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Analysis of space–time discontinuous Galerkin method for nonlinear convection–diffusion problems

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Abstract The paper presents the theory of the discontinuous Galerkin finite element method for the space–time discretization of a nonstationary convection–diffusion initial-boundary value problem with nonlinear convection and linear diffusion. The problem is not singularly perturbed with dominating convection. The discontinuous Galerkin method is applied separately in space and time using, in general, different space grids on different time levels and different polynomial degrees p and q in space and time dicretization. In the space discretization the nonsymmetric, symmetric and incomplete interior and boundary penalty (NIPG, SIPG, IIPG) approximation of diffusion terms is used. The paper is concerned with the proof of error estimates in "$L^2(L^2)$"- and "DG"-norm formed by the "$L^2(H^1)$"-seminorm and penalty terms. A special technique based on the use of the Gauss–Radau interpolation and numerical integration has been used for the derivation of an abstract error estimate. In the "DG"-norm the error estimates are optimal with respect to the size of the space grid.

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They are optimal with respect to the time step, if the Dirichlet boundary condition has behaviour in time as a polynomial of degree $\leq q$.

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1 Introduction

In a number of complex problems from science and technology (aerospace engineering, turbomachinery, oil recovery, meteorology, environmental protection etc.) we meet the requirement to apply new efficient, robust, reliable and highly accurate numerical methods. It is necessary to develop techniques that allow to realize numerical approximations of nonlinear partial differential equations in domains with a complex geometry, whose solutions have a complicated structure.

In many cases, an excellent candidate for the solution of such problems is the discontinuous Galerkin finite element (DGFE) method. of a number of problems.

The DGFE method uses piecewise polynomial approximations of the sought solution on a finite element mesh without any requirement on the continuity between neighbouring elements and can be considered as a generalization of the finite volume and finite element methods. It allows to construct higher order schemes in a natural way and is suitable for the approximation of discontinuous solutions of conservation laws or solutions of singularly perturbed convection–diffusion problems having steep gradients. This method uses advantages of the finite element method and finite volume schemes with an approximate Riemann solver and can be applied on unstructured grids, which are generated for most complex geometries.

The original DGFE method was first used in [47] for the solution of a neutron transport linear equation and analyzed theoretically in [44] and later in [41]. Nearly simultaneously the DGFE techniques were developed for the numerical solution of second-order elliptic problems or parabolic problems [2,57] and a biharmonic problem [6]. Further, the DGFE method was applied to transport-reaction problems [13], nonlinear conservation laws [17,40], convection–diffusion linear or nonlinear problems [11,18,19,35,33], compressible flow [8–10,21,23,36,56], simulation of compressible low Mach number flows at incompressible limit [25,34], solution of incompressible viscous flow [52,55], porous media flow [53], shallow water flow [20], the Hamilton–Jacobi equations [38], the Schrödinger equation [42] and the Maxwell equations [37]. Theoretical analysis of various types of the DGFE method applied to elliptic problems can be found, e.g. in [3–5]. In [48], DGFE analysis is performed in the case of a parabolic problem with a nonlinear diffusion. In [39], analysis of *hp*-version of the DGFE method applied to stationary advection–diffusion-reaction equations is analyzed.

In the discretization of nonstationary problems, one often uses the *space semidiscretization,* also called the *method of lines.* In this approach, the DGFE discretization with respect to space variables is applied only, whereas time remains continuous. This leads to a large system of ordinary differential equations, which can be solved numerically by a suitable ODE solver. (See, e.g., [11,17,24,26,27,48].) In CFD and conservation laws, explicit schemes are often used, which are however conditionally stable. Therefore, it is suitable to apply implicit or semi-implicit methods. In [48] implicit θ-schemes are analyzed, [24] is concerned with the analysis of a semi-implicit linearized scheme for a nonlinear convection–diffusion problem and in [23,34] an efficient semi-implicit method for the solution of the compressible Euler equations was developed. However, these methods have low order of accuracy in time. As for higher-order time discretization methods, we can mention the well-known Crank–Nicolson scheme, which is second-order in time. In computational fluid dynamics, Runge–Kutta methods are very popular (cf. e.g. [19]).However, they are conditionally stable and in connection with the DGFEM the time step is strongly limited by the CFL stability condition. An example of unconditionally stable method is the technique using the backward difference formula (BDF). It was used for the solution of compressible flow, e.g. in [23] and analyzed theoretically in the case of a scalar nonlinear convection–diffusion equation in [28].

The numerical simulation of strongly nonstationary transient problems requires the application of numerical schemes of higher-order of accuracy in space as well as in time. In the paper [7], a time discretization of arbitrary order was proposed and analyzed. Unfortunately, it is applicable to linear parabolic problems only.

One possibility, how to construct unconditionally stable numerical schemes of higher-order of accuracy is to use the discontinuous Galerkin discretization with respect to both space and time. The discontinuous Galerkin time discretization was introduced and analyzed, e.g. in [29] for the solution of ordinary differential equations. In [1,30,31,50,51] the solution of parabolic problems is carried out with the aid of conforming finite elements in space combined with the DG time discretization. See also the monograph [54]. The works [40,56] apply on the other hand the full DG discretization in the space–time domain. This requires to construct the mesh in the space–time cylinder, which may be quite complicated task for 3D problems.

In this paper we are concerned with the space–time discontinuous Galerkin discretization applied separately in space and in time for the numerical solution of a nonstationary nonlinear convection–diffusion equation. The time interval is split into subintervals and on each time level a different space mesh may be used in general. This approach is suitable particularly in the case when the space mesh adaptivity is performed in the course of increasing time. Moreover, the triangulations used for the space discretization may be nonconforming with hanging nodes. In the discontinuous Galerkin formulation we use the nonsymmetric, symmetric or incomplete version of the discretization of the diffusion terms and interior and boundary penalty (i.e., NIPG, SIPG or IIPG versions). For the space and time discretization, piecewise polynomial approximations of different degrees p and q, respectively, are used. The main subject of the paper is the derivation of error estimates of the space–time DGFE method for the nonstationary initial-boundary value problem with nonlinear convection and linear diffusion. We do not consider a singularly perturbed case with dominating convection, but assume that the diffusion coefficient is a fixed positive constant of order $O(1)$. Under the assumption that the triangulations on all time levels are uniformly shape regular,and the exact solution has some regularity properties,error estimates are derived for the space–time DGFE method. These estimates are optimal in time, if the Dirichlet boundary conditions have behaviour in time as a polynomial of degree $\leq q$. In a general case, these estimates become suboptimal.

The structure of the paper is as follows: First, the continuous problem is formulated and the main assumptions are introduced. Further, the discontinuous Galerkin discretization in space and time is described. In the next section, some auxiliary results concerning properties of forms appearing in the definition of the approximate solution are obtained and the abstract error estimate is derived. Then the error estimates of the DG space–time discretization are proven. Finally an outlook of the future work is given.

2 Continuous problem

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded polyhedral domain and $T > 0$. We consider the following initial-boundary value problem: Find $u : Q_T = \Omega \times (0, T) \to I\!R$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \varepsilon \, \Delta \, u = g \quad \text{in } Q_T = \Omega \times (0, T), \tag{2.1}$$

$$u \,\big|_{\partial\Omega \times (0,T)} = u_D, \tag{2.2}$$

$$u(x, 0) = u^0(x), \quad x \in \Omega. \tag{2.3}$$

We assume that $\varepsilon > 0$ and $f_s \in C^1(I\!R), \quad |f_s'| \le C, \; s = 1, \dots, d$. This means that the fluxes f_s are Lipschitz-continuous in $I\!R$.

Using techniques from [49], it is possible to prove the existence and uniqueness of a weak solution to problem (2.1)–(2.3).

We use the standard notation of function spaces (see, e.g. [43]). If ω is a bounded domain, then we define the Lebesgue spaces

$$L^\infty(\omega) = \{\text{measurable functions } \varphi; \; \|\varphi\|_{L^\infty(\omega)} = \text{essup}_{x\in\omega}|\varphi(x)| < \infty\},$$

$$L^2(\omega) = \left\{\text{measurable functions } \varphi; \; \|\varphi\|_{L^2(\omega)} = \left(\int_\omega |\varphi|^2 \, dx\right)^{1/2} < \infty\right\}$$

and the Sobolev space

$$H^k(\omega) = \left\{\varphi \in L^2(\omega); \; \|\varphi\|_{H^k(\omega)} = \left(\sum_{|\alpha|\le k} \|D^\alpha \varphi\|^2_{L^2(\omega)}\right)^{1/2} < \infty\right\},$$

with the seminorm

$$|\varphi|_{H^k(\omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha \varphi\|^2_{L^2(\omega)}\right)^{1/2}.$$

We also use the Bochner spaces. Let X be a Banach space with a norm $\|\cdot\|_X$ and a seminorm $|\cdot|_X$ and let s be an integer. Then we define:

$$C([0,T];X) = \left\{ \varphi : [0,T] \to X, \text{ continuous}, \|\varphi\|_{C([0,T];X)} = \sup_{t\in[0,T]} \|\varphi(t)\|_X < \infty \right\},$$

$$L^2(0,T;X) = \left\{ \varphi : (0,T) \to X, \text{ strongly measurable}, \|\varphi\|^2_{L^2(0,T;X)} = \int_0^T \|\varphi\|^2_X \, dt < \infty \right\},$$

$$H^s(0,T;X) = \left\{ \varphi \in L^2(0,T;X); \|\varphi\|^2_{H^s(0,T;X)} = \int_0^T \sum_{\alpha=0}^{s} \left\| \frac{\partial^\alpha \varphi}{\partial t^\alpha} \right\|^2_X dt < \infty \right\}.$$

Moreover, we set

$$|\varphi|_{C([0,T];X)} = \sup_{t\in[0,T]} |\varphi(t)|_X, \quad |\varphi|_{L^2(0,T;X)} = \left(\int_0^T |\varphi|^2_X \, dt \right)^{1/2},$$

$$|\varphi|_{H^s(0,T;X)} = \left(\int_0^T \left| \frac{\partial^s \varphi}{\partial t^s} \right|^2_X dt \right)^{1/2}.$$

3 Discretization

3.1 Construction of a mesh in Q_T

In the time interval $[0, T]$ we shall construct a partition formed by time instants $0 = t_0 < \cdots < t_M = T$ and denote $I_m = (t_{m-1}, t_m)$, $\overline{I}_m = [t_{m-1}, t_m]$, $\tau_m = t_m - t_{m-1}$. We have $[0, T] = \bigcup_{i=1}^{M} \bar{I}_m$, $I_m \cap I_n = \emptyset$ for $m \neq n$.

For each I_m we consider a partition $\mathcal{T}_{h,m}$ of the closure $\overline{\Omega}$ of the domain Ω into a finite number of closed d-dimensional simplices (triangles for $d = 2$ and tetrahedra for $d = 3$) with mutually disjoint interiors. We shall call $\mathcal{T}_{h,m}$ a triangulation of Ω. We do not require the standard properties of $\mathcal{T}_{h,m}$ used in the finite element method. This means that we admit the so-called hanging nodes (and in 3D also hanging edges). The partitions $\mathcal{T}_{h,m}$ are in general different for different m.

Let $K, K' \in \mathcal{T}_{h,m}$, $K \neq K'$. We say that K and K' are *neighbouring elements*, if the set $\partial K \cap \partial K'$ has positive $(d-1)$-dimensional measure. We say that $\Gamma \subset K$ is a *face* of K, if it is a maximal connected open subset either of $\partial K \cap \partial K'$, where K' is a neighbouring element to K, or of $\partial K \cap \partial\Omega$. By $\mathcal{F}_{h,m}$ we denote the system of all faces of all elements $K \in \mathcal{T}_{h,m}$.Further, we define the set of all boundary faces $\mathcal{F}^B_{h,m} = \{\Gamma \in \mathcal{F}_{h,m};\ \Gamma \subset \partial\Omega\}$ and the set of all inner faces $\mathcal{F}^I_{h,m} = \mathcal{F}_{h,m} \setminus \mathcal{F}^B_{h,m}$.

For each $\Gamma \in \mathcal{F}_{h,m}$ we define a unit normal vector $\boldsymbol{n}_\Gamma$. We assume that for $\Gamma \in \mathcal{F}^B_{h,m}$ the normal $\boldsymbol{n}_\Gamma$ has the same orientation as the outer normal to $\partial\Omega$. For each face $\Gamma \in \mathcal{F}^I_{h,m}$ the orientation of $\boldsymbol{n}_\Gamma$ is arbitrary but fixed. See Fig. 1.

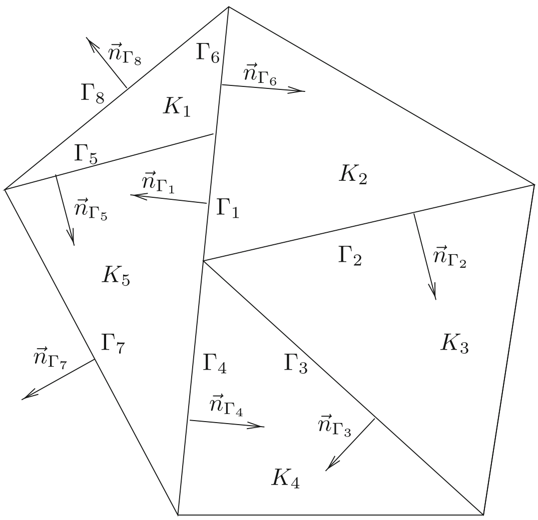

Fig. 1 Example of elements $K_l,\ l = 1, \dots, 5$, and faces $\Gamma_l,\ l = 1, \dots, 8$, with the corresponding normals $\boldsymbol{n}_{\Gamma_l}$

In our further considerations we shall use the following notation. For an element $K \in \mathcal{T}_{h,m}$ we set $h_K = \text{diam}(K),\ h_m = \max_{K \in \mathcal{T}_{h,m}} h_K,\ h = \max_{m=1,\dots,M} h_m$. By ρ_K we denote the radius of the largest d-dimensional ball inscribed into K and by $|K|$ we denote the d-dimensional Lebesgue measure of K. Further, by $d(\Gamma)$ we denote the diameter of $\Gamma \in \mathcal{F}_{h,m}$. Finally, we set $\tau = \max_{m=1,\dots,M} \tau_m$.

3.2 Forms defined on spaces of discontinuous functions

For a function φ defined in $\bigcup_{m=1}^{M} I_m$ we denote

$$\varphi_m^{\pm} = \varphi\,(t_m\pm) = \lim_{t \to t_m \pm} \varphi(t), \quad \{\varphi\}_m = \varphi\,(t_m+) - \varphi\,(t_m-)\,. \tag{3.1}$$

Over a triangulation $\mathcal{T}_{h,m}$ we define the *broken Sobolev spaces*

$$H^k(\Omega, \mathcal{T}_{h,m}) = \{v;\, v|_K \in H^k(K)\ \forall\, K \in \mathcal{T}_{h,m}\} \tag{3.2}$$

equipped with the seminorm

$$|v|_{H^k(\Omega, \mathcal{T}_{h,m})} = \left(\sum_{K \in \mathcal{T}_{h,m}} |v|^2_{H^k(K)} \right)^{1/2}. \tag{3.3}$$

For each face $\Gamma \in \mathcal{F}^I_{h,m}$ there exist two neighbouring elements $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_{h,m}$ such that $\Gamma \subset \partial K_\Gamma^{(L)} \cap \partial K_\Gamma^{(R)}$. We use convention that $\boldsymbol{n}_\Gamma$ is the outer normal to the element $K_\Gamma^{(L)}$ and the inner normal to the element $K_\Gamma^{(R)}$. For $v \in H^1(\Omega, \mathcal{T}_{h,m})$ and

$\Gamma \in \mathcal{F}_{h,m}^I$ we introduce the following notation:

$$\begin{aligned} v|_\Gamma^{(L)} &= \text{the trace of } v|_{K_\Gamma^{(L)}} \quad \text{on } \Gamma, \quad v|_\Gamma^{(R)} = \text{the trace of } v|_{K_\Gamma^{(R)}} \quad \text{on} \Gamma, \\ \langle v \rangle_\Gamma &= \frac{1}{2}\left(v|_\Gamma^{(L)} + v|_\Gamma^{(R)} \right), \quad [v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}. \end{aligned} \tag{3.4}$$

Now, let $\Gamma \in \mathcal{F}_{h,m}^B$ and $K_\Gamma^{(L)} \in \mathcal{T}_{h,m}$ be such an element that $\Gamma \subset K_\Gamma^{(L)} \cap \partial\Omega$. For $v \in H^1(\Omega, \mathcal{T}_{h,m})$ we define $v|_\Gamma^{(R)}$ by extrapolation, i.e. $v|_\Gamma^{(R)} := v|_\Gamma^{(L)} =$ the trace of $v|_{K_\Gamma^{(L)}}$ on Γ.

If $[\cdot]_\Gamma$ and $\langle \cdot \rangle_\Gamma$ appear in an integral $\int_\Gamma \ldots \, \mathrm{d}S$, where $\Gamma \in \mathcal{F}_{h,m}^I$, we usually omit the subscript Γ and write simply $[\cdot]$ and $\langle \cdot \rangle$. Moreover, if $\Gamma \in \mathcal{F}_{h,m}^B$ and $v \in H^1(\Omega, \mathcal{T}_{h,m})$, then $\int_\Gamma v \, \mathrm{d}S$ means $\int_\Gamma v|_\Gamma^{(L)} \, \mathrm{d}S$.

Let $C_W > 0$ be a fixed constant. We introduce the notation

$$\begin{aligned} h(\Gamma) &= \frac{h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}}{2C_W} \quad \text{for } \Gamma \in \mathcal{F}_{h,m}^I, \\ h(\Gamma) &= \frac{h_{K_\Gamma^{(L)}}}{C_W} \quad \text{for } \Gamma \in \mathcal{F}_{h,m}^B. \end{aligned} \tag{3.5}$$

By $(\cdot, \cdot)$ we denote the scalar product in $L^2(\Omega)$ and by $\|\cdot\|$ we denote the norm in $L^2(\Omega)$. If $\overline{u}, \varphi \in H^2(\Omega, \mathcal{T}_{h,m})$, we define the forms

$$\begin{aligned} a_{h,m}(\overline{u}, \varphi) = \varepsilon \sum_{K \in \mathcal{T}_{h,m}} \int_K \nabla \overline{u} \cdot \nabla \varphi \, \mathrm{d}x \\ - \varepsilon \sum_{\Gamma \in \mathcal{F}_{h,m}^I} \int_\Gamma (\langle \nabla \overline{u} \rangle \cdot \boldsymbol{n}_\Gamma [\varphi] + \theta \langle \nabla \varphi \rangle \cdot \boldsymbol{n}_\Gamma [\overline{u}]) \, \mathrm{d}S \\ - \varepsilon \sum_{\Gamma \in \mathcal{F}_{h,m}^B} \int_\Gamma (\nabla \overline{u} \cdot \boldsymbol{n}_\Gamma \varphi + \theta \nabla \varphi \cdot \boldsymbol{n}_\Gamma \overline{u}) \, \mathrm{d}S, \end{aligned} \tag{3.6}$$

$$J_{h,m}(\overline{u}, \varphi) = \sum_{\Gamma \in \mathcal{F}_{h,m}^I} h(\Gamma)^{-1} \int_\Gamma [\overline{u}] [\varphi] \, \mathrm{d}S + \sum_{\Gamma \in \mathcal{F}_{h,m}^B} h(\Gamma)^{-1} \int_\Gamma \overline{u} \, \varphi \, \mathrm{d}S, \tag{3.7}$$

$$A_{h,m} = a_{h,m} + \varepsilon J_{h,m}, \tag{3.8}$$

$$b_{h,m}(\overline{u},\varphi) = -\sum_{K\in\mathcal{T}_{h,m}}\int_K \sum_{s=1}^{d} f_s(\overline{u})\,\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}^I_{h,m}}\int_\Gamma H\left(\overline{u}\big|_\Gamma^{(L)},\overline{u}\big|_\Gamma^{(R)},\boldsymbol{n}_\Gamma\right)[\varphi]_\Gamma\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,m}}\int_\Gamma H\left(\overline{u}\big|_\Gamma^{(L)},\overline{u}\big|_\Gamma^{(L)},\boldsymbol{n}_\Gamma\right)\varphi\big|_\Gamma^{(L)}\,\mathrm{d}S. \tag{3.9}$$

Here H is a numerical flux. We assume that it has the following properties.

(H1) $H(u,v,\boldsymbol{n})$ is defined in $I\!R^2\times B_1$, where $B_1=\{\boldsymbol{n}\in I\!R^d;\,|\boldsymbol{n}|=1\}$, and is *Lipschitz-continuous* with respect to $u,\ v$:

$$|H(u,v,\boldsymbol{n})-H(u^*,v^*,\boldsymbol{n})|\le L_H(|u-u^*|+|v-v^*|),\quad u,v,u^*,v^*\in I\!R,\ \boldsymbol{n}\in B_1.$$

(H2) $H(u,v,\boldsymbol{n})$ is *consistent*:

$$H(u,u,\boldsymbol{n})=\sum_{s=1}^{d} f_s(u)\,n_s,\quad u\in I\!R,\ \boldsymbol{n}=(n_1,\ldots,n_d)\in B_1.$$

(H3) $H(u,v,\boldsymbol{n})$ is *conservative*:

$$H(u,v,\boldsymbol{n})=-H(v,u,-\boldsymbol{n}),\quad u,v\in I\!R,\ \boldsymbol{n}\in B_1.$$

Finally, the right-hand side form is defined on the basis of data:

$$\ell_{h,m}(\varphi)=(g,\varphi)+\varepsilon\sum_{\Gamma\in\mathcal{F}^B_{h,m}}\left(h(\Gamma)^{-1}\int_\Gamma u_D\,\varphi\,\mathrm{d}S-\theta\int_\Gamma \nabla\varphi\cdot\boldsymbol{n}_\Gamma u_D\,\mathrm{d}S\right). \tag{3.10}$$

In the above forms we take $\theta=-1,\ \theta=0,\ \theta=1$ and obtain the nonsymmetric (NIPG), incomplete (IIPG) and symmetric (SIPG) variants of the approximation of the diffusion terms, respectively.

In the space $H^1(\Omega,\mathcal{T}_{h,m})$, the following norm will be used:

$$\|\varphi\|_{\mathrm{DG},m}=\left(\sum_{K\in\mathcal{T}_{h,m}}|\varphi|^2_{H^1(K)}+J_{h,m}(\varphi,\varphi)\right)^{1/2}. \tag{3.11}$$

3.3 Discrete problem

Let $p,\, q \geq 1$ be integers. For each $m = 1, \ldots, M$ we define the finite-dimensional space

$$S^p_{h,m} = \left\{ \varphi \in L^2(\Omega);\, \varphi|_K \in P^p(K)\, \forall\, K \in \mathcal{T}_{h,m} \right\}. \tag{3.12}$$

By Π_m we denote the $L^2(\Omega)$-projection on $S^p_{h,m}$, i.e., if $\varphi \in L^2(\Omega)$, then $\Pi_m \varphi \in S^p_{h,m}$ and

$$(\Pi_m \varphi - \varphi, \psi) = 0, \quad \forall\, \psi \in S^p_{h,m}. \tag{3.13}$$

The approximate solution will be sought in the space

$$S^{p,q}_{h,\tau} = \left\{ \varphi \in L^2(Q_T);\, \varphi\big|_{I_m} = \sum_{i=0}^{q} t^i\, \varphi_i \quad \text{with } \varphi_i \in S^p_{h,m},\ m = 1, \ldots, M \right\}. \tag{3.14}$$

In what follows we shall use the notation $U' = \partial U/\partial t,\, u' = \partial u/\partial t,\, D^{q+1} = \partial^{q+1}/\partial t^{q+1}$.

Definition 1 We say that a function U is an approximate solution of problem (2.1)–(2.3), if $U \in S^{p,q}_{h,\tau}$ and

$$\int_{I_m} \big((U', \varphi) + A_{h,m}(U, \varphi) + b_{h,m}(U, \varphi) \big)\, \mathrm{d}t + \big(\{U\}_{m-1}, \varphi^+_{m-1} \big)$$
$$= \int_{I_m} \ell_{h,m}(\varphi)\, \mathrm{d}t, \forall\, \varphi \in S^{p,q}_{h,\tau}, \quad \forall\, m = 1, \ldots, M, \quad U^-_0 = \Pi_1 u^0. \tag{3.15}$$

It is possible to show that the exact sufficiently regular solution u satisfies the identity

$$\int_{I_m} \big((u', \varphi) + A_{h,m}(u, \varphi) + b_{h,m}(u, \varphi) \big)\, \mathrm{d}t + \big(\{u_{m-1}\}, \varphi^+_{m-1} \big)$$
$$= \int_{I_m} \ell_{h,m}(\varphi)\, \mathrm{d}t, \forall\, \varphi \in S^{p,q}_{h,\tau}, \quad \forall\, m = 1, \ldots, M, \tag{3.16}$$

if we set $u(0-) = u(0)$.

Remark 1 It is also possible to consider $q = 0$. In this case, scheme (3.15) represents a version of the backward Euler method. Since it can be analyzed in a similar way as, for example, in [24], we shall be concerned only with $q \geq 1$.

In the error analysis we shall use the $S_{h,\tau}^{p,q}$-interpolation π of functions $v \in H^1(0,T;L^2(\Omega))$ defined by

$$\text{(a)} \quad \pi\, v \in S_{h,\tau}^{p,q}, \tag{3.17}$$
$$\text{(b)} \quad (\pi\, v)\,(t_m-) = \Pi_m\, v(t_m-),$$
$$\text{(c)} \quad \int\limits_{I_m} (\pi v - v,\, \varphi^*)\,\mathrm{d}t = 0, \quad \forall\, \varphi^* \in S_{h,\tau}^{p,q-1}, \quad \forall\, m = 1,\ldots,M.$$

In [33], Lemma 4, it was proven that πu is uniquely determined. Moreover, by [33], Lemma 9,

$$\pi u|_{I_m} = \pi(\Pi_m u)|_{I_m}. \tag{3.18}$$

Our main goal will be the derivation of the estimation of the error $e = U - u$, which can be expressed in the form

$$e = \xi + \eta, \tag{3.19}$$

where

$$\xi = U - \pi u \in S_{h,\tau}^{p,q}, \quad \eta = \pi u - u. \tag{3.20}$$

Then, in virtue of (3.15) and (3.16),

$$\int\limits_{I_m} \big((\xi', \varphi) + A_{h,m}(\xi, \varphi)\big)\,\mathrm{d}t + \Big(\{\xi_{m-1}\}, \varphi_{m-1}^+\Big) = \int\limits_{I_m} \big(b_{h,m}(u,\varphi) - b_{h,m}(U,\varphi)\big)\,\mathrm{d}t$$
$$- \int\limits_{I_m} \big((\eta', \varphi) + A_{h,m}(\eta, \varphi)\big)\,\mathrm{d}t - \Big(\{\eta\}_{m-1}, \varphi_{m-1}^+\Big), \quad \forall \varphi \in S_{h,\tau}^{p,q}. \tag{3.21}$$

4 Abstract error estimate

In this section we shall be concerned with the derivation of error estimates in terms of interpolation error.

4.1 Assumptions on the triangulation

In our further considerations, by C and c we shall denote positive generic constants, independent of $h, \tau, K, \varepsilon, u, U$, which can attain different values in different places. In the sequel, we shall consider a system of triangulations $\mathcal{T}_{h,m}$, $m = 1,\ldots,M$,

$h \in (0, h_0),\ h_0 > 0$, which is *shape regular* and *locally quasiuniform*:

$$\frac{h_K}{\rho_K} \le C_R, \quad K \in \mathcal{T}_{h,m}, \quad m = 1, \ldots, M, \quad h \in (0, h_0), \tag{4.1}$$

$$h_K \le C_Q h_{K'}, \quad \text{for neighbouring elements } K, K' \in \mathcal{T}_{h,m}. \tag{4.2}$$

Then there exist positive constants C_-, C_+ such that

$$C_- h_K \le h(\Gamma) \le C_+ h_K, \quad \Gamma \in \mathcal{F}_{h,m}, \Gamma \subset K \in \mathcal{T}_{h,m},\ h \in (0, h_0),\ m = 1, \ldots, M. \tag{4.3}$$

4.2 Auxiliary results

In the analysis of the DGFEM we use the following important tools.
Multiplicative trace inequality: There exists a constant $C_M > 0$ independent of v, h, K and M such that

$$\|v\|^2_{L^2(\partial K)} \le C_M \left(\|v\|_{L^2(K)} |v|_{H^1(K)} + h_K^{-1} \|v\|^2_{L^2(K)} \right), \\ v \in H^1(K),\ K \in \mathcal{T}_{h,m},\ h \in (0, h_0),\ m = 1, \ldots, M. \tag{4.4}$$

Inverse inequality: There exists a constant $C_I > 0$ independent of$v$, h, K and M such that

$$|v|_{H^1(K)} \le C_I h_K^{-1} \|v\|_{L^2(K)}, \quad v \in P^p(K),\ K \in \mathcal{T}_{h,m}, h \in (0, h_0),\ m = 1, \ldots, M. \tag{4.5}$$

(For proofs, see, e.g. [12,26].)
Coercivity of the form $A_{h,m}$: It holds

$$A_{h,m}(\xi, \xi) \ge \frac{\varepsilon}{2} \|\xi\|^2_{\mathrm{DG},m} \tag{4.6}$$

provided

$$\begin{aligned} &C_W > 0 \quad \text{for NIPG,} \\ &C_W \ge C_M (1 + C_I)(1 + C_Q) \ \text{ for IIPG,} \\ &C_W \ge 2C_M (1 + C_I)(1 + C_Q) \ \text{for SIPG.} \end{aligned} \tag{4.7}$$

(See, [32].)
Consistency of $b_{h,m}$: For any $\varphi \in S^{p,q}_{h,\tau}$ and $k > 0$,

$$\begin{aligned} \left| b_{h,m}(u, \varphi) - b_{h,m}(U, \varphi) \right| &\le C \|\varphi\|_{\mathrm{DG},m} \left(\|\xi\|^2 + \tilde{\sigma}^2_m(\eta) \right)^{1/2} \\ &\le \frac{\varepsilon}{k} \|\varphi\|^2_{\mathrm{DG},m} + \frac{C}{\varepsilon} \left(\|\xi\|^2 + \tilde{\sigma}^2_m(\eta) \right), \end{aligned} \tag{4.8}$$

where

$$\tilde{\sigma}_m^2(\eta) = \sum_{K\in\mathcal{T}_{h,m}} \left(\|\eta\|^2_{L^2(K)} + h_K^2 |\eta|^2_{H^1(K)} \right). \tag{4.9}$$

(The constant C in the last expression depends, of course, on k.) The proof can be carried out in a similar way as in [22] or [27].

4.3 Derivation of estimates for ξ

Let us substitute $\varphi := \xi$ in (3.21) and analyze individual terms. A simple calculation yields

$$2\int_{I_m} \left(\xi', \xi\right)\,\mathrm{d}t + 2\left(\{\xi\}_{m-1}, \xi^+_{m-1}\right) = \left\|\xi^-_m\right\|^2 - \left\|\xi^-_{m-1}\right\|^2 + \|\{\xi\}_{m-1}\|^2. \tag{4.10}$$

Further, we shall be concerned with estimates of the right-hand side of (3.21). In the same way as in [27], Lemma 9, using Cauchy inequality, multiplicative trace inequality, inverse inequality and Young's inequality, we can show that for $\varphi \in S^{p,q}_{h,\tau}$ and $k > 0$ we have

$$\left|A_{h,m}(\eta, \varphi)\right| \le \frac{\varepsilon}{k}\, \|\varphi\|^2_{\mathrm{DG},m} + C\varepsilon\sigma_m^2(\eta), \tag{4.11}$$

where

$$\sigma_m^2(\eta) = \|\eta\|^2_{\mathrm{DG},m} + \sum_{K\in\mathcal{T}_{h,m}} h_K^2 |\eta|^2_{H^2(K)}. \tag{4.12}$$

Now (3.21), where we set $\varphi := \xi$, relation (4.10) and estimates (4.6), (4.8), (4.11) imply that

$$\begin{aligned} &\left\|\xi^-_m\right\|^2 - \|\xi_{m-1}\|^2 + \left\|\{\xi\}^-_{m-1}\right\|^2 + \varepsilon\int_{I_m} \|\xi\|^2_{\mathrm{DG},m}\,\mathrm{d}t \\ &\quad\le -2\int_{I_m} (\eta', \xi)\,\mathrm{d}t - 2\left(\{\eta\}_{m-1}, \xi^+_{m-1}\right) + \frac{2\varepsilon}{k}\int_{I_m} \|\xi\|^2_{\mathrm{DG},m}\,\mathrm{d}t \\ &\qquad + \frac{C}{\varepsilon}\int_{I_m} \|\xi\|^2\,\mathrm{d}t + C\int_{I_m} \left(\varepsilon\sigma_m^2(\eta) + \frac{1}{\varepsilon}\tilde{\sigma}_m^2(\eta)\right)\mathrm{d}t. \end{aligned} \tag{4.13}$$

Further, we shall be concerned with the expression

$$\int_{I_m} (\eta', \xi)\,\mathrm{d}t + \left(\{\eta\}_{m-1}, \varphi^+_{m-1}\right).$$

Integration by parts yields

$$\int_{I_m} (\eta', \xi)\,\mathrm{d}t = \left(\eta_m^-, \xi_m^-\right) - \left(\eta_{m-1}^+, \xi_{m-1}^+\right) - \int_{I_m} (\eta, \xi')\,\mathrm{d}t. \tag{4.14}$$

Since $\xi' \in S_{h,\tau}^{p,q-1}$, $\xi_j^- \in S_{h,j}^p$ and $\eta = \pi u - u$, the definitions of π and Π_m imply that

$$\left| \int_{I_m} (\eta', \xi)\,\mathrm{d}t + \left(\{\eta\}_{m-1}, \xi_{m-1}^+\right) \right| \leq \frac{1}{2}\,\|\{\xi\}_{m-1}\|^2 + \frac{1}{2}\left\|\eta_{m-1}^-\right\|^2. \tag{4.15}$$

This and (4.13) give

$$\begin{aligned} &\left\|\xi_m^-\right\|^2 - \left\|\xi_{m-1}^-\right\|^2 + \varepsilon\left(1 - \frac{2}{k}\right)\int_{I_m} \|\xi\|_{\mathrm{DG},m}^2\,\mathrm{d}t \\ &\quad\leq \frac{C}{\varepsilon}\int_{I_m} \|\xi\|^2\,\mathrm{d}t + 2\left\|\eta_{m-1}^-\right\|^2 + C\int_{I_m} R_m(\eta)\,\mathrm{d}t, \end{aligned} \tag{4.16}$$

where

$$R_m(\eta) = \varepsilon\sigma_m^2(\eta) + \frac{1}{\varepsilon}\tilde{\sigma}_m^2(\eta). \tag{4.17}$$

In what follows, it will be necessary to estimate the terms with η and $\int_{I_m} \|\xi\|^2\,\mathrm{d}t$.

4.4 Estimation of $\int_{I_m} \|\xi\|^2\,\mathrm{d}t$

By $\mathcal{P}^q$ we shall denote the set of all polynomials in $t \in \mathbb{R}$ of degree $\leq q$. In the interval $(0, 1]$ we shall consider the Gauss–Radau quadrature formula

$$\int_0^1 \varphi(t)\,\mathrm{d}t \approx \sum_{i=1}^{q+1} w_i\,\varphi(\vartheta_i), \tag{4.18}$$

where $0 < \vartheta_1 < \cdots < \vartheta_{q+1} = 1$ are the Radau integration points and $w_i > 0$ are the Radau weights. (We can refer, for example, to formulas from [46] and transform them from the interval $[-1, 1)$ to $(0, 1]$.) Formula (4.18) is transformed to the interval

$(t_{m-1}, t_m]$, which yields

$$\int_{I_m} \varphi(t)\,\mathrm{d}t \approx \tau_m \sum_{i=1}^{q+1} w_i \varphi\left(t^{m,i}\right), \tag{4.19}$$

where $t^{m,i} = t_{m-1} + \tau_m\,\vartheta_i$. Formulas (4.18), (4.19) are exact for polynomials of degree $\leq 2q$. In [1], the following result was proven:

Lemma 1 *Let $p \in \mathcal{P}^q$ and let $\tilde{p} \in \mathcal{P}^q$ be the Lagrange interpolation of the function $\tau_m\, p(t)/(t - t_{m-1})$ at the points $t^{m,i}$, $i = 1, \ldots, q+1$:*

$$\tilde{p}(t^{m,i}) = \tau_m\, p\left(t^{m,i}\right) / \left(t^{m,i} - t_{m-1}\right) = p\left(t^{m,i}\right)\vartheta_i^{-1}, \quad i = 1, \ldots, q+1.$$

Then

$$\int_{I_m} p'\,\tilde{p}\,\mathrm{d}t + p(t_{m-1})\,\tilde{p}(t_{m-1}) = \frac{1}{2}\left(p^2(t_m) + \sum_{i=1}^{q+1} w_i\,\vartheta_i^{-2}\,p^2(t^{m,i})\right). \tag{4.20}$$

Now, by $\tilde{\xi}$ we shall denote the Lagrange interpolation of $\tau_m \xi(t)/(t - t_{m-1})$ at the points $t^{m,i}$, $i = 1, \ldots, q+1$. Then $\tilde{\xi} \in S_{h,\tau}^{p,q}$. In what follows we shall denote

$$\|\xi\|_m^2 = \tau_m \sum_{i=1}^{q+1} w_i\,\vartheta_i^{-1}\left\|\xi(t^{m,i})\right\|^2. \tag{4.21}$$

Let us set $\varphi := \tilde{\xi}$ in (3.21). Then we get

$$\begin{aligned}
&\underbrace{\int_{I_m} (\xi', \tilde{\xi})\,\mathrm{d}t + \left(\xi^+_{m-1}, \tilde{\xi}^+_{m-1}\right)}_{\text{(a)}} + \underbrace{\int_{I_m} A_{h,m}(\xi, \tilde{\xi})\,\mathrm{d}t}_{\text{(b)}} \\
&\quad = \underbrace{\left(\xi^-_{m-1}, \tilde{\xi}^+_{m-1}\right)}_{\text{(c)}} - \underbrace{\int_{I_m} (\eta', \tilde{\xi})\,\mathrm{d}t - \left(\{\eta\}_{m-1}, \tilde{\xi}^+_{m-1}\right)}_{\text{(d)}} - \underbrace{\int_{I_m} A_{h,m}(\eta, \tilde{\xi})\,\mathrm{d}t}_{\text{(e)}} \\
&\qquad + \underbrace{\int_{I_m} \left(b_{h,m}(u, \tilde{\xi}) - b_{h,m}(U, \tilde{\xi})\,\mathrm{d}t\right)}_{\text{(f)}}.
\end{aligned} \tag{4.22}$$

In what follows, we shall analyze individual terms (a)–(f).

(a) By Fubini's theorem and (4.20),

$$\int_{I_m} \left(\xi', \tilde{\xi}\right) \mathrm{d}t + \left(\xi_{m-1}^+, \tilde{\xi}_{m-1}^+\right) = \int_{\Omega} \left(\int_{t_{m-1}}^{t_m} \xi' \, \tilde{\xi} \, \mathrm{d}t + \xi_{m-1}^+ \, \tilde{\xi}_{m-1}^+ \right) \mathrm{d}x$$

$$= \int_{\Omega} \frac{1}{2} \left((\xi_m^-)^2 + \sum_{i=1}^{q+1} w_i \, \vartheta_i^{-2} \left(\xi(t^{m,1})\right)^2 \right) \mathrm{d}x$$

$$= \frac{1}{2} \left(\left\| \xi_m^- \right\|^2 + \sum_{i=1}^{q+1} w_i \, \vartheta_i^{-2} \left\| \xi(t^{m,i}) \right\|^2 \right). \tag{4.23}$$

Hence, since $\vartheta_i^{-1} \geq 1$, using the notation (4.21), we get the inequality

$$\int_{I_m} \left(\xi', \tilde{\xi}\right) \mathrm{d}t + \left(\xi_{m-1}^+, \tilde{\xi}_{m-1}^+\right) \geq \frac{1}{2} \left(\|\xi_m^-\|^2 + \frac{1}{\tau_m} \|\xi\|_m^2 \right). \tag{4.24}$$

(b) We use the following lemma:

Lemma 2 *Under assumptions* (4.7) *we have*

$$\int_{I_m} A_{h,m}(\xi, \tilde{\xi}) \, \mathrm{d}t \geq \frac{\varepsilon}{2} \int_{I_m} \|\xi\|_{\mathrm{DG},m}^2 \mathrm{d}t. \tag{4.25}$$

Proof In view of (3.6) and (3.8),

$$\int_{I_m} A_{h,m}(\xi, \tilde{\xi}) \, \mathrm{d}t = \varepsilon \int_{I_m} \sum_{K \in \mathcal{T}_{h,m}} \int_K \nabla \xi \cdot \nabla \tilde{\xi} \, \mathrm{d}x \mathrm{d}t$$

$$- \varepsilon \int_{I_m} \sum_{\Gamma \in \mathcal{F}_{h,m}^I} \int_{\Gamma} \left(\langle \nabla \xi \rangle \cdot \boldsymbol{n}_\Gamma [\tilde{\xi}] - \theta \langle \nabla \tilde{\xi} \rangle \cdot \boldsymbol{n}_\Gamma [\xi] \right) \mathrm{d}S \mathrm{d}t$$

$$- \varepsilon \int_{I_m} \sum_{\Gamma \in \mathcal{F}_{h,m}^B} \int_{\Gamma} \left(\nabla \xi \cdot \boldsymbol{n}_\Gamma \tilde{\xi} - \theta \nabla \tilde{\xi} \cdot \boldsymbol{n}_\Gamma \xi \, \mathrm{d}S \right) \mathrm{d}t + \varepsilon \int_{I_m} J_{h,m}(\xi, \tilde{\xi}) \, \mathrm{d}t.$$

The expressions $\xi\big|_\Gamma$, $[\xi]_\Gamma$, $\tilde{\xi}\big|_\Gamma$, $[\tilde{\xi}]_\Gamma$, $\nabla \xi$ and $\nabla \tilde{\xi}$ are polynomials in t of degree $\leq q$. Hence, $\int_K \nabla \xi \cdot \nabla \tilde{\xi} \, \mathrm{d}x$, $\int_\Gamma [\xi]_\Gamma [\tilde{\xi}]_\Gamma \, \mathrm{d}S$, $\int_\Gamma \langle \nabla \xi \rangle \cdot \boldsymbol{n} [\tilde{\xi}] \, \mathrm{d}S$, $J_{h,m}(\xi, \tilde{\xi})$, etc. are polynomials in t of degree $\leq 2q$. Therefore, we can express the integrals $\int_{I_m} \cdots \mathrm{d}t$ with the aid of the integration formula (4.19). We also use the relations $\tilde{\xi}(t^{m,i}) = \xi(t^{m,i}) \vartheta_i^{-1}$,

$\nabla\tilde{\xi}(t^{m,i}) = \nabla\xi(t^{m,i})\vartheta_i^{-1}$, $[\tilde{\xi}(t^{m,i})] = [\xi(t^{m,i})\vartheta_i^{-1}]$. Then, by (3.6)–(3.8) we get

$$\int_{I_m} A_{h,m}(\xi,\tilde{\xi})\,\mathrm{d}t = \tau_m \sum_{i=1}^{q+1} \vartheta_i^{-1} w_i A_{h,m}(\xi(t^{m,i}),\xi(t^{m,i})). \tag{4.26}$$

If we use (4.26), (4.6), inequality $\vartheta_i^{-1} \geq 1$ and take into account that $\|\xi\|_{DG,m}^2$ is a polynomial in t of degree $\leq 2q$, we find that under assumption (4.7)

$$\int_{I_m} A_{h,m}(\xi,\tilde{\xi})\,\mathrm{d}t \geq \frac{\varepsilon}{2}\tau_m \sum_{i=1}^{q+1} w_i \,\|\xi(t^{m,i})\|_{DG,m}^2 = \frac{\varepsilon}{2}\int_{I_m} \|\xi\|_{DG,m}^2\,\mathrm{d}t,$$

what we wanted to prove. □

(c) By the Cauchy inequality,

$$\left|\left(\xi_{m-1}^-,\tilde{\xi}_{m-1}^+\right)\right| \leq \left\|\xi_{m-1}^-\right\| \left\|\tilde{\xi}_{m-1}^+\right\|. \tag{4.27}$$

Lemma 3 *There exists a constant c_1 independent of h_K, τ_m, ξ such that*

$$\left\|\tilde{\xi}_{m-1}^+\right\|^2 \leq \frac{c_1}{\tau_m}\,\|\xi\|_m^2 \tag{4.28}$$

Proof The function $\tilde{\xi}$ is the Lagrange interpolant to $\tau_m\,\xi(t)/(t-t_{m-1})$ at points $t^{m,i} = t_{m-1} + \tau_m\,\vartheta_i$, $i = 1,\ldots,q+1$. This means that

$$\tilde{\xi}(t) = \tau_m \sum_{i=1}^{q+1} \frac{\xi(t^{m,i})}{t^{m,i}-t_{m-1}} \prod_{\substack{j=1\\ j\neq i}}^{q+1} \frac{t-t^{m,j}}{t^{m,i}-t^{m,j}} = \tau_m \sum_{i=1}^{q+1} \frac{\xi(t^{m,i})}{\tau_m\,\vartheta_i} \prod_{\substack{j=1\\ j\neq i}}^{q+1} \frac{t-t_{m-1}-\tau_m\,\vartheta_i}{\tau_m(\vartheta_i-\vartheta_j)}.$$

Setting $t = t_{m-1}$, we get

$$\tilde{\xi}_{m-1}^+ = \sum_{i=1}^{q+1} \xi(t^{m,i})\,\vartheta_i^{-1} \prod_{\substack{j=1\\ j\neq i}}^{q+1} \frac{-\vartheta_j}{\vartheta_i-\vartheta_j}$$

and, thus, since $\vartheta_i^{-1} \leq \vartheta_1^{-1}$,

$$\left\| \tilde{\xi}_{m-1}^{+} \right\|^2 \leq C(q) \sum_{i=1}^{q+1} \vartheta_i^{-1} \vartheta_1^{-1} \left\| \xi(t^{m,i}) \right\|^2 \left(\prod_{\substack{j=1 \\ j \neq i}}^{q+1} \frac{\vartheta_j}{\vartheta_i - \vartheta_j} \right)^2$$

$$\leq \tilde{C}(q) \sum_{i=1}^{q+1} \vartheta_i^{-1} \left\| \xi(t^{m,i}) \right\|^2 \left(\prod_{\substack{j=1 \\ j \neq i}}^{q+1} \frac{\vartheta_j}{\vartheta_i - \vartheta_j} \right)^2. \tag{4.29}$$

The Radau weights are defined as

$$w_i = \int_0^1 \prod_{\substack{j=1 \\ j \neq i}}^{q+1} \frac{z - \vartheta_j}{\vartheta_i - \vartheta_j} \, \mathrm{d}z.$$

By [46], $w^* := \min_{i=1,\ldots,q+1} w_i > 0$. Moreover, let us set

$$w^{**} := \max_{i=1,\ldots,q+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{q+1} \frac{\vartheta_j}{\vartheta_i - \vartheta_j} \right)^2.$$

Hence, since $w_i \geq w^*$, using (4.23), we get

$$\left\| \tilde{\xi}_{m-1}^{+} \right\|^2 \leq \tilde{C}(q) \sum_{i=1}^{q+1} \vartheta_i^{-1} \| \xi(t^{m,i}) \|^2 \frac{w^{**} \, w^*}{w^*} \leq c_1 \sum_{i=1}^{q+1} \vartheta_i^{-1} \left\| \xi(t^{m,i}) \right\|^2 w_i = \frac{c_1}{\tau_m} \| \xi \|_m^2,$$

with $c_1 = \tilde{C}(q) w^{**}/w^*$. □

(d) Integration by parts implies that

$$\int_{I_m} \left(\eta', \tilde{\xi} \right) \mathrm{d}t + \left(\{\eta\}_{m-1}, \tilde{\xi}_{m-1}^{+} \right)$$

$$= - \int_{I_m} \left(\eta, \tilde{\xi}' \right) \mathrm{d}t + \left(\eta_m^{-}, \tilde{\xi}_m^{-} \right) - \left(\eta_{m-1}^{+}, \tilde{\xi}_{m-1}^{+} \right)$$

$$+ \left(\eta_{m-1}^{+}, \tilde{\xi}_{m-1}^{+} \right) - \left(\eta_{m-1}^{-}, \tilde{\xi}_{m-1}^{+} \right). \tag{4.30}$$

Since $\tilde{\xi}' \in S_{h,\tau}^{p,q-1}$, $\tilde{\xi}_m^- \in S_{h,m}^p$, in virtue of (3.17), (c) and (4.30) we have

$$\int_{I_m} \left(\eta', \tilde{\xi}\right) \mathrm{d}t + \left(\{\eta\}_{m-1}, \xi_{m-1}^+\right) = -\left(\eta_{m-1}^-, \tilde{\xi}_{m-1}^+\right) \le \|\eta_{m-1}^-\| \left\|\tilde{\xi}_{m-1}^+\right\|. \tag{4.31}$$

(e) We use the following lemma:

Lemma 4 *If* $k > 0$, *then there exists a constant* $C > 0$ *such that*

$$\left|\int_{I_m} A_{h,m}(\eta, \tilde{\xi})\, \mathrm{d}t\right| \le \frac{\varepsilon}{k} \int_{I_m} \|\xi\|_{\mathrm{DG},m}^2 \,\mathrm{d}t + C\,\varepsilon \int_{I_m} \sigma_m^2(\eta)\,\mathrm{d}t. \tag{4.32}$$

Proof Using (4.11) with $\varphi := \tilde{\xi}$, we get

$$\left|\int_{I_m} A_{h,m}(\eta, \tilde{\xi})\, \mathrm{d}t\right| \le \frac{\varepsilon}{k} \int_{I_m} \left\|\tilde{\xi}\right\|_{\mathrm{DG},m}^2 \mathrm{d}t + C\varepsilon \int_{I_m} \sigma_m^2(\eta)\,\mathrm{d}t. \tag{4.33}$$

Now we shall estimate $\int_{I_m} \|\tilde{\xi}\|_{\mathrm{DG},m}^2 \,\mathrm{d}t$. The function $\tilde{\xi}(t) = \sum_{j=0}^q \alpha_j\, t^j$, where $\alpha_j \in S_{h,m}^p$ is the Radau interpolation of the function $\tau_m\, \xi(t)/(t - t_{m-1})$. Hence,

$$\|\tilde{\xi}(t^{m,i})\|_{\mathrm{DG},m}^2 = \|\xi(t^{m,i})\|_{\mathrm{DG},m}^2\, \vartheta_i^{-2}, \quad i = 1, \ldots, q+1,$$

and $\|\tilde{\xi}(t)\|_{\mathrm{DG},m}^2$ is a polynomial in t of degree $\le 2q$. Thus, we get

$$\begin{aligned}\int_{I_m} \left\|\tilde{\xi}(t)\right\|_{\mathrm{DG},m}^2 \mathrm{d}t &= \tau_m \sum_{i=1}^{q+1} w_i\, \vartheta_i^{-2} \left\|\xi(t^{m,i})\right\|_{\mathrm{DG},m}^2 \\ &\le \vartheta_1^{-2} \tau_m \sum_{i=1}^{q+1} w_i \left\|\xi(t^{m,i})\right\|_{\mathrm{DG},m}^2 = \vartheta_1^{-2} \int_{I_m} \|\xi\|_{\mathrm{DG},m}^2 \,\mathrm{d}t.\end{aligned}$$

Hence,

$$\int_{I_m} \left\|\tilde{\xi}\right\|_{\mathrm{DG},m}^2 \mathrm{d}t \le C \int_{I_m} \|\xi\|_{\mathrm{DG},m}^2 \,\mathrm{d}t. \tag{4.34}$$

From (4.33) and (4.34) we get estimate (4.32), which we wanted to prove. □

(f) By (4.8) and (4.34),

$$\left| \int_{I_m} b_{h,m}(u,\tilde{\xi}) - b_{h,m}(U,\tilde{\xi})\,\mathrm{d}t \right| \le \frac{\varepsilon}{k} \int_{I_m} \|\xi\|^2_{\mathrm{DG},m} \mathrm{d}t + \frac{C}{\varepsilon} \left(\int_{I_m} \|\xi\|^2 \mathrm{d}t + \int_{I_m} \tilde{\sigma}^2_m(\eta) \mathrm{d}t \right). \tag{4.35}$$

Now we prove the desired estimate.

Lemma 5 *There exist constants* $C, C^* > 0$ *such that*

$$\int_{I_m} \|\xi\|^2 \,\mathrm{d}t \le C\,\tau_m \left(\left\|\xi^-_{m-1}\right\|^2 + \left\|\eta^-_{m-1}\right\|^2 + \int_{I_m} R_m(\eta)\,\mathrm{d}t \right), \tag{4.36}$$

provided

$$0 < \tau_m \le C^* \varepsilon. \tag{4.37}$$

Proof If we proceed similarly as in the proof of (4.34), using (4.21) and the inequalities $1 \le \vartheta_i^{-1} \le \vartheta_1^{-1}$, we get

$$\begin{aligned} \int_{I_m} \|\xi\|^2 \,\mathrm{d}t &= \tau_m \sum_{i=1}^{q+1} w_i \left\|\xi(t^{m,i})\right\|^2 \le \|\xi\|^2_m, \\ \|\xi\|^2_m &\le \vartheta_1^{-1}\,\tau_m \sum_{i=1}^{q+1} w_i \left\|\xi(t^{m,i})\right\|^2 = \vartheta_1^{-1} \int_{I_m} \|\xi\|^2 \,\mathrm{d}t. \end{aligned} \tag{4.38}$$

Now, (4.22), (4.24), (4.25), (4.27), (4.28), (4.31), (4.32) and (4.35) yield

$$\begin{aligned} &\frac{1}{2}\|\xi^-_m\|^2 + \frac{1}{2}\frac{1}{\tau_m}\|\xi\|^2_m + \frac{\varepsilon}{2}\int_{I_m} \|\xi\|^2_{\mathrm{DG},m}\,\mathrm{d}t \\ &\le \left\|\xi^-_{m-1}\right\| \; \|\xi\|_m \sqrt{\frac{c_1}{\tau_m}} + \left\|\eta^-_{m-1}\right\| \; \|\xi\|_m \sqrt{\frac{c_1}{\tau_m}} + \frac{2\varepsilon}{k}\int_{I_m} \|\xi\|^2_{\mathrm{DG},m}\,\mathrm{d}t \\ &\quad + \frac{C}{\varepsilon}\int_{I_m} \|\xi\|^2\,\mathrm{d}t + C\,\varepsilon \int_{I_m} \sigma^2_m(\eta)\,\mathrm{d}t + \frac{C}{\varepsilon}\int_{I_m} \tilde{\sigma}^2_m(\eta)\,\mathrm{d}t. \end{aligned}$$

This, (4.17), (4.38), Young's inequality and the choice $k := 8$ imply that

$$\left\|\xi_m^-\right\|^2 + \frac{\varepsilon}{2}\int_{I_m} \|\xi\|_{\mathrm{DG},m}^2 \,\mathrm{d}t + \left(\frac{1}{2\tau_m} - \frac{\tilde{C}}{\varepsilon}\right)\int_{I_m} \|\xi\|^2 \,\mathrm{d}t$$
$$\leq C\left(\left\|\xi_{m-1}^-\right\|^2 + \left\|\eta_{m-1}^-\right\|^2 + \int_{I_m} R_m(\eta)\,\mathrm{d}t\right). \qquad (4.39)$$

Let us put $C^* = 1/(4\tilde{C})$, where $\tilde{C}$ is the constant from (4.39), and assume that (4.37) holds. Then $\frac{1}{2\tau_m} - \frac{\tilde{C}}{\varepsilon} \geq \frac{1}{4\tau_m}$ and (4.39) implies (4.36). □

Summarizing estimates (4.16) with $k := 8$ and (4.36), we find that for $m = 1, \ldots, M$,

$$\left\|\xi_m^-\right\|^2 + \frac{\varepsilon}{2}\int_{I_m} \|\xi\|_{\mathrm{DG},m}^2 \,\mathrm{d}t \leq \left(1 + \frac{c}{\varepsilon}\tau_m\right)\left\|\xi_{m-1}^-\right\|^2 + C\left\|\eta_{m-1}^-\right\|^2 + C\int_{I_m} R_m(\eta)\,\mathrm{d}t, \qquad (4.40)$$

with constants $c,\ C > 0$.

Finally, we come to the *abstract error estimate*.

Theorem 1 *Let (4.37) hold. Then there exist constants $C, c > 0$ such that the error $e = U - u$ satisfies the estimate for all $m = 1, \ldots, M$:*

$$\|e_m^-\|^2 + \frac{\varepsilon}{2}\sum_{j=1}^{m}\int_{I_j} \|e\|_{DG,j}^2 \,\mathrm{d}t$$
$$\leq C\exp(ct_m/\varepsilon)\left(\sum_{j=1}^{m} \|\eta_j^-\|^2 + \sum_{j=1}^{m}\int_{I_j} R_j(\eta)\,\mathrm{d}t\right)$$
$$+2\left(\|\eta_m^-\|^2 + \varepsilon\sum_{j=1}^{m}\int_{I_j} \|\eta\|_{DG,j}^2 \,\mathrm{d}t\right). \qquad (4.41)$$

Proof The application of the discrete Gronwall's lemma to (4.40) gives the estimate

$$\left\|\xi_m^-\right\|^2 + \frac{\varepsilon}{2}\sum_{j=1}^{m}\int_{I_j} \|\xi\|_{\mathrm{DG},j}^2 \,\mathrm{d}t$$
$$\leq C\exp(ct_m/\varepsilon)\left(\left\|\xi_0^-\right\|^2 + \sum_{j=1}^{m}\left\|\eta_j^-\right\|^2 + \sum_{j=1}^{m}\int_{I_j} R_j(\eta)\,\mathrm{d}t\right), \qquad (4.42)$$

for $m = 1, \ldots, M$. In view of the definition of U_0^-, we have $\xi_0^- = 0$. Now, if we use the relation $e = \xi + \eta$ and the inequalities

$$\begin{aligned} \|e\|^2 &\leq 2(\|\xi\|^2 + \|\eta\|^2), \\ \|e\|_{DG,j}^2 &\leq 2(\|\xi\|_{DG,j}^2 + \|\eta\|_{DG,j}^2), \end{aligned} \tag{4.43}$$

from (4.42) we immediately get (4.41). □

5 Interpolation error bounds and error estimation in terms of h and τ

This section will be devoted to obtaining error estimates in dependence on the mesh sizes τ and h. They will be obtained on the basis of estimate (4.41), the relations

$$\begin{aligned} e = U - u = \xi + \eta, \qquad \pi\, u\,\big|_{I_m} = \pi\,(\Pi_m u)\big|_{I_m}, \\ \eta\big|_{I_m} = (\pi u - u)\big|_{I_m} = \eta^{(1)} + \eta^{(2)}, \text{ with } \eta^{(1)} = (\Pi_m u - u)\big|_{I_m}, \\ \eta^{(2)} = (\pi(\Pi_m u) - \Pi_m u)\,\big|_{I_m}, \end{aligned} \tag{5.1}$$

and estimates of individual terms on the right-hand side of (4.41) containing η, which will be proven in the sequel. To this end, we assume that the exact solution satisfies the regularity condition

$$u \in H^{q+1}\Big(0, T; H^1(\Omega)\Big) \cap C([0, T]; H^{p+1}(\Omega)) \tag{5.2}$$

and that the meshes satisfy conditions (4.1), (4.2), (4.6) and (4.37). Obviously, $C([0, T]; H^{p+1}(\Omega)) \subset L^2(0, T; H^{p+1}(\Omega))$. Moreover, let

$$\tau_m \geq C h_m^2, \quad m = 1, \ldots, M. \tag{5.3}$$

Let us note that this assumption is not necessary, if the space meshes do not depend on time, i.e. all meshes $\mathcal{T}_{h,m},\ m = 1, \ldots, M$, are identical—see Remark 3.

If $r \geq 1$ is integer and $\mu = \min(r, p)$, then for $m = 1, \ldots, M$ and any $v \in H^{r+1}(\Omega)$we have the standard estimates

$$\begin{aligned} \|\Pi_m v - v\|_{L^2(K)} &\leq C\, h_K^{\mu+1} |v|_{H^{\mu+1}(K)}, \\ |\Pi_m v - v|_{H^1(K)} &\leq C\, h_K^{\mu} |v|_{H^{\mu+1}(K)}, \\ |\Pi_m v - v|_{H^2(K)} &\leq C\, h_K^{\mu-1} |v|_{H^{\mu+1}(K)}, \end{aligned} \tag{5.4}$$

for $K \in \mathcal{T}_{h,m},\ h \in (0, h_0)$ and

$$\begin{aligned} &\text{(a)} \quad \|\Pi_m v\|_{L^2(K)} \leq \|v\|_{L^2(K)} \quad \text{for } v \in L^2(K),\ K \in \mathcal{T}_{h,m},\ h \in (0, h_0), \\ &\text{(b)} \quad |\Pi_m v|_{H^1(K)} \leq C|v|_{H^1(K)} \quad \text{for } v \in H^1(K),\ K \in \mathcal{T}_{h,m},\ h \in (0, h_0). \end{aligned} \tag{5.5}$$

It is possible to find that

$$D^{q+1}(\Pi_m u) = \Pi_m(D^{q+1}u). \tag{5.6}$$

Actually, by (3.13), $\Pi_m u(\cdot, t) \in S^p_{h,m}$ and for all $t \in I_m$,

$$\int_\Omega (\Pi_m u(x,t) - u(x,t))\ \varphi(x)\,\mathrm{d}x = 0, \quad \forall\varphi \in S^p_{h,m}. \tag{5.7}$$

The differentiation with respect to t yields

$$\int_\Omega \Big(D^{q+1}(\Pi_m u(x,t)) - D^{q+1}u(x,t)\Big)\ \varphi(x)\,\mathrm{d}x = 0, \quad \forall\varphi \in S^p_{h,m}. \tag{5.8}$$

Moreover, obviously $D^{q+1}(\Pi_m u(t)) \in S^p_{h,m}$ and thus, (5.6) holds.

Similarly we can prove that

$$D^{q+1}(\nabla\Pi_m u) = \nabla\Pi_m(D^{q+1}u). \tag{5.9}$$

5.1 Time interpolation

Lemma 6 *Let $\varphi \in C((t_{m-1}, t_m], S^p_{h,m})$, $m = 1\ldots, M$. Then for each $x \in K,\ K \in \mathcal{T}_{h,m},\ t \in I_m,\ m = 1,\ldots,M$ we have*

$$\pi\,\varphi(x,t) = \tilde{P}_m\varphi(x,t), \tag{5.10}$$

where $\tilde{P}_m$ is defined in the following way: For $\omega \in C\,((t_{m-1}, t_m])$,

$$\begin{aligned}
&\text{(a)}\quad \tilde{P}_m\omega \in \mathcal{P}^q(I_m),\\
&\text{(b)}\quad \int_{I_m}\Big(\tilde{P}_m\omega(t) - \omega(t)\Big)\,t^j\,\mathrm{d}t = 0, \quad \forall j = 0,\ldots,q-1,\\
&\text{(c)}\quad \tilde{P}_m\omega(t_m-) = \omega(t_m-).
\end{aligned} \tag{5.11}$$

Proof Let $m \in \{1,\ldots,M\}$. From the definition of the operators π and $\tilde{P}_m$ it follows that for each $K \in \mathcal{T}_{h,m}$ the functions $\pi\varphi$ and $\tilde{P}_m\varphi$ are on $K \times I_m$ polynomials of degree $\leq q$ in $t \in I_m$ and of degree $\leq p$ in $x \in K$. Moreover, $\pi\varphi(x, t_m-) = \varphi(x, t_m-) = \tilde{P}_m\varphi(x, t_m-)$ for all $x \in K$. Obviously, condition (3.17), (c) is equivalent to

$$\int_{I_m}\left(\int_K (\pi\varphi(x,t) - \varphi(x,t))\,\sigma(x)\,\mathrm{d}x\right) t^j\,\mathrm{d}t = 0,$$
$$\forall j = 0,\ldots,q-1, \quad \forall\sigma \in P^p(K), \quad \forall K \in \mathcal{T}_{h,m}. \tag{5.12}$$

Further, by (5.11), for any $K \in \mathcal{T}_{h,m}$,

$$\int_{I_m} \left(\tilde{P}_m\varphi(x,t) - \varphi(x,t) \right) t^j \, \mathrm{d}t = 0, \quad \forall j = 0, \ldots, q-1, \quad \forall x \in K. \tag{5.13}$$

Let $\sigma \in P^p(K)$. Then (5.13) and Fubini's theorem imply that

$$\begin{aligned} 0 &= \int_K \left(\int_{I_m} (\tilde{P}_m\varphi(x,t) - \varphi(x,t))\, t^j \, \mathrm{d}t \right) \sigma(x) \, \mathrm{d}x \\ &= \int_{I_m} \left(\int_K (\tilde{P}_m\varphi(x,t) - \varphi(x,t))\sigma(x) \, \mathrm{d}x \right) t^j \, \mathrm{d}t, \\ &\qquad \forall j = 0, \ldots, q-1, \quad \forall \sigma \in P^p(K), \quad \forall K \in \mathcal{T}_{h,m}. \end{aligned} \tag{5.14}$$

Comparing (5.14) with (5.12) and taking into account the fact that the operator π is uniquely determined by conditions (3.17), we immediately get (5.10). □

Lemma 7 *If $\omega \in H^{q+1}(I_m)$, then*

$$\left\| \tilde{P}_m\omega - \omega \right\|^2_{L^2(I_m)} \le C\, \tau_m^{2q+2} \left\| D^{q+1}\omega \right\|^2_{L^2(I_m)}, \tag{5.15}$$

where $C > 0$ is a constant independent of ω, m and τ_m.

Similar result can be found in several works. For example, estimate (5.15) can be obtained as a consequence of Corollary 3.13 in [11].

Lemmas 6 and 7 imply that for $\varphi \in H^{q+1}(I_m, S^p_{h,m})$ we have

$$\begin{aligned} &\|\pi\varphi(x,\cdot) - \varphi(x,\cdot)\|^2_{L^2(I_m)} \le C\, \tau_m^{2q+2} \left\| D^{q+1}\varphi(x,\cdot) \right\|^2_{L^2(I_m)}, \\ &x \in K,\ K \in \mathcal{T}_{h,m},\ m = 1, \ldots, M. \end{aligned} \tag{5.16}$$

5.2 Estimates of terms with η

Our further goal is to estimate the expressions

$$\|\eta_m^-\|^2, \int_{I_m} \|\eta\|^2_{L^2(K)} \, \mathrm{d}t, \int_{I_m} |\eta|^2_{H^1(K)} \, \mathrm{d}t, \ h_K^2 \int_{I_m} |\eta|^2_{H^2(K)} \, \mathrm{d}t, \int_{I_m} J_{h,m}(\eta,\eta)\, dt.$$

By (5.1),

$$\begin{aligned} \|\eta\|^2_{L^2(K)} &\le 2\|\eta^{(1)}\|^2_{L^2(K)} + 2\|\eta^{(2)}\|^2_{L^2(K)}, \\ |\eta|^2_{H^s(K)} &\le 2|\eta^{(1)}|^2_{H^s(K)} + 2|\eta^{(2)}|^2_{H^s(K)},\ s = 1, 2. \end{aligned} \tag{5.17}$$

Lemma 8 *The following estimates hold for* $K \in \mathcal{T}_{h,m},\ m = 1, \ldots, M$:

$$\|\eta_m^-\|^2 \le Ch^{p+1}|u(t_m)|_{H^{p+1}(\Omega)}, \tag{5.18}$$

$$\int_{I_m} \|\eta^{(1)}\|^2_{L^2(K)}\mathrm{d}t \le C\, h_K^{2(p+1)}|u|^2_{L^2(I_m,H^{p+1}(K))}, \tag{5.19}$$

$$\int_{I_m} |\eta^{(1)}|^2_{H^1(K)}\mathrm{d}t \le C\, h_K^{2p}|u|^2_{L^2(I_m,H^{p+1}(K))}, \tag{5.20}$$

$$h_K^2 \int_{I_m} |\eta^{(1)}|^2_{H^2(K)}\mathrm{d}t \le C\, h_K^{2p}|u|^2_{L^2(I_m,H^{p+1}(K))}. \tag{5.21}$$

Proof It is enough to use (5.4). □

The derivation of estimates of terms with $\eta^{(2)}$ is more complicated.

Lemma 9 *For* $K \in \mathcal{T}_{h,m},\ m = 1, \ldots, M$, *we have*

$$\int_{I_m} \|\eta^{(2)}\|^2_{L^2(K)}\mathrm{d}t \le C\, \tau_m^{2(q+1)}|u|^2_{H^{q+1}(I_m,L^2(K))}, \tag{5.22}$$

$$\int_{I_m} |\eta^{(2)}|^2_{H^1(K)}\mathrm{d}t \le C\, \tau_m^{2(q+1)}|u|^2_{H^{q+1}(I_m,H^1(K))}, \tag{5.23}$$

$$h_K^2 \int_{I_m} |\eta^{(2)}|^2_{H^2(K)}\mathrm{d}t \le C\, \tau_m^{2(q+1)}|u|^2_{H^{q+1}(I_m,H^1(K))}. \tag{5.24}$$

Proof (a) The use of Fubini's theorem and (5.10), (5.6), (5.16), (5.5), (a) and (5.15) yield the relations

$$\begin{aligned}
\int_{I_m} \|\eta^{(2)}\|^2_{L^2(K)} &= \int_K \|\tilde{P}_m(\Pi_m u) - \Pi_m u\|^2_{L^2(I_m)}\mathrm{d}x \\
&\le C\, \tau_m^{2q+2} \int_{I_m} \left(\int_K |\, D^{q+1}(\Pi_m u)|^2 \mathrm{d}x \right) \mathrm{d}t \\
&\le C\, \tau_m^{2q+2} \int_{I_m} \left(\int_K |D^{q+1} u|^2 \mathrm{d}x \right) \mathrm{d}t = C\, \tau_m^{2q+2}|u|^2_{H^{q+1}(I_m,L^2(K))}.
\end{aligned}$$

(b) Further, due to Fubini's theorem, (5.10), (5.16), (5.9) and (5.5), (b), we find that

$$\begin{aligned}
\int_{I_m} |\eta^{(2)}|_{H^1(K)} \mathrm{d}t &= \int_{I_m} \left(\int_K \left| \nabla \left(\Pi_m u - \tilde{P}_m(\Pi_m u) \right) \right|^2 \mathrm{d}x \right) \mathrm{d}t \\
&\le C\, \tau_m^{2q+2} \int_K |\nabla(\Pi_m u)|^2_{H^{q+1}(I_m)} \mathrm{d}x \\
&= C\, \tau_m^{2q+2} \int_K \left(\int_{I_m} \left| D^{q+1} \nabla(\Pi_m u) \right|^2 \mathrm{d}t \right) \mathrm{d}x \\
&= C\, \tau_m^{2q+2} \int_{I_m} \left(\int_K \left| \nabla(\Pi_m D^{q+1} u) \right|^2 \mathrm{d}x \right) \mathrm{d}t \\
&\le C\, \tau_m^{2q+2} \int_{I_m} \left| D^{q+1} u \right|^2_{H^1(K)} \mathrm{d}t \\
&= C\, \tau_m^{2q+2} |u|^2_{H^{q+1}(I_m, H^1(K))}.
\end{aligned}$$

(c) Using a similar process as in (b) and (4.5), we find that

$$\begin{aligned}
\int_{I_m} |\eta^{(2)}|_{H^2(K)} \mathrm{d}t &\le C\, \tau_m^{2q+2} \int_{I_m} \left| \Pi_m \left(D^{q+1} u \right) \right|^2_{H^2(K)} \mathrm{d}t \\
&\le C\, \tau_m^{2q+2} h_K^{-2} \int_{I_m} \left| D^{q+1} u \right|^2_{H^1(K)} \mathrm{d}t \\
&= C\, \tau_m^{2q+2} h_K^{-2} |u|_{H^{q+1}(I_m, H^1(K))}.
\end{aligned}$$

This yields (5.24). □

Finally, we shall be concerned with the estimation of $\int_{I_m} J_{h,m}(\eta, \eta)\, \mathrm{d}t$. It holds

$$\begin{aligned}
&J_{h,m}(\eta, \eta) \\
&\le C \left(J_{h,m}(\Pi_m u - u, \Pi_m u - u) + J_{h,m}(\pi(\Pi_m u) - \Pi_m u, \pi(\Pi_m u) - \Pi_m u) \right).
\end{aligned} \tag{5.25}$$

Using the multiplicative trace inequality (4.4) and (5.4), in the same way as in [22] we get

$$\int_{I_m} J_{h,m} \left(\Pi_m u - u, \Pi_m u - u \right) \mathrm{d}t \le C\, h^{2p} \, |u|^2_{L^2(I_m, H^{p+1}(\Omega))}. \tag{5.26}$$

Further, we shall estimate the expression

$$\int_{I_m} J_{h,m}\left(\pi(\Pi_m u) - \Pi_m u,\ \pi(\Pi_m u) - \Pi_m u\right)\,\mathrm{d}t.$$

Lemma 10 *Let the Dirichlet data $u_D = u_D(x,t)$ have the behaviour in t as a polynomial of degree $\le q$:*

$$u_D(x,t) = \sum_{j=0}^{q} \psi_j(x)\,t^j, \tag{5.27}$$

where $\psi_j \in H^{p+1/2}(\partial\Omega)$ for $j = 0,\dots,q$. Then

$$\int_{I_m} J(\pi(\Pi_m u) - \Pi_m u, \pi(\Pi_m u) - \Pi_m u)\,\mathrm{d}t \le C\tau_m^{2q+2}|u|^2_{H^{q+1}(I_m,H^1(\Omega))}, \quad m = 1,\dots,M. \tag{5.28}$$

For general data u_D, if there exists a constant $\overline{C} > 0$ such that $\tau_m \le \overline{C}h_K$ for all $K \in \mathcal{T}_{h,m},\ h \in (0,h_0)$ and $m = 1,\dots,M$, then

$$\int_{I_m} J(\pi(\Pi_m u) - \Pi_m u, \pi(\Pi_m u) - \Pi_m u)\,\mathrm{d}t \le C\tau_m^{2q}|u|^2_{H^{q+1}(I_m,H^1(\Omega))}, \quad m = 1,\dots,M. \tag{5.29}$$

Proof We proceed in two steps.

(I) Let $\Gamma \in \mathcal{F}^I_{h,m}$, i. e. $\Gamma \subset \Omega$. If we set $\varphi := \Pi_m u$ and use the relation $[\tilde{P}_m\varphi - \varphi] = \tilde{P}_m[\varphi] - [\varphi]$ and estimate (5.16), we find that

$$\int_{I_m}\left(\int_\Gamma [\pi(\Pi_m u) - \Pi_m u]^2\,\mathrm{d}S\right)\mathrm{d}t = \int_\Gamma \left\|\tilde{P}_m[\varphi(x,\cdot)] - [\varphi(x,\cdot)]\right\|^2_{L^2(I_m)}\mathrm{d}S \le C\,\tau_m^{2q+2}\int_\Gamma \left\|D^{q+1}[\varphi(x,\cdot)]\right\|^2_{L^2(I_m)}\mathrm{d}S. \tag{5.30}$$

If we take into account that

$$D^{q+1}[\varphi(x,\cdot)] = [D^{q+1}\varphi(x,\cdot)], \quad [D^{q+1}u] = 0, \tag{5.31}$$

and use Fubini's theorem, we obtain

$$\begin{aligned}\int_{I_m}\left(\int_\Gamma [\pi(\Pi_m u) - \Pi_m u]^2\,\mathrm{d}S\right)\mathrm{d}t &= \int_\Gamma\left(\int_{I_m} [\pi(\Pi_m u) - \Pi_m u]^2\,\mathrm{d}t\right)\mathrm{d}S\\ &\le C\,\tau_m^{2q+2}\int_\Gamma\left(\int_{I_m}\left|D^{q+1}[\varphi(x,t)]\right|^2\,\mathrm{d}t\right)\mathrm{d}S\\ &= C\,\tau_m^{2q+2}\int_{I_m}\left(\int_\Gamma [D^{q+1}(\Pi_m u - u)]^2\,\mathrm{d}S\right)\mathrm{d}t.\end{aligned} \tag{5.32}$$

The application of the multiplicative trace inequality implies that

$$\begin{aligned}\sum_{\Gamma\in\mathcal{F}_{h,m}^I}\int_\Gamma\left[D^{q+1}(\Pi_m u - u)\right]^2\,\mathrm{d}S &\le C\sum_{K\in\mathcal{T}_{h,m}}\left\|D^{q+1}(\Pi_m u - u)\right\|^2_{L^2(\partial K)}\\ &\le C\sum_{K\in\mathcal{T}_{h,m}}\left(\left\|D^{q+1}(\Pi_m u - u)\right\|_{L^2(K)}\right.\\ &\quad\times\left|D^{q+1}(\Pi_m u - u)\right|_{H^1(K)}\\ &\quad\left.+\,h_K^{-1}\left\|D^{q+1}(\Pi_m u - u)\right\|^2_{L^2(K)}\right).\end{aligned} \tag{5.33}$$

By (5.6),

$$D^{q+1}(\Pi_m u - u) = \Pi_m(D^{q+1}u) - D^{q+1}u. \tag{5.34}$$

In virtue of (5.2),$D^{q+1}u \in L^2(I_m, H^1(\Omega))$. This and the approximation properties (5.4) of Π_m imply that

$$\begin{aligned}\|\Pi_m(D^{q+1}u) - D^{q+1}u\|_{L^2(K)} &\le Ch_K|D^{q+1}u|_{H^1(K)},\\ |\Pi_m(D^{q+1}u) - D^{q+1}u|_{H^1(K)} &\le C|D^{q+1}u|_{H^1(K)}.\end{aligned} \tag{5.35}$$

Summarizing (4.3), (5.32), (5.33), (5.34) and (5.35), we get

$$\begin{aligned}&\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}_{h,m}^I} h(\Gamma)^{-1}\int_\Gamma [\pi(\Pi_m u) - \Pi_m u]^2\,\mathrm{d}S\right)\mathrm{d}t\\ &\quad\le C\,\tau_m^{2q+2}\int_{I_m}\sum_{K\in\mathcal{T}_{h,m}}\left|D^{q+1}u\right|^2_{H^1(K)}\,\mathrm{d}t = C\,\tau_m^{2q+2}|u|^2_{H^{q+1}(I_m,H^1(\Omega))}.\end{aligned} \tag{5.36}$$

(II) In what follows, we shall assume that $\Gamma \in \mathcal{F}^B_{h,m}$, i.e. $\Gamma \subset \partial\Omega \cap \partial K$ for some $K \in \mathcal{T}_{h,m}$, and estimate the expression

$$\beta := \int_{I_m} \left(h(\Gamma)^{-1} \int_{\Gamma} |\pi(\Pi_m u) - \Pi_m u|^2 \mathrm{d}S \right) \mathrm{d}t. \tag{5.37}$$

Proceeding in a similar way as above, we find that

$$\begin{aligned} \beta &\le C\, \tau_m^{2q+2} h(\Gamma)^{-1} \int_{\Gamma} \left\| D^{q+1}(\Pi_m u) \right\|^2_{L^2(I_m)} \mathrm{d}S \\ &= C\, \tau_m^{2q+2} h(\Gamma)^{-1} \int_{I_m} \left(\int_{\Gamma} \left| \Pi_m(D^{q+1}u) \right|^2 \mathrm{d}S \right) \mathrm{d}t. \end{aligned} \tag{5.38}$$

If we apply the multiplicative trace inequality and use the assumption that $\tau_m \le \overline{C} h_K$ for all $K \in \mathcal{T}_{h,m}$, we get

$$\int_{I_m} \left(\sum_{\Gamma \in \mathcal{F}^I_{h,m}} h(\Gamma)^{-1} \int_{\Gamma} |\pi(\Pi_m u) - \Pi_m u|^2 \,\mathrm{d}S \right) \mathrm{d}t \le C\, \tau_m^{2q} |u|^2_{H^{q+1}(I_m, H^1(\Omega))}. \tag{5.39}$$

Now let us assume that the Dirichlet data $u_D = u_D(x,t)$ satisfy (5.27). Then $D^{q+1}u|_{\partial\Omega} = D^{q+1}u_D = 0$. This and (5.38) imply that

$$\beta \le C\, \tau_m^{2q+2} \int_{I_m} \left(h(\Gamma)^{-1} \int_{\Gamma} \left| \Pi_m(D^{q+1}u) - D^{q+1}u \right|^2 \mathrm{d}S \right) \mathrm{d}t. \tag{5.40}$$

Again we use the multiplicative trace inequality and estimates (5.35) and get the estimate

$$\int_{I_m} \left(\sum_{\Gamma \in \mathcal{F}^B_{h,m}} h(\Gamma)^{-1} \int_{\Gamma} |\pi(\Pi_m u) - \Pi_m u|^2 \,\mathrm{d}s \right) \mathrm{d}t \le C\, \tau_m^{2q+2} |u|^2_{H^{q+1}(I_m, H^1(\Omega))}. \tag{5.41}$$

Finally, summarizing estimates (5.36), (5.39) and (5.41), we get (5.28) and (5.29). □

5.3 Main results

In this section we shall conclude the analysis of the error estimate.

Theorem 2 *Let u be the exact solution satisfying the regularity condition* (5.2) *of problem* (2.1)–(2.3) *with Dirichlet data u_D defined by* (5.27). *Let U be the approximate solution to problem* (2.1)–(2.3) *obtained by scheme* (3.15) *over spatial meshes $\mathcal{T}_{h,m}$ and time partition I_m, $m = 1, \dots, M$, satisfying conditions* (4.1), (4.2), (4.37) *and* (5.3). *Then there exist constants $C, c > 0$ independent of $h, \tau, m, \varepsilon, u$ such that*

$$\|e_m^-\| + \frac{\varepsilon}{2}\sum_{j=1}^{m}\int_{I_j}\|e\|_{DG,j}^2\,\mathrm{d}t$$
$$\le C\exp(ct_m/\varepsilon)\left(\left(h^{2p}|u|^2_{L^2(0,T;H^{p+1}(\Omega))} + \tau^{2q+2}|u|_{H^{q+1}(0,T;H^1(\Omega))}\right)\left(\varepsilon + \frac{1}{\varepsilon}\right)\right.$$
$$\left. + h^{2p}|u|^2_{C([0,T]+H^{p+1}(\Omega))}\right), \quad m = 1, \dots, M,\ h \in (0, h_0) \tag{5.42}$$

and

$$\|e\|^2_{L^2(Q_T)} \le C\left(h^{2p+2} + e^{cT/\varepsilon}h^{2p}\right)|u|^2_{C([0,T],H^{p+1}(\Omega))}$$
$$+ C\left(\varepsilon + \frac{1}{\varepsilon}\right)\left(1 + e^{cT/\varepsilon}\right)$$
$$\times\left(h^{2p}|u|^2_{L^2(0,T;H^{p+1}(\Omega))} + \tau^{2q+2}|u|^2_{H^{q+1}(0,T;H^1(\Omega))}\right)$$
$$+ C\left(h^{2p+2}|u|^2_{L^2(0,T;H^{p+1}(\Omega))} + \tau^{2q+2}|u|^2_{H^{q+1}(0,T;L^2(\Omega))}\right),$$
$$m = 1, \dots, M,\ h \in (0, h_0). \tag{5.43}$$

Proof (1) In order to prove (5.42), we start from (4.41) and estimate the terms containing η. In virtue of (4.17), (4.12), (4.9),

$$R_j(\eta) = \varepsilon\,\sigma_j^2(\eta) + \frac{1}{\varepsilon}\,\tilde{\sigma}_j^2(\eta)$$
$$= \varepsilon\left(\sum_{K\in\mathcal{T}_{h,j}}\left(|\eta|^2_{H^1(K)} + h_K^2|\eta|^2_{H^2(K)}\right) + J_{h,j}(\eta,\eta)\right)$$
$$+ \frac{1}{\varepsilon}\sum_{K\in\mathcal{T}_{h,j}}\left(\|\eta\|^2_{L^2(K)} + h_K^2|\eta|^2_{H^1(K)}\right). \tag{5.44}$$

Now, (5.44) together with (5.17) and Lemmas 8 and 9 yield the estimate

$$\int_{I_j} R_j(\eta)\,\mathrm{d}t \le C\left(\varepsilon + \frac{1}{\varepsilon}\right)\sum_{K\in\mathcal{T}_{h,j}}\left(h_K^{2p}|u|^2_{L^2(I_j,H^{p+1}(K))} + \tau_j^{2q+2}|u|^2_{H^{q+1}(I_j,H^1(K))}\right). \tag{5.45}$$

This and the inequalities $\tau_j \le \tau$, $h_K \le h_j \le h$ lead to

$$\int_{I_j} R_j(\eta)\,\mathrm{d}t \le C\left(\varepsilon + \frac{1}{\varepsilon}\right)\left(h^{2p}|u|^2_{L^2(I_j,H^{p+1}(\Omega))} + \tau^{2q+2}|u|^2_{H^{q+1}(I_j,H^1(\Omega))}\right). \tag{5.46}$$

Similarly, we get

$$\int_{I_j} \|\eta\|_{DG,j}\,\mathrm{d}t \le C\,\varepsilon \sum_{K\in\mathcal{T}_{h,j}} h_K^{2p}|u|^2_{L^2(I_j,H^{p+1}(K))} \le C\,\varepsilon\, h^{2p}|u|_{L^2(I_j,H^{p+1}(\Omega))}. \tag{5.47}$$

Further, by (5.18) and (5.3),

$$\sum_{j=1}^{m} \|\eta_j^-\|^2 \le C\sum_{j=1}^{M} \tau_j\, h_j^{2p}|u(t_j)|^2_{H^{p+1}(\Omega)} \le C\,T h^{2p}|u|^2_{C([0,T];H^{p+1}(\Omega))}. \tag{5.48}$$

Finally, using (4.42) and (5.46)–(5.48), we arrive at estimate (5.42).

(2) It follows from (4.36) and (4.43) that

$$\int_0^T \|e\|^2\,\mathrm{d}t \le C\sum_{m=1}^{M}\tau_m\left(\|\xi_{m-1}^-\|^2 + \|\eta_{m-1}^-\|^2 + \int_{I_m} R_m(\eta)\,\mathrm{d}t\right) + 2\int_0^T \|\eta\|^2\,\mathrm{d}t. \tag{5.49}$$

Now we use (4.42) with $m := m-1 < M$, $\xi_0 = 0$, $\eta_0^- = \Pi_1 u^0 - u^0$ and get

$$\begin{aligned}\|e\|^2_{L^2(Q_T)} = \int_0^T \|e\|^2\,\mathrm{d}t \le C\sum_{m=1}^{M}\tau_m\Bigg(\|\eta_{m-1}^-\|^2 + \int_{I_m} R_m(\eta)\,\mathrm{d}t \\ + C\,e^{cT/\varepsilon}\left(\sum_{j=1}^{M}\|\eta_j^-\|^2 + \sum_{j=1}^{M}\int_{I_j} R_j(\eta)\,\mathrm{d}t\right)\Bigg) + 2\|\eta\|^2_{L^2(Q_T)}.\end{aligned} \tag{5.50}$$

Estimating the individual terms in (5.50) depending on η, with the aid of (5.45), (5.48), Lemmas 8, 9 and 10 and the relation $\sum_{m=1}^{M}\tau_m = T$, we obtain (5.43). □

Remark 2 As we see, estimates (5.42) and (5.43) are not uniform with respect to $\varepsilon \to 0$. Just on the contrary, the constant in this estimate behaves as $C\exp(cT/\varepsilon)$, which blows up to ∞ as $\varepsilon \to 0$. This is a consequence of the application of Young's inequality used for the treatment of nonlinear terms and Gronwall's lemma. The question, how to avoid this bad behaviour of the error estimate, remains open.

Using estimate (5.29) from Lemma 10, we conclude that in the case of general Dirichlet data u_D, under the additional assumption that $\tau_m \le \overline{C} h_K$, $K \in \mathcal{T}_{h,m}$, $h \in (0, h_0)$, we get suboptimal error estimates of order $O(\tau^q)$ with respect to the time step τ instead of optimal error estimates of order $O(\tau^{q+1})$ obtained in the case, when u_D is given by (5.27).

Remark 3 The case of identical meshes on all time levels. If all meshes $\mathcal{T}_{h,m}$, $m = 1 \dots, M$, are identical, which means that $\mathcal{T}_{h,m} = \mathcal{T}_h$ for all $m = 1, \dots, M$, then all spaces $S^p_{h,m}$ and forms $a_{h,m}, b_{h,m}, \dots$ are also identical: $S^p_{h,m} = S^p_h$, $a_{h,m} = a_h, b_{h,m} = b_h, \dots$ for all $m = 1, \dots, M$. This implies that $\{\xi\}_{m-1} \in S^p_h$ and by (3.20), (3.17), (a), and (3.13), we have $(\eta^-_{m-1}, \{\xi\}_{m-1}) = 0$. Hence, by (4.14),

$$\int\limits_{I_m} (\eta', \xi)\, \mathrm{d}t + \left(\{\eta\}_{m-1}, \xi^+_{m-1}\right) = 0. \tag{5.51}$$

Moreover, similarly it is possible to show that the expression $\sum_{j=1}^m \|\eta^-_j\|^2$ does not appear in estimate (4.42) and instead of (4.41) we get the estimate

$$\begin{aligned} &\|e^-_m\|^2 + \frac{\varepsilon}{2} \sum_{j=1}^m \int\limits_{I_j} \|e\|^2_{DG,j}\, \mathrm{d}t \\ &\quad \le C \exp(ct_m/\varepsilon) \left(\sum_{j=1}^m \int\limits_{I_j} R_j(\eta)\, \mathrm{d}t \right. \\ &\qquad \left. + 2\|\eta^-_m\|^2 + 2\varepsilon \sum_{j=1}^m \int\limits_{I_j} \|\eta\|^2_{DG,j}\, \mathrm{d}t \right), \quad m = 1, \dots, M. \end{aligned} \tag{5.52}$$

From this we deduce that in the case of identical meshes on all time levels the assertion of Theorem 2 is valid without assumption (5.3).

6 Appendix: Alternative proof of Lemma 7

Lemma 11 *Let* $z \in C^\infty([0, 1])$, $z(0) = 0$ *and*

$$\int\limits_0^1 \vartheta^i z(x)\, \mathrm{d}\vartheta = 0, \quad i = 0, \dots, q-1. \tag{6.1}$$

Then

$$\|z\|_{L^2(0,1)} \le C\, \|D^{q+1} z\|_{L^2(0,1)}, \tag{6.2}$$

where C is a constant independent of the function z.

Proof Let us develop the function with the aid of the Taylor formula with integral remainder:

$$z(\vartheta) = z(0) + \cdots + \frac{z^{(q)}(0)}{q!}\,\vartheta^q + \int_0^{\vartheta} \frac{(\vartheta-\tau)^q}{q!} z^{(q+1)}(\tau)\,\mathrm{d}\tau, \quad \vartheta \in [0,1]. \tag{6.3}$$

In the space $L^2(0,1)$ we choose an orthonormal system of polynomials $\varphi_i, \;\; i = 0, 1, \ldots,$ such thatφ_i is a polynomial of degree i and $\varphi_i(0) \neq 0$. (At the end of Appendix we show how this system can be constructed.) Obviously,

$$\int_0^1 \vartheta^i z(\vartheta)\,\mathrm{d}\vartheta = 0, \;\; i = 0,\ldots,q-1 \quad \Longleftrightarrow \quad \int_0^1 \varphi_i(\vartheta) z(\vartheta)\,\mathrm{d}\vartheta = 0, \;\; i = 0,\ldots,q-1. \tag{6.4}$$

In virtue of the properties of the system $\varphi_i, \;\; i = 0, 1, \ldots,$ the expansion (6.3) can be written in the form

$$z(\vartheta) = \sum_{i=0}^{q} c_i \varphi_i(\vartheta) + \int_0^{\vartheta} \frac{(\vartheta-\tau)^q}{q!} z^{(q+1)}(\tau)\,\mathrm{d}\tau, \quad \vartheta \in [0,1], \tag{6.5}$$

where c_i are constants depending on the values $z(0), z'(0), \ldots, z^{(q)}(0)$. From assumption (6.3) and equivalence (6.4) for $j = 0, \ldots, q-1$ we get

$$0 = \int_0^1 \varphi_j(\vartheta) z(\vartheta)\,\mathrm{d}\vartheta = c_j + \int_0^1 \varphi_j(\vartheta) \int_0^{\vartheta} \frac{(\vartheta-\tau)^q}{q!} z^{(q+1)}(\tau)\,\mathrm{d}\tau\,\mathrm{d}\vartheta.$$

The use of Fubini's theorem yields

$$c_j = -\int_0^1 \varphi_j(\vartheta) \int_0^{\vartheta} \frac{(\vartheta-\tau)^q}{q!} z^{(q+1)}(\tau)\,\mathrm{d}\tau\,\mathrm{d}\vartheta = -\int_0^1 \psi_j(\tau) z^{(q+1)}(\tau)\,\mathrm{d}\tau, \tag{6.6}$$

where

$$\psi_j(\tau) = \frac{1}{q!} \int_{\tau}^1 \varphi_j(\vartheta)(\vartheta-\tau)^q\,\mathrm{d}\vartheta, \quad j = 0,\ldots,q-1.$$

Since $\varphi_q(0) \neq 0$, from the assumption that $z(0) = 0$ and expansion (6.5) we get

$$c_q = -\frac{1}{\varphi_q(0)} \sum_{i=0}^{q-1} c_i \varphi_i(0) = \int_0^1 \psi_q(\tau) z^{(q+1)}(\tau)\, \mathrm{d}\tau$$

with

$$\psi_q(\tau) = \frac{1}{\varphi_q(0)} \sum_{j=0}^{q-1} \varphi_j(0) \psi_j(\tau).$$

Substituting in expansion (6.5) for $c_i,\ \ i = 0, \ldots, q$,we find that

$$z(\vartheta) = \int_0^1 k(\vartheta, \tau) z^{(q+1)}(\tau)\, \mathrm{d}\tau, \tag{6.7}$$

where

$$k(\vartheta, \tau) = \begin{cases} \frac{(\vartheta-\tau)^q}{q!} + \varphi_q(\vartheta)\psi_q(\tau) - \sum_{i=0}^{q-1} \varphi_i(\vartheta)\psi_i(\tau) & \text{for} \quad 1 \geq \vartheta > \tau \geq 0, \\ \varphi_q(\vartheta)\psi_q(\tau) - \sum_{i=0}^{q-1} \varphi_i(\vartheta)\psi_i(\tau) & \text{for} \quad 1 \geq \tau > \vartheta \geq 0. \end{cases}$$

The function $k(\vartheta, t)$ is continuous on the set $[0, 1] \times [0, 1]$ and from (6.7) we get (6.2). □

Lemma 12 *Let* $z \in H^{q+1}(0, 1)$*,* $z(0) = 0$ *and*

$$\int_0^1 \vartheta^i z(\vartheta)\, \mathrm{d}\vartheta = 0, \quad i = 0, \ldots, q-1. \tag{6.8}$$

Then

$$\|z\|_{L^2(0,1)} \leq C\, \|D^{q+1} z\|_{L^2(0,1)}, \tag{6.9}$$

where $C > 0$ *is a constant independent of the function* s.

Proof The space $C^\infty([0, 1])$ is dense in $H^{q+1}(0, 1)$. Therefore, there exists a sequence $z_n \in C^\infty([0, 1]$ such that

$$\lim_{n\to\infty} \|z_n - z\|_{H^{q+1}(0,1)} = 0.$$

This and Lemma 11 imply that

$$\|z_n\|_{L^2(0,1)} \to \|z\|_{L^2(0,1)}, \ \|D^{q+1} z_n\|_{L^2(0,1)} \to \|D^{q+1} z\|_{L^2(0,1)} \quad \text{as } n \to \infty,$$
$$\|z_n\|_{L^2(0,1)} \le C \|D^{q+1} z_n\|_{L^2(0,1)}, \quad n = 1, 2, \ldots .$$

Hence, (6.9) holds. □

Now, we can finish the proof of Lemma 7. We transform the reference interval [0, 1] onto the interval $[t_{m-1}, t_m]$by the mapping

$$t = t_m - \vartheta \tau_m, \quad \vartheta \in [0, 1]. \tag{6.10}$$

If $\omega \in H^{q+1}(I_m)$ and $s(\vartheta) = \omega(t_m - \vartheta \tau_m)$, then $s \in H^{q+1}(0, 1)$ and

$$(\tilde{P}_m \omega)(t_m - \vartheta \tau_m) = (Ps)(\vartheta), \tag{6.11}$$

where the operator P is defined by

$$\begin{aligned} &\text{(a)} \quad Ps \in \mathcal{P}^q(0, 1), \\ &\text{(b)} \quad \int_0^1 (Ps(\vartheta) - s(\vartheta))\, \vartheta^j \, \mathrm{d}\vartheta = 0 \quad \forall j = 0, \ldots, q-1, \\ &\text{(c)} \quad Ps(0+) = s(0+). \end{aligned} \tag{6.12}$$

Moreover, if we set

$$Z_m(t) = \tilde{P}_m \omega(t) - \omega(t), \ t \in (t_{m-1}, t_m), \quad z(\vartheta) = Ps(\vartheta) - s(\vartheta), \ \vartheta \in (0, 1), \tag{6.13}$$

we have

$$\begin{aligned} &z(\vartheta) = Z_m(t_m - \vartheta \tau_m), \quad D^{q+1} z(\vartheta) = (-1)^{q+1} \, \tau_m^{q+1} \\ &D^{q+1} Z_m(t_m - \vartheta \tau_m), \quad \vartheta \in (0, 1). \end{aligned} \tag{6.14}$$

We can see that $z \in H^{q+1}(0, 1), \ z(0) = 0,$

$$\begin{aligned} \|z\|^2_{L^2(0,1)} &= \frac{1}{\tau_m} \|Z_m\|^2_{L^2(I_m)}, \\ \|D^{q+1} z\|^2_{L^2(0,1)} &= \tau_m^{2q+1} \|D^{q+1} Z_m\|^2_{L^2(I_m)}. \end{aligned} \tag{6.15}$$

and

$$\int_0^1 z(\vartheta) \vartheta^j \, \mathrm{d}\vartheta = 0 \quad \text{for } j = 0, \ldots, q-1.$$

By Lemma 12, the function z satisfies (6.9) and, in virtue of (6.15), estimate (5.15) holds, what we wanted to prove. □

Finally, we show how to construct in the space $L^2(0,1)$ the system of orthonormal polynomials $\varphi_i,\ i = 0, 1, \ldots,$ such that φ_i is a polynomial of degree i satisfying $\varphi_i(0) \neq 0$. It is possible to put

$$\varphi_i(\vartheta) = \sqrt{2}\, P_i(2\vartheta - 1), \quad i = 0, 1, \ldots,$$

where P_i is the Legendre polynomial of degree i, defined as

$$P_i(\vartheta) = \sqrt{i + \frac{1}{2}}\,\frac{1}{2^i\, i!}\frac{\mathrm{d}^i}{\mathrm{d}\vartheta^i}\,(\vartheta^2 - 1)^i, \quad i = 0, 1, \ldots.$$

The system $P_i,\ i = 0, 1, \ldots,$is a complete orthonormal basis in the space $L^2(-1,1)$. It is possible to verify that $\varphi_i,\ i = 0, 1, \ldots,$ form a complete orthonormal basis in $L^2(0,1)$ and

$$\varphi_i(\vartheta) = \sqrt{2i+1}\,\frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}\vartheta^i}\,(\vartheta^2 - \vartheta)^i, \quad \vartheta \in [0,1].$$

Since

$$\frac{\mathrm{d}^i}{\mathrm{d}\vartheta^i}\,(\vartheta^2 - \vartheta)^i|_{\vartheta=0} = (-1)^i\, i!,$$

for all $i = 0, 1, \ldots$ we have

$$\varphi_i(0) = (-1)^i\,\sqrt{2i+1} \neq 0.$$

Conclusion

In this paper we have presented an analysis of error estimates of the space–time discontinuous Galerkin discretization of an initial-boundary value problem for a nonstationary convection–diffusion equation with nonlinear convection and Dirichlet boundary condition. In the space discretization, we use polynomial approximations of degree $p \geq 1$ and NIPG, IIPG and SIPG versions of the diffusion terms are considered. In time the discontinuous approximations of degree $q \geq 1$, in general $q \neq p$, are used. On different time levels, different space meshes may be used. Under the assumption that the Dirichlet data behave in time as polynomials of degree $\leq q$, the derived estimates in $L^2(H^1)$-norm are optimal in space and time. The error estimate in $L^2(L^2)$-norm is optimal in time, but suboptimal in space. In the case of general Dirichlet data, the error estimates are suboptimal in time. The technique applied in this paper can be extended to the case the discontinuous Galerkin time semidiscretization combined with the *hp* discontinuous Galerkin space discretization.The error estimates have been obtained with the aid of a "parabolic machinery" for problems with "dominating diffusion". This means that our technique is not applicable to conservation laws and does not allow to obtain results similar to [14–16], where the finite difference or finite volume methods were used.

There are the following subjects for further work:

- derivation of optimal error estimates in space and time in the case of the SIPG method and general Dirichlet data,
- numerical realization of the discrete problem and the demonstration of results by numerical experiments,
- analysis of the effect of numerical integration in space and time integrals,
- extension of the results to problems with nonlinear convection as well as diffusion,
- analysis of the combination of the time DGFEM with other space DG discretizations, as e.g. the LDG method (cf. [11,18]),
- application of the space–time DGFEM to the numerical solution of some technically relevant problems, as, e.g. interaction of compressible flow with structures.

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