Numer. Math. (2011) 118:531–548
DOI 10.1007/s00211-010-0353-0

Numerische
Mathematik

Spline spaces on TR-meshes with hanging vertices

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Received: 21 May 2010 / Revised: 19 October 2010 / Published online: 14 December 2010

Abstract Polynomial spline spaces defined on mixed meshes consisting of triangles and rectangles are studied for the C^0 case. These include triangulations with hanging vertices as well as T-meshes. In addition to dimension formulae, explicit basis functions are constructed, and their supports and stability are discussed. The approximation power of the spaces is also treated.

Mathematics Subject Classification (2000) 41A15 · 65D07 · 65N30

1 Introduction

Spaces of bivariate piecewise polynomials (splines) defined on meshes consisting of triangles or rectangles play an important role in approximation theory and numerical analysis, particularly in the finite-element method. For many applications it is important to create local refinements of the mesh. Standard local refinement algorithms are designed to avoid introducing hanging vertices. However, allowing such vertices leads to much simpler refinement algorithms, and produces fewer subtriangles or subrectangles. Consequently, meshes with hanging vertices have started to attract attention in the finite element literature, see e.g. [1–3,6,14].

Spline spaces defined on meshes with hanging vertices do not seem to have been treated in the current spline literature, with the exception of some dimension results for very special meshes of rectangles called T-meshes, see [4,5,11] and references therein. Our aim in this paper is to initiate the study of such spaces by discussing dimension, the construction of basis functions, and approximation power. In this paper we focus

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on C^0 splines. Spaces of splines with higher order smoothness are more complicated (see Remark 1), and will be treated in a separate paper.

While in most applications it suffices to work with meshes containing only triangles or rectangles, it was suggested already in [8] that at times it is useful to use mixed meshes involving both. Hence, for the sake of generality, we work with so-called TR-meshes (see Definition 2.1) which allow both triangles and rectangles. Our TR-meshes include both T-meshes and triangulations with hanging vertices as special cases.

The paper is organized as follows. We introduce TR-meshes and spline spaces defined on them in Sect. 2. Properties of TR-meshes are explored in Sect. 3, while in Sect. 4 we explain how to use Bernstein–Bézier methods to work with polynomial splines on TR-meshes. Section 5 contains the main result on dimension. The construction of bases and approximation power are discussed in Sects. 6 and 7, respectively. We conclude the paper with some remarks and references.

2 Basic definitions

In this section we define the splines of interest in this paper. First we introduce the meshes.

2.1 TR-meshes

We work with mixed meshes in the plane that consist of both triangles and rectangles.

Definition 2.1 Let $\triangle := \{\Omega_i\}_{i=1}^N$ be a collection of triangles and rectangles such that the interior of the domain $\Omega := \cup\Omega_i$ is connected. In addition, suppose that any distinct pair Ω_i, Ω_j can intersect each other only at points on their boundaries. Then we call $\triangle$ a TR-mesh.

This definition does not require that the rectangles have edges oriented with the Cartesian axes, and the domains may have holes. Moreover, it also allows what we call hanging vertices.

Definition 2.2 A vertex v of a triangle or rectangle in a TR mesh is called a hanging vertex provided it lies in the interior of an edge of another triangle or rectangle.

Figure 1 (left) shows an example of a TR-mesh with hanging vertices. Note that in the finite-element literature, hanging vertices are usually referred to as hanging nodes.

TR-meshes include both T-meshes and triangulations with hanging vertices as special cases, see Fig. 2.

2.2 Splines on TR-meshes

Before defining the spaces of splines of interest in this paper, we need to introduce the spaces of polynomials to be used. Given a positive integer d, let

$$\mathcal{P}_d := \text{span}\{x^i y^j\}_{0\le i+j\le d}.$$

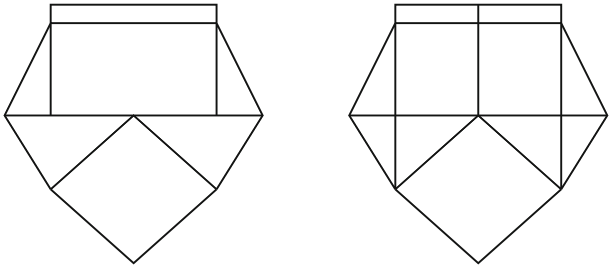

Fig. 1 A TR-mesh with hanging vertices, and its completion

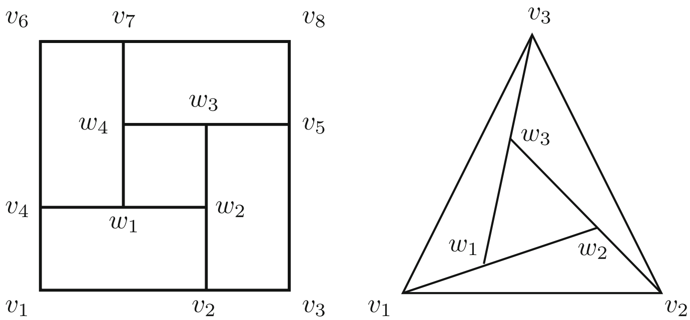

Fig. 2 A T-mesh and a triangulation with hanging vertices

This is the usual space of polynomials of total degree d used in working with splines on triangulations. We shall use it here on our triangles. However, for rectangles we want to use tensor-product polynomials. Since we are allowing the rectangles to have an arbitrary orientation with respect to the x–y axis, we need to work with a space which takes account of the orientation. Given a rectangle R, we define a local coordinate system with origin at one of the corners of R, and with axes obtained by extending the two edges of R that pass through this corner. Then we define

$$\mathcal{P}_{d,d}^{R} := \operatorname{span}\{\tilde{x}^i \tilde{y}^j\}_{0 \le i,j \le d}, \tag{2.1}$$

where $\tilde{x}$, $\tilde{y}$ are the local coordinates for R. If R is a rectangle with edges aligned with the x–y axes, then $\mathcal{P}_{d,d}^{R}$ is just the usual space of tensor-product polynomials spanned by $\{x^i y^j\}_{0 \le i,j \le d}$.

We are now ready to define our spline spaces. Given integers $0 \le r \le d$, we define the space of polynomial splines of degree d and smoothness r associated with the TR-mesh $\triangle$ by

$$\begin{aligned} \mathcal{S}_d^r(\triangle) := \Big\{ s \in C^r(\Omega) : s|_{\Omega_i} \in \mathcal{P}_d, \text{ if } \Omega_i \text{ is a triangle}, \\ \text{and } s|_{\Omega_i} \in \mathcal{P}_{d,d}^{\Omega_i}, \text{ if } \Omega_i \text{ is a rectangle} \Big\}. \end{aligned} \tag{2.2}$$

If the mesh contains only triangles, and there are no hanging vertices, $\mathcal{S}_d^r(\triangle)$ is just the space of ordinary splines studied in detail in the literature, see e.g. [10]. If the mesh contains only rectangles with edges aligned with the axes, $\mathcal{S}_d^r(\triangle)$ is the space of splines on T-meshes considered in several recent papers, see e.g. [4,5,11]. In this paper we restrict ourselves to spline spaces with $r = 0$.

3 Properties of TR-meshes

In this section we discuss several important properties of TR-meshes, including composite edges, binary trees, and completions.

3.1 Composite edges

It is important to clarify what we mean by an edge of a TR-mesh. If $e := \langle v, w\rangle$ is a line segment of $\triangle$ connecting two vertices v, w of $\triangle$ such that there are no vertices lying in the interior of e, then we call e an edge segment. If $e := \langle v, w\rangle$ is a line segment of $\triangle$ connecting two vertices v, w of $\triangle$ such that all vertices lying in the interior of e are hanging vertices, and if e cannot be extended to a longer line segment with the same property, then we say that e is a composite edge of $\triangle$. Composite edges can consist of one or more edge segments. Note that some composite edges are edges of rectangles or triangles in $\triangle$, but not all edges of rectangles or triangles are composite edges. For example, in Fig. 2 (left), $\langle v_4, w_1\rangle$ is an edge of a rectangle, but is not a composite edge since it can be extended to $\langle v_4, w_2\rangle$, which is a composite edge. Similarly, in Fig. 2 (right), $\langle v_1, w_1\rangle$ is an edge of a triangle, but it is not a composite edge, whereas $\langle v_1, w_2\rangle$ is.

3.2 Binary trees of vertices

Suppose w is a hanging vertex of $\triangle$, and let e_w be the composite edge containing w in its interior. Then we can write w as a convex combination of the endpoints w_1, w_2 of e_w. We can represent this geometrically with a graph connecting w to w_1 and w_2. Now if either w_1 or w_2 is also a hanging vertex, we can repeat this process to create a binary tree $\mathcal{T}$ associated with w.

Algorithm 3.1 *Let w be a hanging vertex of $\triangle$.*

1) *Put w on a stack.*
2) *Repeat until the stack is empty:*
   a) *Pick the top entry u on the stack. Find the ends u_1, u_2 of the composite edge containing u in its interior.*
   b) *Put u_1 and u_2 on the tree as children of u, and remove u from the stack.*
   c) *If u_1 is a hanging vertex that has not appeared earlier in the tree, add it to the top of the stack. Do the same for u_2.*

We call w the root of the tree, and refer to each of the split vertices in the tree as nodes. We refer to the vertices at the end of paths through the tree as leaves. Figure 3 shows an example of the tree associated with the hanging vertex w_1 in the triangulation with hanging vertices shown in Fig. 2 (right). For this tree, w_1 is the root, and $v_1, v_2, v_3,$

$$w_1 < \begin{matrix} v_1 \\ w_2 < \begin{matrix} v_2 \\ w_3 < \begin{matrix} v_3 \\ w_1 \end{matrix} \end{matrix} \end{matrix}$$

Fig. 3 The tree corresponding to vertex w_1 in Fig. 2 (*right*)

and w_1 are the leaves. Here w_1 is both the root and a leaf. In general, if u is a hanging vertex that is a leaf of a tree, then it also appears earlier in the tree at least once.

Lemma 3.2 *Every binary tree $\mathcal{T}$ defined in Algorithm 3.1 must contain at least one leaf which is a nonhanging vertex of $\triangle$.*

Proof Suppose all leaves of $\mathcal{T}$ are hanging vertices. Then all vertices of $\mathcal{T}$ are hanging vertices, since by definition all nodes of the tree must be hanging vertices. Now let u be a vertex of $\mathcal{T}$ with minimal y-coordinate. If there is more than one, let u be the one which also has minimal x-coordinate. The first time u appears in the tree, it was expanded as a strictly convex combination of the endpoints of the composite edge e which contains u in the interior. If e is not horizontal, then one of these endpoints has a smaller y-coordinate than u, contradicting the choice of u. If e is horizontal, then one of the endpoints has the same y-coordinate as u, but a smaller x-coordinate. This is again a contradiction of the choice of u, and we conclude that there must be a nonhanging vertex of $\triangle$ which is a leaf of $\mathcal{T}$. □

3.3 Completions of TR-meshes

Given a TR-mesh $\triangle$, we can refine it to create a TR-mesh $\bar{\triangle}$ without hanging vertices. We call $\bar{\triangle}$ a completion of $\triangle$.

Algorithm 3.3 *Repeat until there are no hanging vertices left on the current TR-mesh: Pick a hanging vertex w.*

1) *if w lies in the interior of an edge $\langle v_1, v_2\rangle$ of a triangle $T := \langle v_1, v_2, v_3\rangle$, insert the edge $\langle w, v_3\rangle$,*
2) *If w lies in the interior of an edge $\langle v_1, v_2\rangle$ of a rectangle $R := \langle v_1, v_2, v_3, v_4\rangle$, insert the edge $e := \langle w, z\rangle$ where z is a point on $\langle v_3, v_4\rangle$ and e is parallel to $\langle v_2, v_3\rangle$.*

Note that in carrying out step 2) we may introduce a new hanging vertex. This happens for example in forming the completion of the TR-mesh in Fig. 1 (left). However, it is easy to see that the algorithm always finishes in a finite number of steps. It is easy to give examples to show that there may be more than one completion of a given TR-mesh.

4 The Bernstein–Bézier representation of splines on TR-meshes

The key tool for analyzing spaces of polynomial splines on triangulations is the Bernstein–Bézier representation, see [10]. In this section we show how this representation can be extended to work with splines on TR-meshes.

4.1 The Bernstein–Bézier representation of a spline on an edge

Let s be a spline in $\mathcal{S}_d^0(\triangle)$, where $\triangle$ is a TR-mesh, and let $e := \langle v_1, v_2\rangle$ be a composite edge of $\triangle$. If e consists of several edge segments, $s|_e$ may appear to be a piecewise polynomial. However, since every hanging vertex is in the interior of an edge of some triangle or rectangle, it follows that $s|_e$ is in fact a single polynomial p on all of e.

We now recall how to represent a univariate polynomial p in Bernstein–Bézier form relative to an arbitrary line segment $e := \langle v_1, v_2\rangle$ in $\mathbb{R}^2$. Each point v on e can be written as $v = b_1 v_1 + b_2 v_2$, where $b_1 + b_2 = 1$ and $b_1, b_2 \geq 0$. Given d, let

$$B_i^{d,e}(v) := \binom{d}{i} b_1^{d-i} b_2^i, \qquad i = 0, \ldots, d, \tag{4.1}$$

be the usual univariate Bernstein polynomials associated with the interval e. At times it is more convenient to index these in terms of the domain points

$$\mathcal{D}_{d,e} := \left\{ \xi_i^{d,e} := \frac{(d-i)v_1 + i v_2}{d} \right\}, \qquad i = 0, \ldots, d, \tag{4.2}$$

associated with the edge e. For each $i = 0, \ldots, d$, $B_i^{d,e}$ can be identified with $B_\xi^{d,e}$, where $\xi = \xi_i^{d,e}$. Now we can write p in the form

$$p(v) = \sum_{i=0}^{d} c_i B_i^{d,e}(v) = \sum_{\xi \in \mathcal{D}_{d,e}} c_\xi B_\xi^{d,e}(v). \tag{4.3}$$

Note that $p(v_1) = c_0$ and $p(v_2) = c_d$. We call the coefficients in (4.3) the B-coefficients of p.

Lemma 4.1 *Suppose p is a polynomial written in the form (4.3). Then for any v in the interior of e,*

$$\min_{0 \leq i \leq d} c_i \leq p(v) \leq \max_{0 \leq i \leq d} c_i. \tag{4.4}$$

Moreover, both inequalities are strict unless all c_i have the same value, in which case p is a constant.

Proof This follows from the fact that the Bernstein polynomials $B_i^{d,e}$ are nonnegative on e and form a partition of unity. □

4.2 The Bernstein–Bézier representation of a spline on a triangle

We recall some facts about the Bernstein–Bézier form of a bivariate polynomial. For more details, see [10]. Fix a triangle $T := \langle v_1, v_2, v_3\rangle$, where the v_i's appear in counterclockwise order. For every $v \in \mathbb{R}^2$, let b_1, b_2, b_3 be its barycentric coordinates relative to T. Then the corresponding Bernstein–Bézier basis polynomials associated with d and T are the functions

$$B_{ijk}^{d,T} := \frac{d\,!}{i\,!\;j\,!\;k\,!} b_1^i b_2^j b_3^k, \qquad i + j + k = d. \tag{4.5}$$

It is common practice to index them with the set of domain points

$$\mathcal{D}_{d,T} := \left\{ \xi_{ijk}^T := \frac{i v_1 + j v_2 + k v_3}{d} \right\}_{i+j+k=d},$$

associated with d and T.

Now suppose $s \in \mathcal{S}_d^0(\triangle)$, and let $p = s|_T$. Then p has a unique representation in Bernstein–Bézier form as

$$p = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}^{d,T} = \sum_{\xi \in \mathcal{D}_{d,T}} c_\xi B_\xi^{d,T}. \tag{4.6}$$

The real numbers $\{c_\xi\}_{\xi \in \mathcal{D}_{d,T}} = \{c_{ijk}^T\}_{i+j+k=d}$ are called the B-coefficients of p relative to T, see Sect. 2.2 of [10]. The properties of the Bernstein basis polynomials ensure that for the vertex v_1 of T, $p(v_1) = c_{d00}^T$, with similar assertions at the other vertices.

4.3 The Bernstein–Bézier representation of a spline on a rectangle

Fix a rectangle $R := \langle v_1, v_2, v_3, v_4 \rangle$, where the v_i's appear in counterclockwise order. Given $v \in \mathbb{R}^2$, let u be the orthogonal projection of v onto the edge $e_1 := \langle v_1, v_2 \rangle$, and let w be the orthogonal projection of v onto the edge $e_2 := \langle v_2, v_3 \rangle$. Suppose $s \in \mathcal{S}_d^0(\triangle)$, and let $p = s|_R$. Then p has a unique representation in the form

$$p(v) = \sum_{i=0}^{d} \sum_{j=0}^{d} c_{ij}^R B_i^{d,e_1}(u) B_j^{d,e_2}(w). \tag{4.7}$$

We call the real numbers $\{c_{ij}^R\}_{i,j=0}^d$ the B-coefficients of p relative to R. The properties of the Bernstein basis polynomials ensure that for the vertex v_1 of R, $g(v_1) = c_{00}^R$, with similar assertions at the other vertices. For later use, we define the set of domain points associated with d and R to be the set

$$\mathcal{D}_{d,R} := \left\{ \xi_{ij}^R := \frac{(d-i)(d-j)v_1 + i(d-j)v_2 + ijv_3 + (d-i)jv_4}{d^2} \right\}_{i,j=0}^{d}. \tag{4.8}$$

Then for each $\xi = \xi_{ij}^R \in \mathcal{D}_{d,R}$, we define $B_\xi^{d,R}(v) := B_i^{d,e_1}(u)B_j^{d,e_2}(w)$, and rewrite (4.7) as

$$p = \sum_{\xi \in \mathcal{D}_{d,R}} c_\xi^R B_\xi^{d,R}. \tag{4.9}$$

4.4 The Bernstein–Bézier representation of a spline on a TR-mesh

Give a TR-mesh $\triangle$, let $\mathcal{V}_{NH}$ be the set of all nonhanging vertices of $\triangle$, and let $\mathcal{E}_c$ be the set of all composite edges. Then for every composite edge e, triangle T, and rectangle R, we define

$$\begin{aligned}
\mathcal{D}_{d,e}^0 &:= \left\{\xi \in \mathcal{D}_{d,e} : \xi \text{ is in the interior of } e\right\},\\
\mathcal{D}_{d,T}^0 &:= \left\{\xi \in \mathcal{D}_{d,T} : \xi \text{ is in the interior of } T\right\},\\
\mathcal{D}_{d,R}^0 &:= \left\{\xi \in \mathcal{D}_{d,R} : \xi \text{ is in the interior of } R\right\}.
\end{aligned}$$

Let

$$\mathcal{D}_{d,\triangle} := \mathcal{V}_{NH} \cup \bigcup_{e\in\mathcal{E}_c} \mathcal{D}_{d,e}^0 \cup \bigcup_{T\in\triangle} \mathcal{D}_{d,T}^0 \cup \bigcup_{R\in\triangle} \mathcal{D}_{d,R}^0. \tag{4.10}$$

We call this the set of **domain points associated with** $\mathcal{S}_d^0(\triangle)$. The cardinality of this set is

$$n_d = V_{NH} + (d-1)E_c + \binom{d-1}{2}N_T + (d-1)^2 N_R, \tag{4.11}$$

where V_{NH}, E_c, N_T, N_R are the numbers of nonhanging vertices, composite edges, triangles, and rectangles of $\triangle$, respectively.

Fix $\xi \in \mathcal{D}_{d,\triangle}$. Then we define a corresponding linear functional γ_ξ defined on $\mathcal{S}_d^0(\triangle)$ as follows:

1) if ξ lies on a composite edge e of $\triangle$, let $\gamma_\xi s$ be the B-coefficient of $s|_e$ corresponding to ξ in (4.3),
2) if ξ lies in the interior of a triangle T of $\triangle$, let $\gamma_\xi s$ be the B-coefficient of $s|_T$ corresponding to ξ in (4.6),
3) if ξ lies in the interior of a rectangle R of $\triangle$, let $\gamma_\xi s$ be the B-coefficient of $s|_R$ corresponding to ξ in (4.9).

We call the set $\{\gamma_\xi s\}_{\xi\in\mathcal{D}_{d,\triangle}}$ the **B-coefficients of** s.

Theorem 4.2 *If* $\triangle$ *is a TR-mesh with no hanging vertices, then* $\dim \mathcal{S}_d^0(\triangle) = n_d$.

Proof We claim that $\mathcal{S}_d^0(\triangle)$ is in one-to-one correspondence with the set $\mathcal{D}_{d,\triangle}$. Indeed, suppose we assign arbitrary values c_ξ for each $\xi \in \mathcal{D}_{d,\triangle}$. This defines a univariate polynomial p_e of degree d associated with each composite edge e, a polynomial

$p_T \in \mathcal{P}_d$ associated with each triangle T, and a tensor-product polynomial $p_R \in \mathcal{P}^R_{d,d}$ associated with each rectangle R. It is easy to see that these polynomial pieces join together to form a continuous function on all of Ω. Conversely, given any $s \in \mathcal{S}^0_d(\triangle)$, it has a unique set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$. □

4.5 The space $\mathcal{S}^0_d(\triangle)$ as a subspace of $\mathcal{S}^0_d(\bar{\triangle})$

The key to working with $\mathcal{S}^0_d(\triangle)$ on a general TR-mesh is the observation that $\mathcal{S}^0_d(\triangle)$ is a subspace of $\mathcal{S}^0_d(\bar{\triangle})$, where $\bar{\triangle}$ is a completion of $\triangle$. To see this, suppose $e_1, \ldots, e_m$ are the edges added to $\triangle$ to form $\bar{\triangle}$ in Algorithm 3.3. Given two polynomial pieces on a pair of adjoining triangles or a pair of adjoining rectangles, we can make them be a single polynomial by forcing them to join together with C^d smoothness across the common edge. Thus,

$$\mathcal{S}^0_d(\triangle) = \left\{ s \in \mathcal{S}^0_d(\bar{\triangle}) : s \text{ is } C^d \text{ across the edge } e_i \text{ for } i = 1, \ldots, m \right\}.$$

Lemma 4.3 *Suppose $T_1 := \langle v_1, v_2, v_3 \rangle$ and $T_2 := \langle v_1, v_3, v_4 \rangle$ are two triangles such that v_2, v_3, v_4 are collinear and the triangles share the edge $e := \langle v_1, v_3 \rangle$. Let s be a C^0 spline of degree d on the triangulation $T_1 \cup T_2$. Then we can make s reduce to a single polynomial on $T_1 \cup T_2$ by enforcing $\binom{d+1}{2}$ linear conditions on its B-coefficients.*

Proof To make s be a polynomial on $T_1 \cup T_2$, we can force the two polynomials $s|_{T_1}$ and $s|_{T_2}$ to join with C^d smoothness across the edge e. To get C^1 smoothness across the edge requires d conditions, see Sect. 2.10 of [10]. For C^2 smoothness, $d-1$ additional conditions are required. Continuing, we see that the total number of conditions for C^d smoothness is $\binom{d+1}{2}$. □

Lemma 4.4 *Suppose $R_1 := \langle v_1, v_2, v_3, v_4 \rangle$ and $R_2 := \langle v_1, v_5, v_6, v_2 \rangle$ are two rectangles sharing the edge $e := \langle v_1, v_2 \rangle$. Let s be a C^0 spline on $R_1 \cup R_2$. Then we can make s reduce to a single polynomial on $R_1 \cup R_2$ by enforcing $d(d+1)$ linear conditions on its B-coefficients. If $s|_{\langle v_4, v_5 \rangle}$ is already a polynomial, we can drop d of these conditions.*

Proof The result follows easily from the fact that smoothness conditions between rectangular patches reduce to a set of smoothness conditions between univariate Bernstein polynomials defined on the knot lines perpendicular to the common edge, see e.g. p. 135 of [12]. □

5 Dimension of $\mathcal{S}^0_d(\triangle)$

We begin by establishing a lower bound for the dimension of $\mathcal{S}^0_d(\triangle)$, where $\triangle$ is an arbitrary TR-mesh. Recall that n_d is the cardinality of the set $\mathcal{D}_{d,T}$ as given by the formula in (4.11).

Theorem 5.1 *For any TR-mesh $\triangle$ and any $d \geq 1$,*

$$\dim \mathcal{S}_d^0(\triangle) \geq n_d. \tag{5.1}$$

Proof Suppose $\bar{\triangle}$ is a completion of $\triangle$, and let $e_1, \dots, e_m$ be the edges inserted in Algorithm 3.3 to form it. Set $\triangle_0 = \triangle$, and let $\triangle_i$ be the TR-mesh where the first i edges have been inserted. We know by Theorem 4.2 that (5.1) holds for $\triangle_m$. We now proceed by induction, and suppose that (5.1) holds for $\triangle_i$ with $i > 0$. We need to show it also holds for $\triangle_{i-1}$. Let $\delta_{NH}, \delta_C, \delta_T$, and δ_R be the change in the numbers of nonhanging vertices, composite edges, triangles, and rectangles in going from $\triangle_{i-1}$ to $\triangle_i$. Then the change in the formula is equal to

$$\delta_i := \delta_{NH} + (d-1)\delta_C + \binom{d-1}{2}\delta_T + (d-1)^2\delta_R.$$

On the other hand, the space $\mathcal{S}_d^0(\triangle_{i-1})$ is obtained from $\mathcal{S}_d^0(\triangle_i)$ by forcing k_i smoothness conditions across the ith inserted edge e_i, where k_i is as in Lemmas 4.3 and 4.4. We shall show that $\delta_i = k_i$, which implies (5.1) holds for $\mathcal{S}_d^0(\triangle_{i-1})$. There are two cases:
Case 1. $e_i \in \triangle_i$ is in the interior of a triangle $T \in \triangle_{i-1}$. In this case $\delta_{NH} = 1, \delta_C = 2$, $\delta_T = 1$, and $\delta_R = 0$, which gives $\delta_i = 1 + 2(d-1) + \binom{d-1}{2} = \binom{d+1}{2} = k_i$.

Case 2. $e_i \in \triangle_i$ is in the interior of a rectangle in $\triangle_{i-1}$. In this case $\delta_T = 0$, and we have two subcases:

a) Neither end of e_i is a hanging vertex of $\triangle_i$. In this case $\delta_{NH} = 2$, $\delta_C = 3$, and $\delta_R = 1$, which gives $\delta_i = 2 + 3(d-1) + (d-1)^2 = d(d+1) = k_i$.
b) One end of e_i is a hanging vertex of $\triangle_i$. In this case $\delta_{NH} = 1$, $\delta_C = 2$, and $\delta_R = 1$, which gives $\delta_i = 1 + 2(d-1) + (d-1)^2 = d^2 = k_i$.

This completes the proof. □

The following lemma is the key tool for proving our dimension result as well as important properties of the basis functions introduced in Sect. 6. Let $\mathcal{V}_H$ be the set of hanging vertices of $\triangle$.

Lemma 5.2 *Suppose $s \in \mathcal{S}_d^0(\triangle)$ is a spline with B-coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$. Then for any $\eta \in \mathcal{V}_H$,*

$$\min_{\xi \in \mathcal{D}_{d,\triangle}} c_\xi \leq s(\eta) \leq \max_{\xi \in \mathcal{D}_{d,\triangle}} c_\xi. \tag{5.2}$$

Proof We prove only the lower inequality as the upper one is similar. Let $\alpha := \min_{\xi \in \mathcal{D}_{d,\triangle}} c_\xi$, and suppose $\zeta \in \mathcal{V}_H$ is such that $s(\zeta) = \min_{\eta \in \mathcal{V}_H} s(\eta) < \alpha$. Suppose e is the composite edge containing ζ in its interior. Let β_e be the minimum of the B-coefficients associated with domain points in $\mathcal{D}_{d,e}$. Then by Lemma 4.1, $s(\zeta) \geq \beta_e$, and strict inequality holds unless all the B-coefficients associated with domain points in $\mathcal{D}_{d,e}$ have the same value. We have $\beta_e \geq s(\zeta)$ by the choice of ζ. This leads to a contradiction if either end of e is a nonhanging vertex, since the coefficients associated with nonhanging vertices are greater than or equal to $\alpha > s(\zeta)$. So assume both ends of e are hanging vertices. Then we can repeat this argument on these vertices. This is equivalent to working our way out the branches of the binary tree associated with ζ. But by Lemma 3.2, any such tree must contain at least one leaf which is a nonhanging vertex, so we eventually arrive at a contradiction. □

We are ready to prove our main result on the dimension of the spline spaces $\mathcal{S}_d^0(\triangle)$.

Theorem 5.3 *Suppose $\triangle$ is a TR-mesh, and let $d \geq 1$. Then the dimension of $\mathcal{S}_d^0(\triangle) = n_d$, where n_d is given by the formula in (4.11).*

Proof In view of Theorem 5.1, it suffices to show $\dim \mathcal{S}_d^0(\triangle) \leq n_d$. First we show that

$$\gamma_\xi s = 0, \text{ for all } \xi \in \mathcal{D}_{d,\triangle} \qquad \Rightarrow \qquad s \equiv 0, \tag{5.3}$$

for all $s \in \mathcal{S}_d^0(\triangle)$, where the γ_ξ's are the linear functionals introduced in Sect. 4.4. Given such a spline s, Lemma 5.2 shows that $s(\eta) = 0$ for all $\eta \in \mathcal{V}_H$. This implies that all B-coefficients of s associated with domain points on composite edges of $\triangle$ must be zero, which in turn implies that $s \equiv 0$ on all such edges. Now for every triangle T, $\gamma_\xi s = 0$ for all $\xi \in \mathcal{D}_{d,T}^0$, while for every rectangle R, $\gamma_\xi s = 0$ for all $\xi \in \mathcal{D}_{d,R}^0$. We conclude that $s \equiv 0$.

Now suppose that $n := \dim \mathcal{S}_d^0(\triangle) > n_d$, and let $\phi_1, \ldots, \phi_n$ be a basis for $\mathcal{S}_d^0(\triangle)$. Suppose we number the $\{\gamma_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$ as $\gamma_1, \ldots, \gamma_{n_d}$. Then since $n_d < n$, there exists a nontrivial solution $c \in \mathbb{R}^n$ of the linear system $Ac = 0$, where A is the $n_d \times n$ matrix with entries $\gamma_i \phi_j$. But then $s = \sum_{j=1}^n c_j \phi_j$ is a nontrivial spline with $\gamma_i s = 0$ for all $i = 1, \ldots, n_d$, which is impossible by (5.3). □

This theorem shows that the space $\mathcal{S}_d^0(\triangle)$ is in 1–1 correspondence with the set of domain points $\mathcal{D}_{d,\triangle}$, or in other words each spline $s \in \mathcal{S}_d^0(\triangle)$ is uniquely determined by setting its coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$.

6 A basis for $\mathcal{S}_d^0(\triangle)$

We now show how to construct a basis for $\mathcal{S}_d^0(\triangle)$. For each $\xi \in \mathcal{D}_{d,\triangle}$, let ϕ_ξ be the spline in $\mathcal{S}_d^0(\triangle)$ whose B-coefficients are given by

$$\gamma_\eta \phi_\xi = \delta_{\xi\eta}, \qquad \text{all } \eta \in \mathcal{D}_{d,\triangle}. \tag{6.1}$$

Theorem 6.1 *The set of splines $\{\phi_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$ are linearly independent and form a basis for $\mathcal{S}_d^0(\triangle)$. Moreover, for all $\xi \in \mathcal{D}_{d,\triangle}$,*

$$\phi_\xi(x, y) \geq 0, \qquad \textit{all } (x, y) \in \Omega \tag{6.2}$$

and these basis functions form a partition of unity*, i.e.,*

$$\sum_{\xi \in \mathcal{D}_{d,\triangle}} \phi_\xi \equiv 1, \qquad \text{on } \Omega. \tag{6.3}$$

Proof Suppose $\sum_{\xi \in \mathcal{D}_{d,\triangle}} c_\xi \phi_\xi \equiv 0$. Then given any $\eta \in \mathcal{D}_{d,\triangle}$, applying the linear functional γ_η defined in Sect. 4.4, we see that $c_\eta = 0$. This shows the linear independence. Since we already know that $\dim \mathcal{S}_d^0(\triangle) = \#\mathcal{D}_{d,\triangle}$, it follows that $\{\phi_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$ form a basis. Now fix ξ and consider ϕ_ξ. Then $c_\xi = 1$ and $c_\eta = 0$ for all other η in $\mathcal{D}_{d,\triangle}$. Using Lemma 5.2, it follows that $0 \le \phi_\xi(\zeta) \le 1$ for every hanging vertex ζ. Now using the fact that the Bernstein basis functions in the representations (4.3), (4.6), and (4.9) associated with an edge, triangle, or rectangle are nonnegative and form a partition of unity, we conclude that $0 \le \phi_\xi(x, y) \le 1$. Finally, suppose s is the spline with coefficients $c_\xi = 1$ for all $\xi \in \mathcal{D}_{d,\triangle}$. Then using Lemma 5.2 again, we see that $s(\zeta) = 1$ for every hanging vertex ζ. But then by the partition of unity properties of the Bernstein basis functions in the representations (4.3), (4.6), and (4.9), we conclude that $s \equiv 1$. □

We now discuss the supports $\sigma(\phi_\xi)$ of these basis functions. For any composite edge e of $\triangle$, we define $\text{star}(e)$ to be the set of triangles and rectangles which share an edge segment with e.

Definition 6.2 Let ξ be a domain point, and let e be a composite edge of $\triangle$. Then we say that the edge e depends on ξ provided either ξ lies on e, or there exists a sequence of composite edges $e_1, \ldots, e_m$ with $e = e_m$ such that $\xi \in e_1$, and one end of e_{i+1} is in the interior of e_i, for each $i = 1, \ldots, m-1$.

For each domain point $\xi \in \mathcal{D}_{d,\triangle}$, we write $\mathcal{A}_\xi$ for the set of all composite edges that depend on ξ.

Theorem 6.3 *For each $\xi \in \mathcal{D}_{d,\triangle}$, the support $\sigma(\phi_\xi)$ is contained in*

1) $\bigcup_{\tilde{e} \in \mathcal{A}_\xi} \text{star}(\tilde{e})$*, if $\xi \in \mathcal{D}_{d,e}$ for some composite edge e,*
2) *T, if $\xi \in \mathcal{D}_{d,T}^0$ for some triangle T,*
3) *R, if $\xi \in \mathcal{D}_{d,R}^0$ for some rectangle R.*

Proof Suppose ξ lies on composite edge e_1. Then since $c_\xi = 1$, $\phi_\xi|_{e_1}$ is a nontrivial polynomial on e_1. Now if e_2 is another composite edge with one end lying on e_1, it follows that the polynomial $\phi_\xi|_{e_2}$ may also be nontrivial. Continuing, we see that ϕ_ξ may have nonzero values on all edges in $\mathcal{A}_\xi$, and it follows that ϕ_ξ can have nonzero values on the set in 1). On the other hand, if e is a composite edge which is not in this chain, then ϕ_ξ must have zero values at all points on that edge, and it follows that $\phi_\xi \equiv 0$ outside the set in 1). Now suppose ξ lies in some triangle or rectangle. Then ϕ_ξ must be zero at all points on the edges of $\triangle$, and the assertions 2) and 3) follow. □

If $\triangle$ is a TR-mesh without hanging vertices, all of the basis functions in this theorem have local support. In particular, if ξ lies at a vertex, then ϕ_ξ has support on the set of

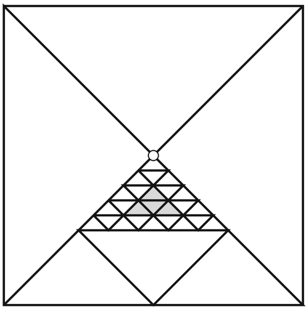

Fig. 4 A locally refined triangulation with hanging vertices

triangles and rectangles with a vertex at ξ. If ξ lies in the interior of an edge e, then ϕ_ξ has support on the triangles or rectangles containing that edge.

As soon as we allow hanging vertices, the supports of some of the basis functions can become very large. For example, in the TR-mesh in Fig. 4, the basis function corresponding to the domain point marked with an open circle has support on the entire domain minus the shaded region. On the other hand, if we choose the mesh with some care, we can ensure that on parts of the domain, the spline space does not involve basis functions with large supports. For example, if T is one of the triangles in the shaded region in Fig. 4, then all of the basis functions ϕ overlapping T (i.e. with $\sigma(\phi) \cap T = T$) have small supports. Thus, on this region we can expect better approximation of smooth functions than in other parts of Ω.

7 Approximation power

In Sect. 5.7 of [10] it is shown that if $\mathcal{S}$ is a spline space defined on an ordinary triangulation and $\mathcal{S}$ has a stable local basis, then $\mathcal{S}$ will exhibit optimal order approximation properties. In this section we show how the methods used there can be adapted to give local bounds for approximation of smooth functions by splines on TR-meshes. To prove our results, we now define a quasi-interpolation operator Q mapping $L_1(\Omega)$ into $\mathcal{S}_d^0(\triangle)$. Let

$$Qf := \sum_{\xi \in \mathcal{D}_{d,\triangle}} (\lambda_\xi f)\, \phi_\xi, \tag{7.1}$$

with linear functionals λ_ξ defined as follows:

1) Suppose $\xi \in \mathcal{D}_{d,e}$ for some composite edge e of $\triangle$, and suppose A_ξ is the triangle or rectangle that has the longest edge contained in e. If A_ξ is a rectangle, let T_ξ be the triangle with one edge in e obtained by inserting one diagonal in A_ξ. If A_ξ is a triangle, let $T_\xi = A_\xi$. Let $F_\xi f$ be the averaged Taylor expansion of f based on the largest disk contained in T_ξ, see Sect. 1.5 of [10]. Set $\lambda_\xi f$ to be the B-coefficient corresponding to ξ in the expansion (4.3) of $F_\xi f|_e$ in terms of univariate Bernstein polynomials on e.

2) Suppose $\xi \in \mathcal{D}^0_{d,T_\xi}$ for some triangle T_ξ. Let $F_\xi f$ be the averaged Taylor expansion of f based on the largest disk contained in T_ξ. Set $\lambda_\xi f$ to be the B-coefficient of F_ξ corresponding to ξ in the expansion (4.6).
3) Suppose $\xi \in \mathcal{D}^0_{d,R}$ for some rectangle R of $\triangle$. Split R into two triangles by drawing in one diagonal, and let T_ξ be such that $\xi \in T_\xi$. Let $F_\xi f$ be the averaged Taylor expansion of f based on the largest disk contained in T_ξ. Set $\lambda_\xi f$ to be the B-coefficient of F_ξ in the expansion (4.9) corresponding to ξ.

By construction, Q is a linear operator, and $Qp = p$ for all $p \in \mathcal{P}_d$. Suppose A is either a triangle or a rectangle in $\triangle$. We need an estimate for the size of $\|Qf\|_{q,A}$. Let

$$\Gamma_A := \{\xi \in \mathcal{D}_{d,\triangle} : \sigma(\phi_\xi) \cap A = A\}, \tag{7.2}$$

and set

$$\Omega_A := \bigcup_{\xi \in \Gamma_A} \sigma(\phi_\xi). \tag{7.3}$$

Let α_A be a bound on the number of hanging vertices that can lie on any composite edge in Ω_A. Let $a(A)$ be the area of A, and let $|\Omega_A|$ be the diameter of the smallest disk containing Ω_A. Define

$$\beta_A := \max_{\xi \in \Gamma_A} \frac{|T_\xi|}{\rho_\xi}, \tag{7.4}$$

and

$$\delta_A := \max_{\xi \in \Gamma_A} \frac{a(A)}{a(T_\xi)}, \tag{7.5}$$

where ρ_ξ is the diameter of the largest disk in T_ξ.

Theorem 7.1 *Let A be either a rectangle or triangle of $\triangle$, and suppose $f \in L_q(\Omega_A)$ for some $1 \le q \le \infty$. Then*

$$\|Qf\|_{q,A} \le K_0 \|f\|_{q,\Omega_A}, \tag{7.6}$$

where the constant K_0 depends only on d, α_A, β_A, and δ_A.

Proof The proof is similar to that of Theorem 5.17 in [10]. We give details only for $1 \le q < \infty$ as the proof for $q = \infty$ is similar and simpler. Fix $\xi \in \mathcal{D}_{d,\triangle}$ and let T_ξ be the triangle arising in steps 1)–3) of the construction of λ_ξ. If ξ is as in steps 2) or 3), it follows immediately from the stability of the B-form (see Theorem 2.6 of [10]) that

$$|\lambda_\xi f| \le K_1 \|F_\xi f\|_{\infty,T_\xi}, \tag{7.7}$$

where K_1 is a constant depending only on d. The situation is slightly more complicated if $\xi \in \mathcal{D}_{d,e}$. The bound (7.7) holds for the coefficients of $F_\xi f$ relative to the triangle T_ξ. However, we define λ_ξ to be the coefficient of $F_\xi f|_e$ relative to the Bernstein basis on e. Since the length of e can be up to $\alpha_A + 1$ times the length of the edge of T_ξ lying on e, it follows that (7.7) holds, but with a constant that depends on α_A.

Now using Theorem 1.1 and Lemma 1.6 of [10], we get

$$|\lambda_\xi f| \le \frac{K_2}{a(T_\xi)^{1/q}} \|f\|_{q,T_\xi}, \tag{7.8}$$

where K_2 depends only on d, α_A, and $|T_\xi|/\rho_\xi$ which is bounded by β_A. Now for any point (x, y) in A, using (6.2) and (6.3), we have

$$|Qf(x,y)| \le \sum_{\xi\in\Gamma_A} |\lambda_\xi f||\phi_\xi(x,y)| \le \max_{\xi\in\Gamma_A} |\lambda_\xi f| \sum_{\xi\in\Gamma_A} \phi_\xi(x,y) = \max_{\xi\in\Gamma_A} |\lambda_\xi f|.$$

Inserting (7.8), taking the qth power, and integrating over A, we get (7.6). □

Let A be either a rectangle or triangle in $\triangle$, and let

$$\kappa_A := \frac{|\Omega_A|}{\rho_A},$$

where ρ_A is the diameter of the largest disk that can be inscribed in A. We state our results in terms of the classical Sobolev spaces $W_q^{m+1}(\Omega_A)$ with the usual semi-norm $|f|_{m+1,q,\Omega_A}$, see [10].

Theorem 7.2 *For every $f \in W_q^{m+1}(\Omega_A)$ with $0 \le m \le d$ and $1 \le q \le \infty$,*

$$\|D_x^\nu D_y^\mu (f - Qf)\|_{q,A} \le K\ |\Omega_A|^{m+1-\nu-\mu}\ |f|_{m+1,q,\Omega_A}, \tag{7.9}$$

for all $0 \le \nu + \mu \le m + 1$. If Ω_A is convex, then K depends only on d, α_A, β_A, δ_A, and κ_A. If Ω_A is not convex, then K also depends on the Lipschitz constant of the boundary of Ω_A.

Proof By Theorem 1.9 of [10], there exists a polynomial $p \in \mathcal{P}_m$ such

$$\|D_x^i D_y^j (f - p)\|_{q,\Omega_A} \le K_3 |\Omega_A|^{m+1-i-j} |f|_{m+1,q,\Omega_A}, \tag{7.10}$$

for $0 \le i + j \le m + 1$. If Ω_A is convex, then K_3 depends only on d and the ratio $|\Omega_A|/\rho_{\Omega_A}$, where ρ_{Ω_A} is the diameter of the largest disk that can be inscribed in Ω_A. This ratio is bounded by κ_A. If Ω_A is not convex, then K_3 also depends on the Lipschitz constant of the boundary of Ω_A.

Now fix $0 \le \nu + \mu \le m + 1$. Since Q reproduces polynomials in $\mathcal{P}_d$,

$$\|D_x^\nu D_y^\mu (f - Qf)\|_{q,A} \le \|D_x^\nu D_y^\mu (f - p)\|_{q,A} + \|D_x^\nu D_y^\mu Q(f - p)\|_{q,A}.$$

In view of (7.10), it suffices to estimate the second term. Suppose A is a triangle. Then using the Markov inequality (cf. Theorem 1.2 in [10]) on the polynomial $Q(f-p)|_A$, we have

$$\|D_x^\nu D_y^\mu Q(f-p)\|_{q,A} \le \frac{K_4}{\rho_A^{\nu+\mu}}\|Q(f-p)\|_{q,A} \le \frac{K_0K_4}{\rho_A^{\nu+\mu}}\|f-p\|_{q,\Omega_A}$$
$$\le \frac{K_0K_3K_4|\Omega_A|^{m+1}}{\rho_A^{\nu+\mu}}|f|_{m+1,q,\Omega_A},$$

where the constant K_4 depends only on d. Now the ratio $|\Omega_A|/\rho_A = \kappa_A$, and combining these inequalities immediately leads to (7.9). When A is a rectangle, we apply the Markov inequality separately on the two triangles obtained by splitting A along a diagonal. □

The constants appearing in Theorem 7.2 are needed to state the result for very general TR-partitions. However, for practical TR-meshes it is usually easy to estimate them. For example, if we refine one triangle T into smaller triangle as shown in Fig. 4 and restrict our attention to a shaded triangle A, then $\alpha_A = 0$, $\delta_A = 1$, and β_A and κ_A are controlled by the smallest angle in the original triangle T. This is the kind of hierarchical refinement often used in practice, see e.g. [5,7] and the finite-element literature.

8 Remarks

Remark 1 It is known (cf. [10]) that even on ordinary triangulations, the dimension of spline spaces with smoothness $r > 0$ depends not only on the combinatorics of the partition (i.e. the numbers of vertices and edges), but on the exact geometry (i.e. the slopes of the edges attached to each vertex). This complication carries over to splines on TR-meshes. In view of this and the likelihood that splines of smoothness $r = 0$ are of the most interest to finite element practitioners, we have left the discussion of $r > 0$ to a separate paper.

Remark 2 Suppose R is a rectangle whose local coordinate system is obtained by rotating the x–y axis in the counterclockwise direction by an angle $0 \le \theta < \pi/2$. Then the space of tensor-product polynomials defined in (2.1) can be written as

$$\mathcal{P}_{d,d}^R := \text{span}\{(x\cos\theta - y\sin\theta)^i(x\sin\theta + y\cos\theta)^j\}_{0\le i,j\le d}.$$

From this expression, it is clear that $\mathcal{P}_{d,d}^R \subset \mathcal{P}_{2d}$.

Remark 3 For the special case of T-meshes, our results extend the existing theory in several ways: 1) we give a new and simple proof of the dimension of the spaces $\mathcal{S}_d^0(\triangle)$, taking careful account of the possibility of cycles of hanging vertices; 2) we explicitly construct nonnegative basis functions that form a partition of unity, and explore when they are local; 3) we establish a local approximation result. We believe all of our results are new for the special case of triangulations with hanging vertices.

Remark 4 We emphasize that Theorem 7.2 is a local approximation result. This can be used effectively in practice. For example, if the spline space $\mathcal{S}_d^0(\triangle)$ does not have enough approximation power in some triangle or rectangle A of a given TR-mesh, we can locally refine A as shown in Fig. 4 for a triangle. This gives a much smaller mesh size in the estimate (7.9) for all subtriangles and subrectangles contained in the refinement of A that do not touch the boundary of A.

Remark 5 The results here can be extended to somewhat more general partitions where we allow parallelograms in place of rectangles. In this case the space $\mathcal{P}_{d,d}^R$ associated with a parallelogram R should be defined in terms of tensor products of local affine coordinates.

Remark 6 The natural analog of TR-meshes in 3D would consist of a mix of tetrahedra and rectangular boxes (with edges not necessarily aligned with the axes). Such meshes would include 3D T-meshes, which already present new challenges. These topics are beyond the scope of this paper.

Remark 7 Peter Alfeld (see his web site at http://www.math.utah.edu/~pa) has written a Java program that is capable of numerically computing the dimension of bivariate spline spaces on regular triangulations. This code can be used to investigate spline spaces on triangulations with hanging vertices by working with a completion of $\triangle$ and enforcing C^d smoothness across all edges that were inserted to form the completion. He also has a similar Java program for working with splines on T-meshes.

Remark 8 As shown in [13], having a minimal determining set for a spline space can be very useful for numerical computations. This is part of the reason why we have given explicit minimal determining sets for the spaces considered here.

Remark 9 The spaces of splines discussed here are polynomials of total degree on each triangle, and tensor-product polynomials on each rectangle. Using total degree polynomials on all subdomains gives different spaces, which can even be defined for general rectilinear partitions. For results on the dimension of these types of spaces, see [9].

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