Numer. Math. (2011) 118:429–456
DOI 10.1007/s00211-011-0363-6

Numerische Mathematik

Galerkin and Runge–Kutta methods: unified formulation, a posteriori error estimates and nodal superconvergence

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Received: 14 August 2010 / Published online: 25 February 2011
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Abstract We unify the formulation and analysis of Galerkin and Runge–Kutta methods for the time discretization of parabolic equations. This, together with the concept of reconstruction of the approximate solutions, allows us to establish a posteriori superconvergence estimates for the error at the nodes for all methods.

Mathematics Subject Classification (2000) 65M15 · 65M50

1 Introduction

In this paper, we consider a unified formulation of the most popular implicit single-step time-stepping methods. Based on this formulation, we derive a posteriori estimates

Georgios Akrivis was partially supported by the Institute of Applied and Computational Mathematics, FORTH, Greece. Charalambos Makridakis was partially supported by the RTN-network HYKE, HPRN-CT-2002-00282, the EU Marie Curie Dev. Host Site, HPMD-CT-2001-00121 and the program Pythagoras of EPEAEK II. Ricardo H. Nochetto was partially supported by NSF grants DMS-0505454 and DMS-0807811.

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and in particular superconvergence estimates at the nodes of the time partition. We consider Runge–Kutta (RK for short) schemes, in particular interpolatory implicit RK (collocation or perturbed collocation schemes), as well as the Continuous and the Discontinuous Galerkin methods (cG and dG for short). We formulate the methods, cast them into a unified abstract method, and carry out the a posteriori error analysis for *linear* equations in a Hilbert space setting: Seek $u : [0, T] \to D(A)$ satisfying

$$\begin{cases} u'(t) + Au(t) = f(t), \quad 0 < t < T, \\ u(0) = u^0, \end{cases} \tag{1.1}$$

with A a positive definite, self-adjoint, linear operator on a Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with domain $D(A)$ dense in H, and a given forcing term $f : [0, T] \to H$.

This paper is a continuation of our previous work on a posteriori error estimation via appropriate reconstructions of the approximate solutions with focus on different methods: dG and Runge–Kutta–Radau methods [19]; Crank–Nicolson [3]; and cG schemes and RK collocation methods [4]. Here we provide a unified treatment of essentially all single-step time-stepping schemes. Notably, with the aid of reconstructions, we even cast cG and dG schemes into a unified formulation, a connection we explore here.

For previous a posteriori results using various discretization methods which are covered by the formulation presented in this paper we refer, e.g., to [3,4,9,10,12,13,16,19,22]. Regarding a posteriori superconvergence results at the time nodes we refer to [9] where fully discrete schemes combined with dG piecewise linear time discretization methods were considered.

1.1 Notation

For convenience, we use the notation $F(t, v) = Av - f(t)$ to describe the time-stepping methods for (1.1).

To discretize (1.1) we consider piecewise polynomial functions in arbitrary partitions $0 = t^0 < t^1 < \cdots < t^N = T$ of $[0, T]$, and let $J_n := (t^{n-1}, t^n]$ and $k_n := t^n - t^{n-1}$. We denote by $\mathcal{V}_q^{\rm d}, q \in \mathbb{N}_0$, the space of possibly *discontinuous* functions at the nodes t^n that are piecewise polynomials of degree at most q in time in each subinterval J_n, i.e., $\mathcal{V}_q^{\rm d}$ consists of functions $g : [0, T] \to D(A)$ of the form

$$g|_{J_n}(t) = \sum_{j=0}^{q} t^j w_j, \quad w_j \in D(A),$$

without continuity requirements at the nodes t^n; the elements of $\mathcal{V}_q^{\rm d}$ are taken continuous to the left at the nodes t^n. Let $\mathcal{V}_q(J_n)$ consist of the restrictions to J_n of the elements of $\mathcal{V}_q^{\rm d}$. The spaces $\mathcal{H}_q^{\rm d}$ and $\mathcal{H}_q(J_n)$ are defined analogously by requiring $w_j \in H$. Furthermore, let the spaces $\mathcal{V}_q^{\rm c}$ and $\mathcal{H}_q^{\rm c}$ consist of the *continuous* elements of $\mathcal{V}_q^{\rm d}$ and $\mathcal{H}_q^{\rm d}$, respectively. For $v \in \mathcal{V}_q^{\rm d}$ we let $v^n := v(t^n)$, $v^{n+} := \lim_{t \downarrow t^n} v(t)$.

1.2 The general discretization method

To describe the discretization method we consider two operators: Π_{q-1} will be a *projection operator* to piecewise polynomials of degree $q-1$,

$$\Pi_{q-1} : C([0,T]; H) \to \oplus_{n=1}^{N} \mathcal{H}_{q-1}(J_n) \tag{1.2}$$

that does not enforce continuity at $\{t^n\}_{n=1}^{N}$. In addition, $\widetilde{\Pi}_q : \mathcal{H}_q(J_n) \to \mathcal{H}_\ell(J_n)$ is an operator mapping polynomials of degree q to polynomials of degree ℓ, with $\ell = q$ or $\ell = q-1$, depending on the example. Note that to avoid confusion, Π_{q-1} and $\widetilde{\Pi}_q$ are defined in a reference time interval and then transformed into J_n.

With the aid of these operators we define the time discrete approximation U to the solution u of (1.1) as follows: We seek $U \in \mathcal{V}_q^{\mathrm{c}}$ satisfying the initial condition $U(0) = u^0$ as well as the pointwise equation

$$U'(t) + \Pi_{q-1} F(t, \widetilde{\Pi}_q U(t)) = 0 \quad \forall t \in J_n. \tag{1.3}$$

An equivalent Galerkin formulation is

$$\int_{J_n} \big[\langle U', v\rangle + \langle \Pi_{q-1} F(t, \widetilde{\Pi}_q U(t)), v\rangle\big]\, dt = 0 \quad \forall v \in \mathcal{V}_{q-1}(J_n), \tag{1.4}$$

for $n = 1, \dots, N$. We already considered methods of this form in [4].

Recall, in particular, that the continuous Galerkin (cG) method is just

$$\int_{J_n} \big[\langle U', v\rangle + \langle F(t, U(t)), v\rangle\big]\, dt = 0 \quad \forall v \in \mathcal{V}_{q-1}(J_n), \tag{1.5}$$

i.e., $\Pi_{q-1} := P_{q-1}$, with P_ℓ denoting the (local) L^2 orthogonal projection operator onto $\mathcal{H}_\ell(J_n)$, for each n,

$$\int_{J_n} \langle P_\ell w, v\rangle\, ds = \int_{J_n} \langle w, v\rangle\, ds \quad \forall v \in \mathcal{H}_\ell(J_n).$$

It follows from (1.5) that $U \in \mathcal{V}_q^{\mathrm{c}}$ satisfies also the following pointwise equation

$$U'(t) + P_{q-1} F(t, U(t)) = 0 \quad \forall t \in J_n. \tag{1.6}$$

This method is indeed the simplest one described in (1.3), with $\Pi_{q-1} = P_{q-1}$, $\widetilde{\Pi}_q = I$.

One thus may view the class of methods (1.4) as a sort of numerical integration applied to the continuous Galerkin method. In Sect. 2 we will see that (1.4) covers all important implicit single-step time stepping methods. In particular

- the *cG method* with $\Pi_{q-1} := P_{q-1}$, and $\widetilde{\Pi}_q = I$ (the identity);
- the *RK collocation methods* (RK-C) with $\Pi_{q-1} := I_{q-1}$ and $\widetilde{\Pi}_q = I$, with I_{q-1} denoting the interpolation operator at the collocation points;
- all other *interpolatory RK methods* with $\Pi_{q-1} := I_{q-1}$, and appropriate $\widetilde{\Pi}_q$ (with $\ell = q$) described below in (2.23);
- the *dG method* with $\Pi_{q-1} := P_{q-1}$ and $\widetilde{\Pi}_q = I_{q-1}$, where I_{q-1} is the interpolation operator at the Radau points $0 < \tau_1 < \cdots < \tau_q = 1$ (so $\ell = q - 1$).

1.3 Superconvergence condition

The single-step time-stepping schemes we consider in this paper are associated to q *pairwise distinct* points $\tau_1, \dots, \tau_q \in [0, 1]$. These points are transformed to the interval J_n as

$$t^{n,i} := t^{n-1} + \tau_i k_n, \quad i = 1, \dots, q. \tag{1.7}$$

Since we are interested in Galerkin and RK methods attaining higher order accuracy at the nodes $\{t^n\}_{n=1}^N$ than globally, a phenomenon known as *superconvergence*, we assume that the points $\tau_1, \dots, \tau_q$ satisfy the orthogonality condition

$$\int_0^1 \prod_{i=1}^q (\tau - \tau_i) v(\tau)\, d\tau = 0 \quad \forall v \in \mathbb{P}_r \tag{1.8}$$

for some $1 \le r \le q - 1$, whence $q \ge 2$. Condition (1.8) is satisfied if and only if every element of $\mathbb{P}_{q+r}$ and its Lagrange interpolant at $\tau_1, \dots, \tau_q$ have the same integral, i.e., if the interpolatory quadrature formula with nodes $\tau_1, \dots, \tau_q$ integrates the elements of $\mathbb{P}_{q+r}$ exactly. For cG and collocation methods, the maximal superconvergence order is then $\mathcal{O}(k^{q+1+r})$. From now on, for these methods, we let

$$p := q + 1 + r \tag{1.9}$$

be the superconvergence order at the nodes, and call it the *superorder*.

Since $q \ge 2$, we exclude the Crank–Nicolson–Galerkin scheme ($q = 1$) from our discussion of cG methods. This is natural, since it is well known that the Crank–Nicolson–Galerkin scheme yields second order approximations, both globally and at the nodes. We also exclude the backward Euler and Crank–Nicolson methods from the present analysis of RK methods for the same reason; they yield first and second order approximations, respectively, both globally and at the nodes.

The superconvergence order p might be reduced compared to (1.9) for the Perturbed Collocation methods considered in Sect. 2.4. In that case, although (1.8) still holds, we make use only of part of it as in [21], namely orthogonality with respect to $\mathbb{P}_{r'}$, $1 \le r' \le r$. Then our assumption on the superorder p will be

$$p \ge q + 1 + r'. \tag{1.10}$$

1.4 Reconstruction

Let projection operators $\widehat{\Pi}_q$ onto $\mathcal{H}_q(J_n), n = 1, \dots, N$, satisfy the fundamental property that $\widehat{\Pi}_q w$ agrees with $\Pi_{q-1} w$ at $t^{n,i}$:

$$(\widehat{\Pi}_q - \Pi_{q-1})w(t^{n,i}) = 0, \quad i = 1, \dots, q, \quad \forall w \in C([0,T]; H). \tag{1.11}$$

For each $n = 1, \dots, N$, the *reconstruction* $\widehat{U} \in \mathcal{H}_{q+1}(J_n)$ of U is given by

$$\widehat{U}(t) := U(t^{n-1}) - \int_{t^{n-1}}^{t} \widehat{\Pi}_q \left[A\widetilde{\Pi}_q U(s) - f(s)\right] ds \quad \forall t \in J_n. \tag{1.12}$$

In view of (1.8) for $v(\tau) = 1$ and (1.11), we obtain $\widehat{U}(t^n) = U^n$ and conclude that $\widehat{U}$ is *continuous*. Differentiation of $\widehat{U}$ yields

$$\widehat{U}'(t) = -\widehat{\Pi}_q[A\widetilde{\Pi}_q U(t) - f(t)] = -\widehat{\Pi}_q F(t, \widetilde{\Pi}_q U(t)) \quad \forall t \in J_n, \tag{1.13}$$

which has a similar structure to (1.3). The idea behind (1.12) is not new: this reconstruction was introduced in [4] for cG and collocation methods, with $\widetilde{\Pi}_q = I$ and so $\widehat{\Pi}_q \widetilde{\Pi}_q U = \widehat{\Pi}_q U$. The operator $\widehat{\Pi}_q$ is here chosen as follows, depending on *discrete compatibility conditions* such as (1.22) (further discussed in Sect. 6):

- for the *cG methods*, $\widehat{\Pi}_q$ is either P_q or an interpolation operator at $q+1$ Gauss points, q from J_n and one from an adjacent interval, applied to P_{q-1}; see (6.4);
- for the *dG methods*, $\widehat{\Pi}_q = P_q$, whence $\widehat{\Pi}_q I_{q-1} U = I_{q-1} U$;
- for the *RK-C methods*, $\widehat{\Pi}_q$ is an interpolation operator at the q collocation points of J_n plus another point either in J_n or a collocation point in an adjacent interval;
- for the *perturbed RK-C methods*, $\widehat{\Pi}_q$ is the first option for RK-C methods.

If $v \in \mathcal{H}_r(J_n)$, then $\langle(\widehat{\Pi}_q - \Pi_{q-1})w, v\rangle \in \mathbb{P}_{q+r}$ and vanishes at $t^{n,i}$, $i = 1, \dots, q$. Thus, (1.8) leads to the orthogonality property

$$\int_{J_n} \langle(\widehat{\Pi}_q - \Pi_{q-1})w(s), v(s)\rangle \, ds = 0 \qquad \forall w \in C([0,T]; H), \quad v \in \mathcal{H}_r(J_n), \tag{1.14}$$

for $n = 1, \dots, N$, which will play a central role in the superconvergence analysis. To see why, subtract (1.13) from (1.3) to get

$$\widehat{U}' - U' = (\widehat{\Pi}_q - \Pi_{q-1})(f - A\widetilde{\Pi}_q U), \tag{1.15}$$

whence, in view of (1.11),

$$(\widehat{U} - U)'(t^{n,i}) = 0, \quad i = 1, \dots, q. \tag{1.16}$$

Moreover, we observe that (1.14) and (1.15) yield the orthogonality relation

$$\int_{J_n} \langle (\widehat{U} - U)', v \rangle dt = 0 \quad \forall v \in \mathcal{H}_r(J_n).$$

Integrating by parts and using the fact that $\widehat{U} - U$ vanishes at t^{n-1} and t^n, we arrive at the first abstract orthogonality condition with $r \geq 1$

$$\int_{J_n} \langle \widehat{U} - U, v \rangle \, dt = 0 \quad \forall v \in \mathcal{H}_{r-1}(J_n). \tag{1.17}$$

The second one is a further assumption on Π_{q-1}, namely for all $V \in \mathcal{H}_q(J_n)$,

$$\int_{J_n} \langle V - \Pi_{q-1} V, v \rangle dt = 0 \quad \forall v \in \mathcal{H}_{r-1}(J_n), \tag{1.18}$$

which, in view of (1.14), yields

$$\int_{J_n} \langle \widehat{\Pi}_q V - V, v \rangle dt = 0 \quad \forall v \in \mathcal{H}_{r-1}(J_n). \tag{1.19}$$

Condition (1.18) is verified by both cG and dG methods, for which $\Pi_{q-1} = P_{q-1}$, as well as by RK methods, for which $\Pi_{q-1} = I_{q-1}$; see Sect. 1.2.

Remark 1.1 (Roots of $\widehat{\Pi}_q V - V$) We notice for later use that for cG, dG and RK methods there holds

$$\Pi_{q-1} V(t^{n,i}) = V(t^{n,i}), \quad i = 1, \ldots, q, \tag{1.20}$$

for all $V \in \mathcal{H}_q(J_n)$. This is obvious for RK methods, since $\Pi_{q-1} = I_{q-1}$. It also holds for cG and dG methods since $V - \Pi_{q-1} V = V - P_{q-1} V$ is a multiple of the Legendre polynomial of degree q. Combining (1.20) with (1.11), we obtain

$$(\widehat{\Pi}_q V - V)(t^{n,i}) = 0, \quad i = 1, \ldots, q, \tag{1.21}$$

for all $V \in \mathcal{H}_q(J_n)$. □

1.5 Superconvergence estimates

For cG it is known that the error decays *a priori* with optimal rate $\mathcal{O}(k^{2q})$ at the nodes (thus $p = 2q$) [5], provided restrictive compatibility conditions of the form [1,6–8,22]

$$f \in D(A^{\rho}), \quad U^0 \in D(A^{\rho+1}) \tag{1.22}$$

hold for $1 \le \rho \le q-1$. In Subsect. 4.1 we establish the following *a posteriori* analogue of these results for the error $e := u - U$ by using duality (see Theorem 4.1)

$$|e(t^n)| \le C_I L_n \max_{1\le m\le n} \left(k_m^\rho |A^{\rho-1}\widehat{R}|_{L^\infty(J_m)} \right), \tag{1.23}$$

in terms of the residual $\widehat{R} := \widehat{U}' + A\widehat{U} - f$ with $\widehat{\Pi}_q = P_q$; we denote by $|\cdot|$ the norm of H. Note that a compatibility condition of the form (1.22) is implicitly assumed in (1.23) provided $|A^{\rho-1}\widehat{R}|_{L^\infty(J_n)}$ is bounded. We examine this discrete regularity in Sect. 6 and show that the alternative choice of $\widehat{\Pi}_q$ requires, instead of (1.22),

$$f \in D(A^\rho), \quad U^0 \in D(A^\rho). \tag{1.24}$$

Compared with the bound in $L^\infty([0,T];H)$ in Theorem 2.1 in [4], this additional regularity of $\widehat{R}$ yields the asserted extra power ρ of k_m in (1.23) at the nodes t^n. Such estimate is valid for dG as well, but the order of the residual $\widehat{R}$ is in general at most q instead of $q+1$ for cG; see Theorem 4.2 and Remarks 4.2 and 4.4.

In contrast, *superconvergence order* at the nodes for RK methods is the *classical order* of the method in the standard terminology of RK methods [14,15]. Since the seminal work of Crouzeix [8], it is known that this order is limited by requiring nontrivial conditions of the form (1.22) which may fail to be fulfilled in applications [7,17,18,22]. This lack of superconvergence at the nodes is usually called *order reduction* in the literature [7,22]. A result similar to (1.23) is established in Theorem 5.1 for collocation methods. The proof follows along the same lines as (1.23) but additional difficulties arise due to the *quadrature* effect inherent to collocation methods. Note that we avoid time derivatives of f in the final estimate. In fact, quadrature errors are quantified by higher interpolation errors of the form

$$k_m^j |A^{j-1}(f - \widehat{I}_{p-j-1} f)|_{L^\infty(J_m)}, \quad j = 1, \ldots, p-q-2, \tag{1.25}$$

in the final estimate. A similar *multiorder* splitting has been proposed in [20] to avoid the explicit use of the Bramble–Hilbert lemma. Similar results hold in the perturbed collocation case assuming that the final superconvergence order is given.

The paper is organized as follows: in Sect. 2, we cast the classes of single-step schemes into the unified formulation. In Sect. 3, we develop an error representation formula, which exploits the nature of the unified formulation and simplifies the forthcoming analysis. Sections 4 and 5 are devoted to proving a posteriori error bounds that account for nodal *superconvergence* for Galerkin and RK methods, respectively. They hinge on compatibility properties of the discrete solution U, which are further explored in Sect. 6 for all the methods.

2 Casting single-step schemes into the unified formulation

In this section, we cast various single-step time-stepping schemes, in particular interpolatory RK methods and the cG and dG methods, into the unified formulation.

It is instructive to start with the cG method, since all other methods are obtained as perturbations using appropriate operators and quadrature.

2.1 The cG method

We already described the cG method in Sect. 1.2; see (1.5) and (1.6). Here we recall some of its properties.

A natural norm for error estimates for parabolic type equations is the $L^\infty([0,T];H)$ norm. Since U is piecewise polynomial of degree q, the highest possible order of convergence in $L^\infty([0,T];H)$ is $q+1$. This is indeed the order of the cG method:

$$\max_{0\le t\le T}|u(t)-U(t)| = O(k^{q+1}) \tag{2.1}$$

with $k := \max_n k_n$ provided u is sufficiently smooth; see [5] and [22, p. 206–207]. Higher accuracy can be obtained at the nodes $\{t^n\}_{n=1}^N$ for ODEs. In particular the maximal order of the cG method at the nodes is $p=2q$, i.e.,

$$\max_n |u(t^n)-U(t^n)| = O(k^{2q}). \tag{2.2}$$

This superconvergence phenomenon for cG is well understood, [5,2]: it is due to the relation between cG methods, the q Gauss points $\tau_1,\dots,\tau_q$, and their orthogonality property (1.8) with $r=q-1$. This property is the basis of our a posteriori analysis at the nodes in Sect. 5. We stress, however, that superconvergence is not just a consequence of extra regularity but of compatibility conditions; see Sect. 6.1.

2.2 The dG method

The time discrete dG$(q-1)$ approximation V to the solution u of (1.1) is defined as follows: we seek $V\in\mathcal{V}_{q-1}^{\mathrm{d}}$ such that $V(0)=u(0)$, and

$$\int_{J_n}\big[\langle V',v\rangle+\langle F(t,V),v\rangle\big]\,dt+\langle V^{n-1+}-V^{n-1},v^{n-1+}\rangle=0, \tag{2.3}$$

for all $v\in\mathcal{V}_{q-1}(J_n)$, $n=1,\dots,N$. The dG$(q-1)$ method gives a convergence rate $O(k^q)$ in $L^\infty([0,T];H)$ and the superorder

$$\max_n |u(t^n)-V(t^n)| = O(k^{2q-1}) \tag{2.4}$$

provided certain compatibility conditions hold for u [22, Chapter 12]; see Sect. 6.2.

The approximations in the unified formulation (1.3), as well as in its variational counterpart (1.4), are *continuous* piecewise polynomials; in contrast, the dG approximations may be *discontinuous*. Therefore, in order to cast the dG method into the unified formulation we first need to associate discontinuous piecewise polynomials to continuous ones. To this end, we let $0 < \tau_1 < \cdots < \tau_q = 1$ be the abscissae of the Radau quadrature formula in the interval [0, 1]; this formula integrates exactly polynomials of degree at most $2q - 2$. The Radau nodes $t^{n,i} \in J_n$ satisfy (1.7). Now, we introduce an invertible linear operator $\tilde{I}_q : \mathcal{V}^{\mathrm{d}}_{q-1} \to \mathcal{V}^{\mathrm{c}}_q$ as follows: To every $v \in \mathcal{V}^{\mathrm{d}}_{q-1}$ we associate an element $\tilde{v} := \tilde{I}_q v \in \mathcal{V}^{\mathrm{c}}_q$ defined by locally interpolating at the Radau nodes and at t^{n-1} in each subinterval J_n, i.e., $\tilde{v}|_{J_n} \in \mathcal{V}_q(J_n)$ is such that

$$\begin{cases} \tilde{v}(t^{n-1}) = v(t^{n-1}), \\ \tilde{v}(t^{n,i}) = v(t^{n,i}), \quad i = 1, \ldots, q. \end{cases} \tag{2.5}$$

We call $\tilde{v}$ a *reconstruction* of v [19], and use this notation throughout this subsection. Exploiting the exactness of the Radau integration rule and the fact that $\tilde{v}$ and v coincide at the q Radau points in J_n, we deduce (for $q \geq 2$)

$$\int_{J_n} \langle \tilde{v} - v, w' \rangle \, dt = 0 \quad \forall v, w \in \mathcal{V}_{q-1}(J_n), \tag{2.6}$$

i.e., for $v, w \in \mathcal{V}_{q-1}(J_n)$,

$$\int_{J_n} \langle \tilde{v}', w \rangle \, dt = \int_{J_n} \langle v', w \rangle \, dt + \langle v^{n-1+} - v^{n-1}, w^{n-1+} \rangle; \tag{2.7}$$

this relation will prove useful in the sequel. This reconstruction was introduced in [19] as the main tool in the a posteriori error analysis of dG methods. Conversely, if $\tilde{v} \in \mathcal{V}^{\mathrm{c}}_q$ is given and I_{q-1} is the interpolation operator at the Radau nodes $t^{n,i}$, i.e., $(I_{q-1}\varphi)(t^{n,i}) = \varphi(t^{n,i})$, $i = 1, \ldots, q$, we can recover v locally via interpolation, i.e., $v = I_{q-1}\tilde{v}$ in J_n; furthermore, $v(0) = \tilde{v}(0)$. Thus, $I_{q-1} = \tilde{I}_q^{-1}$.

Using the *dG reconstruction* $\widetilde{V} \in \mathcal{V}^{\mathrm{c}}_q$ of $V \in \mathcal{V}^{\mathrm{d}}_{q-1}$, from (2.3) and (2.7) we obtain

$$\int_{J_n} \left[\langle \widetilde{V}', v \rangle + \langle F(t, V), v \rangle \right] dt = 0 \quad \forall v \in \mathcal{V}_{q-1}(J_n), \tag{2.8}$$

$n = 1, \ldots, N$. Obviously, (2.8) can be written in the form

$$\int_{J_n} \left[\langle \widetilde{V}', v \rangle + \langle F(t, I_{q-1}\widetilde{V}), v \rangle \right] dt = 0 \quad \forall v \in \mathcal{V}_{q-1}(J_n), \tag{2.9}$$

$n = 1, \ldots, N$. It is easily seen that the variational formulation (2.9) for the reconstruction $\widetilde{V}$ can in turn be equivalently written as a pointwise equation, namely

$$\widetilde{V}'(t) + P_{q-1} F\left(t, (I_{q-1}\widetilde{V})(t)\right) = 0 \quad \forall t \in J_n. \tag{2.10}$$

Obviously (2.10) is of the form of (1.3) with $\Pi_{q-1} = P_{q-1}$ and $\widetilde{\Pi}_q = I_{q-1}$, with I_{q-1} being the interpolation operator at the Radau points. To avoid confusion in the forthcoming analysis we denote by $U = \widetilde{V} \in \mathcal{V}_q^{\mathrm{c}}$ the continuous in time approximation associated to the dG method and rewrite (2.10) in the form of (1.3):

$$U'(t) + P_{q-1}F\left(t, (I_{q-1}U)(t)\right) = 0 \quad \forall t \in J_n, \tag{2.11}$$

with P_{q-1} and I_{q-1} as above. We emphasize that $U = \widetilde{V}$ is continuous whereas the standard dG approximation V for dG is not. We may thus wonder how the reconstruction $\widehat{U}$ of (1.12) with $\widehat{\Pi}_q = P_q$ relates to U. A simple calculation, employing (1.12) and (2.11), reveals the following interesting property for all $t \in J_n$:

$$\begin{aligned}\widehat{U}(t) &= U(t^{n-1}) - \int_{t^{n-1}}^{t} [AV - P_q f]ds \\ &= U(t) + \int_{t^{n-1}}^{t} (P_q - P_{q-1}) f ds.\end{aligned} \tag{2.12}$$

2.3 RK and collocation methods

For $q \in \mathbb{N}$, a $q-$stage RK method is described by the constants $a_{ij}, b_i, \tau_i, i, j = 1, \dots, q$, arranged in a Butcher tableau,

$$\begin{array}{ccc|c} a_{11} & \dots & a_{1q} & \tau_1 \\ \vdots & & \vdots & \vdots \\ a_{q1} & \dots & a_{qq} & \tau_q \\ \hline b_1 & \dots & b_q & \end{array}.$$

Given an approximation U^{n-1} to $u(t^{n-1})$, the n-th step of the RK method applied to (1.1) that yields the approximation U^n to $u(t^n)$ is

$$\begin{cases} U^{n,i} = U^{n-1} - k_n \sum_{j=1}^{q} a_{ij} F(t^{n,j}, U^{n,j}), \quad i = 1, \dots, q, \\ U^n \ \ = U^{n-1} - k_n \sum_{i=1}^{q} b_i F(t^{n,i}, U^{n,i})\,; \end{cases} \tag{2.13}$$

here $U^{n,i}$ are the intermediate stages and approximate $u(t^{n,i})$ with $t^{n,i}$ given by (1.7).

Let $\tilde{p}$ and $\tilde{s}$ be the largest integers such that

$$\begin{cases} \sum_{i=1}^{q} b_i \tau_i^{\ell} = \frac{1}{\ell+1}, & \ell = 0, \dots, \tilde{p}-1, \\ \sum_{j=1}^{q} a_{ij} \tau_j^{\ell} = \frac{\tau_i^{\ell+1}}{\ell+1}, & \ell = 0, \dots, \tilde{s}-1, \quad i = 1, \dots, q. \end{cases} \tag{2.14}$$

We throughout assume that $\tau_1, \dots, \tau_q$ are pairwise distinct and that the RK method is interpolatory, i.e., $\tilde{p} \geq q$. The *stage order* of the RK method is $s := \min(\tilde{s}, \tilde{p})$. It is known, [14, Theorem 7.7], that a $q-$stage RK method is equivalent to a collocation method with the same nodes if and only if its stage order $s \geq q$. The *classical* (*nonstiff*) *order* of the method is the largest integer p such that after one step of the RK method, with $y^{n-1} := y(t^{n-1})$, there holds $y(t^n) - y^n = O(k_n^{p+1})$ for smooth solutions y of ODEs with bounded derivatives; p is the *superorder* of RK. For collocation methods we have $p = \tilde{p}$, while for general RK methods $p \leq \tilde{p}$.

It is a simple matter to check that finding $U \in \mathcal{V}_q^{\mathrm{c}}$ such that

$$U'(t^{n,i}) + F\left(t^{n,i}, U(t^{n,i})\right) = 0, \quad i = 1, \dots, q, \tag{2.15}$$

for $n = 1, \dots, N$ (collocation method), is equivalent to the RK method with

$$a_{ij} := \int_0^{\tau_i} L_j(\tau) d\tau, \quad b_i := \int_0^1 L_i(\tau) d\tau, \quad i, j = 1, \dots, q,$$

with $L_1, \dots, L_q$ the Lagrange polynomials of degree $q-1$ associated with the nodes $\tau_1, \dots, \tau_q$, in the sense that $U(t^{n,i}) = U^{n,i}$, $i = 1, \dots, q$, and $U(t^n) = U^n$; see [14, Theorem 7.6]. If I_{q-1} denotes the (local) interpolation operator

$$I_{q-1} v \in \mathcal{H}_{q-1}(J_n): \quad (I_{q-1} v)(t^{n,i}) = v(t^{n,i}), \quad i = 1, \dots, q, \tag{2.16}$$

then (2.15) can be written equivalently as follows because U' and $I_{q-1} F$ are polynomials of degree $q-1$ in each interval J_n:

$$U'(t) + I_{q-1} F(t, U(t)) = 0 \quad \forall t \in J_n. \tag{2.17}$$

Thus the *RK Collocation* (*RK-C*) class (2.17) is a subclass of (1.3) with $\Pi_{q-1} = I_{q-1}$ and $\widetilde{\Pi}_q = I$.

As in the case of cG we are interested in estimating the $L^\infty([0, T]; H)$ norm of the error. If U is piecewise polynomial of degree q, then the highest possible order of convergence in $L^\infty([0, T]; H)$ is $q+1$, namely

$$\max_{0 \leq t \leq T} |u(t) - U(t)| = O(k^{q+1}) \tag{2.18}$$

with $k := \max_n k_n$. This is indeed the order of the RK-C class provided that the *classical order* p of the method satisfies $p \geq q+1$. Estimators of optimal order in $L^\infty([0,T];H)$ in this case were derived in our recent paper [4]. The classical order for ODEs is the convergence order observed at the nodes $\{t^n\}_{n=1}^N$:

$$\max_n |u(t^n) - U(t^n)| = O(k^p). \tag{2.19}$$

Recall that the classical order of the RK-C method is $p > q$, if and only if the nodes $\tau_1, \ldots, \tau_q$ satisfy the orthogonality condition (1.8) for $r \geq 0$, where $r = p-q-1$, [14, Theorem 7.8]. If $p > q+1$, then classical order p of the RK-C method corresponds to the maximal *superconvergence order* at the nodes $\{t^n\}_{n=1}^N$ (*superorder* for short) [14,15]. The superorder is reduced unless nontrivial compatibility conditions of the form (1.22) are valid for $1 \leq \rho \leq r$. We explore their discrete counterpart in Sect. 6.3 and apply them to derive a posteriori error estimates in Sect. 5.1; compare with the a priori results in [17,18]. We refer to, e.g., [4], for three well-known important classes of collocation methods, namely the RK Gauss–Legendre, the RK Radau IIA, and the RK Lobatto IIIA methods.

We now discuss the *collocation reconstruction*. According to Sect. 1.4 there are two variants depending on the choice of $\widehat{\Pi}_q$. The first one, called $\widehat{I}_q$, is the *extended* interpolation operator defined on continuous functions v over J_n by

$$\widehat{I}_q v \in \mathcal{H}_q(J_n): \quad (\widehat{I}_q v)(t^{n,i}) = v(t^{n,i}), \quad i = 0, \ldots, q, \tag{2.20}$$

where $t^{n,0} \neq t^{n,i}$, $i = 1, \ldots, q$, and $t^{n,0} \in J_n$; compare with (2.16). An immediate by-product of (2.14) is (1.14) with $\widehat{\Pi}_q = \widehat{I}_q$ and $\Pi_{q-1} = I_{q-1}$. We now define a *RK-C reconstruction* $\widehat{U} \in \mathcal{H}_{q+1}(J_n)$ of the approximation U by

$$\widehat{U}(t) = U(t^{n-1}) - \int_{t^{n-1}}^{t} \left[AU(s) - \widehat{I}_q f(s)\right] ds \quad \forall t \in J_n; \tag{2.21}$$

comparing with (1.12) we see that $\widehat{\Pi}_q = \widehat{I}_q$ and $\widetilde{\Pi}_q = I$. If $\tau_q = 1$ and $\tau_1 > 0$, then the natural choice would be $t^{n,0} = t^{n-1}$. This is what happens with the RK Radau IIA methods (with $\tau_q = 1$) for $q > 1$; see also Sect. 6.3.

The second alternative for $\widehat{\Pi}_q$ exploits the fact that the discrete compatibility at collocation points is better than at the nodes and picks $t^{n,0} \notin J_n$; see Sect. 6.3:

- Case $\tau_q < 1$ (t^n is not a collocation point): If $n = 1$, let $t^{1,0} = t^{2,1}$ be the first collocation node in J_2. If $n > 1$, let $t^{n,0} = t^{n-1,q}$ be the last collocation node in J_{n-1}.
- Case $\tau_q = 1$ (t^n is a collocation point): If $n > 1$, let $t^{n,0} = t^{n-1} \in J_n$. If $n = 1$ let $t^{1,0} = 0$, provided that $AU^0 - f(0) \in D(A^\rho)$, or as in the case $\tau_q < 1$, let $t^{1,0} = t^{2,1}$.

The *RK-C reconstruction* $\widehat{U}$ is defined according to (1.12) with $\widetilde{\Pi}_q = I$:

$$\widehat{U}(t) = U(t^{n-1}) - \int_{t^{n-1}}^{t} \widehat{\Pi}_q \left[AU(s) - f(s)\right] ds \quad \forall t \in J_n. \tag{2.22}$$

2.4 Interpolatory RK and perturbed collocation methods

We consider here q-stage RK methods with stage order $s < q$, because $s \geq q$ corresponds to collocation RK methods (see Sect. 2.3), and follow Nørsett and Wanner [21] to cast them into the unified form (1.3). To this end, we introduce the operator $\widetilde{\Pi}_q : \mathcal{H}_q(J_n) \to \mathcal{H}_q(J_n)$ given by

$$\widetilde{\Pi}_q v(t) = v(t) + \sum_{j=1}^{q} N_j \left(\frac{t - t^{n-1}}{k_n}\right) v^{(j)}(t^{n-1})\, k_n^j\,, \quad t \in J_n. \tag{2.23}$$

Here $N_j \in \mathbb{P}_{q-1}$ are given polynomials. For $\tau_1, \ldots, \tau_q \in [0, 1]$ pairwise distinct, the corresponding perturbed collocation method is: Seek $U \in \mathcal{V}_q^{\mathrm{c}}$ such that

$$U'(t^{n,i}) + F\left(t^{n,i}, \left(\widetilde{\Pi}_q U\right)\left(t^{n,i}\right)\right) = 0, \quad i = 1, \ldots, q, \tag{2.24}$$

for $n = 1, \ldots, N$. Since U' and $I_{q-1}F$ are polynomials of degree $q-1$ in each interval J_n, it follows that (2.24) is equivalently written as

$$U'(t) + I_{q-1}F(t, \widetilde{\Pi}_q U(t)) = 0 \quad \forall t \in J_n, \tag{2.25}$$

with I_{q-1} the interpolation operator of (2.16).

It is proved in [21] that *each interpolatory RK method with pairwise different* $\tau_1, \ldots, \tau_q$ *is equivalent to a perturbed collocation method*. Note that, for a given RK method, the polynomials N_j needed in the definition (2.23) of $\widetilde{\Pi}_q$ can be explicitly constructed. Thus all interpolatory RK schemes can be written in the form (1.3) with $\Pi_{q-1} = I_{q-1}$ and $\widetilde{\Pi}_q$ as in (2.23). Since we consider collocation methods separately in Sect. 2.3, we assume that $\widetilde{\Pi}_q \neq I$ for perturbed collocation methods.

Assessing both the convergence order in the $L^\infty([0, T]; H)-$norm and the superorder is not obvious due to the presence of $\widetilde{\Pi}_q$. The stage order is $s < q$ if and only if $N_j = 0$, $j = 1, \ldots, s$, [21]; thus $\|v - \widetilde{\Pi}_q v\|_{L^\infty([0,T);H)} = O(k_n^{s+1})$ and the order in $L^\infty([0, T); H)$ is $s + 1$. In the perturbed collocation case we assume throughout, that the superorder p satisfies $p \geq q + r' + 1$, for some r', $1 \leq r' \leq r$, [21], where r is the full orthogonality polynomial order in (1.8). Since the order of the residual of (2.24) is just $s + 1$, we resort to the *perturbed collocation reconstruction* to derive superconvergence estimates at the nodes, namely $\widehat{U} \in \mathcal{H}_{q+1}(J_n)$ defined by

$$\widehat{U}(t) := U(t^{n-1}) - \int_{t^{n-1}}^{t} \left[A\widetilde{\Pi}_q U(s) - \widehat{I}_q f(s)\right] ds \quad \forall t \in J_n, \tag{2.26}$$

with

$$\widetilde{\Pi}_q v(t) = v(t) + \sum_{j=s+1}^{q} N_j \left(\frac{t - t^{n-1}}{k_n}\right) v^{(j)}(t^{n-1})\, k_n^j\,, \quad t \in J_n. \tag{2.27}$$

In Sect. 5.2 we develop such an analysis upon using only part of (1.8), i.e., orthogonality with respect to $\mathbb{P}_{r'}$, $1 \le r' \le \min\{r, 2s - q + 1\}$, and the following orthogonality properties for N_j [21, Theorem 10]

$$\int_0^1 N_j(\tau)v(\tau)\, d\tau = 0 \quad \forall v \in \mathbb{P}_{q+r'-j}, \quad j = s+1, \ldots, q. \tag{2.28}$$

3 Nodal error representation formula

To avoid repetitions, in this section we derive a nodal error representation formula for the unified method (1.3). It will be used in Sect. 4 for Galerkin methods and in Sect. 5 for RK schemes.

Let $\widehat{R}$ be the residual of $\widehat{U}$,

$$\widehat{R}(t) := \widehat{U}'(t) + A\widehat{U}(t) - f(t). \tag{3.1}$$

Subtracting (3.1) from the differential equation in (1.1), we obtain the equation

$$\hat{e}'(t) + A\hat{e}(t) = -\widehat{R}(t), \tag{3.2}$$

for the error $\hat{e} := u - \widehat{U}$, which we rewrite in the form

$$\hat{e}'(t) + A\hat{e}(t) = R_{\widehat{U}}(t) + R_{\widetilde{\Pi}_q}(t) + R_{\widehat{\Pi}_q}(t) + R_f(t) \tag{3.3}$$

with

$$R_{\widehat{U}} := A(U - \widehat{U}), \quad R_{\widehat{\Pi}_q} := A(\widehat{\Pi}_q - I)\widetilde{\Pi}_q U, \quad R_f := f - \widehat{\Pi}_q f, \tag{3.4}$$

and

$$R_{\widetilde{\Pi}_q}(t) := A(\widetilde{\Pi}_q U - U). \tag{3.5}$$

We set $\widehat{R}_I := R_{\widehat{U}} + R_{\widehat{\Pi}_q} + R_f$, and observe that $R_{\widehat{\Pi}_q}$ vanishes when $\widehat{\Pi}_q$ is a projector over $\mathcal{H}_q(J_n)$ whereas $R_{\widetilde{\Pi}_q}$ vanishes when $\widetilde{\Pi}_q = I$.

We resort to a duality argument. For $n \in \{1, \dots, N\}$, we let ζ be the solution of

$$\begin{cases} -\zeta' + A\zeta = 0 \quad \text{in} \quad (0, t^n), \\ \zeta(t^n) = \hat{e}(t^n). \end{cases} \tag{3.6}$$

Then, the following strong stability estimate holds true [22, Lemma 12.5]

$$\max_{t \in J_n} |\zeta(t)| + \int_0^{t^{n-1}} |\zeta'(t)|\, dt \le L_n |\hat{e}(t^n)| \tag{3.7}$$

with

$$L_n := \left(\log \frac{t^n}{k_n}\right)^{1/2} + 1. \tag{3.8}$$

We use (3.3) and (3.6), as well as $\hat{e}(0) = 0$, to get the error representation formula

$$\begin{aligned} |e(t^n)|^2 = |\hat{e}(t^n)|^2 &= \int_0^{t^n} \langle \hat{e}, \zeta \rangle'\, dt = \int_0^{t^n} \big[\langle \hat{e}', \zeta \rangle + \langle \hat{e}, \zeta' \rangle\big]\, dt \\ &= \int_0^{t^n} \big[\langle \hat{e}', \zeta \rangle + \langle \hat{e}, A\zeta \rangle\big]\, dt = \int_0^{t^n} \langle \hat{e}' + A\hat{e}, \zeta \rangle\, dt = -\int_0^{t^n} \langle \widehat{R}, \zeta \rangle\, dt. \end{aligned}$$

We now wonder about possible orthogonality properties of the right-hand side. In view of (1.17) and (1.19), we easily infer that

$$\int_{J_n} \langle A(\widehat{U} - U), v \rangle\, dt = \int_{J_n} \langle A(\widehat{\Pi}_q - I)\widetilde{\Pi}_q U, v \rangle dt = 0 \tag{3.9}$$

for all $v \in \mathcal{H}_{r-1}(J_n)$. We recall the range of r for cG and RK-C. For cG we have $r = q - 1$, so our assumption excludes the case $q = 1$, namely the Crank–Nicolson–Galerkin method. In this case, however, the superorder $2q$ coincides with the order $q + 1$; therefore, in the sequel we assume $q \ge 2$ for cG. On the other hand, the superorder p of RK-C satisfies $q + 2 \le p \le 2q$. Since the interpolatory quadrature formula with nodes $\tau_1, \dots, \tau_q$ integrates polynomials of degree at most $p - 1$ exactly (see (1.8) and (1.9)), and $R_{\widehat{U}} \in \mathcal{H}_{q+1}$ vanishes at $t^{n,1}, \dots, t^{n,q}$, we get (1.17) with $r = p - q - 1 \ge 1$.

We now make use of (3.9) to rewrite $|e(t^n)|^2$ as follows

$$|e(t^n)|^2 = V + Q + \widetilde{V} \tag{3.10}$$

with

$$V := \sum_{m=1}^{n} \int_{J_m} \langle \widehat{R}_I(t), \zeta - P_{\rho-1}\zeta \rangle \, dt, \quad Q := \sum_{m=1}^{n} \int_{J_m} \langle R_f, P_{\rho-1}\zeta \rangle \, dt,$$

$$\widetilde{V} := \sum_{m=1}^{n} \int_{J_m} \langle R_{\widetilde{\Pi}_q}(t), \zeta \rangle \, dt,$$

where $P_{\rho-1}\zeta$ is the orthogonal projection of ζ on $\mathcal{H}_{\rho-1}(J_m)$, for $1 \le \rho \le r$ and $m = 1, \dots, n$. The first term V is the variational component of the error whereas the second term Q is the quadrature part of it. To estimate term V we assume in Sect. 4 and 5 the following *compatibility condition* for all $t \in [0, T]$

$$U(t) \in D(A^{\rho}), \widehat{R}_I(t) \in D(A^{\rho-1}), \quad 1 \le \rho \le r, \tag{3.11}$$

and further explore its validity in Sect. 6. The last term $\widetilde{V}$ accounts for the perturbation due to $\widetilde{\Pi}_q$ and vanishes when $\widetilde{\Pi}_q = I$, in which case $\widehat{R}_I = -\widehat{R}$.

Both terms Q and $\widetilde{V}$ may vanish. For the Galerkin methods, both cG and dG, we have $\Pi_{q-1} = P_{q-1}$ whence, utilizing (1.14), we deduce for $1 \le \rho \le r$

$$\begin{aligned} \int_{J_m} \langle R_f, P_{\rho-1}\zeta \rangle dt &= \int_{J_m} \langle f - \widehat{\Pi}_q f, P_{\rho-1}\zeta \rangle dt \\ &= \int_{J_m} \langle f - P_{q-1} f, P_{\rho-1}\zeta \rangle dt = 0. \end{aligned} \tag{3.12}$$

The second variational term $\widetilde{V}$ vanishes for cG and collocation methods because $\widetilde{\Pi}_q = I$. In contrast, we account for Q in Sect. 5 for RK methods and for $\widetilde{V}$ in Sect. 4.2 for dG and 5.2 for perturbed collocation schemes.

4 Nodal superconvergence for Galerkin schemes

In this section we establish a posteriori estimates for the errors at the nodes for Galerkin methods. Our point of departure is the nodal error representation formula (3.10) along with the fact that $Q = 0$ for both cG and dG, as shown in (3.12).

The first variational term V in (3.10) can be handled in a unified manner for both Galerkin and RK methods; we examine it next. We split V as $V = V_1 + V_2$ with

$$V_1 := \sum_{m=1}^{n-1} \int_{J_m} \langle \widehat{R}_I, \zeta - P_{\rho-1}\zeta \rangle \, dt, \quad V_2 := \int_{J_n} \langle \widehat{R}_I, \zeta - P_{\rho-1}\zeta \rangle \, dt. \tag{4.1}$$

To estimate V_1, we observe that $\zeta = A^{-1}\zeta'$ and

$$\left| \int_{J_m} \langle \widehat{R}_I, \zeta - P_{\rho-1}\zeta \rangle \, dt \right| \leq \int_{J_m} |A^{\rho-1}\widehat{R}_I| \, |A^{-(\rho-1)} \left(\zeta - P_{\rho-1}\zeta\right)| \, dt.$$

Consequently,

$$\begin{aligned} \left| \int_{J_m} \langle \widehat{R}_I, \zeta - P_{\rho-1}\zeta \rangle dt \right| &\leq |A^{\rho-1}\widehat{R}_I|_{L^\infty(J_m)} \int_{J_m} |A^{-(\rho-1)} \left(\zeta - P_{\rho-1}\zeta\right)| dt \\ &\leq C_I k_m^\rho |A^{\rho-1}\widehat{R}_I|_{L^\infty(J_m)} \int_{J_m} |A^{-(\rho-1)}\zeta^{(\rho)}| \, dt \\ &= C_I k_m^\rho |A^{\rho-1}\widehat{R}_I|_{L^\infty(J_m)} \int_{J_m} |\zeta'| \, dt, \end{aligned}$$

where C_I is a suitable interpolation constant depending on ρ. This implies

$$|V_1| \leq C_I \max_{1\leq m\leq n-1} \left(k_m^\rho |A^{\rho-1}\widehat{R}_I|_{L^\infty(J_m)}\right) \int_0^{t^{n-1}} |\zeta'| \, dt. \tag{4.2}$$

Similarly,

$$\begin{aligned} |V_2| &\leq \int_{J_n} |A^{\rho-1}\widehat{R}_I| \, |A^{-(\rho-1)} \left(\zeta - P_{\rho-1}\zeta\right)| \, dt \\ &\leq k_n |A^{-(\rho-1)} \left(\zeta - P_{\rho-1}\zeta\right)|_{L^\infty(J_n)} \, |A^{\rho-1}\widehat{R}_I|_{L^\infty(J_n)} \\ &\leq C_I k_n^\rho |A^{-(\rho-1)}\zeta^{(\rho-1)}|_{L^\infty(J_n)} \, |A^{\rho-1}\widehat{R}_I|_{L^\infty(J_n)}, \end{aligned}$$

whence

$$|V_2| \leq C_I k_n^\rho \max_{t\in J_n} |\zeta(t)| \, |A^{\rho-1}\widehat{R}_I|_{L^\infty(J_n)}. \tag{4.3}$$

4.1 Nodal superconvergence for cG

We now focus on cG, which means $\rho \leq r = q-1$. The derivation above thus becomes the following result.

Theorem 4.1 (Nodal superconvergence for cG) *Let $q \geq 2$ and $\widehat{R} \in D(A^{\rho-1})$ hold for some $1 \leq \rho \leq q-1$ and all $t \in [0, T]$. Then, the following a posteriori error* *estimate is valid for the cG method of order q*

$$|e(t^n)| \le C_I L_n \max_{1\le m\le n} k_m^{\rho} |A^{\rho-1}\widehat{R}|_{L^\infty(J_m)}, \tag{4.4}$$

where C_I is an interpolation constant depending on ρ and L_n is given in (3.8).

Proof It suffices to insert (3.7) into (4.2) and (4.3) and observe that $\widehat{R}_I = -\widehat{R}$. □

Corollary 4.1 (Explicit error estimate) *If χ_q, φ_{q+1} are the polynomials*

$$\chi_q(\tau) := \prod_{i=1}^{q}(\tau - \tau_i), \quad \varphi_{q+1}(\tau) := (q+1)\int_0^{\tau} \chi_q(s)\, ds,$$

then the following representation formulas are valid

$$\widehat{U}(t) - U(t) = \frac{1}{(q+1)!} k_n^{q+1} \widehat{U}^{(q+1)} \varphi_{q+1}\left(\frac{t-t^{n-1}}{k_n}\right),$$
$$\widehat{\Pi}_q U(t) - U(t) = k_n^{q+1} W_q \chi_q \left(\frac{t-t^{n-1}}{k_n}\right)$$

for a suitable function $W_q \in D(A^{\rho})$ which vanishes if $\widehat{\Pi}_q = P_q$. Moreover, if (3.11) holds, so does the following a posteriori estimate

$$|e(t^n)| \le C_I L_n \max_{1\le m\le n} \Big(k_m^{q+\rho+1} \alpha_q |A^{\rho} \widehat{U}^{(q+1)}|_{L^\infty(J_m)} + k_m^{q+\rho+1} \beta_q |A^{\rho} W_q|_{L^\infty(J_m)} + k_m^{\rho} |A^{\rho-1}(f - P_q f)|_{L^\infty(J_n)} \Big) \tag{4.5}$$

with C_I, L_n as in Theorem 4.1 and constants α_q, β_q given by

$$\alpha_q := \frac{1}{(q+1)!} \max_{0\le\tau\le 1} |\varphi_{q+1}(\tau)|, \quad \beta_q := \max_{0\le\tau\le 1} |\chi_q(\tau)|.$$

Proof We argue as in [4, Theorem 2.2]. It suffices to observe that (1.16) and (1.21) translate into

$$(\widehat{U}' - U')(t) = \frac{1}{q!} \widehat{U}^{(q+1)} k_n^q \chi_q \left(\frac{t-t^{n-1}}{k_n}\right),$$
$$(\widehat{\Pi}_q U - U)(t) = W_q k_n^q \chi_q \left(\frac{t-t^{n-1}}{k_n}\right).$$

Integration in time of the first term gives

$$\widehat{U}(t) - U(t) = \frac{1}{(q+1)!} k_n^{q+1} \widehat{U}^{(q+1)} \varphi_{q+1}\left(\frac{t-t^{n-1}}{k_n}\right), \tag{4.6}$$

and splitting the residual $\widehat{R}$ according to (3.4) completes the proof. □

Remark 4.1 (Roots of $\widehat{U} - U$) Relation (4.6) applies to all methods considered in this paper. In particular cases, $\chi_q, \varphi_{q+1}, \alpha_q, \beta_q$, and the roots of $\widehat{U} - U$ can be given explicitly. For instance, for the Galerkin schemes and the RK Gauss–Legendre method, $\tau_1, \ldots, \tau_q$ are the Gauss points in (0, 1) and the roots of $\widehat{U} - U$ are the $q + 1$ Lobatto points in J_n; see Sect. 3.1 in [4].

Remark 4.2 (Rate of convergence) If f admits $q + 1$ time derivatives, then all three terms in (4.5) are of the same order. Therefore, we realize that $1 \le \rho \le q - 1$ is the additional convergence rate at the nodes, so that the superorder becomes $q+\rho+1 \le 2q$ depending on the degree of data compatibility (3.11).

4.2 Nodal superconvergence for dG

We recall that $\widehat{\Pi}_q = P_q$ and $\widetilde{\Pi}_q = I_{q-1}$ for dG, whence $R_{\widehat{\Pi}_q} = 0$ and $R_{\widetilde{\Pi}_q} \neq 0$. On the other hand, (2.12) implies the orthogonality property (3.9) with $r = q - 1$. So it remains to estimate $R_{\widetilde{\Pi}_q}$.

Theorem 4.2 (Nodal superconvergence for dG) *Let* $q \ge 2$ *and* $\widehat{R} \in D(A^{\rho-1})$ *hold for some* $1 \le \rho \le q - 1$ *and all* $t \in [0, T]$. *Then, the following a posteriori error estimate is valid for the* dG$(q - 1)$ *method*

$$|e(t^n)| \le C_I L_n \max_{1\le m\le n} k_m^{\rho} |A^{\rho-1}\widehat{R}|_{L^\infty(J_m)}. \tag{4.7}$$

Proof First, we observe that according to (2.6)

$$\int_{J_n} \langle R_{\widetilde{\Pi}_q}, v\rangle\, dt = \int_{J_n} \langle A(I_{q-1}U - U), v\rangle\, dt = 0 \tag{4.8}$$

for $v \in \mathcal{V}_{q-2}(J_n)$. This orthogonality property is similar to (3.9) with $r = q - 1$. Consequently, we can combine terms $\widetilde{V}$ with V in (3.10), split $V + \widetilde{V}$ as in (4.1), and proceed thereafter as we did with term V. This completes the proof. □

Remark 4.3 (Explicit error estimate) An explicit error estimate for dG can be easily established. It suffices to combine Corollary 5.1 with the representation of the interpolation error $U - I_{q-1}U$ in terms of $U^{(q)}$.

Remark 4.4 (Rate of convergence) Notice that the highest possible order of the residual $\widehat{R}$ for dG is q in (4.7), whereas it is $q + 1$ in (4.4) for cG. The difference is due to the fact that in the dG case the residual $\widehat{R}$ contains $R_{\widetilde{\Pi}_q} = A(U - I_{q-1}U)$.

5 Nodal superconvergence for Runge–Kutta methods

In this section we establish a posteriori estimates for the nodal error for RK methods under the restrictive *compatibility condition* (3.11). Since estimating the variational term V is similar to (4.4) and (4.5), it remains to deal with the quadrature term Q in (3.10) and, in addition for perturbed RK, with the variational term $\widetilde{V}$.

5.1 Nodal superconvergence for RK-C methods

We recall that the superorder p of the RK-C method satisfies $q+2 \le p \le 2q$, whence $1 \le \rho \le r = p-q-1$.

If $P_{-1}\zeta = 0$, then the telescopic decomposition $P_{\rho-1}\zeta = \sum_{j=0}^{\rho-1}(P_j - P_{j-1})\zeta$ gives

$$\int_{J_m} \langle R_f, P_{\rho-1}\zeta\rangle\, dt = \sum_{j=0}^{\rho-1} \int_{J_m} \langle f - \widehat{I}_q f, (P_j - P_{j-1})\zeta\rangle\, dt. \tag{5.1}$$

Let $\hat{t}^{m,j} \in J_m$, with $j = 1,\dots,\rho$, be pairwise distinct and different from $t^{n,i}$, with $i = 0,\dots,q$. Let $\widehat{I}_\ell$ be the following interpolation operators of order ℓ with $\ell = q+1,\dots,q+\rho$, defined on continuous functions v on $[0,T]$ and values on $\mathcal{H}_\ell(J_m)$:

$$(\widehat{I}_\ell v)(\sigma) = v(\sigma), \quad \sigma = t^{m,i}, \hat{t}^{m,j}, \quad i = 0,\dots,q, \quad j = 1,\dots,\ell-q.$$

A simple but crucial consequence of the orthogonality condition (1.8) reads

$$\int_{J_m} \langle (\widehat{I}_{q+\rho-j} - \widehat{I}_q) f, (P_j - P_{j-1})\zeta\rangle\, dt = 0,$$

because the total polynomial degree is $q+\rho \le q+r = p-1$ and $(\widehat{I}_{q+\rho-j} f - \widehat{I}_q f)(t)$ vanishes at the nodes $t = t^{m,i}, i = 1,\dots,q$, whence it contains the factor $\prod_{i=1}^{q}(t - t^{n,i})$. Consequently, (5.1) becomes

$$\int_{J_m} \langle R_f, P_{\rho-1}\zeta\rangle\, dt = \sum_{j=0}^{\rho-1} \int_{J_m} \langle f - \widehat{I}_{q+\rho-j} f, (P_j - P_{j-1})\zeta\rangle\, dt. \tag{5.2}$$

Now, for $m = 1,\dots,n-1$,

$$\begin{aligned}
&\left| \int_{J_m} \langle f - \widehat{I}_{q+\rho-j} f, (P_j - P_{j-1})\zeta\rangle\, dt \right| \\
&\quad \le \int_{J_m} |A^{j-1}(f - \widehat{I}_{q+\rho-j} f)|\, |A^{-(j-1)}(P_j - P_{j-1})\zeta|\, dt \\
&\quad \le |A^{j-1}(f - \widehat{I}_{q+\rho-j} f)|_{L^\infty(J_m)} \int_{J_m} |A^{-(j-1)}(P_j - P_{j-1})\zeta|\, dt \\
&\quad \le C_I k_m^j |A^{j-1}(f - \widehat{I}_{q+\rho-j} f)|_{L^\infty(J_m)} \int_{J_m} |A^{-(j-1)}\zeta^{(j)}|\, dt
\end{aligned}$$

and thus

$$\left| \int_{J_m} \langle f - \widehat{I}_{q+\rho-j} f, (P_j - P_{j-1})\zeta \rangle \, dt \right| \\ \leq C_I k_m^j |A^{j-1}(f - \widehat{I}_{q+\rho-j} f)|_{L^\infty(J_m)} \int_{J_m} |\zeta'| \, dt. \tag{5.3}$$

Similarly,

$$\left| \int_{J_n} \langle f - \widehat{I}_{q+\rho-j} f, (P_j - P_{j-1})\zeta \rangle \, dt \right| \\ \leq C_I k_n^j |A^{j-1}(f - \widehat{I}_{q+\rho-j} f)|_{L^\infty(J_n)} \, |\zeta|_{L^\infty(J_n)}. \tag{5.4}$$

Invoking (5.2), (5.3) and (5.4), in conjunction with (3.7), we obtain

$$|Q| \leq C_I L_n |\hat{e}(t^n)| \sum_{j=0}^{\rho-1} \max_{1 \leq m \leq n} \left(k_m^j |A^{j-1} \left(f - \widehat{I}_{q+\rho-j} f \right) |_{L^\infty(J_m)} \right). \tag{5.5}$$

Theorem 5.1 (Superorder) *Let the superorder p of a $q-$stage RK-C method satisfy $p \in \{q+2, \ldots, 2q\}$ and let $\widehat{R} \in D(A^{\rho-1})$ for $1 \leq \rho \leq r = p - q - 1$. Then the following a posteriori error estimate is valid at the nodes $\{t^n\}_{n=1}^N$*

$$|e(t^n)| \leq C_I L_n \left(\mathcal{E}_1 + \mathcal{E}_2 \right),$$

with

$$\mathcal{E}_1 = \max_{1 \leq m \leq n} \left(k_m^\rho |A^{\rho-1} \widehat{R}|_{L^\infty(J_m)} \right),$$

$$\mathcal{E}_2 = \sum_{j=0}^{\rho-1} \max_{1 \leq m \leq n} \left(k_m^j |A^{j-1} \left(f - \widehat{I}_{q+\rho-j} f \right) |_{L^\infty(J_m)} \right).$$

In addition, as in Corollary 4.1, the estimator $\mathcal{E}_1$ has the explicit expression

$$\mathcal{E}_1 = \max_{1 \leq m \leq n} \Big(k_m^{q+\rho+1} \alpha_q |A^\rho \widehat{U}^{(q+1)}|_{L^\infty(J_m)} \\ + k_m^{q+\rho+1} \beta_q |A^\rho W_q|_{L^\infty(J_m)} + k_m^\rho |A^{\rho-1}(f - \widehat{I}_q f)|_{L^\infty(J_m)} \Big).$$

Remark 5.1 (Rate of convergence) The order of these estimators is $q+\rho+1 \leq p \leq 2q$, provided the solution u and forcing function f are sufficiently smooth.

5.2 Nodal superconvergence for perturbed collocation methods

The only difference with the RK-C case is the presence of the residual $R_{\widetilde{\Pi}_q}$ and corresponding term $\widetilde{V}$ in (3.10). To estimate $\widetilde{V}$, we use (2.27) to obtain

$$\begin{aligned}\widetilde{V} &= \sum_{m=1}^{n} \int_{J_m} \langle R_{\widetilde{\Pi}_q}, \zeta\rangle \, dt \\ &= \sum_{m=1}^{n} \int_{J_m} \sum_{j=s+1}^{q} k_m^j N_j \left(\frac{t-t^{m-1}}{k_m}\right) \langle A\, U^{(j)}(t^{m-1}), \zeta\rangle \, dt,\end{aligned}$$

i.e., in view of the orthogonality assumption (2.28), $\widetilde{V} = \sum_{j=s+1}^{q} \widetilde{V}_j$ with

$$\widetilde{V}_j := \sum_{m=1}^{n} k_m^j \int_{J_m} N_j \left(\frac{t-t^{m-1}}{k_m}\right) \langle A\, U^{(j)}(t^{m-1}), \zeta - P_{\rho_j}\zeta\rangle \, dt,$$

$j = s+1, \dots, q$, and $\rho_j := q + \rho - j \le q + r - j$. In analogy to (4.1), we write $\widetilde{V}_j = \widetilde{V}_j^1 + \widetilde{V}_j^2$ with

$$\begin{cases} \widetilde{V}_j^1 := \sum\limits_{m=1}^{n-1} k_m^j \int\limits_{J_m} N_j \left(\frac{t-t^{m-1}}{k_m}\right) \langle A\, U^{(j)}(t^{m-1}), \zeta - P_{\rho_j}\zeta\rangle \, dt \\ \widetilde{V}_j^2 := k_n^j \int\limits_{J_n} N_j \left(\frac{t-t^{n-1}}{k_n}\right) \langle A\, U^{(j)}(t^{n-1}), \zeta - P_{\rho_j}\zeta\rangle \, dt. \end{cases} \tag{5.6}$$

To estimate $\widetilde{V}_j^1$, we proceed as in the proof of (4.2) to obtain

$$\begin{aligned}&\left| \int_{J_m} \langle A\, U^{(j)}(t^{m-1}), \zeta - P_{\rho_j}\zeta\rangle \, dt \right| \\ &\quad \le \int_{J_m} |A^{\rho_j+1}\, U^{(j)}(t^{m-1})|\, |A^{-\rho_j} \left(\zeta - P_{\rho_j}\zeta\right)| \, dt \\ &\quad \le C k_m^{\rho_j+1} |A^{\rho_j+1}\, U^{(j)}|_{L^\infty(J_m)} \int_{J_m} |A^{-\rho_j} \zeta^{(\rho_j+1)}| \, dt \\ &\quad = C k_m^{\rho_j+1} |A^{\rho_j+1}\, U^{(j)}|_{L^\infty(J_m)} \int_{J_m} |\zeta'| \, dt;\end{aligned}$$

therefore

$$|\widetilde{V}_j^1| \leq C \max_{1\leq m\leq n-1} \left(k_m^{q+\rho+1} |A^{q+\rho+1-j} U^{(j)}|_{L^\infty(J_m)}\right) \int_0^{t^{n-1}} |\zeta'| dt, \tag{5.7}$$

with a constant C proportional to $\|N_j\|_{L^\infty(0,1)}$. Furthermore, as in (4.3), we get

$$\begin{aligned} |\widetilde{V}_j^2| &\leq C_I k_n^{j+\rho_j} |A^{-\rho_j} \zeta^{\rho_j}|_{L^\infty(J_n)} \int_{J_n} |A^{\rho_j+1} U^{(j)}| \, dt \\ &\leq C_I k_n^{q+\rho+1} \max_{t\in J_n} |\zeta(t)| \, |A^{q+\rho+1-j} \, U^{(j)}|_{L^\infty(J_n)}, \end{aligned} \tag{5.8}$$

with C as in (5.7). Consequently, combining the above arguments with the analysis of Sect. 4.1 and 5.1 we conclude the final a posteriori bound for Perturbed-RK-C methods.

Theorem 5.2 (Superorder for Perturbed-RK-C) *Let $p > q$ be the superorder and $s < q$ be the stage order of a $q-$stage Perturbed-RK-C method. Let r', $1 \leq r' \leq r$, with r defined in (1.8), satisfy*

$$r' = \min\{2s - q + 1, p - q - 1\}.$$

Let (2.28), and the compatibility condition (3.11) hold for $1 \leq \rho \leq r'$. Then the following a posteriori error estimate is valid at the nodes $\{t^n\}_{n=1}^N$

$$|e(t^n)| \leq C_I L_n \left(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3\right),$$

where C_I is an interpolation constant, L_n is given by (3.8), $\mathcal{E}_1$ and $\mathcal{E}_2$ are defined in Theorem 5.1 with r' in the place of r, and $\mathcal{E}_3$ is given by

$$\mathcal{E}_3 = C \max_{1\leq m\leq n} \left(k_m^{q+\rho+1} \max_{s+1\leq j\leq q} |A^{q+\rho+1-j} U^{(j)}|_{L^\infty(J_m)}\right),$$

with a constant C_3 that depends on $\max_{s+1\leq j\leq q} \|N_j\|_{L^\infty(0,1)}$.

Remark 5.2 (Order of perturbed collocation methods) Our assumptions in Theorem 5.2 mimic those of the a priori theory [21, Theorem 10]: let $N_j = 0$, $j = 1, \ldots, s$, $N_j \in \mathbb{P}_j$, $j = s+1, \ldots, q$, let (1.8) and (2.28) be valid for $r' \in \{1, \ldots, r\}$ and $2(s+1) \geq q + r'$. Then *the (classical) order p of the perturbed collocation method is at least $q + r' + 1$* [21, Theorem 10]. Therefore, our Theorem 5.2 gives an *a posteriori* counterpart of [21, Theorem 10] with the same formal order.

6 Discrete compatibility

The nodal superconvergence estimates of the previous two sections require rather stringent *discrete compatibility* conditions of U and its reconstruction $\widehat{U}$ that mimic the a priori error analysis [8], [22, Chapter 12], [5]:

$$U(t), \widehat{U}(t) \in D(A^\rho), \quad \widehat{R}_I(t) \in D(A^{\rho-1}), \quad 1 \leq \rho \leq r. \tag{6.1}$$

Note that for periodic boundary conditions the compatibility conditions are natural; indeed, compatibility and smoothness requirements coincide in this case. In this section we discuss (6.1) in detail for the Galerkin and collocation methods. We leave out of our discussion Perturbed Collocation methods since, as the simpler Collocation case suggests, the derivation of compatibility conditions is heavily case dependent. Notice though that these conditions might be derived following the reasoning of Sect. 6.3 once a given method is at hand.

6.1 Compatibility for cG

The solution U of (1.6) satisfies

$$U'(t^{n,i}) + AU(t^{n,i}) = P_{q-1}f(t^{n,i}), \quad i = 1, \dots, q, \tag{6.2}$$

because $P_{q-1}U(t^{n,i}) = U(t^{n,i})$ at the Gauss points $t^{n,i}$. The regularity of the nodal values U^n for cG is not better than that of U^0 whereas the smoothness of $U(t^{n,i})$ is better. We quantify this statement now and exploit it below.

Lemma 6.1 (Discrete compatibility for cG) *If* $\rho \geq 1$, $f(t) \in D(A^{\rho-1})$ *for all* $t \in [0, T]$ *and* $U^0 \in D(A^\rho)$*, then* $U(t) \in D(A^\rho)$ *for all* $t \in [0, T]$*. If in addition* $f(t) \in D(A^\rho)$ *for all* $t \in [0, T]$*, then* $U(t^{n,i}) \in D(A^{\rho+1})$ *for all* $1 \leq i \leq q$ *and* $1 \leq n \leq N$*.*

Proof We use a nested induction argument in ρ and n. Suppose that the first assertion is true for $\rho \geq 1$. This implies that $U'(t) \in D(A^\rho)$ for all $t \in [0, T]$ because $U(t)$ is a polynomial in time with coefficients in $D(A^\rho)$. On the other hand, $f(t) \in D(A^\rho)$ yields $P_{q-1}f(t) \in D(A^\rho)$ for all $t \in [0, T]$. This, in conjunction with (6.2), translates into $U(t^{n,i}) \in D(A^{\rho+1})$ and yields the first assertion.

Since we only have q Gauss points, this is not enough to conclude regularity of $U(t)$, which is a polynomial of degree $\leq q$. To do so, we employ an induction argument in n, and assume $U^{n-1} \in D(A^{\rho+1})$. This is true for $n = 1$ by assumption. If this is true for $n \geq 1$, then $U(t)$ belongs to $D(A^{\rho+1})$ at $q + 1$ distinct points of J_n, whence $U(t) \in D(A^{\rho+1})$ for all $t \in J_n$ and so $U^n \in D(A^{\rho+1})$. This concludes the induction argument in n and leads to the first assertion for $\rho + 1$. □

We now recall the definition of $\widehat{U} \in \mathcal{H}^{c}_{q+1}$

$$\widehat{U}(t) := U(t^{n-1}) - \int_{t^{n-1}}^{t} \left[A\widehat{\Pi}_q U(s) - \widehat{\Pi}_q f(s)\right] ds \quad \forall t \in J_n, \tag{6.3}$$

where the interpolation operator $\widehat{\Pi}_q$ must satisfy the fundamental property (1.11). The first choice $\widehat{\Pi}_q = P_q$ gives $P_p U = U$, which is as smooth as U^0 but not better.

Corollary 6.1 (L^2-projection) *Let $\widehat{\Pi}_q = P_q$ in (6.3). If $f(t) \in D(A^\rho)$ for all $t \in [0, T]$ and $U^0 \in D(A^{\rho+1})$, then condition (6.1) holds.*

Proof Applying Lemma 6.1, we deduce $U(t) \in D(A^{\rho+1})$ for all $t \in [0, T]$. We next combine (6.3), namely $\widehat{U}(t) \in D(A^\rho)$ for all $t \in [0, T]$, with (3.4) to obtain (6.1). □

In contrast, we construct as in [4] an alternative operator $\widehat{\Pi}_q$. Given a piecewise smooth function v, let $\widehat{I}_q v \in \mathcal{V}_q(J_n)$ be the restriction to J_n of the polynomial of degree $\le q$ that interpolates v at the q Gauss points $t^{n,i}$ of J_n, plus $t^{n,0}$,

$$\widehat{I}_q v(t^{n,i}) = v(t^{n,i}), \quad i = 0, \ldots, q,$$

the latter being the last Gauss point $t^{n-1,q}$ of J_{n-1} if $n > 1$ or the first Gauss point $t^{2,1}$ of J_2 if $n = 1$. To define $\widehat{\Pi}_q v$, we first project v onto $\mathcal{V}_{q-1}$ and next interpolate the resulting piecewise polynomial of degree $\le q - 1$

$$\widehat{\Pi}_q v := \widehat{I}_q(P_{q-1} v). \tag{6.4}$$

Since $\Pi_{q-1} = P_{q-1}$, the fundamental property (1.11) holds. Moreover, since $P_{q-1}|_{\mathbb{P}_q} = I_{q-1}$, $\widehat{\Pi}_q U$ interpolates U at the $q + 1$ Gauss points $t^{n,i}$, $i = 0, \ldots, q$, and thus improves the discrete compatibility relative to Corollary 6.1.

Corollary 6.2 (Interpolation at Gauss points) *Let $\widehat{\Pi}_q$ in (6.3) be defined by (6.4). If $f(t) \in D(A^\rho)$ for all $t \in [0, T]$ and $U^0 \in D(A^\rho)$, then condition (6.1) holds.*

Proof Since $f(t) \in D(A^\rho)$, we have $P_{q-1} f \in D(A^\rho)$ and thus $\widehat{\Pi}_q f \in D(A^\rho)$. Apply Lemma 6.1 to infer that $U(t^{n,i}) \in D(A^{\rho+1})$ at all Gauss points $t^{n,i}$. Hence, (6.3) implies $\widehat{U}(t) \in D(A^\rho)$ for all $t \in [0, T]$ and (3.4) completes the proof. □

6.2 Compatibility for dG

Since dG is a dissipative scheme, the nodal values U^n are smoother than U^0 (smoothing effect); see Lemma 6.2. We start with the pointwise counterpart of (6.2), namely (2.10), which we write in the form

$$\widetilde{V}'(t^{n,i}) + AV(t^{n,i}) = P_{q-1} f(t^{n,i}), \quad i = 1, \ldots, q. \tag{6.5}$$

Note the presence of both $\widetilde{V} = U$ and $V = I_{q-1}\widetilde{V}$ in (6.5), which however coincide at the q Radau points $t^{n,i}$; moreover $\widetilde{V}(t^{n-1}) = V(t^{n-1})$ according to (2.5). Recall that I_{q-1} is the interpolation operator at the q Radau points in J_n.

Lemma 6.2 (Discrete compatibility for dG) *If $\rho \ge 1$, $f(t) \in D(A^{\rho-1})$ for all $t \in [0,T]$ and $U^0 \in D(A^\rho)$, then $V(t), \widetilde{V}(t) \in D(A^\rho)$ for all $t \in [0,T]$. Moreover, if also $f(t) \in D(A^\rho)$ for all $t \in [0,T]$, then $V(t^{n,i}) \in D(A^{\rho+1})$ at all Radau points $\{t^{n,i}\}_{i=1}^q$, $V(t) \in D(A^{\rho+1})$ for all $t \in [0,T]$, and $U^n = V(t^n) \in D(A^{\rho+1})$ for all $n = 1,\dots,N$.*

Proof Proceed as in Lemma 6.1 and use the fact that the last Radau point in J_n coincides with the rightmost node $t^{n,q} = t^n$. □

Recall that the dG reconstruction $\widehat{U} \in \mathcal{V}^{\mathrm{c}}_{q+1}$ reads

$$\widehat{U}(t) = U(t^{n-1}) - \int_{t^{n-1}}^{t} \big[A I_{q-1}U(s) - P_q f(s)\big]\,ds, \quad \forall t \in J_n. \tag{6.6}$$

Corollary 6.3 (Interpolation at Radau points) *If $f(t) \in D(A^\rho)$ for all $t \in [0,T]$ and $U^0 \in D(A^\rho)$, then condition (6.1) holds for dG.*

Proof Apply Lemma 6.2 to deduce $I_{q-1}U(t) = V(t) \in D(A^{\rho+1})$ and $U(t) = \widetilde{V}(t) \in D(A^\rho)$ for all $t \in [0,T]$. Since $U(t^{n-1}) \in D(A^\rho)$ for $n \ge 1$, (6.6) yields (6.1). □

6.3 Compatibility for RK-C

The collocation method reads

$$U'(t^{n,i}) + AU(t^{n,i}) = f(t^{n,i}), \quad i = 1,\dots,q, \tag{6.7}$$

according to (2.17). The *RK-C reconstruction* $\widehat{U} \in \mathcal{H}_{q+1}(J_n)$ is given by

$$\widehat{U}(t) = U(t^{n-1}) - \int_{t^{n-1}}^{t} \widehat{I}_q(AU - f)(s)\,ds \quad \forall t \in J_n\,, \tag{6.8}$$

with $\widehat{I}_q$ the operator defined on (6.4) that interpolates at the collocation points $\{t^{n,i}\}_{i=1}^q$ of J_n plus an extra point which may or may not belong to J_n.

The compatibility conditions for RK-C depend partly on the particular choice of the collocation points and the resulting smoothing properties of the methods. In all cases the following simple observation will be important.

Lemma 6.3 (Discrete compatibility for RK-C) *Let $\rho \ge 1$. If $U^0, f(t) \in D(A^\rho)$ for all $t \in [0,T]$, then $U(t) \in D(A^\rho)$ for all $t \in [0,T]$ and for all $1 \le n \le N$*

$$U^{n,i} \in D(A^{\rho+1}), \quad i = 1,\dots,q. \tag{6.9}$$

If, in addition,

$$U^{n,0} \in D(A^{\rho+1}), \quad n = 1, \dots, N, \tag{6.10}$$

then

$$\widehat{I}_q(AU - f)(t), \quad \widehat{U}(t) \in D(A^{\rho}) \quad \forall t \in [0, T]. \tag{6.11}$$

Proof Let $t \in J_n$. Since $U(t) \in D(A^{\rho})$ and is a polynomial, so is $U'(t)$. This, in conjunction with the regularity of f and (6.8), implies (6.9). If, in addition, (6.10) is valid, then $\widehat{I}_q$ interpolates functions of $D(A^{\rho+1})$ at $q+1$ distinct points, whence $\widehat{I}_q U(t) \in D(A^{\rho+1})$ for all $t \in [0, T]$. Therefore, (6.11) follows immediately. □

Next we find conditions on the data U^0, f and the reconstruction $\widehat{U}$ which guarantee $\widehat{U} \in D(A^{\rho})$ depending on the method and nodes $\{t^{n,i}\}_{i=0}^{q}$. To this end, we proceed as in [4, Sect. 5.1], namely we examine two cases in accordance with the location of node τ_1 i.e., whether $\tau_1 > 0$ or $\tau_1 = 0$.

Corollary 6.4 (Case $\tau_1 > 0$) *Let $\rho \geq 1$, and U^0, $f(t) \in D(A^{\rho})$ for all $t \in [0, T]$. If $\tau_q < 1$, then let $t^{1,0} = t^{2,1}$ be the first collocation node in J_2, and $t^{n,0} = t^{n-1,q}$ be the last collocation node in J_{n-1} if $n > 1$. If $\tau_q = 1$, then let $t^{1,0} = t^{2,1}$ and $t^{n,0} = t^{n-1}$, for $n \geq 2$. In both cases, (6.1) holds.*

Proof We apply Lemma 6.3 to obtain $U^{n,i} \in D(A^{\rho+1})$, $i = 1, \dots, q$. So, it remains to check the validity of (6.10). The two choices of $t^{n,0}$, depending on whether $\tau_q < 1$ or not, guarantee that (6.10) holds, and Lemma 6.3 along with (6.8) lead to (6.1). □

Corollary 6.5 (Case $\tau_1 = 0$, $\tau_q = 1$) *Let $\rho \geq 1$, $U^0 \in D(A^{\rho+1})$ and $f(t) \in D(A^{\rho})$ for all $t \in [0, T]$. If $t^{1,0} = t^{2,1}$ is the first collocation node in J_2, and $t^{n,0} = t^{n-1,q}$ is the last collocation node in J_{n-1} if $n > 1$, then (6.1) holds.*

Proof Since $\tau_1 = 0$, $\tau_q = 1$, the solution $U(t)$ is continuous at the nodes t^n, and so is $U'(t)$ in view of (6.7). Arguing as in Lemma 6.1, we infer that $U(t) \in D(A^{\rho})$ for all $t \in [0, T]$ and $U^{n,i} \in D(A^{\rho+1})$. Applying Lemma 6.3 together with (6.8), we obtain (6.1) as desired. □

It is worth observing that, for $\tau_1 = 0$ the compatibility of $U(t^{n,i})$ is not better than that of U^0. In particular, the extra regularity $U^0 \in D(A^{\rho+1})$ guarantees $U'(0) = -AU^0 + f(0) \in D(A^{\rho})$, a necessary condition to infer that $U(t) \in D(A^{\rho})$ for $t \in J_1$.

If in addition $\tau_q < 1$, then the compatibility of U^n is worse than that of U^{n-1}. Indeed, arguing as in Lemma 6.1 we see that the regularity of $U(t)$ is one degree less than that at the collocation points $U(t^{n,i})$, and in particular at $t = t^n \neq t^{n,i}$. The explicit Euler scheme is an instructive example.

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