehat{F}\in\widehat{\mathcal{E}}$ which vanish $(n-i)$ times at c. One can prove that such a nested sequence is necessarily of the form $\mathcal{E}_i=ECP(w_0,\dots,w_i)$, $0\le i\le n$, where $(w_0,\dots,w_n)$ is a system of piecewise weight functions on $([a,b],\mathcal{T})$. The latter arguments already provided us with infinitely many systems of piecewise weight functions on $([a,b],\mathcal{T})$, such that $\mathcal{E}=ECP(w_0,\dots,w_n)$. Thanks to Theorem 5.3, we can now state how to obtain all of them, that is, how to build all nested sequences of ECP-spaces on $([a,b],\mathcal{T})$ contained in a given one.

Theorem 6.3 *Let $\mathcal{E}$ be an $(n+1)$-dimensional ECP-space on $([a,b],\mathcal{T})$. Then, all possible systems $(w_0,\dots,w_n)$ of piecewise weight functions on $([a,b],\mathcal{T})$, such that $\mathcal{E}=ECP(w_0,\dots,w_n)$ are obtained as follows. Given an integer p, $0\le p\le n$, assume that we already have built $w_0,\dots,w_{p-1}$ so that $DL_{p-1}\mathcal{E}$ is an $(n-p+1)$-dimensional ECP-space on $([a,b],\mathcal{T})$, with $DL_{-1}\mathcal{E}:=\mathcal{E}$. In $DL_{p-1}\mathcal{E}$ select a Bernstein-like basis $(V_0^p,\dots,V_{n-p}^p)$ relative to (a,b). Then, all possible piecewise weight functions w_p are obtained as*

$$w_p:=\sum_{i=0}^{n-p}\alpha_i^pV_i^p,\quad \textit{with any positive } \alpha_0^p,\dots,\alpha_{n-p}^p,\quad 0\le p\le n.$$

Observe that, at each step $p\ge1$, the Bernstein-like basis $(V_0^p,\dots,V_{n-p}^p)$ relative to (a,b) can be obtained from the Bernstein basis in $L_{p-1}\mathcal{E}$ via Lemma 4.4. As a consequence, all successive Bernstein-like bases can be derived from the initial Bernstein basis relative to (a,b) in the space $\mathcal{E}$ itself (see proof of Theorem 4.1). Note that, when $q=0$, under the assumption of Theorem 6.3, $\mathcal{E}$ is an EC-space on $[a,b]$. As a special case, Theorem 6.3 provides us with all systems of weight functions on $[a,b]$ such that $\mathcal{E}=EC(w_0,\dots,w_n)$. We thus recover the results obtained in [25].

6.2 New type of shape parameters

Let us come back to the general situation of a bi-infinite knot-vector $\mathbb{K}$. Theorem 4.2 provided us with ALL possible PEC-spline spaces which are good for design, thus giving us the answer to Question 1 (see the introduction). We now consider a slightly different problem: we would like to determine ALL possible PEC-spline spaces good for design, but with prescribed section-spaces.

In other words we would like to give an answer to Question 2: for given section-spaces, we want to determine all sequences of connection matrices leading to spline spaces good for design. In theory, they must be chosen so as to ensure existence of blossoms in the corresponding spline space (see Definition 2.9). The problem is that, in practice, existence of blossoms is not easy to check, especially in high dimensions.

Now that we have all results of the previous sections at our disposal, with the help of formula (17) we can state the following theorem:

Theorem 6.4 *Let the sequence $\mathbb{E}_k, k \in \mathbb{Z}$, of section-spaces be given, each $\mathbb{E}_k \subset C^n([t_k, t_{k+1}])$ being an $(n+1)$-dimensional space containing constants such that the space $D\mathbb{E}_k$ is an EC-space on $[t_k, t_{k+1}]$. A bi-infinite knot-vector $\mathbb{K}$ being given, let $\mathbb{S}$ denote the PEC-spline space composed of all splines with kth sections in $\mathbb{E}_k, k \in \mathbb{Z}$, and with connection conditions (2) at the knots. For simplicity of the statement, assume that $m_k \le n$ for all k. The sequences of connection matrices $M_k, k \in \mathbb{Z}$, for which the spline space $\mathbb{S}$ is good for design are all sequences of block diagonal matrices $M_k := (1, \overline{M}_k)$, of the form*

$$\overline{M}_k := \Gamma_{n-m_k-1}(w_1^k, \ldots, w_{n-m_k}^k; t_k^+). \Gamma_{n-m_k-1}(w_1^{k-1}, \ldots, w_{n-m_k}^{k-1}; t_k^-)^{-1}, \quad k \in \mathbb{Z}, \tag{40}$$

where, for each $k \in \mathbb{Z}$, $(w_1^k, \ldots, w_n^k)$ is any system of weight functions on $[t_k^+, t_{k+1}^-]$ such that $\mathbb{E}_k = EC(\mathbb{1}_k, w_1^k, \ldots, w_n^k)$.

The important point is that we know how to build all such systems of weight functions as a special case of Theorem 6.3. Choose any positive real numbers

$$\alpha_{k,i}^p, \quad 0 \le i \le n-p, \; 1 \le p \le n, \; k \in \mathbb{Z}.$$

From $p = 1$ up to $p = n$, set

$$w_p^k := \sum_{i=0}^{n-p} \alpha_{k,i}^p V_{k,i}^p, \quad k \in \mathbb{Z}, \tag{41}$$

where $(V_{k,0}^p, \ldots, V_{k,n-p}^p)$ is a given Bernstein-like basis in the space $DL_{p-1}^k \mathbb{E}_k$, for instance the one deduced from the Bernstein basis of $L_{p-1}^k \mathbb{E}_k$ via Lemma 4.4. Note that it depends on $w_1^k, \ldots, w_{p-1}^k$, that is, of the real numbers $\alpha_{k,i}^j, 0 \le i \le n-j, 1 \le j \le p-1$. This provides us with shape parameters of a new type. Indeed, the α's are free positive parameters which determine the weight functions, and each matrix $\overline{M}_k$ is then obtained from them via (40) and (41). It depends on the free positive parameters $\alpha_{k-1,i}^p, \alpha_{k,i}^p, 0 \le i \le n-p, , 0 \le p \le n-m_k$, relative to the two section-spaces $\mathbb{E}_{k-1}$ and $\mathbb{E}_k$. Due to the length of the paper we will not investigate more the resulting form of $\overline{M}_k$.

Remark 6.5 In [15], for $n = 3$ and $m_k = 1$ for all k, the geometrical definition of blossoms enabled us to achieve practical necessary and sufficient conditions for their existence. In other words, we managed to produce all suitable matrices $\overline{M}_k$, lower triangular of order two and with positive diagonals, leading to a good spline space. Apart from this special case or for $n \le 2$, up to now the only way we could build PEC-spline spaces good for design was to choose them in the class $\widehat{\mathcal{C}}$ introduced in

Remark 4.3. Assuming each section-space $\mathbb{E}_k$ to be defined by a given system of weight functions, *i.e.*,

$$\mathbb{E}_k := EC(\mathbb{1}_k, W_1^k, \dots, W_n^k), \quad k \in \mathbb{Z},$$

and taking (15) into acount, the corresponding connection matrices were thus of the form:

$$\begin{aligned}\overline{M}_k := \Gamma_{n-m_k-1}(W_1^k, \dots, W_{n-m_k}^k; t_k^+). N_k \\ .\Gamma_{n-m_k-1}(W_1^{k-1}, \dots, W_{n-m_k}^{k-1}; t_k^-)^{-1}, \quad k \in \mathbb{Z},\end{aligned} \tag{42}$$

where N_k was any totally positive regular lower triangular matrix of order $(n - m_k)$. Although no totally positive matrix appears in (40), the class of all matrices $\overline{M}_k$ of the form (40) is bigger than that of all matrices of the form (42). This is due to the fact that we consider all other possible systems of weight functions associated with the same section-spaces. The "difference" between the two classes may be significant, as shown in [15].

6.3 An elementary example

Let us start with the following immediate consequence of Theorem 4.2.

Proposition 6.6 *Let $\mathbb{S}$ be a PW-spline space on $(I, \mathbb{T})$ assumed to be good for design. Let $(w_1, \dots, w_p)$ be any system of piecewise weight functions on $(I, \mathbb{T})$, with $w_i \in PC^{n+p-i}(I, \mathbb{T})$ for $1 \le i \le p$, and let $L_0 = Id, L_1, \dots, L_p$ be the piecewise differential operators associated with $(\mathbb{1}, w_1, \dots, w_p)$. Then, the set $\widehat{\mathbb{S}}$ composed of all piecewise functions $\widehat{S}$ on $(I, \mathbb{T})$ such that*

(1) *$L_p\widehat{S}$ belongs to $\mathbb{S}$,*
(2) *for each $k \in \mathbb{Z}$ and for each i, $0 \le i \le \min(p, n+p-m_k)$, $L_i\widehat{S}(t_k^+) = L_i\widehat{S}(t_k^-)$,*

is a PEC-spline space on $(I, \mathbb{T})$ which good for design.

Proof If $\mathbb{S}$ has $(n + 1)$-dimensional section-spaces, Theorem 4.2 ensures the existence of a system $(w_{p+1}, \dots, w_{p+n})$ of piecewise weight functions on $(I, \mathbb{T})$ such that $\mathbb{S} = ECPS(\mathbb{1}, w_{p+1}, \dots, w_{p+n}; \mathbb{K})$. Then it can be easily checked that $\widehat{\mathbb{S}} = ECPS(\mathbb{1}, w_1, \dots, w_p, w_{p+1}, \dots, w_{p+n}; \mathbb{K})$. □

For a given sequence of section-spaces, it may be interesting to know if choosing $\overline{M}_k := \mathcal{I}_{n-m_k}$ (identity matrix of order $(n - m_k)$) for all $k \in \mathbb{Z}$ leads to a spline space which is good for design. To illustrate the previous subsection we take the section-spaces in the class studied in [5].

Proposition 6.7 *For any $k \in \mathbb{Z}$, let $\mathbb{U}_k \subset C^1([t_k, t_{k+1}])$ be a two-dimensional EC-space on $[t_k, t_{k+1}]$. Assuming that $n \ge 2$, let $\mathbb{S}$ be the piecewise spline space composed of all piecewise functions S on $(I, \mathbb{T})$ such that*

- *for each $k \in \mathbb{Z}$, the restriction of S to $[t_k^+, t_{k+1}^-]$ belongs to the EC-space $\mathbb{E}_k := \{F \in C^n([t_k, t_{k+1}]) \mid D^{n-1}F \in \mathbb{U}_k\}$;*
- *for $k = 1, \ldots, q$, S is C^{n-m_k} at t_k.*

If $m_k \geq 1$ for all $k \in \mathbb{Z}$, then the spline space $\mathbb{S}$ is good for design.

Proof Without loss of generality we can assume that $m_k = 1$ for all $k \in \mathbb{Z}$. According to Proposition 6.6, we can also limit ourselves to considering the case $n = 2$.

For each integer k, since $\mathbb{U}_k$ is a two-dimensional EC-space on $[t_k, t_{k+1}]$, there exist infinitely many ways to write it as $\mathbb{U}_k = EC(w_{n-1}^k, w_n^k)$. Any such choice implies that $\mathbb{E}_k = (\mathbb{1}, w_1^k, w_2^k)$. Let $L_0^k = \mathrm{Id}$, L_1^k, L_2^k be the associated differential operators on $C^n([t_k, t_{k+1}])$. The problem is thus the following: can we choose the weight functions w_1^k, $k \in \mathbb{Z}$, so that

$$L_1^k S(t_k^+) = L_1^{k-1} S(t_k^-) \quad \text{for all } k \in \mathbb{Z} \Leftrightarrow S'(t_k^+) = S'(t_k^-) \quad \text{for all } k \in \mathbb{Z}? \tag{43}$$

Since $L_1^k S(t_k^+) := S'(t_k^+)/w_1^k(t_k^+)$ and $L_1^{k-1} S(t_k^-) := S'(t_k^-)/w_1^{k-1}(t_k^-)$, and S is C^{n-1} at t_k, the question turns to be: can we choose the weight functions so that

$$w_1^k(t_k^+) = w_1^{k-1}(t_k^-), \quad k \in \mathbb{Z}? \tag{44}$$

For each $k \in \mathbb{Z}$, let $(B_{k,0}, B_{k,1})$ be the basis of $\mathbb{U}_k$ defined by

$$B_{k,0}(t_k^+) = B_{k,1}(t_{k+1}^-) = 1, \quad B_{k,0}(t_{k+1}^-) = B_{k,1}(t_k^+) = 0.$$

It is a Bernstein-like basis relative to (t_k, t_{k+1}). We know that the weights w_1^k must be chosen as

$$w_1^k = \alpha_{k,0} B_{k,0} + \alpha_{k,1} B_{k,1},$$

with positive $\alpha_{k,0}, \alpha_{k,1}$. Accordingly, to obtain (44) we just have to require $\alpha_{k,0} := \alpha_{k-1,1}$ for each $k \in \mathbb{Z}$. Whence the claimed result. □

Example 6.8 For any $k \in \mathbb{Z}$, assume that $\mathbb{U}_k$ is spanned on $[t_k, t_{k+1}]$ either by the two functions $\cosh x$, $\sinh x$, or by the two functions $\cos x$, $\sin x$. In the first case $\mathbb{U}_k$ is always an EC-space on $[t_k, t_{k+1}]$. In the second one, for $\mathbb{U}_k$ to be an EC-space on $[t_k, t_{k+1}]$ it is necessary and sufficient to require that

$$h_k := t_{k+1} - t_k < \pi. \tag{45}$$

Then, for any positive n, we denote by $\mathbb{E}_k$ the $(n+1)$-dimensional EC-space spanned on $[t_k, t_{k+1}]$ either by the functions $1, x, \ldots, x^{n-2}, \cosh x, \sinh x$ (hyperbolic space) or by the functions $1, x, \ldots, x^{n-2}, \cos x, \sin x$ (trigonometric space). Let us require that condition (45) be satisfied for each integer k such that $\mathbb{E}_k$ is a trigonometric space. Then, according to Proposition 6.7, the set of all C^{n-1} functions on I with kth sections in $\mathbb{E}_k$ is automatically a PEC-spline space good for design. Hyperbolic and trigonometric spaces having complementary shape effects, it is especially interesting to mix them. This shows the interest of the present example. Illustrations can be found in [15] in the case $n = 3$.

Remark 6.9 The previous example can be used to emphasise that the condition provided by Proposition 6.7 is only a sufficient one. Let us illustrate this with simple knots and $n = 3$. In that case, a necessary and sufficient condition can be found in [15], from which we showed that it was actually sufficient to prescribe

$$\begin{aligned} &h_k \le 3\pi/2 \quad \text{if the } k\text{th section is trigonometric and next to an hyperbolic section,} \\ &h_k + h_{k-1} < 2\pi \quad \text{if the } k\text{th and the } (k-1)\text{th sections are both trigonometric.} \end{aligned} \tag{46}$$

Proposition 6.6 shows that the conditions (46) are sufficient to obtain C^{n-1} mixed hyperbolic/trigonometric splines good for design not simply for $n = 3$, but even for any $n \ge 3$.

7 Integral recurrence relations for B-splines

In consequence of our results, in any PW-spline space good for design, B-splines do satisfy integral recurrence relations (Theorem 7.1 below). This fact is sufficiently important to deserve a brief outline of its proof.

Indeed, any system $(w_1, \ldots, w_n)$ of piecewise weight functions on $(I, \mathbb{T})$ leads to integral relations in any associated spline space $ECPS(\mathbb{1}, w_1, \ldots, w_n; \mathbb{K})$. Though proved in [23] in a slightly different context (finite number of interior knots in a closed bounded interval), all results of [23] remain true in the present one. Let us observe that we were not far from integral recurrence relations in (35). However, once we have piecewise weight functions and associated piecewise differential operators, it is interesting to obtain them in connection with a very nice *differentiation formula for blossoms*. Select any strictly increasing function $U_1 \in ECP(\mathbb{1}, w_1) \subset ECP(\mathbb{1}, w_1, \ldots, w_n)$, and let u_1 denote its blossoms in n variables, *i.e.*, the blossom of U_1 considered as an element of $ECP(\mathbb{1}, w_1, \ldots, w_n)$. Then, given any $F \in ECP(\mathbb{1}, w_1, \ldots, w_n)$, the blossoms $f^{\{1\}}$ of $L_1 F := DF/w_1 \in ECP(\mathbb{1}, w_2, \ldots, w_n)$ (defined on I^{n-1}), can be calculated as follows

$$f^{\{1\}}(x_1, \ldots, x_{n-1}) := \frac{f(x_1, \ldots, x_{n-1}, z) - f(x_1, \ldots, x_{n-1}, y)}{u_1(x_1, \ldots, x_{n-1}, z) - u_1(x_1, \ldots, x_{n-1}, y)}, \tag{47}$$

where y, z denote any two distinct points in I. For a detailed proof of (47), we refer to [23] and other references therein, limiting ourselves to mentioning the two points which it is based on:

- firstly, the geometrical definition of blossoms;
- secondly, the fact that the blossom u_1 is strictly increasing in each variable on I^n.

Let us now recall the main steps from (47) towards integral recurrence relations. Due to (47), if P_ℓ, $\ell \in \Lambda_n(\mathbb{K})$, are the poles of a spline $S \in ECPS(\mathbb{1}, w_1, \dots, w_n; \mathbb{K})$, then the poles $P_\ell^{\{1\}}$, $\ell \in \Lambda_{n-1}(\mathbb{K})$, of the spline $L_1 S \in ECPS(\mathbb{1}, w_2, \dots, w_n; \mathbb{K})$ can be calculated as follows

$$P_\ell^{\{1\}} = \frac{P_\ell - P_{\ell-1}}{u_{1,\ell} - u_{1,\ell-1}}, \quad \ell \in \Lambda_{n-1}(\mathbb{K}), \tag{48}$$

where $u_{1,\ell} := u_1(\xi_{\ell+1}, \dots, \xi_{\ell+n})$, $\ell \in \Lambda_n(\mathbb{K})$, are the poles of U_1 considered as an element of $ECPS(\mathbb{1}, w_1, \dots, w_n; \mathbb{K})$. Let us denote by $N_\ell^{\{1\}}$, $\ell \in \Lambda_{n-1}(\mathbb{K})$, the B-spline basis in

$$L_1\,(ECPS(\mathbb{1}, w_1, \dots, w_n; \mathbb{K})) = ECPS(\mathbb{1}, w_2, \dots, w_n; \mathbb{K}).$$

For a given $j \in \Lambda_n(\mathbb{K})$, applying (48) with $P_\ell := \delta_{\ell,j}$, $\ell \in \Lambda_n(\mathbb{K})$, yields

$$L_1 N_j = \frac{1}{u_{1,j} - u_{1,j-1}} N_j^{\{1\}} - \frac{1}{u_{1,j+1} - u_{1,j}} N_{j+1}^{\{1\}}, \tag{49}$$

given that the quantity $\dfrac{1}{u_{1,\ell} - u_{1,\ell-1}} N_\ell^{\{1\}}$ is to be replaced by 0 whenever $\ell \notin \Lambda_{n-1}(\mathbb{K})$. This eventually yields:

$$N_\ell(x) = \frac{\int_{\xi_\ell}^x w_1(t) N_\ell^{\{1\}}(t)\,dt}{u_{1,\ell} - u_{1,\ell-1}} - \frac{\int_{\xi_{\ell+1}}^x w_1(t) N_{\ell+1}^{\{1\}}(t)\,dt}{u_{1,\ell+1} - u_{1,\ell}}, \quad \ell \in \Lambda_n(\mathbb{K}), \tag{50}$$

with the convention that, for any index $\ell \notin \Lambda_{n-1}(\mathbb{K})$,

$$\frac{\int_{\xi_\ell}^x w_k(t) N_\ell^{\{k\}}(t)\,dt}{u_{1,\ell} - u_{1,\ell-1}} := 0 \text{ if } x \le \xi_\ell^-, \quad := 1 \quad \text{if } x \ge \xi_\ell^+. \tag{51}$$

From the support and normalisation properties of B-splines it is then easy to derive the following interesting property of the poles of U_1:

$$u_{1,\ell} - u_{1,\ell-1} = \int_{\xi_\ell}^{\xi_{\ell+n}} w_1(t) N_\ell^{\{1\}}(t)\,dt, \quad \ell \in \Lambda_{n-1}(\mathbb{K}). \tag{52}$$

Replacing all numerators in (53) by the appropriate integrals yields the usual form of the first step of integral recurrence relations.

As a result of the previous reminder and of Theorems 3.12 and 4.2, we can therefore state, with $w_i^* := w_{n+1-i}$, $1 \le i \le n$:

Theorem 7.1 *For $1 \le i \le n$, consider a positive piecewise function $w_i^* \in PC^{i-1}(I, \mathbb{T})$. For any $x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-]$, and any $\ell \in \Lambda_0(\mathbb{K})$, let us set:*

$$N_\ell^0(x) := 1 \quad \text{when } \xi_\ell^+ \le x \le \xi_{\ell+1}^-, \quad := 0 \quad \text{otherwise},$$

and then, for $1 \le k \le n$,

$$N_\ell^k(x) := \frac{\int_{\xi_\ell}^x w_k^*(t) N_\ell^{k-1}(t)\, dt}{\int_{\xi_\ell}^{\xi_{\ell+k}} w_k^*(t) N_\ell^{k-1}(t)\, dt} - \frac{\int_{\xi_{\ell+1}}^x w_k^*(t) N_{\ell+1}^{k-1}(t)\, dt}{\int_{\xi_{\ell+1}}^{\xi_{\ell+k+1}} w_k^*(t) N_{\ell+1}^{k-1}(t)\, dt}, \quad \ell \in \Lambda_k(\mathbb{K}), \quad (53)$$

with a convention similar to (51). Then, the set $\mathbb{S}$ composed of all piecewise functions on $(I, \mathbb{T})$ defined by $S(x) := \sum_{\ell\in\Lambda_n(\mathbb{K})} a_\ell N_\ell^n(x)$, $x \in \cup_{k\in\mathbb{Z}}[t_k^+, t_{k+1}^-]$, where a_ℓ, $\ell \in \Lambda_n(\mathbb{K})$, are any real numbers, is well defined. It is a PW-spline space based on $\mathbb{K}$ which is good for design and the sequence N_ℓ^n, $\ell \in \Lambda_n(\mathbb{K})$, forms its B-spline basis. This procedure yields all PW-spline spaces based on $\mathbb{K}$ which are good for design.

Remark 7.2 Formulæ (53) are the ones used by Bister and Prautzsch [2] in order to build B-spline-type bases in the general framework of *integral positive locally Lebesgue integrable functions* $w_1^*, \dots, w_n^*$, in the sense that their integrals on any compact interval with positive length are positive (see also Theorem 2.17 in [3]). Apparently the authors of [2] have never exploited their integral approach in the large interesting subclass considered in Theorem 7.1 (*i.e.*, the case where $(\mathbb{1}, w_n^*, \dots, w_1^*)$ is a system of piecewise weight functions on $(I, \mathbb{T})$) leading to PEC-spline spaces. We could actually have used their results as a key-step towards existence of a B-spline basis in any spline space of the form $ECPS(\mathbb{1}, w_n^*, \dots, w_1^*)$. Nevertheless, we did prefer to explain the latter existence via the fact that the space $ECP(w_1, \dots, w_n)$ is an ECP-space on $(I, \mathbb{T})$, due to a generalised version of Rolle's (see Sect. 3). We took this opportunity to precisely describe the matrices connecting the left/right ordinary derivatives at the knots. The main point of our approach is that it enabled us to take advantage of the powerfulness of blossoms both in the space $ECP(\mathbb{1}, w_n^* \dots, w_1^*)$ and in all spline spaces based on it. On the other hand, we would like to emphasise that the difficult part of Theorem 7.1 (stated in its last sentence) is a totally new result. It could not be expected from Bister–Prautzsch's work. It is a consequence of our Theorem 4.2 and, therefore, of the forcefulness of blossoms.

As a consequence of Theorem 4.8 we can even state:

Theorem 7.3 *In any piecewise PW-spline space $\mathbb{S}$ which is good for design, there exist infinitely many ways to obtain the B-spline basis by means of integral recurrence relations of the type (53).*

Remark 7.4 In any PW-spline space satisfying (i) of Theorem 5.3, differentiation/integration formulæ for B-spline-like bases can be derived from the ones for B-splines after division by an appropriate positive function w_0.

8 Conclusion

In this article we achieved a simple practical description of all spaces of geometrically continuous piecewise Chebyshevian splines which can be used for geometric design, with prescribed section-spaces or not. Once more blossoms and their three fundamental properties proved to be extremely efficient tools. The crucial part of the article consisted in showing that any such spline space, supposed to be good for design, can be associated with infinitely many piecewise differential operators which are the analogues of the nth order ordinary differentiation for ordinary polynomial splines. Its proof strongly involved knot insertion and B-spline bases which are direct products of the three properties of blossoms.

By way of conclusion, let us mention that the existence of such operators is important in many ways. We have already seen that it guarantees differentiation/integration formulæ for B-splines, but it can also serve for other purposes, e.g. for simultaneous approximation.

References

1. Barry, P.J.: de Boor-Fix dual functionals and algorithms for Tchebycheffian B-splines curves. Constr. Approx. **12**, 385–408 (1996)
2. Bister, D., Prautzsch, H.: A new approach to Tchebycheffian B-splines. In: Curves and Surfaces with Applications in CAGD, pp. 35–41. Vanderbilt University Press, Nashville (1997)
3. Bosner, T.: Knot insertion algorithms for Chebyshev splines, PhD thesis, Department of Mathematics, University of Zagreb (2006)
4. Buchwald, B., Mühlbach, G.: Construction of B-splines for generalized spline spaces generated from local ECT-systems. J. Comput. Appl. Math. **159**, 249–267 (2003)
5. Costantini, P., Lyche, T., Manni, C.: On a class of weak Tchebycheff systems. Numer. Math. **101**, 333–354 (2005)
6. Dyn, N., Edelman, A., Micchelli, C.A.: On locally supported basis functions for the representation of geometrically continuous curves. Analysis **7**, 313–341 (1987)
7. Dyn, N., Micchelli, C.A.: Piecewise polynomial spaces and geometric continuity of curves. Numer. Math. **54**, 319–337 (1988)
8. Goodman, T.N.T.: Properties of β-splines. J. Approx. Theory **44**, 132–153 (1985)
9. Goodman, T.N.T.: Shape preserving representations. In: Lyche, T., Schumaker, L.L. (eds.) Mathematical Methods in Computer Aided Geometric Design, pp. 333–357. Academic Press, New York (1989)
10. Karlin, S.: Total Positivity. Stanford University Press, Stanford (1968)
11. Karlin, S.J., Studden, W.J.: Tchebycheff Systems: With Applications in Analysis and Statistics. Wiley Interscience, New York (1966)
12. Koch, P.E., Lyche, T. : Exponential B-splines in tension. In: Chui, C.K., Schumaker, L.L., Ward, J.D. (eds.) Approximation Theory VI, pp. 361–364. Academic Press, New York (1990)
13. Lyche, T.: A recurrence relation for Chebyshevian B-splines. Constr. Approx. **1**, 155–178 (1985)
14. Mazure, M.-L.: Blossoming: a geometrical approach. Constr. Approx. **15**, 33–68 (1999)
15. Mazure, M.-L.: Chebyshev splines beyond total positivity. Adv. Comput. Math. **14**, 129–156 (2001)
16. Mazure, M.-L.: B-spline bases and osculating flats: one result of H-P. Seidel revisited. ESAIM-Math. Model. Numer. Anal. **36**, 1177–1186 (2002)
17. Mazure, M.-L.: Blossoms and optimal bases. Adv. Comp. Math. **20**, 177–203 (2004)
18. Mazure, M.-L.: On the equivalence between existence of B-spline bases and existence of blossoms. Constr. Approx. **20**, 603–624 (2004)
19. Mazure, M.-L.: Chebyshev spaces and Bernstein bases. Constr. Approx. **22**, 347–363 (2005)
20. Mazure, M.-L.: Towards existence of piecewise Chebyshevian B-spline bases. Numer. Alg. **39**, 399–414 (2005)
21. Mazure, M.-L.: Extended Chebyshev piecewise spaces characterised via weight functions. J. Approx. Theory **145**, 33–54 (2007)

22. Mazure, M.-L.: Understanding recurrence relations for Chebyshevian B-splines via blossoms. J. Comput. Appl. Math. **219**, 457–470 (2008)
23. Mazure, M.-L.: On differentiation formulæ for Chebyshevian Bernstein and B-spline bases. Jaén J. Approx. **1**, 111–143 (2009)
24. Mazure, M.-L.: Ready-to-blossom bases and the existence of geometrically continuous piecewise Chebyshevian B-splines. CRAS **347**, 829–834 (2009)
25. Mazure, M.-L.: Finding all systems of weight functions associated with a given extended Chebyshev space. J. Approx. Theory **163**, 363–376 (2011)
26. Mazure, M.-L., Laurent, P.-J.: Piecewise smooth spaces in duality: application to blossoming. J. Approx. Theory **98**, 316–353 (1999)
27. Mühlbach, G.: ECT-B-splines defined by generalized divided differences. J. Comput. Appl. Math. **187**, 96–122 (2006)
28. Mühlbach, G., Tang, Y.: Computing ECT-B-splines recursively. Numer. Algebra **41**, 35–78 (2006)
29. Pottmann, H.: The geometry of Tchebycheffian splines. Comput. Aided Geom. Design **10**, 181–210 (1993)
30. Pottmann, H.: A geometric approach to variation diminishing free-form curve schemes. In: Peña, J.M. (ed.) Shape Preserving Representations in Computer-Aided Geometric Design, pp. 119–131. Nova Science Publication (1999)
31. Prautzsch, H.: B-Splines with Arbitrary Connection Matrices. Constr. Approx. **20**, 191–205 (2004)
32. Ramshaw, L.: Blossoms are polar forms. Comput. Aided Geom. Design **6**, 323–358 (1989)
33. Schumaker, L.L.: Spline Functions. Wiley Interscience, New York (1981)
34. Seidel, H.-P.: New algorithms and techniques for computing with geometrically continuous spline curves of arbitrary degree. Math. Model. Numer. Anal. **26**, 149–176 (1992)