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A second order accuracy for a full discretized time-dependent Navier–Stokes equations by a two-grid scheme

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Abstract We study a second-order two-grid scheme fully discrete in time and space for solving the Navier–Stokes equations. The two-grid strategy consists in discretizing, in the first step, the fully non-linear problem, in space on a coarse grid with mesh-size H and time step Δt and, in the second step, in discretizing the linearized problem around the velocity u_H computed in the first step, in space on a fine grid with mesh-size h and the same time step. The two-grid method has been applied for an analysis of a first order fully-discrete in time and space algorithm and we extend the method to the second order algorithm. This strategy is motivated by the fact that under suitable assumptions, the contribution of u_H to the error in the non-linear term, is measured in the L^2 norm in space and time, and thus has a higher-order than if it were measured in the H^1 norm in space. We present the following results: if $h^2 = H^3 = (\Delta t)^2$, then the global error of the two-grid algorithm is of the order of h^2, the same as would have been obtained if the non-linear problem had been solved directly on the fine grid.

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1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^2$ with a polygonal boundary $\partial\Omega$ and let $]0,T[$ be a given time-interval. Consider the following Navier–Stokes problem for an incompressible fluid

$$\frac{\partial u}{\partial t}(x,t) - \nu\Delta u(x,t) + u(x,t)\cdot\nabla u(x,t) + \nabla p(x,t) = f(x,t) \quad \text{in } \Omega\times]0,T[, \tag{1.1}$$

with the incompressibility condition

$$\operatorname{div} u(x,t) = 0 \quad \text{in } \Omega\times]0,T[, \tag{1.2}$$

the homogeneous Dirichlet boundary condition

$$u(x,t) = 0 \quad \text{on } \partial\Omega\times]0,T[, \tag{1.3}$$

and the initial condition

$$u(x,0) = 0 \quad \text{in } \Omega, \tag{1.4}$$

where u and p represent, respectively, the velocity and the pressure of the fluid. All the quantities are taken at the point (x,t) where $x = (x_i)_{1\le i\le 2} \in \mathbb{R}^2$ denotes the position and $t \in [0,T]$ the time. We suppose that the fluid density is a constant ($\rho = 1$); f denotes the external forces applied to the fluid and ν is the viscosity. The notations $u\cdot\nabla u$, Δu and $\operatorname{div} u$ mean:

$$u\cdot\nabla u = \sum_{i=1}^{2} u_i \frac{\partial u}{\partial x_i}, \quad \Delta u = \sum_{i=1}^{2} \frac{\partial^2 u}{\partial^2 x_i} \quad \text{and} \quad \operatorname{div} u = \sum_{i=1}^{2} \frac{\partial u_i}{\partial x_i}.$$

The term $u\cdot\nabla u$ is the convection term and $\nu\Delta u$ is the diffusion one.

The purpose of this article is to solve by a second-order, in time and space, two-grid scheme, on a coarse grid and a fine grid, the non-stationary incompressible Navier–Stokes problem and to show that the two-grid algorithm's global error is similar to the error of the direct resolution of the non-linear problem on a fine grid. The two-grid strategy is a general method for solving a non-linear Partial Differential Equation (PDE), depending or not in time, with solution u. This technique consists on what follows: In a first step, we discretize the fully non-linear PDE on a coarse grid of mesh-size H and we compute an approximate solution u_H. Then, in a second step, we linearize the PDE around u_H and we discretize the linearized problem on a fine grid of mesh-size h; let u_h^{lin} be the corresponding solution. Then, under suitable assumptions, we can prove that if h, H and the time step Δt are well-chosen, the global error of the two-grid algorithm $\|u - u_h^{lin}\|$ has the same order as the error $\|u - u_h\|$ that would have been obtained if the non-linear problem had been directly discretized on the fine grid.

Two-grid discretizations have been widely applied to linear and non-linear elliptic boundary value problems: Xu in [24–26] has pioneered their development. These methods have been extended to the steady Navier–Stokes equations, cf. for instance the work of Layton in [14], Layton and Lenferink in [15] and Girault and Lions in [8]. Also, this method has been applied to the time-dependent Navier–Stokes problem, cf. Girault and Lions [9] in which they analyze a semi-discrete algorithm, Abboud and Sayah in [2] and Abboud et al. in [3] for an analysis of a first order fully-discrete in time and space algorithm and in [1] for a numerical analysis of a second-order totally discrete in time and space scheme.

This two-grid algorithm is closer to the nonlinear Galerkin methods developed by Foias and Al [7], of which one finds a good description in the work of Marion and Temam [18–20] and Ait Ou Amni and Marion [5]. An essential difference between NLG and the two-grid method is that, in the latter, one calculates first u_H and then u_h, i.e at each step, the two calculations are uncoupled, while this calculation is not uncoupled in NLG.

Noting that the velocity u depends on the time variable t, setting $u(t) = u(x, t)$, $L^2_0(\Omega) = \{q \in L^2(\Omega); \int_\Omega q \ dx = 0\}$ and assuming that f belongs to $L^2(0, T; H^{-1}(\Omega)^2)$, it is well-known that (1.1) and (1.2) has the following variational formulation in $]0, T[$: Find $u(t) \in H^1_0(\Omega)^2$, such that in the sense of distributions on $]0, T[$,

$$\forall v \in H^1_0(\Omega)^2, \quad \frac{d}{dt}(u(t), v) + \nu(\nabla u(t), \nabla v) + (u(t) \cdot \nabla u(t), v) - (p(t), \operatorname{div} v) = \langle f(t), v \rangle, \tag{1.5}$$

$$\forall q \in L^2_0(\Omega), \quad (q, \operatorname{div} u(t)) = 0, \tag{1.6}$$

and

$$u(0) = 0. \tag{1.7}$$

Furthermore, this problem has one and only one solution u in $L^\infty(0, T; L^2(\Omega)^2) \cap L^2(0, T; H^1(\Omega)^2)$ and p in the dual space of $W^{1,1}_0(0, T; L^2_0(\Omega))$ (see e.g. Lions in [16] and Ladyzenskaya in [13]).
In addition, we have the following regularity result:

Theorem 1.1 *If Ω is convex and $f \in L^2(0, T; L^2(\Omega)^2)$, then*

$$u \in L^\infty\big(0, T; H^1(\Omega)^2\big) \cap L^2\big(0, T; H^2(\Omega)^2\big) \quad \textit{and} \quad p \in L^2\big(0, T; H^1(\Omega)\big). \tag{1.8}$$

For discretizing (1.5)–(1.7), let $\eta > 0$ be a discretization parameter in space and for each η, let $\mathcal{T}_\eta$ be a corresponding regular (or non-degenerate) family of triangulations of $\overline{\Omega}$, consisting of triangles such that any two triangles are either disjoint or share a vertex or an entire side. For an arbitrary triangle κ, we denote by η_κ the diameter of κ and by ρ_κ the diameter of the circle inscribed in κ. Then η denotes the maximum of

η_κ and we assume that $\mathcal{T}_\eta$ is regular in the sense of Ciarlet [6]: there exists a constant σ independent of η such that

$$\sup_{\kappa \in \mathcal{T}_\eta} \frac{\eta_\kappa}{\rho_\kappa} = \sigma_\kappa \leq \sigma. \tag{1.9}$$

Let X_η and M_η be a "stable" pair of finite-element spaces for discretizing the velocity u and the pressure p, stable in the sense that it satisfies a uniform discrete inf–sup condition: there exists a constant $\beta^\star \geq 0$, independent of η, such that

$$\forall q_\eta \in M_\eta, \quad \sup_{v_\eta \in X_\eta} \frac{1}{|v_\eta|_{H^1(\Omega)}} \int_\Omega q_\eta \operatorname{div} v_\eta dx \geq \beta^\star \|q_\eta\|_{L^2(\Omega)}. \tag{1.10}$$

Let $\mathbb{P}_\kappa$ denote the space of polynomials with total degree less than or equal to κ. For a second-order two-grid scheme, we choose the Taylor–Hood finite-element, where in each triangle κ, each component of the velocity is a polynomial of $\mathbb{P}_2$ and the pressure p is a polynomial of $\mathbb{P}_1$. Therefore, the finite-element spaces are:

$$X_\eta = \left\{ v_\eta \in C^0(\overline{\Omega})^2;\ \forall \kappa \in \mathcal{T}_\eta,\ v_{\eta|_\kappa} \in \mathbb{P}_2^2, v_{\eta|\partial\Omega} = 0 \right\}, \tag{1.11}$$

$$M_\eta = \left\{ q_\eta \in C^0(\overline{\Omega});\ \forall \kappa \in \mathcal{T}_\eta,\ q_{\eta|_\kappa} \in \mathbb{P}_1, \int_\Omega q_\eta dx = 0 \right\}. \tag{1.12}$$

There exists an approximation operator $P_\eta \in \mathcal{L}(H_0^1(\Omega)^2; X_\eta)$ such that (see Girault and Raviart in [10]):

$$\forall v \in H_0^1(\Omega)^2, \quad \forall q_\eta \in M_\eta, \quad \int_\Omega q_\eta \operatorname{div}(P_\eta(v) - v) dx = 0, \tag{1.13}$$

and for $k = 0$ or 1,

$$\forall v \in \left[H^{1+k}(\Omega) \cap H_0^1(\Omega) \right]^2, \quad \|P_\eta(v) - v\|_{L^2(\Omega)} \leq C\eta^{1+k} |v|_{H^{1+k}(\Omega)}, \tag{1.14}$$

and for all $r \geq 2$, $k = 0$ or 1,

$$\forall v \in \left[W^{1+k,r}(\Omega) \cap H_0^1(\Omega) \right]^2, \quad |P_\eta(v) - v|_{W^{1,r}(\Omega)} \leq C_r \eta^k |v|_{W^{1+k,r}(\Omega)}. \tag{1.15}$$

In addition, as M_η contains all polynomials of degree one, there exists an operator $r_\eta \in \mathcal{L}(L_0^2(\Omega); M_\eta)$, such that for any real number $s \in [0, 2]$,

$$\forall q \in H^s(\Omega) \cap L_0^2(\Omega), \quad \|r_\eta(q) - q\|_{L^2(\Omega)} \leq C\eta^s |q|_{H^s(\Omega)}. \tag{1.16}$$

To discretize in time, we divide the interval $[0, T]$ into N subintervals of equal length $k = \frac{T}{N}$, with grid-points $t^n = nk, 0 \le n \le N$.

With these spaces, we propose the following two-grid scheme for discretizing (1.5)–(1.7). We use two regular triangulations of $\overline{\Omega}$: a coarse triangulation $\mathcal{T}_H$ and a fine one $\mathcal{T}_h$, that for practical purposes, is a refinement of $\mathcal{T}_H$. On each of these, we define the same stable pair of finite-element spaces, (X_H, M_H) and (X_h, M_h) such that $X_H \subset X_h$ and $M_H \subset M_h$. At each time step, we solve (1.17)–(1.18) and (1.19)–(1.20) below. The two-grid algorithm reads:

- **Step one** (non-linear problem on coarse grid): Knowing u_H^{n-1} and u_H^n, find (u_H^{n+1}, p_H^{n+1}) with values in $X_H \times M_H$, solution of

$$
\begin{aligned}
\forall v_H \in X_H, \quad & \frac{1}{2\Delta t}\big(3u_H^{n+1} - 4u_H^n + u_H^{n-1}, v_H\big) + \nu(\nabla u_H^{n+1}, \nabla v_H) \\
& + \big(u_H^{n+1} \cdot \nabla u_H^{n+1}, v_H\big) + \frac{1}{2}\Big(\operatorname{div} u_H^{n+1}, u_H^{n+1} \cdot v_H\Big) \\
& - \big(p_H^{n+1}, \operatorname{div} v_H\big) = \big(f^{n+1}, v_H\big), \qquad (1.17) \\
& \forall q_H \in M_H, \quad \big(q_H, \operatorname{div} u_H^{n+1}\big) = 0. \qquad (1.18)
\end{aligned}
$$

- **Step two** (linearized problem on fine grid): Knowing (u_H^{n+1}, p_H^{n+1}), find (u_h^{n+1}, p_h^{n+1}) with values in $X_h \times M_h$ solution of

$$
\begin{aligned}
\forall v_h \in X_h, \quad & \frac{1}{2\Delta t}\Big(3u_h^{n+1} - 4u_h^n + u_h^{n-1}, v_h\Big) + \nu\Big(\nabla u_h^{n+1}, \nabla v_h\Big) \\
& + \Big(u_H^{n+1} \cdot \nabla u_h^{n+1}, v_h\Big) - \Big(p_h^{n+1}, \operatorname{div} v_h\Big) = \Big(f^{n+1}, v_h\Big), \qquad (1.19) \\
& \forall q_h \in M_h, \quad \Big(q_h, \operatorname{div} u_h^{n+1}\Big) = 0. \qquad (1.20)
\end{aligned}
$$

By assumption, $u_H^0 = u_h^0 = 0$ and u_H^1 and u_h^1 are computed by solving one iteration of an Euler scheme. In both (1.17) and (1.19), f^{n+1} is a suitable approximation of f at time t^{n+1}. The purpose of this two-grid algorithm is to reduce the time of computation for both velocity and pressure.

In the sequel, we shall take $(\Delta t)^2$ of the order of H^3: there exist constants α_1 and α_2 that do not depend on H and Δt such that

$$\alpha_1 H^3 \le (\Delta t)^2 \le \alpha_2 H^3.$$

The remainder of this article is organized as follows: In Sect. 2, we present some conventions and notations that will be used throughout the article. In Sect. 3, we present a first error estimate for the fully-discrete Step One, then in sect. 4 we establish a duality argument based on the backward semi-discrete Stokes system and we derive some uniform bounds that allow us to prove the Stokes problem's error estimate in $L^2(\Omega\times]0, T[)^2$, then we apply it to the Navier–Stokes problem. We also prove a "superconvergence" result for the non-linear part. Finally, the pressure is estimated in

Sect. 5 and the error estimation for the solution of Step Two is studied in Sect. 6. In Sect. 7, we confirm this results with numerical tests.

2 Preliminaries

To begin with, we present some conventions and notations that will be used throughout the article. As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a, b[$ with values in a functional space, say X (cf. [17]). More precisely, let $\|.\|_X$ denote the norm of X; then for any r, $1 \leq r \leq \infty$, we define

$$L^r(a, b; X) = \left\{ f \text{ mesurable in }]a, b[; \int_a^b \|f(t)\|_X^r \, dt < \infty \right\}$$

equipped with the norm

$$\|f\|_{L^r(a,b;X)} = \left(\int_a^b \|f(t)\|_X^r \, dt \right)^{1/r},$$

with the usual modifications if $r = \infty$. It is a Banach space if X is a Banach space.

Let (k_1, k_2) denote a pair of non-negative integers, set $|k| = k_1 + k_2$ and define the partial derivative ∂^k by $\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2}}$. Here X is usually a Sobolev space, such as (cf. [4] or Nečas [21]): for any non-negative integer m and number $r \geq 1$,

$$W^{m,r}(\Omega) = \left\{ v \in L^r(\Omega); \partial^k v \in L^r(\Omega), \forall |k| \leq m \right\}.$$

This space is equipped with the seminorm

$$|v|_{W^{m,r}(\Omega)} = \left[\sum_{|k|=m} \int_\Omega |\partial^k v|^r dx \right]^{1/r},$$

and is a Banach space for the norm

$$\|v\|_{W^{m,r}(\Omega)} = \left[\sum_{0 \leq |k| \leq m} |v|^r_{W^{k,r}(\Omega)} \right]^{1/r},$$

with the usual extension when $r = \infty$. When $r = 2$, this space is the Hilbert space $H^m(\Omega)$. In particular, the scalar product of $L^2(\Omega)$ is denoted by $(\, . \, , \, . \,)$. Similarly, $L^2(a, b; H^m(\Omega))$ is a Hilbert space and in particular $L^2(a, b; L^2(\Omega))$ coincides with $L^2(\Omega \times]a, b[)$.

For functions that vanish on the boundary, we recall Poincaré's inequality: there exists a constant $\mathcal{P}$ such that

$$\forall v \in H_0^1(\Omega), \quad \|v\|_{L^2(\Omega)} \leq \mathcal{P}|v|_{H^1(\Omega)}. \tag{2.1}$$

More generally, recall the inequalities of Sobolev imbeddings in two dimensions: for each $r \in [2, \infty[$, there exits a constant S_r such that

$$\forall v \in H_0^1(\Omega), \quad \|v\|_{L^r(\Omega)} \leq S_r|v|_{H^1(\Omega)}, \tag{2.2}$$

where

$$|v|_{H^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)} \ . \tag{2.3}$$

When $r = 2$, (2.2) reduces to Poincaré's inequality and S_2 is Poincaré's constant. The case $r = \infty$ is excluded and is replaced by: for any $r > 2$, there exists a constant M_r such that

$$\forall v \in W_0^{1,r}(\Omega), \quad \|v\|_{L^\infty(\Omega)} \leq M_r|v|_{W^{1,r}(\Omega)}. \tag{2.4}$$

We also have in dimension 2,

$$\|g\|_{L^4(\Omega)} \leq 2^{1/4} \ \|g\|_{L^2(\Omega)}^{1/2} \|\nabla g\|_{L^2(\Omega)}^{1/2} \ . \tag{2.5}$$

Owing to (2.1), we use the seminorm $|.|_{H^1(\Omega)}$ as a norm on $H_0^1(\Omega)$ and we use it to define the norm of the dual space $H^{-1}(\Omega)$ of $H_0^1(\Omega)$:

$$\|f\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v \rangle}{|v|_{H^1(\Omega)}},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Also, we recall the spaces we introduced at the beginning:

$$V = \left\{v \in H_0^1(\Omega)^2; \text{div } v = 0 \text{ in } \Omega\right\} \quad \text{and} \quad L_0^2(\Omega) = \left\{q \in L^2(\Omega); \int_\Omega q \ dx = 0\right\},$$

and the orthogonal complement of V in $H_0^1(\Omega)^2$:

$$V^\perp = \left\{v \in H_0^1(\Omega)^2; \forall w \in V, (\nabla v, \nabla w) = 0\right\}.$$

The results of this article are based on the identity:

$$2\left(a^{n+1}, 3a^{n+1} - 4a^n + a^{n-1}\right) = |a^{n+1}|^2 + |2a^{n+1} - a^n|^2 + |\delta^2 a^n|^2 - |a^n|^2 - |2a^n - a^{n-1}|^2, \tag{2.6}$$

where

$$\delta^2 a^n = a^{n+1} - 2a^n + a^{n-1}. \tag{2.7}$$

3 Error estimates for the solution of step one

The results in this paragraph are written for the non-linear scheme (1.17) and (1.18).

To simplify, we denote by η the mesh parameter. First of all, we prove the existence and the uniqueness of the solution of (1.17) and (1.18).

Lemma 3.1 (Stability) *Let u_η^{n+1} be a solution of* (1.17) and (1.18) *with the initial datas u_η^0 and $u_\eta^1 \in V_\eta$; We have*

$$\sup_{2\le n\le N} \|u_\eta^n\|_{L^2(\Omega)} + \sup_{2\le n\le N} \|2u_\eta^n - u_\eta^{n-1}\|_{L^2(\Omega)} + \sqrt{2\nu}\left(\sum_{n=2}^{N} \Delta t\ \|\nabla u_\eta^n\|^2_{L^2(\Omega)}\right)^{1/2}$$
$$+\left(\sum_{n=1}^{N-1} \|\delta^2 u_\eta^n\|^2_{L^2(\Omega)}\right)^{1/2} \le C\left(\frac{2S_2^2}{\nu}\sum_{n=2}^{N} \Delta t\ \|f^n\|^2_{L^2(\Omega)} + \|u_\eta^1\|^2_{L^2(\Omega)}\right.$$
$$\left. + \|2u_\eta^1 - u_\eta^0\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Proof We take the scalar product in $L^2(\Omega)$ of (1.17) by $4\Delta t u_\eta^{n+1}$, use (2.6) and sum the result over $1 \le n \le m-1$. □

The stability of (1.17) and (1.18) results from the following a priori estimation:

Lemma 3.2 (Uniqueness) *The scheme* (1.17) *and* (1.18) *has a solution for all $\nu > 0$, all initial data $u_\eta^0, u_\eta^1 \in V_\eta$ and for all data $f \in \mathcal{C}^0([0, T]; L^2(\Omega)^2)$. The solution is unique for Δt sufficiently small.*

Proof For all $1 \le n \le N-1$, the problem (1.17) and (1.18) is a square system of algebraic non-linear equations in finite dimension. Due to the anti-symmetrisation of the non-linear term, we prove, by the theorem of the saddle point of Brouwer and the inf–sup condition, that for all $1 \le n \le N-1$, the problem has at least a solution (u_η^n, p_η^n). For the unicity, we consider two solutions $(u_\eta^{(1)}, p_\eta^{(1)})$ and $(u_\eta^{(2)}, p_\eta^{(2)})$. Their difference (w_η^n, p_η^n) satisfies:

$$\forall v_\eta \in V_\eta, \quad \frac{1}{2\Delta t}\left(3w_\eta^{n+1} - 4w_\eta^n + w_\eta^{n-1}, v_\eta\right) + \nu\left(\nabla w_\eta^{n+1}, \nabla v_\eta\right)$$
$$+\left(w_\eta^{n+1}\cdot\nabla u_\eta^{(1)n+1}, v_\eta\right) + \left(u_\eta^{(2)n+1}\cdot w_\eta^{n+1}, v_\eta\right)$$
$$+\frac{1}{2}\left(\operatorname{div} w_\eta^{n+1}, u_\eta^{(1)n+1}\cdot v_\eta\right) + \frac{1}{2}\left(\operatorname{div} u_\eta^{(2)n+1}, w_\eta^{n+1}\cdot v_\eta\right) = 0.$$

By using the identity (2.6) and choosing $v_\eta = w_\eta^{n+1}$, we obtain

$$\frac{1}{4\Delta t}\Big(\|w_\eta^{n+1}\|^2_{L^2(\Omega)} + \|2w_\eta^{n+1} - w_\eta^n\|^2_{L^2(\Omega)} + \|\delta^2 w_\eta^n\|^2_{L^2(\Omega)}$$
$$- \|w_\eta^n\|^2_{L^2(\Omega)} - \|2w_\eta^n - w_\eta^{n-1}\|^2_{L^2(\Omega)}\Big)$$
$$+\nu|w_\eta^{n+1}|^2_{H^1(\Omega)} - \|w_\eta^{n+1}\|^2_{L^4(\Omega)}\,|u_\eta^{(1)n+1}|_{H^1(\Omega)}$$
$$-\frac{1}{2}|w_\eta^{n+1}|_{H^1(\Omega)}\,\|w_\eta^{n+1}\|_{L^4(\Omega)}\|u_\eta^{(1)n+1}\|_{L^4(\Omega)} \le 0.$$

Due to the fact that in finite dimension, all the norms are equivalent, summing the precedent inequality from $n = 1$ to $m - 1$, and using Lemma 3.1 and $w_\eta^0 = w_\eta^1 = 0$, we obtain

$$\|w_\eta^m\|^2_{L^2(\Omega)} + \nu\sum_{n=2}^{m}\Delta t|w_\eta^n|^2_{H^1(\Omega)} + \sum_{n=1}^{m-1}\|\delta^2 w_\eta^n\|^2_{L^2(\Omega)} + \|2w_\eta^m - w_\eta^{m-1}\|^2_{L^2(\Omega)}$$
$$\le C\Delta t\sum_{n=1}^{m}\|w_\eta^n\|^2_{L^2(\Omega)}, \tag{3.1}$$

with a constant C that depends on η but does not depend on Δt. For the last term of the sum of the right-hand side, we write:

$$w_\eta^m = \delta^2 w_\eta^{m-1} + 2w_\eta^{m-1} - w_\eta^{m-2}.$$

Then

$$\|w_\eta^m\|^2_{L^2(\Omega)} \le 2\left(\|\delta^2 w_\eta^{m-1}\|^2_{L^2(\Omega)} + \|2w_\eta^{m-1} - w_\eta^{m-2}\|^2_{L^2(\Omega)}\right),$$

and for Δt sufficiently small, the term

$$2C\Delta t\left(\|\delta^2 w_\eta^{m-1}\|^2_{L^2(\Omega)} + \|2w_\eta^{m-1} - w_\eta^{m-2}\|^2_{L^2(\Omega)}\right)$$

can be absorbed by the term in the left-hand side of the inequality. Applying Gronwall's lemma, we obtain $w_\eta^n = \mathbf{0}$ then, the inf–sup condition implies $p_\eta^n = 0$, $2 \le n \le N$. □

In the next proposition, we will establish the error estimate for the solution computed by one iteration of Euler's scheme $(u_\eta^1 - u(\Delta t), p_\eta^1 - p(\Delta t))$:

Proposition 3.3 *Suppose that* $u'' \in \mathcal{C}^0(0,T;L^2(\Omega)^2)$, $u(\Delta t) \in H^3(\Omega)^2$ *and* $p(\Delta t) \in H^2(\Omega)$, *the error of the solution computed by one iteration of Euler's scheme satisfies the following estimations, for* $\Delta t \le k_0 > 0$ *sufficiently small,*

$$\begin{aligned} &\frac{1}{2}\ \|u_\eta^1 - u(\Delta t)\|^2_{L^2(\Omega)} + \frac{\nu\Delta t}{2}|u_\eta^1 - u(\Delta t)|^2_{H^1(\Omega)} \\ &\quad\le \frac{(\Delta t)^4}{4}\ \|u''\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + C(\Delta t)\eta^4\left(|u(\Delta t)|^2_{H^3(\Omega)} + |p(\Delta t)|^2_{H^2(\Omega)}\right) \\ &\qquad + C\eta^6|u(\Delta t)|^2_{H^3(\Omega)}, \end{aligned} \tag{3.2}$$

and

$$(\Delta t)^{1/2}\ \|p(\Delta t) - p_\eta^1\|_{L^2(\Omega)} \le C\left((\Delta t)^{3/2} + \eta^2 + \frac{\eta^3}{\sqrt{\Delta t}}\right), \tag{3.3}$$

where u'' *means the second derivative in time of* u.

Proof Due to the regularity assumption of u, and forall x, there exists $\theta \in]0,1[$ that depends on x such that

$$0 = u_0 = u(\Delta t) - (\Delta t)u'(\Delta t) + \frac{1}{2}(\Delta t)^2 u''(\theta\Delta t),$$

and u_η^1 satisfies the following error equation

$$\begin{aligned} \forall v_\eta \in V_\eta,\quad &\frac{1}{\Delta t}\left(u_\eta^1 - u(\Delta t), v_\eta\right) + \nu\left(\nabla(u_\eta^1 - u(\Delta t)), \nabla v_\eta\right) \\ &= \frac{\Delta t}{2}\left(u''(\theta\Delta t), v_\eta\right) - \left(p(\Delta t) - r_\eta p(\Delta t), \operatorname{div} v_\eta\right) \\ &\quad + \left(u(\Delta t)\cdot\nabla u(\Delta t) - u_\eta^1\cdot\nabla u_\eta^1, v_\eta\right) - \frac{1}{2}\left(\operatorname{div} u_\eta^1, u_\eta^1\cdot v_\eta\right). \end{aligned} \tag{3.4}$$

Setting $v_\eta = v_\eta^1 = u_\eta^1 - P_\eta u(\Delta t)$ and $\varphi_\eta^1 = P_\eta u(\Delta t) - u(\Delta t)$, we obtain

$$\begin{aligned} \frac{1}{\Delta t}\ \|v_\eta^1\|^2_{L^2(\Omega)} + \nu|v_\eta^1|^2_{H^1(\Omega)} &= \frac{\Delta t}{2}\left(u''(\theta\Delta t), v_\eta^1\right) + \left(r_\eta p(\Delta t) - p(\Delta t), \operatorname{div} v_\eta^1\right) \\ &\quad - \left(v_\eta^1\cdot\nabla u_\eta^1, v_\eta^1\right) - \frac{1}{2}\left(\operatorname{div} v_\eta^1, u_\eta^1\cdot v_\eta^1\right) - \left(\varphi_\eta^1\cdot\nabla u_\eta^1, v_\eta^1\right) \\ &\quad - \frac{1}{2}\left(\operatorname{div}\varphi_\eta^1, u_\eta^1\cdot v_\eta^1\right) - \left(u(\Delta t)\cdot\nabla\varphi_\eta^1, v_\eta^1\right) \\ &\quad - \frac{1}{\Delta t}(\varphi_\eta^1, v_\eta^1) - \nu\left(\nabla\varphi_\eta^1, \nabla v_\eta^1\right). \end{aligned} \tag{3.5}$$

Then (3.2) follows readily by applying the error approximation of P_η.

For the pressure, we have

$$\begin{aligned}
&\big(r_\eta p(\Delta t) - p(\Delta t), \operatorname{div} v_\eta\big) + \Big(p_\eta^1 - r_\eta p(\Delta t), \operatorname{div} v_\eta\Big) \\
&= \frac{1}{\Delta t}\Big(u_\eta^1 - u(\Delta t), v_\eta\Big) + \nu\Big(\nabla(u_\eta^1 - u(\Delta t)), \nabla v_\eta\Big) \\
&\quad -\frac{\Delta t}{2}\big(u''(\theta \Delta t), v_\eta\big) - \Big(u(\Delta t)\cdot\nabla u(\Delta t) - u_\eta^1\cdot\nabla u_\eta^1, v_\eta\Big) \\
&\quad +\frac{1}{2}\Big(\operatorname{div} u_\eta^1, u_\eta^1\cdot v_\eta\Big), \qquad (3.6)
\end{aligned}$$

and owing to the inf–sup condition (1.10), there exists $v_\eta \in V_\eta^\perp$ such that

$$\Big(p_\eta^1 - r_\eta p(\Delta t), \operatorname{div} v_\eta\Big) = \|p_\eta^1 - r_\eta p(\Delta t)\|_{L^2(\Omega)}^2 \quad \text{and} \quad |v_\eta|_{H^1(\Omega)} \leq \frac{1}{\beta^\star}\, \|p_\eta^1 - r_\eta p(\Delta t)\|_{L^2(\Omega)},$$

with $\beta^\star > 0$ independent of η. Then, by applying (3.2), we obtain (3.3). □

The next result, stated in Lemma 3.4, is a standard error estimate. We give the proof for the sake of completeness.

Lemma 3.4 *Let X_η and M_η be defined by (1.11) and (1.12) and approximate f^{n+1} by $f^{n+1} = f(t^{n+1})$. At each time step, (1.17) and (1.18) has a solution u_η^{n+1} and this solution is unique if Δt is sufficiently small. Suppose that $u \in L^2(0,T;H^3(\Omega)^2)$, $u' \in L^2(0,T;H^2(\Omega)^2)$, $u^{(3)} \in L^2(\Omega\times]0,T[)^2$ and $p \in L^2(0,T;H^2(\Omega))$, there exist a constant C that does not depend on η and Δt and a constant $k_0 > 0$ that does not depend on η such that, for all $\Delta t \leq k_0$,*

$$\begin{aligned}
&\sup_{1\leq n\leq N} \|u_\eta^n - u(t^n)\|_{L^2(\Omega)} + \left(\sum_{n=1}^{N-1} \|\delta^2(u_\eta^n - u(t^n))\|_{L^2(\Omega)}^2\right)^{1/2} \\
&\quad + \sqrt{\nu}\left(\sum_{n=1}^{N-1} \Delta t |u_\eta^{n+1} - u(t^{n+1})|_{H^1(\Omega)}^2\right)^{1/2} \leq C(\eta^2 + (\Delta t)^2). \qquad (3.7)
\end{aligned}$$

Proof Setting $v_\eta^n = u_\eta^n - P_\eta u(t^n)$ and $\varphi_\eta^n = P_\eta u(t^n) - u(t^n)$, $0 \leq n \leq N$, we substruct (1.17) and (1.1) taken at $t = t^{n+1}$ and by using the following second-order backward finite difference scheme

$$\frac{\partial u}{\partial t}(t^{n+1}) = \frac{3u(t^{n+1}) - 4u(t^n) + u(t^{n-1})}{2\Delta t} + \mathcal{O}((\Delta t)^2), \qquad (3.8)$$

we have

$$\left|u'(t+\Delta t) - \frac{3u(t+\Delta t) - 4u(t) + u(t-\Delta t)}{2\Delta t}\right| \le \frac{(\Delta t)^{3/2}}{2\sqrt{3}} \, \|u^{(3)}\|_{L^2(t-\Delta t;t+\Delta t)}, \tag{3.9}$$

and by summing the result over $1 \le n \le m-1$, we obtain:

$$\begin{aligned}
&\|v_\eta^m\|^2_{L^2(\Omega)} + \|2v_\eta^m - v_\eta^{m-1}\|^2_{L^2(\Omega)} + \sum_{n=1}^{m-1} \|\delta^2 v_\eta^n\|^2_{L^2(\Omega)} + 4\nu \sum_{n=1}^{m-1} \Delta t |v_\eta^{n+1}|^2_{H^1(\Omega)} \\
&\le \left(\|v_\eta^1\|^2_{L^2(\Omega)} + \|2v_\eta^1 - v_\eta^0\|^2_{L^2(\Omega)}\right) + 2\left|\sum_{n=1}^{m-1} \left(3\varphi_\eta^{n+1} - 4\varphi_\eta^n + \varphi_\eta^{n-1}, v_\eta^{n+1}\right)\right| \\
&\quad + 4\nu \left|\sum_{n=1}^{m-1} \Delta t \left(\nabla\varphi_\eta^{n+1}, \nabla v_\eta^{n+1}\right)\right| \\
&\quad + 4\left|\sum_{n=1}^{m-1} \Delta t \left(p(t^{n+1}) - r_\eta p(t^{n+1}), \operatorname{div} v_\eta^{n+1}\right)\right| \\
&\quad + 4\sum_{n=1}^{m-1} \Delta t \frac{(\Delta t)^{3/2}}{2\sqrt{3}} \, \|u^{(3)}\|_{L^2(t^{n-1};t^{n+1})} \|v_\eta^{n+1}\|_{L^2(\Omega)} \\
&\quad + 4\left|\sum_{n=1}^{m-1} \Delta t \left(u(t^{n+1}) \cdot \nabla u(t^{n+1}) - u_\eta^{n+1} \cdot \nabla u_\eta^{n+1}, v_\eta^{n+1}\right)\right. \\
&\qquad \left. - \frac{1}{2}\sum_{n=1}^{m-1} \Delta t \left(\operatorname{div} u_\eta^{n+1}, u_\eta^{n+1} . v_\eta^{n+1}\right)\right|.
\end{aligned} \tag{3.10}$$

Let us study the terms of the right hand side of (3.10) denoted by $((t_{rhs})_i)_{1\le i\le 7}$. The first term $(t_{rhs})_1$ is bounded as in Proposition 3.3.

To study the second term, we have

$$\frac{3\varphi_\eta^{n+1} - 4\varphi_\eta^n + \varphi_\eta^{n-1}}{2\Delta t} = P_\eta u'(t^{n+1}) - u'(t^{n+1}) + R_2,$$

with

$$|R_2| \le \frac{(\Delta t)^{3/2}}{2\sqrt{3}} \, \|P_\eta u^{(3)} - u^{(3)}\|_{L^2(t^{n-1};t^{n+1})}.$$

Hence, by assuming that P_η is stable in the norm L^2 (cf. [9]), we have

$$
\begin{aligned}
|(t_{rhs})_2| &= \left|4\sum_{n=1}^{m-1}\Delta t\left(\frac{3\varphi_\eta^{n+1}-4\varphi_\eta^n+\varphi_\eta^{n-1}}{2\Delta t}, v_\eta^{n+1}\right)\right|\\
&\le \frac{C\eta^4}{2\varepsilon_2}\,\|u'\|^2_{L^\infty(0,T;H^2(\Omega)^2)} + \frac{C(\Delta t)^4}{2\varepsilon_2}\,\|u^{(3)}\|^2_{L^2(\Omega\times]0,T[)^2}\\
&\quad +\frac{\varepsilon_2}{2}\sum_{n=1}^{m-1}\Delta t|v_\eta^{n+1}|^2_{H^1(\Omega)}.
\end{aligned}
$$

The third term is bounded as follows:

$$
\begin{aligned}
|(t_{rhs})_3| &= \left|4\nu\sum_{n=1}^{m-1}\Delta t\left(\nabla\varphi_\eta^{n+1}, \nabla v_\eta^{n+1}\right)\right|\\
&\le \frac{2C\nu\eta^4}{\varepsilon_3}\,\|u\|^2_{L^2(0,T;H^3(\Omega)^2)} + 2\nu\varepsilon_3\sum_{n=1}^{m-1}\Delta t|v_\eta^{n+1}|^2_{H^1(\Omega)}.
\end{aligned}
$$

For the pressure contribution, we have:

$$
\begin{aligned}
|(t_{rhs})_4| &= \left|4\sum_{n=1}^{m-1}\Delta t\left(p(t^{n+1})-r_\eta p(t^{n+1}), \operatorname{div} v_\eta^{n+1}\right)\right|\\
&\le \frac{2C\eta^4}{\varepsilon_4}\,\|p\|^2_{L^2(0,T;H^2(\Omega))} + 2\varepsilon_4\sum_{n=1}^{m-1}\Delta t|v_\eta^{n+1}|^2_{H^1(\Omega)}.
\end{aligned}
$$

The fifth term is treated as follows:

$$
\begin{aligned}
|(t_{rhs})_5| &= 4\sum_{n=1}^{m-1}\Delta t\,\frac{(\Delta t)^{3/2}}{2\sqrt{3}}\,\|u^{(3)}\|_{L^2(t^{n-1};t^{n+1})}\|v_\eta^{n+1}\|_{L^2(\Omega)}\\
&\le \frac{(\Delta t)^4 S_2^2}{3\varepsilon_5}\,\|u^{(3)}\|^2_{L^2(\Omega\times]0,T[)^2} + \varepsilon_5\sum_{n=1}^{m-1}\Delta t|v_\eta^{n+1}|^2_{H^1(\Omega)}.
\end{aligned}
$$

Let us consider now the non-linear terms, $(t_{rhs})_6 + (t_{rhs})_7$, which are treated like follows:

$$
\begin{aligned}
&\left(-u(t^{n+1})\cdot\nabla u(t^{n+1})+u_\eta^{n+1}\cdot\nabla u_\eta^{n+1},\, v_\eta^{n+1}\right)+\frac{1}{2}\left(\operatorname{div} u_\eta^{n+1},\, u_\eta^{n+1}\cdot v_\eta^{n+1}\right)\\
&=-\left(v_\eta^{n+1}\cdot\nabla v_\eta^{n+1},\, P_\eta u(t^{n+1})\right)-\frac{1}{2}\left(\operatorname{div} v_\eta^{n+1},\, P_\eta u(t^{n+1})\cdot v_\eta^{n+1}\right)\\
&\quad-\left(\varphi_\eta^{n+1}\cdot\nabla v_\eta^{n+1},\, P_\eta u(t^{n+1})\right)\\
&\quad-\frac{1}{2}\left(\operatorname{div}\varphi_\eta^{n+1},\, P_\eta u(t^{n+1})\cdot v_\eta^{n+1}\right)-\left(u(t^{n+1})\cdot\nabla v_\eta^{n+1},\, \varphi_\eta^{n+1}\right). \qquad (3.11)
\end{aligned}
$$

The study of the three terms in the right-hand side of (3.11), denoted by $((t_{rhs})_{67}).j$, $j=1,2,3$, will end the proof. Setting

$$
C_1=\sup_n |u(t^n)|_{H^1(\Omega)},
$$

applying on one hand

$$
\begin{aligned}
&\int_\Omega \operatorname{div}(v_\eta^{n+1}-u(t^{n+1}))u(t^{n+1})\cdot\varphi_\eta^{n+1}dx\\
&\quad=-\int_\Omega (v_\eta^{n+1}-u(t^{n+1}))\cdot\nabla u(t^{n+1})\cdot\varphi_\eta^{n+1}dx\\
&\qquad-\int_\Omega (v_\eta^{n+1}-u(t^{n+1}))\cdot\nabla\varphi_\eta^{n+1}\cdot u(t^{n+1})dx, \qquad (3.12)
\end{aligned}
$$

and on the other hand

$$
ab\le\frac{a^p}{p}+\frac{b^{p'}}{p'},\quad \text{avec } \frac{1}{p}+\frac{1}{p'}=1, \qquad (3.13)
$$

we have

$$
\begin{aligned}
|((t_{rhs})_{67}).1| &= \left|4\sum_{n=1}^{m-1}\Delta t\left((v_\eta^{n+1}\cdot\nabla v_\eta^{n+1},\, P_\eta u(t^{n+1}))\right.\right.\\
&\quad\left.\left.+\frac{1}{2}(\operatorname{div} v_\eta^{n+1},\, P_\eta u(t^{n+1})\cdot v_\eta^{n+1})\right)\right|\\
&\le\frac{3S_4C_1\sqrt{2}}{2\varepsilon_6^4}\sum_{n=1}^{m-1}\Delta t\ \|v_\eta^{n+1}\|^2_{L^2(\Omega)}+\frac{9S_4C_1\varepsilon_6^{4/3}}{2\sqrt{2}}\sum_{n=1}^{m-1}\Delta t|v_\eta^{n+1}|^2_{H^1(\Omega)},
\end{aligned}
$$

$$|((t_{rhs})_{67}).2| = \left|4\sum_{n=1}^{m-1}\Delta t\left((\varphi_\eta^{n+1}\cdot\nabla v_\eta^{n+1}, P_\eta u(t^{n+1})) + \frac{1}{2}(\operatorname{div}\varphi_\eta^{n+1}, P_\eta u(t^{n+1})\cdot v_\eta^{n+1})\right)\right|$$
$$\leq 3S_4^2 C C_1\left\{\frac{\eta^4}{\varepsilon_7}\,\|u\|^2_{L^2(0,T;H^3(\Omega)^2)} + \varepsilon_7\sum_{n=1}^{m-1}\Delta t|v_\eta^{n+1}|^2_{H^1(\Omega)}\right\},$$

and

$$|((t_{rhs})_{67}).3| = \left|4\sum_{n=1}^{m-1}\Delta t\left(u(t^{n+1})\cdot\nabla v_\eta^{n+1}, \varphi_\eta^{n+1}\right)\right|$$
$$\leq \frac{S_4^2 C_1}{2}\left\{\frac{C^2\eta^4}{\varepsilon_8}\,\|u\|^2_{L^2(0,T;H^3(\Omega)^2)} + \varepsilon_8\sum_{n=1}^{m-1}\Delta t|v_\eta^{n+1}|^2_{H^1(\Omega)}\right\}.$$

After a suitable choice of ε_i, (3.10) becomes

$$\|v_\eta^m\|^2_{L^2(\Omega)} + \|2v_\eta^m - v_\eta^{m-1}\|^2_{L^2(\Omega)} + \sum_{n=1}^{m-1}\|\delta^2 v_\eta^n\|^2_{L^2(\Omega)} + \nu\sum_{n=1}^{m-1}\Delta t|v_\eta^{n+1}|^2_{H^1(\Omega)}$$
$$\leq \alpha(\Delta t)^4 + \beta\eta^4 + \xi\sum_{n=1}^{m-1}\Delta t\;\|v_\eta^{n+1}\|^2_{L^2(\Omega)}, \tag{3.14}$$

where α, β and ξ are constants that do not depend on η and Δt.

Then after applying Gronwall's lemma and for Δt sufficiently small, the result follows from this inequality:

$$\sup_{1\leq n\leq N}\|v_\eta^n\|_{L^2(\Omega)} + \sup_{1\leq n\leq N}\|2v_\eta^n - v_\eta^{n-1}\|_{L^2(\Omega)} + \left(\sum_{n=1}^{N-1}\|\delta^2 v_\eta^n\|^2_{L^2(\Omega)}\right)^{1/2}$$
$$+\sqrt{\nu}\left(\sum_{n=1}^{N-1}\Delta t|v_\eta^{n+1}|^2_{H^1(\Omega)}\right)^{1/2} \leq \alpha'(\Delta t)^2 + \beta'\eta^2. \tag{3.15}$$

Finally, (3.7) follows by applying a triangular inequality and the P_η's properties. □

Remark 3.5 We suppose that there exist two constants α' and $\gamma' > 0$ that do not depend on η and Δt such that

$$\alpha'\eta^3 \leq (\Delta t)^2 \leq \gamma'\eta^3, \tag{3.16}$$

which means that $(\Delta t)^2$ is of the same order of η^3.

4 Some error estimates for the Stokes problem

The error estimate of order two in $L^2(\Omega\times]0,T[)^2$, that will be established in the next section, is based on a duality argument for the transient Stokes problem:

$$\frac{\partial v}{\partial t}(x,t) - \nu\Delta v(x,t) + \nabla q(x,t) = g(x,t) \text{ in } \Omega\times]0,T[, \tag{4.1}$$

$$\operatorname{div} v(x,t) = 0 \text{ in } \Omega\times]0,T[,\ v(x,t) = 0 \text{ on } \partial\Omega\times]0,T[,\ v(x,0) = 0 \quad \text{in } \Omega. \tag{4.2}$$

The fully-discrete scheme for (4.1) and (4.2) is: Find $(v_\eta^{n+1}, q_\eta^{n+1})$ with values in $X_\eta \times M_\eta$, for each $1 \le n \le N-1$, solution of:

$$\forall z_\eta \in X_\eta,\quad \frac{1}{2\Delta t}\left(3v_\eta^{n+1} - 4v_\eta^n + v_\eta^{n-1}, z_\eta\right) + \nu\left(\nabla v_\eta^{n+1}, \nabla z_\eta\right) - \left(q_\eta^{n+1}, \operatorname{div} z_\eta\right) = (g^{n+1}, z_\eta), \tag{4.3}$$

$$\forall \lambda_\eta \in M_\eta,\quad \left(\lambda_\eta, \operatorname{div} v_\eta^{n+1}\right) = 0. \tag{4.4}$$

These equations are completed by initial conditions similar to the Navier–Stokes problem's ones.

This linear problem (4.3) and (4.4) has a unique solution, owing to the inf–sup condition (1.10), without any restriction on Δt. This solution satisfies the following error estimates in norm $L^\infty(0,T;L^2(\Omega)^2)$ and $L^2(0,T;H^1(\Omega)^2)$: We prove, first of all, that the initial value v_η^1, as in the Navier–Stokes problem, satisfies:
If $v(\Delta t) \in H^3(\Omega)^2$, $q(\Delta t) \in H^2(\Omega)$ and $v'' \in \mathcal{C}^0([0,T];L^2(\Omega)^2)$, then

$$\begin{aligned}&\|v_\eta^1 - v(\Delta t)\|_{L^2(\Omega)}^2 + \nu\Delta t|v_\eta^1 - v(\Delta t)|_{H^1(\Omega)}^2\\ &\quad\le \frac{(\Delta t)^4}{2}\,\|v''\|_{L^\infty(0,T;L^2(\Omega)^2)}^2 + C(\Delta t)\eta^4\left(|v(\Delta t)|_{H^3(\Omega)}^2 + |q(\Delta t)|_{H^2(\Omega)}^2\right)\\ &\qquad + C\eta^6|v(\Delta t)|_{H^3(\Omega)}^2.\end{aligned} \tag{4.5}$$

Secondly, in the general case, we have the following result (the proof is similar to the one of Lemma 3.4, but simpler because of the absence of the convection term).

Lemma 4.1 *Let (v,q) and (v_η^n, q_η^n) be the respective solution of (4.1)–(4.2) and (4.3)–(4.4). In addition to the precedent hypotheses, we suppose that g is regular enough in space and in time, $v \in L^2(0,T;H^3(\Omega)^2)$, $v' \in L^2(0,T;H^2(\Omega)^2)$, $v^{(3)} \in L^2(\Omega\times]0,T[)^2$ and $q \in L^2(0,T;H^2(\Omega))$. There exists a constant C that does not depend on η and Δt such that*

$$\sup_{1\le n\le N} \|v_\eta^n - v(t^n)\|_{L^2(\Omega)} + \sup_{1\le n\le N} \|2(v_\eta^n - v(t^n)) - (v_\eta^{n-1} - v(t^{n-1}))\|_{L^2(\Omega)}$$
$$+\left(\sum_{n=1}^{N-1} \|\delta^2(v_\eta^n - v(t^n))\|^2_{L^2(\Omega)}\right)^{1/2} + \sqrt{\nu}\left(\sum_{n=1}^{N} \Delta t |v_\eta^n - v(t^n)|^2_{H^1(\Omega)}\right)^{1/2}$$
$$\le C(\eta^2 + (\Delta t)^2). \tag{4.6}$$

In addition, the solution $(v_\eta^{n+1}, q_\eta^{n+1})$ of (4.3) and (4.4) satisfies an error estimate in $L^\infty(0, T; H^1(\Omega)^2)$. To simplify, we introduce the following notation

$$\delta^1 a^n = \frac{3a^{n+1} - 4a^n + a^{n-1}}{2\Delta t}. \tag{4.7}$$

The proof is based on the following Stokes projection: $\forall(u, p) \in V \times L^2_0(\Omega)$, $S_\eta(u) \in V_\eta$ satisfies

$$\forall v_\eta \in V_\eta, \quad \nu(\nabla(S_\eta(u) - u), \nabla v_\eta) = -(p, \operatorname{div} v_\eta). \tag{4.8}$$

The operator S_η satisfies the following inequalities:

Lemma 4.2 *Let* $(u, p) \in V \times L^2_0(\Omega)$. $S_\eta(u)$ *defined by* (4.8) *satisfies*

$$|S_\eta(u) - u|_{H^1(\Omega)} \le 2|P_\eta(u) - u|_{H^1(\Omega)} + \frac{1}{\nu}\, \|r_\eta(p) - p\|_{L^2(\Omega)}. \tag{4.9}$$

If in addition Ω *is convex, there exists a constant* C *that does not depend on* η *such that*

$$\|S_\eta(u) - u\|_{L^2(\Omega)} \le C\eta \left(|S_\eta(u) - u|_{H^1(\Omega)} + \|r_\eta(p) - p\|_{L^2(\Omega)}\right). \tag{4.10}$$

Lemma 4.3 *In addition to the hypotheses of Lemma 4.1, suppose that* $v' \in \mathcal{C}^0(0, T; H^2(\Omega)^2)$, $v'' \in L^2(0, T; H^1(\Omega)^2)$, $v^{(3)} \in L^2(\Omega\times]0, T[)^2$, $q' \in \mathcal{C}^0(0, T; H^1(\Omega))$ *and* $q'' \in L^2(\Omega\times]0, T[)$. *Then, if* Ω *is convex, there exists a constant* C *that does not depend on* η *and* Δt *such that*

$$\sup_{1\le n\le N} |v_\eta^n - v(t^n)|_{H^1(\Omega)} + \sup_{1\le n\le N-1} |2(v_\eta^{n+1} - v(t^{n+1})) - (v_\eta^n - v(t^n))|_{H^1(\Omega)}$$
$$+\left(\sum_{n=1}^{N-1} \Delta t\ \|\delta^1(v_\eta^n - v(t^n))\|^2_{L^2(\Omega)}\right)^{1/2} + \left(\sum_{n=1}^{N-1} |\delta^2(v_\eta^n - v(t^n))|^2_{H^1(\Omega)}\right)^{1/2}$$
$$\le C\left(\eta^2 + (\Delta t)^{3/2} + \frac{\eta^3}{\sqrt{\Delta t}}\right). \tag{4.11}$$

Proof Setting $\varphi(t) = v(t) - S_\eta v(t)$, $\varphi_\eta^i = \varphi(t^i)$ and $e_\eta^i = v_\eta^i - S_\eta v(t^i)$ and applying (3.9) to (4.3), we obtain

$$\forall z_\eta \in V_\eta, \quad (\delta^1 e_\eta^n, z_\eta) + \nu(\nabla e_\eta^{n+1}, \nabla z_\eta) = (\delta^1 \varphi_\eta^n, z_\eta) + R_3, \tag{4.12}$$

where

$$|R_3| \leq \frac{(\Delta t)^{3/2}}{2\sqrt{3}} \; \|v^{(3)}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega)^2)} \|z_\eta\|_{L^2(\Omega)} .$$

Taking the scalar product by $z_\eta = z_\eta^{n+1} = \frac{3e_\eta^{n+1}-4e_\eta^n+e_\eta^{n-1}}{2\Delta t}$, summing over $1 \leq n \leq m-1$, and applying Jensen's inequality, (4.12) becomes

$$\begin{aligned} &\frac{1}{2}\sum_{n=1}^{m-1} \Delta t \; \|z_\eta^{n+1}\|_{L^2(\Omega)}^2 + \frac{\nu}{4}\left(|e_\eta^m|_{H^1(\Omega)}^2 + |2e_\eta^m - e_\eta^{m-1}|_{H^1(\Omega)}^2 + \sum_{n=1}^{m-1} |\delta^2 e_\eta^n|_{H^1(\Omega)}^2 \right) \\ &\leq \frac{5\nu}{4}|e_\eta^1|_{H^1(\Omega)}^2 + \frac{(\Delta t)^4}{3} \; \|v^{(3)}\|_{L^2(0,T;L^2(\Omega)^2)}^2 + \sum_{n=1}^{m-1} \Delta t \; \|\delta^1 \varphi_\eta^n\|_{L^2(\Omega)}^2 . \end{aligned} \tag{4.13}$$

Then (4.11) follows readily by applying (4.5), (4.9) and (4.10). □

The parabolic duality argument (cf. [23]) consists in defining the solution (w^{n-1}, λ^{n-1}) of the backward semi-discrete Stokes system of second order in time :

$$\begin{aligned} &-\frac{w^{n+1} - 4w^n + 3w^{n-1}}{2\Delta t} + \nu\Delta w^{n-1} - \nabla\lambda^{n-1} = v_\eta^{n-1} - v(t^{n-1}) \quad \text{in } \Omega, \\ &1 \leq n \leq N+1, \end{aligned} \tag{4.14}$$

$$\operatorname{div} w^{n-1} = 0 \quad \text{in } \Omega, \quad 1 \leq n \leq N+1, \tag{4.15}$$

$$w^{n-1}|_{\partial\Omega} = 0, \quad 1 \leq n \leq N+1, \tag{4.16}$$

$$w^{N+2} = 0, \quad w^{N+1} = 0 \quad \text{in } \Omega. \tag{4.17}$$

For each $n, 0 \leq n \leq N$, the Stokes problem (4.14)–(4.17) has a unique solution $w^n \in H_0^1(\Omega)^2, \lambda^n \in L_0^2(\Omega)$, (cf. [10,22]).

The next lemma establishes basic estimates for the velocity w^n of the backward semi-discrete Stokes problem (4.14)–(4.17).

Lemma 4.4 *Standard arguments give the uniform bounds:*

$$\begin{aligned} &\sup_{0\leq n\leq N} \|w^n\|_{L^2(\Omega)} + \sup_{1\leq n\leq N+1} \|2w^{n-1} - w^n\|_{L^2(\Omega)} + \sqrt{2\nu}\left(\sum_{n=0}^{N} \Delta t |w^n|_{H^1(\Omega)}^2\right)^{1/2} \\ &+\sum_{n=1}^{N+1} \|\delta^2 w^n\|_{L^2(\Omega)}^2 \leq \sqrt{\frac{2S_2}{\nu}} \left(\sum_{n=0}^{N} \Delta t \; \|v(t^n) - v_\eta^n\|_{L^2(\Omega)}^2\right)^{1/2}, \end{aligned} \tag{4.18}$$

where S_2 is the constant of Poincars inequality, and

$$\sqrt{\frac{\nu}{2}}\sup_{0\le n\le N}|w^n|_{H^1(\Omega)}+\sqrt{\frac{\nu}{2}}\left(\sum_{n=1}^{N+1}|\delta^2 w^n|^2_{H^1(\Omega)}\right)^{1/2}+\sqrt{\frac{\nu}{2}}\sup_{0\le n\le N}|2w^n-w^{n+1}|_{H^1(\Omega)}$$
$$+\left(\sum_{n=1}^{N+1}\Delta t\ \|\frac{w^{n+1}-4w^n+3w^{n-1}}{2\Delta t}\|^2_{L^2(\Omega)}\right)^{1/2}\le\left(\sum_{n=0}^{N}\Delta t\ \|v(t^n)-v^n_\eta\|^2_{L^2(\Omega)}\right)^{1/2}. \tag{4.19}$$

If Ω is convex, then $\forall\ 0\le n\le N$, $w^n\in H^2(\Omega)^2$, $\lambda^n\in H^1(\Omega)$ and (4.19) implies the uniform bound

$$\left(\sum_{n=0}^{N}\Delta t\left(|w^n|^2_{H^2(\Omega)}+|\lambda^n|^2_{H^1(\Omega)}\right)\right)^{1/2}\le C\left(\sum_{n=0}^{N}\Delta t\ \|v(t^n)-v^n_\eta\|^2_{L^2(\Omega)}\right)^{1/2} \tag{4.20}$$

with a constant C independent of Δt and η.

Proof For the first inequality, we take the scalar product of (4.14) with $z=4\Delta t w^{n-1}$, and we use the incompressibility condition. Multiplying the result by Δt, summing it over n from $m+1$ to $N+1$, and applying the Poincars inequality, we obtain for any $\varepsilon>0$

$$\|w^m\|^2_{L^2(\Omega)}+\|2w^m-w^{m+1}\|^2_{L^2(\Omega)}+4\nu\sum_{n=m}^{N}\Delta t|w^n|^2_{H^1(\Omega)}+\sum_{n=m+1}^{N+1}\|\delta^2 w^n\|^2_{L^2(\Omega)}$$
$$\le\frac{2}{\varepsilon}\sum_{n=m}^{N}\Delta t\ \|v(t^n)-v^n_\eta\|^2_{L^2(\Omega)}+2\varepsilon S_2\sum_{n=m}^{N}\Delta t|w^n|^2_{H^1(\Omega)},$$

where S_2 is Poincars constant. Then (4.18) follows after the suitable choice of $\varepsilon=\frac{\nu}{S_2}$.

Similarly, for the second inequality, we take the scalar product of (4.14) with $z=\frac{w^{n+1}-4w^n+3w^{n-1}}{2\Delta t}$, we multiply the equation by Δt and sum it over n. We obtain (4.19) by choosing $\varepsilon=\frac{1}{2\Delta t}$.

Now, we assume that Ω is convex. Since (4.14)–(4.17) is a steady Stokes problem with right-hand side $-\frac{w^{n+1}-4w^n+3w^{n-1}}{2\Delta t}+(v(t^{n-1})-v^{n-1}_\eta)$, we have $w^n\in H^2(\Omega)^2, \lambda^n\in H^1(\Omega)$ (cf. [11]) and (4.19) implies also the uniform bound (4.20). □

From now on, we assume that Ω is convex. Using these inequalities, the next theorem establishes that the error satisfies an estimate of order two in $L^2(\Omega\times]0,T[)^2$.

Theorem 4.5 *If* $g \in L^2(\Omega\times]0,T[)^2$, $v \in L^2(0,T;H^3(\Omega)^2)$, $q \in L^2(0,T;H^2(\Omega))$, $v' \in L^2(0,T;H^2(\Omega)^2)$ *and* $v^{(3)} \in L^2(\Omega\times]0,T[)^2$, *then there exists a constant* C, *that depends on the norm of* v, v', $v^{(3)}$ *and* q, *but not on* η *and* Δt *such that*

$$\left(\sum_{n=0}^{N} \Delta t \; \|v_\eta^n - v(t^n)\|^2_{L^2(\Omega)}\right)^{1/2} \le C\left(\eta^3 + (\Delta t)^2 + \eta(\Delta t)^2\right). \tag{4.21}$$

In particular, if (3.16) *holds, then*

$$\left(\sum_{n=0}^{N} \Delta t \; \|v_\eta^n - v(t^n)\|^2_{L^2(\Omega)}\right)^{1/2} \le C\eta^3. \tag{4.22}$$

Proof Let $e^{n-1} = v_\eta^{n-1} - v(t^{n-1})$. Taking the scalar product of (4.14) by e^{n-1}, summing over n form 1 to $N+1$ and applying a discrete integration by parts, we obtain

$$\begin{aligned}\sum_{n=0}^{N} \Delta t \; \|e^n\|^2_{L^2(\Omega)} = &\sum_{n=1}^{N-1}\left\{-\frac{1}{2}\left(3e^{n+1} - 4e^n + e^{n-1}, P_\eta w^{n+1}\right)\right.\\ &\left.- \nu\Delta t\left(\nabla P_\eta w^{n+1}, \nabla e^{n-1}\right)\right\}\\ &-\frac{1}{2}\sum_{n=1}^{N+1}\left(3e^{n+1} - 4e^n + e^{n-1}, w^{n+1} - P_\eta w^{n+1}\right)\\ &-\nu\sum_{n=1}^{N+1}\Delta t\left(\nabla(w^{n-1} - P_\eta w^{n-1}), \nabla e^{n-1}\right)\\ &+\sum_{n=1}^{N+1}\Delta t\left(\lambda^{n-1} - r_\eta\lambda^{n-1}, \operatorname{div} e^{n-1}\right)\\ &-\left\{\frac{3}{2}(w^1, e^1) + \nu\Delta t\left(\nabla e^1, \nabla P_\eta w^1\right)\right\}.\end{aligned} \tag{4.23}$$

Denote the terms in the right-hand side of (4.23) by $(W_{RH})_j$, $j = 1, \ldots, 5$. The first term is treated as follows:

$$\begin{aligned}|(W_{RH})_1| \le &\left|\sum_{n=1}^{N}\Delta t\left(q(t^{n+1}) - r_\eta q(t^{n+1}), \operatorname{div}(P_\eta w^{n+1} - w^{n+1})\right)\right|\\ &+\frac{P}{\sqrt{3}}(\Delta t)^2\;\|v^{(3)}\|_{L^2(\Omega\times]0,T[)^2}\left(\sum_{n=1}^{N-1}\Delta t|P_\eta w^{n+1}|^2_{H^1(\Omega)}\right)^{1/2}\end{aligned}$$

$$\leq \left[C\eta^3 \ \|q\|_{L^2(0,T;H^1(\Omega))} + \frac{P}{\sqrt{3}}(\Delta t)^2 \ \|v^{(3)}\|_{L^2(\Omega\times]0,T[)^2} \right]$$
$$\times \left(\sum_{n=0}^{N} \Delta t \ \|e^n\|^2_{L^2(\Omega)} \right)^{1/2}.$$

The second term is bounded as follows:

$$|(W_{RH})_2| \leq \left(\sum_{n=1}^{N-1} \Delta t \ \|\delta^1 e^n\|^2_{L^2(\Omega)} \right)^{1/2} \left(\sum_{n=1}^{N-1} \Delta t \ \|w^{n+1} - P_\eta w^{n+1}\|^2_{L^2(\Omega)} \right)^{1/2}$$
$$\leq C\eta^2((\Delta t)^{3/2} + \eta^2) \left(\sum_{n=0}^{N} \Delta t \ \|e^n\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Owing to Lemma 4.1, the third and fourth terms can be bounded by:

$$|(W_{RH})_3| \leq C\eta \left(\sum_{n=0}^{N} \Delta t |e^n|^2_{H^1(\Omega)} \right)^{1/2} \left(\sum_{n=0}^{N} \Delta t |w^n|^2_{H^2(\Omega)} \right)^{1/2}$$
$$\leq C\eta((\Delta t)^2 + \eta^2) \left(\sum_{n=0}^{N} \Delta t \ \|e^n\|^2_{L^2(\Omega)} \right)^{1/2},$$

and

$$|(W_{RH})_4| \leq C\eta \left((\Delta t)^2 + \eta^2 \right) \left(\sum_{n=0}^{N} \Delta t \ \|e^n\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Finally, the last term can be written as follows:

$$|(W_{RH})_5| = -\frac{3}{2}(w^1 - P_\eta w^1, e^1) - \frac{3}{2}\left[(P_\eta w^1, e^1) + \nu \Delta t (\nabla e^1, \nabla P_\eta w^1) \right]$$
$$+ \frac{\nu}{2} \Delta t (\nabla e^1, \nabla P_\eta w^1).$$

Let us consider the terms between square brackets and write the error equation at time t^1 : there exists $\theta \in]0, 1[$ such that

$$(e^1, P_\eta w^1) + \nu \Delta t (\nabla e^1, \nabla P_\eta w^1) = \Delta t \left(r_\eta q(\Delta t) - q(\Delta t), \operatorname{div} P_\eta w^1 \right)$$
$$- \frac{(\Delta t)^2}{2} \left(v''(\theta \Delta t), P_\eta w^1 \right),$$

then

$$
\begin{aligned}
&\left|(e^1, P_\eta w^1) + \nu \Delta t (\nabla e^1, \nabla P_\eta w^1)\right| \\
&\leq C \left[(\Delta t)\eta^2 |q(\Delta t)|_{H^2(\Omega)} + \frac{(\Delta t)^2}{2} \, \|v''\|_{L^\infty(0,T;L^2(\Omega)^2)}\right] \times \left(\sum_{n=0}^{N} \Delta t \, \|e^n\|^2_{L^2(\Omega)}\right)^{1/2}.
\end{aligned}
$$

The first and last parts of $(W_{RH})_5$ are bounded by using (4.5).

Substituting these inequalities into (4.23), we obtain (4.21). In addition, if (3.16) holds, then (4.21) implies (4.22). □

Now, we split $u_\eta^n - u(t^n)$ into a linear contribution, $v_\eta^n - u(t^n)$, and a non-linear one $u_\eta^n - v_\eta^n$. Here v_η^{n+1} is the solution of the Stokes problem (4.3) and (4.4) with $g = f - u \cdot \nabla u$. Therefore, $v = u$ and v_η^{n+1} solves the discrete problem $\forall w_\eta \in V_\eta$,

$$
\begin{aligned}
&\frac{\left(3v_\eta^{n+1} - 4v_\eta^n + v_\eta^{n-1}, w_\eta\right)}{2\Delta t} + \nu(\nabla v_\eta^{n+1}, \nabla w_\eta) - (q_\eta^{n+1}, \operatorname{div} w_\eta) \\
&\quad = \left(f(t^{n+1}) - u(t^{n+1}) \cdot \nabla u(t^{n+1}), w_\eta\right).
\end{aligned}
\tag{4.24}
$$

Therefore, Theorem 4.5 gives

Corollary 4.6 *Suppose that u satisfies the hypotheses on v in Theorem 4.5 and that $f \in \mathcal{C}^0([0,T]; L^2(\Omega)^2)$, then*

$$
\left(\sum_{n=0}^{N} \Delta t \, \|v_\eta^n - u(t^n)\|^2_{L^2(\Omega)}\right)^{1/2} \leq C\left(\eta^3 + (\Delta t)^2 + \eta(\Delta t)^2\right), \tag{4.25}
$$

with another constant $C(f, u, p, \nu, T)$ that does not depend on η nor on Δt.

Furthermore, if p' belongs to $L^2(0,T; H^1(\Omega))$, Lemma 4.3 implies that

$$
\sup_{0 \leq n \leq N} |v_\eta^n - u(t^n)|_{H^1(\Omega)} \leq C\left(\eta^2 + (\Delta t)^{3/2} + \frac{\eta^3}{\sqrt{\Delta t}}\right). \tag{4.26}
$$

On the other hand, we prove the following "superconvergence" result for the non-linear part.

Theorem 4.7 *Under the assumptions of Corollary 4.6 and if $p' \in L^2(0,T; H^1(\Omega))$, $u' \in L^2(0,T; H^1(\Omega)^2)$ and $u \in L^\infty(0,T; W^{1,4}(\Omega)^2)$ then there exists a constant C*

that does not depend on η *and* Δt, *such that*

$$\sup_{0\le n\le N}\|v_\eta^n-u_\eta^n\|_{L^2(\Omega)}+\sup_{1\le n\le N}\|2(v_\eta^n-u_\eta^n)-(v_\eta^{n-1}-u_\eta^{n-1})\|_{L^2(\Omega)}$$
$$+\left(\sum_{n=1}^{N-1}\|\delta^2(v_\eta^n-u_\eta^n)\|_{L^2(\Omega)}^2\right)^{1/2}+\left(\sum_{n=0}^{N-1}\Delta t|v_\eta^{n+1}-u_\eta^{n+1}|_{H^1(\Omega)}^2\right)^{1/2}$$
$$\le C(\eta^3+(\Delta t)^2). \tag{4.27}$$

Proof In one hand, we take the difference between (4.24) and (1.17). We split the non-linear term as follows:

$$u_\eta^{n+1}\cdot\nabla u_\eta^{n+1}+\frac{1}{2}\operatorname{div}u_\eta^{n+1}u_\eta^{n+1}-u(t^{n+1})\cdot\nabla u(t^{n+1})$$
$$=-\varphi_\eta^{n+1}\cdot\nabla u_\eta^{n+1}-\frac{1}{2}\operatorname{div}\varphi_\eta^{n+1}\cdot u_\eta^{n+1}-v_\eta^{n+1}\cdot\nabla\varphi_\eta^{n+1}-\frac{1}{2}\operatorname{div}v_\eta^{n+1}\varphi_\eta^{n+1}$$
$$+(v_\eta^{n+1}-u(t^{n+1}))\cdot\nabla(v_\eta^{n+1}-u(t^{n+1}))$$
$$+\frac{1}{2}\operatorname{div}(v_\eta^{n+1}-u(t^{n+1}))(v_\eta^{n+1}-u(t^{n+1}))$$
$$+(v_\eta^{n+1}-u(t^{n+1}))\cdot\nabla u(t^{n+1})+\frac{1}{2}\operatorname{div}(v_\eta^{n+1}-u(t^{n+1}))u(t^{n+1})$$
$$+u(t^{n+1})\cdot\nabla(v_\eta^{n+1}-u(t^{n+1})).$$

On the other hand, we multiply the resultant equation by φ_η^{n+1} and sum it over $n=1,\dots,m-1$. We obtain:

$$\frac{1}{2}\sum_{n=1}^{m-1}\left(\varphi_\eta^{n+1},3\varphi_\eta^{n+1}-4\varphi_\eta^n+\varphi_\eta^{n-1}\right)+\nu\sum_{n=1}^{m-1}\Delta t|\varphi_\eta^{n+1}|_{H^1(\Omega)}^2$$
$$=\sum_{n=1}^{m-1}\Delta t\left\{\left(-\varphi_\eta^{n+1}\cdot\nabla u_\eta^{n+1},\varphi_\eta^{n+1}\right)-\frac{1}{2}\left(\operatorname{div}\varphi_\eta^{n+1},u_\eta^{n+1}\cdot\varphi_\eta^{n+1}\right)\right\}$$
$$+\sum_{n=1}^{m-1}\Delta t\left(u(t^{n+1})\cdot\nabla(v_\eta^{n+1}-u(t^{n+1})),\varphi_\eta^{n+1}\right)$$
$$+\sum_{n=1}^{m-1}\Delta t\left\{\left((v_\eta^{n+1}-u(t^{n+1}))\cdot\nabla(v_\eta^{n+1}-u(t^{n+1})),\varphi_\eta^{n+1}\right)\right.$$
$$\left.+\frac{1}{2}\left(\operatorname{div}\,(v_\eta^{n+1}-u(t^{n+1})),(v_\eta^{n+1}-u(t^{n+1}))\cdot\varphi_\eta^{n+1}\right)\right\}$$
$$+\sum_{n=1}^{m-1}\Delta t\left\{\left((v_\eta^{n+1}-u(t^{n+1}))\cdot\nabla u(t^{n+1}),\varphi_\eta^{n+1}\right)\right.$$
$$\left.+\frac{1}{2}\left(\operatorname{div}\,(v_\eta^{n+1}-u(t^{n+1})),u(t^{n+1})\cdot\varphi_\eta^{n+1}\right)\right\}. \tag{4.28}$$

The left-hand side of (4.28) can be written as follows:

$$\frac{1}{4}\ \|\varphi_\eta^m\|^2_{L^2(\Omega)} - \frac{1}{4}\ \|\varphi_\eta^1\|^2_{L^2(\Omega)} + \frac{1}{4}\ \|2\varphi_\eta^m - \varphi_\eta^{m-1}\|^2_{L^2(\Omega)} - \frac{1}{4}\ \|2\varphi_\eta^1 - \varphi_\eta^0\|^2_{L^2(\Omega)}$$
$$+ \frac{1}{4}\sum_{n=1}^{m-1} \|\delta^2\varphi_\eta^n\|^2_{L^2(\Omega)} + \nu \sum_{n=1}^{m-1} \Delta t |\varphi_\eta^{n+1}|^2_{H^1(\Omega)}.$$

We note $(U_{RH})_i, i = 1, \ldots, 4$, the terms in the right-hand side of (4.28). For the first term, setting $C_1 = \sup_n |u_\eta^n|_{H^1(\Omega)}$, we can write

$$|(U_{RH})_1| \le \frac{C_1}{2}\left\{\left(\sqrt{2}\varepsilon_1 + \frac{2^{1/4}3S_4\varepsilon_2^{4/3}}{8}\right)\sum_{n=1}^{m-1} \Delta t|\varphi_\eta^{n+1}|^2_{H^1(\Omega)}\right.$$
$$\left. + \left(\frac{\sqrt{2}}{\varepsilon_1} + \frac{2^{1/4}S_4}{8\varepsilon_2^4}\right)\sum_{n=1}^{m-1} \Delta t\ \|\varphi_\eta^{n+1}\|^2_{L^2(\Omega)}\right\}.$$

Setting $C_3 = \sup_n \|u(t^{n+1})\|_{L^\infty(\Omega)}$ and due to Corollary 4.6, the second term is bounded as follows:

$$|(U_{RH})_2| \le \frac{CC_3}{2\varepsilon_3}\left(\eta^6 + (\Delta t)^4 + (\Delta t)^4\eta^2\right) + \frac{C_3\varepsilon_3}{2}\sum_{n=1}^{m-1} \Delta t|\varphi_\eta^{n+1}|^2_{H^1(\Omega)}.$$

For the third term, we use Lemma 4.1 and (4.26) and we obtain:

$$|(U_{RH})_3| \le \frac{CS_4^2}{2\varepsilon_4}\left(\eta^8 + (\Delta t)^7 + (\Delta t)^3\eta^4\right) + \frac{3S_4^2\varepsilon_4}{4}\sum_{n=1}^{m-1} \Delta t|\varphi_\eta^{n+1}|^2_{H^1(\Omega)}.$$

In order to bound the last term, we use the well-known formula (3.12):

$$\left((v_\eta^{n+1} - u(t^{n+1}))\cdot\nabla u(t^{n+1}), \varphi_\eta^{n+1}\right) + \frac{1}{2}\left(\text{div}\ (v_\eta^{n+1} - u(t^{n+1})), u(t^{n+1})\cdot\varphi_\eta^{n+1}\right)$$
$$= \frac{1}{2}\left((v_\eta^{n+1} - u(t^{n+1}))\cdot\nabla u(t^{n+1}), \varphi_\eta^{n+1}\right)$$
$$-\frac{1}{2}\left((v_\eta^{n+1} - u(t^{n+1}))\cdot\nabla\varphi_\eta^{n+1}, u(t^{n+1})\right), \quad (4.29)$$

we set $C_2 = \sup_{1\le n\le N} |u(t^{n+1})|_{W^{1,4}(\Omega)}$ and we obtain:

$$|(U_{RH})_4| \le \frac{C(C_2S_4 + C_3)}{4\varepsilon_5}\left(\eta^6 + (\Delta t)^4 + \eta^2(\Delta t)^4\right)$$
$$+ \frac{(S_4C_2 + C_3)\varepsilon_5}{4}\sum_{n=1}^{m-1} \Delta t|\varphi_\eta^{n+1}|^2_{H^1(\Omega)}.$$

Finally, we still have to estimate φ_η^1 :

$$\|\varphi_\eta^1\|^2_{L^2(\Omega)} + \nu\Delta t|\varphi_\eta^1|^2_{H^1(\Omega)} = \Delta t \left|\left(u_\eta^1 \cdot \nabla u_\eta^1 - u(\Delta t)\cdot u(\Delta t), \varphi_\eta^1\right)\right|.$$

The non-linear term is splitted as the general one. The first part is bounded by:

$$\frac{C_1}{2}\left\{\left(\sqrt{2}\varepsilon_6 + \frac{2^{1/4}3S_4\varepsilon_7^{4/3}}{8}\right)\Delta t|\varphi_\eta^1|^2_{H^1(\Omega)} + \left(\frac{\sqrt{2}}{\varepsilon_6} + \frac{2^{1/4}S_4}{8\varepsilon_7^4}\right)\Delta t\ \|\varphi_\eta^1\|^2_{L^2(\Omega)}\right\},$$

and if Δt is sufficiently small, these terms are absorbed by the left-hand side of (4.28). In the second part, we obtain:

$$\Delta t\ \|v_\eta^1 - u(t^1)\|^2_{L^2(\Omega)} \leq C\left(\eta^6 + (\Delta t)^4\right),$$

and in the third one:

$$\begin{aligned}&\Delta t|v_\eta^1 - u(\Delta t)|_{H^1(\Omega)}\ \|v_\eta^1 - u(\Delta t)\|_{L^4(\Omega)}\ |\varphi_\eta^1|_{H^1(\Omega)}\\ &\leq \frac{1}{2}\left(\varepsilon_8\Delta t|\varphi_\eta^1|^2_{H^1(\Omega)}\right) + \frac{1}{\varepsilon_8}C\left(\eta^8 + \eta^6(\Delta t)^{3/2} + \frac{\eta^9}{\sqrt{\Delta t}} + \eta^4(\Delta t)^{7/2}\right).\end{aligned}$$

In the last part, we obtain

$$\Delta t\ \|v_\eta^1 - u(t^1)\|^2_{L^2(\Omega)} \leq C\left(\eta^6 + (\Delta t)^4\right).$$

Then (4.27) follows readily by applying these results. □

Combining Corollary 4.6 and Theorem 4.7, we obtain:

Corollary 4.8 *Under the assumptions of Theorem 4.7, there exists a constant C that does not depend on η and Δt, such that*

$$\left(\sum_{n=0}^{N}\Delta t\ \|u(t^n) - u_\eta^n\|^2_{L^2(\Omega)}\right)^{1/2} \leq C(\eta^3 + (\Delta t)^2), \tag{4.30}$$

In particular, if (3.16) holds, then

$$\left(\sum_{n=1}^{N-1}\Delta t\ \|u(t^{n+1}) - u_\eta^{n+1}\|^2_{L^2(\Omega)}\right)^{1/2} \leq C\eta^3. \tag{4.31}$$

5 An estimate for the pressure

The results of the preceding section allow one to establish an error estimate for the pressure. We start with a general bound.

Lemma 5.1 *Under the assumptions of Lemma 3.4, let* $(u(t^{n+1}), p(t^{n+1}))$ *and* $(u_\eta^{n+1}, p_\eta^{n+1})$ *be the respective solution of* (1.1)–(1.4) *and* (1.17)–(1.18). *We have*

$$\left(\sum_{n=1}^{N-1} \Delta t\ \|p_\eta^{n+1} - r_\eta p(t^{n+1})\|_{L^2(\Omega)}^2\right)^{1/2}$$
$$\leq \frac{1}{\beta^\star}\left\{C_1(\eta^2 + (\Delta t)^2) + C_2(\Delta t)^2\ \|u^{(3)}\|_{L^2(\Omega\times]0,T[)^2}\right.$$
$$\left. + C_3\eta^2\ \|p\|_{L^2(0,T;H^2(\Omega))} + S_2\left(\sum_{n=1}^{N} \Delta t\ \|\delta^1(u_\eta^n - u(t^n))\|_{L^2(\Omega)}^2\right)^{1/2}\right\}, \tag{5.1}$$

where $\beta^\star$ *is the constant of the inf–sup condition* (1.10) *and the coefficients* C_i, $1 \leq i \leq 3$, *are independent of* η *and* Δt.

Proof Let us substruct the non-linear terms and set $e_\eta^i = u_\eta^i - u(t^i)$. We obtain

$$u(t^{n+1})\cdot\nabla u(t^{n+1}) - u_\eta^{n+1}\cdot\nabla u_\eta^{n+1} - \frac{1}{2}\operatorname{div} u_\eta^{n+1}u_\eta^{n+1}$$
$$= -u(t^{n+1})\cdot\nabla e_\eta^{n+1} - e_\eta^{n+1}\cdot\nabla u_\eta^{n+1} - \frac{1}{2}\operatorname{div} e_\eta^{n+1}u_\eta^{n+1}.$$

Then, for all $w_\eta^n \in X_\eta$ and due to (3.9), we have

$$\sum_{n=1}^{N-1}\Delta t\left(p_\eta^{n+1} - r_\eta p(t^{n+1}), \operatorname{div} w_\eta^{n+1}\right)$$
$$= \sum_{n=1}^{N-1}\Delta t\left(\frac{3e_\eta^{n+1} - 4e_\eta^n + e_\eta^{n-1}}{2\Delta t}, w_\eta^{n+1}\right) + \nu\sum_{n=1}^{N-1}\Delta t\left(\nabla e_\eta^{n+1}, \nabla w_\eta^{n+1}\right)$$
$$+ \sum_{n=1}^{N-1} R_1 + \sum_{n=1}^{N-1}\Delta t\left(u(t^{n+1})\cdot\nabla e_\eta^{n+1}, w_\eta^{n+1}\right)$$
$$+ \sum_{n=1}^{N-1}\Delta t\left\{\left(e_\eta^{n+1}\cdot\nabla u_\eta^{n+1}, w_\eta^{n+1}\right) + \frac{1}{2}\left(\operatorname{div} e_\eta^{n+1}u_\eta^{n+1}, w_\eta^{n+1}\right)\right\}$$
$$+ \sum_{n=1}^{N-1}\Delta t\left(p(t^{n+1}) - r_\eta p(t^{n+1}), \operatorname{div} w_\eta^{n+1}\right). \tag{5.2}$$

Owing to the inf–sup condition (1.10), there exists a function $w_\eta \in V_\eta^\perp$ such that

$$\left(\operatorname{div} w_\eta,\, p_\eta^{n+1} - r_\eta p(t^{n+1})\right) = \|p_\eta^{n+1} - r_\eta p(t^{n+1})\|^2_{L^2(\Omega)} \ \text{ and } |w_\eta|_{H^1(\Omega)} \le \frac{1}{\beta^\star}\, \|p_\eta^{n+1} - r_\eta p(t^{n+1})\|_{L^2(\Omega)}\,.$$

Let $(P_{RH})_i, i = 1, \ldots, 6$, denote the terms of the right-hand side of (5.2).

We deduce by standard arguments:

$$|(P_{RH})_1| \le S_2 \left(\sum_{n=1}^{N-1} \Delta t\ \|\delta^1 e_\eta^n\|^2_{L^2(\Omega)}\right)^{1/2} \left(\sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|^2_{H^1(\Omega)}\right)^{1/2},$$

$$|(P_{RH})_2| \le \nu \left(\sum_{n=1}^{N-1} \Delta t |P_\eta u(t^{n+1}) - u(t^{n+1})|^2_{H^1(\Omega)}\right)^{1/2} \left(\sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|^2_{H^1(\Omega)}\right)^{1/2}$$
$$\le C_1(\eta^2 + (\Delta t)^2) \left(\sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|^2_{H^1(\Omega)}\right)^{1/2},$$

$$|(P_{RH})_3| \le C_2 (\Delta t)^2\ \|u^{(3)}\|_{L^2(\Omega\times]0,T[)^2} \left(\sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|^2_{H^1(\Omega)}\right)^{1/2}.$$

The fourth and fifth terms $(P_{RH})_4$, $(P_{RH})_5$ are bounded as follows:

$$|(P_{RH})_4| \le S_4^2 (\sup_t |u(t)|_{H^1(\Omega)}) \left(\sum_{n=1}^{N-1} \Delta t |e_\eta^{n+1}|^2_{H^1(\Omega)}\right)^{1/2} \left(\sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|^2_{H^1(\Omega)}\right)^{1/2}$$
$$\le C_3(\eta^2 + (\Delta t)^2) \left(\sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|^2_{H^1(\Omega)}\right)^{1/2}.$$

$$|(P_{RH})_5| = \frac{1}{2} \left| \sum_{n=0}^{N-1} \Delta t \left\{ (e_\eta^{n+1} \cdot \nabla u_\eta^{n+1}, w_\eta^{n+1}) - (e_\eta^{n+1} \cdot \nabla w_\eta^{n+1}, u_\eta^{n+1}) \right. \right|$$
$$\le S_4^2 (\sup |u_\eta^{n+1}|_{H^1(\Omega)}) \left(\sum_{n=1}^{N-1} \Delta t |e_\eta^{n+1}|^2_{H^1(\Omega)}\right)^{1/2} \left(\sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|^2_{H^1(\Omega)}\right)^{1/2}$$
$$\le C_4(\eta^2 + (\Delta t)^2) \left(\sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|^2_{H^1(\Omega)}\right)^{1/2},$$

and the last term is bounded as follows:

$$|(P_{RH})_6 \le C_5 \eta^2\ \|p\|_{L^2(0,T;H^2(\Omega))} \left(\sum_{n=1}^{N-1} \Delta t |w_\eta^{n+1}|^2_{H^1(\Omega)}\right)^{1/2}.$$

Then (5.1) follows easily by substituting these inequalities into (5.2). □

We have to estimate $(\sum_{n=1}^{N-1} \Delta t \ \|\delta^1(u_\eta^n - u(t^n))\|_{L^2(\Omega)}^2)^{1/2}$. This estimate is proven assuming the triangulation satisfies a milder regularity property than uniform regularity (1.9): there exists a constante $\tilde{\tau}$ that does not depend on η or Δt such that

$$\rho_{\min} \geq \tilde{\tau}\eta^5, \quad \text{where } \rho_{\min} = \inf_{\kappa \in \mathcal{T}_\eta} \rho_\kappa. \tag{5.3}$$

More precisely, this assumption is used in proving that u_η^n is bounded in $L^\infty(0, T; W^{1,5/2}(\Omega)^2)$:

Lemma 5.2 *Under the assumptions of Theorem 4.7 and if $\mathcal{T}_\eta$ satisfies (5.3), there exists a constant C that depends neither on η nor on Δt, such that*

$$\sup_n |u_\eta^n|_{W^{1,5/2}(\Omega)} \leq C. \tag{5.4}$$

Proof We refer to [2] for the sketch of this proof. □

Lemma 5.3 *Under the assumptions of Theorem 4.7, there exists a constant $C = C(u, u', u^{(3)})$ that does not depend on η and Δt, such that*

$$\begin{aligned}
&\left(\sum_{n=1}^{N-1} \Delta t \ \|\delta^1(u_\eta^n - u(t^n))\|_{L^2(\Omega)}^2\right)^{1/2} + \sqrt{\nu} \sup_{1\leq n\leq N} |u_\eta^n - u(t^n)|_{H^1(\Omega)} \\
&\quad + \sqrt{\nu} \sup_{1\leq n\leq N} \left|2(u_\eta^n - u(t^n)) - (u_\eta^{n-1} - u(t^{n-1}))\right|_{H^1(\Omega)} \\
&\quad + \sqrt{\nu} \left(\sum_{n=1}^{N-1} |\delta^2(u_\eta^n - u(t^n))|_{H^1(\Omega)}^2\right)^{1/2} \\
&\leq C\left(\eta^2 + (\Delta t)^{3/2} + \frac{\eta^3}{\sqrt{\Delta t}}\right).
\end{aligned} \tag{5.5}$$

Proof The proof is similar to that of Lemma 5.1. By taking $e_\eta^i = u_\eta^i - S_\eta u(t^i)$, $\varphi_\eta^i = u(t^i) - S_\eta u(t^i)$ and the test function $w_\eta = w_\eta^{n+1} = \delta^1 e_\eta^n$:

$$\sum_{n=1}^{m-1} \Delta t \ \|\delta^1 e_\eta^n\|^2_{L^2(\Omega)} + \frac{\nu}{2}\Bigg(|e_\eta^m|^2_{H^1(\Omega)} - |e_\eta^1|^2_{H^1(\Omega)} + |2e_\eta^m - e_\eta^{m-1}|^2_{H^1(\Omega)}$$
$$- |2e_\eta^1 - e_\eta^0|^2_{H^1(\Omega)} + \sum_{n=1}^{m-1} |\delta^2 e_\eta^n|^2_{H^1(\Omega)} \Bigg)$$
$$\leq \nu \left| \sum_{n=1}^{m-1} \Delta t (\nabla \varphi_\eta^{n+1}, \nabla \delta^1 e_\eta^n) \right| + \left| \sum_{n=1}^{m-1} \Delta t (p(t^{n+1}), \operatorname{div} \delta^1 e_\eta^n) \right|$$
$$+ \sum_{n=1}^{m-1} \Delta t (\delta^1 \varphi_\eta^n, \delta^1 e_\eta^n) + \sum_{n=1}^{m-1} R_1$$
$$+ \sum_{n=1}^{m-1} \Delta t \left\{ \left(u(t^{n+1}) \cdot \nabla u(t^{n+1}) - u_\eta^{n+1} \cdot \nabla u_\eta^{n+1}, \delta^1 e_\eta^n \right) \right.$$
$$\left. - \frac{1}{2} \left(\operatorname{div} u_\eta^{n+1} u_\eta^{n+1}, \delta^1 e_\eta^n \right) \right\}, \tag{5.6}$$

with

$$u(t^{n+1}) \cdot \nabla u(t^{n+1}) - u_\eta^{n+1} \cdot \nabla u_\eta^{n+1} - \frac{1}{2} \operatorname{div} u_\eta^{n+1} u_\eta^{n+1}$$
$$= u(t^{n+1}) \cdot \nabla (u(t^{n+1}) - u_\eta^{n+1}) + (u(t^{n+1}) - u_\eta^{n+1}) \cdot \nabla u_\eta^{n+1}$$
$$- \frac{1}{2} \operatorname{div}(u(t^{n+1}) - u_\eta^{n+1}) u_\eta^{n+1}.$$

Due to the definition of the operator S_η, we only have to estimate the three last terms $(V_{RH})_i, i = 1, \ldots, 3$, in the right-hand side of (5.6).

The first one is bounded as precedently as follows:

$$|(V_{RH})_1| = \left| \sum_{n=1}^{m} \Delta t (\delta^1 \varphi_\eta^n, \delta^1 e_\eta^n) \right|$$
$$\leq \frac{C}{2\varepsilon_1} \left\{ \eta^4 \left(\|u'\|^2_{L^\infty(0,T;H^2(\Omega)^2)} + \|p'\|^2_{L^\infty(0,T;H^1(\Omega))} \right) \right.$$
$$\left. + (\Delta t)^2 \eta^2 \left(\|u''\|^2_{L^2(0,T;H^1(\Omega)^2)} + \|p''\|_{L^2(\Omega \times]0,T[)} \right) \right\}$$
$$+ \frac{\varepsilon_1}{2} \sum_{n=1}^{m-1} \Delta t \ \|\delta^1 e_\eta^n\|^2_{L^2(\Omega)} .$$

The second term is bounded as follows:

$$|(V_{RH})_2| = \left|\sum_{n=1}^{m-1} R_1\right| \leq \frac{C(\Delta t)^4}{2\varepsilon_2} \; \|u^{(3)}\|^2_{L^2(\Omega\times]0,T[)^2} + \frac{\varepsilon_2}{2}\sum_{n=1}^{m-1} \Delta t \; \|\delta^1 e^n_\eta\|^2_{L^2(\Omega)}.$$

For the last term, it is splitted into two parts that we treat successively. The first part is treated as follows:

$$\begin{aligned}
&\left|\sum_{n=1}^{m-1} \Delta t \left(u(t^{n+1}) \cdot \nabla(u(t^{n+1}) - u^{n+1}_\eta), \delta^1 e^n_\eta\right)\right| \\
&\quad \leq \frac{C \; \|u\|_{L^\infty(\Omega\times]0,T[)^2}}{2}\left(\frac{C'}{\varepsilon_3}(\eta^4 + (\Delta t)^4) + \varepsilon_3 \sum_{n=1}^{m-1} \Delta t \; \|\delta^1 e^n_\eta\|^2_{L^2(\Omega)}\right),
\end{aligned}$$

and for the second part, we notice that:

$$\begin{aligned}
\|(u(t^{n+1}) - u^{n+1}_\eta) \cdot \nabla u^{n+1}_\eta\|_{L^2(\Omega)} &\leq |u^{n+1}_\eta|_{W^{1,5/2}(\Omega)} \; \|u(t^{n+1}) - u^{n+1}_\eta\|_{L^{10}(\Omega)} \\
&\leq S_{10}|u^{n+1}_\eta|_{W^{1,5/2}(\Omega)}|u(t^{n+1}) - u^{n+1}_\eta|_{H^1(\Omega)},
\end{aligned}$$

then it is bounded as follows:

$$\begin{aligned}
&\left|\sum_{n=1}^{m-1} \Delta t \left(\left((u(t^{n+1}) - u^{n+1}_\eta) \cdot \nabla u^{n+1}_\eta, \delta^1 e^n_\eta\right) + \frac{1}{2}\left(\operatorname{div}(u(t^{n+1}) - u^{n+1}_\eta) u^{n+1}_\eta, \delta^1 e^n_\eta\right)\right)\right| \\
&\leq \left(\frac{C}{2} + S_{10}\right) \sup_n |u^n_\eta|_{W^{1,5/2}(\Omega)} \sum_{n=1}^{m-1} \Delta t \left| u(t^{n+1}) - u^{n+1}_\eta \right|_{H^1(\Omega)} \; \|\delta^1 e^n_\eta\|_{L^2(\Omega)} \\
&\leq C''\left(\frac{C'}{\varepsilon_4}(\eta^4 + (\Delta t)^4) + \varepsilon_4 \sum_{n=1}^{m-1} \Delta t \; \|\delta^1 e^n_\eta\|^2_{L^2(\Omega)}\right).
\end{aligned}$$

Then by setting $C_1 = \|u\|_{L^\infty(\Omega\times]0,T[)^2}$, the last term of the right-hand side of (5.6) is bounded by:

$$\left(\frac{C_1 C'}{2\varepsilon_3} + \frac{C'' C'}{\varepsilon_4}\right)\left(\eta^4 + (\Delta t)^4\right) + \left(\frac{C_1 \varepsilon_3}{2} + C''\varepsilon_4\right)\sum_{n=1}^{m-1} \Delta t \; \|\delta^1 e^n_\eta\|^2_{L^2(\Omega)}.$$

Finally the initial data are bounded due to Proposition 3.3. Then, choosing suitably the parameters ε_i, Eq. (5.6) becomes

$$\left(\sum_{n=1}^{m-1} \Delta t \; \|\delta^1(u_\eta^n - S_\eta u(t^n))\|^2_{L^2(\Omega)}\right)^{1/2} + \sqrt{\nu} \sup_{1\le n\le N} |u_\eta^n - S_\eta u(t^n)|_{H^1(\Omega)}$$
$$+\sqrt{\nu} \sup_{1\le n\le N} \left|2(u_\eta^n - S_\eta u(t^n)) - (u_\eta^{n-1} - S_\eta u(t^{n-1}))\right|_{H^1(\Omega)}$$
$$+\sqrt{\nu}\left(\sum_{n=1}^{N-1} |\delta^2(u_\eta^n - S_\eta u(t^n))|^2_{H^1(\Omega)}\right)^{1/2}$$
$$\le C\left(\eta^2 + (\Delta t)^{3/2} + \frac{\eta^3}{\sqrt{\Delta t}}\right).$$

Finally (5.5) follows readily from this result and by applying a triangular inequality and S_η's properties. □

From these three lemmas, we easily derive an estimate of the pressure.

Theorem 5.4 *Under the assumptions of Lemma 5.1, there exists a constant C that does not depend on η nor on Δt, such that*

$$\left(\sum_{n=1}^{N} \Delta t \; \|p(t^n) - p_\eta^n\|^2_{L^2(\Omega)}\right)^{1/2} \le C\left(\eta^2 + (\Delta t)^{3/2} + \frac{\eta^3}{\sqrt{\Delta t}}\right). \quad (5.7)$$

6 Error estimate for the solution of step two

We assume at this stage that we know the solution u_H^{n+1} of the first step. Then at each time step, the second step (1.19) and (1.20) is a square system of linear equations in finite dimension, and if Δt is small enough, it has a unique solution. First, we will establish the error estimate for the solution computed by one step of Euler's scheme $(u_h^1 - u(\Delta t), p_h^1 - p(\Delta t))$:

Proposition 6.1 *The error of the solution computed by one iteration of Euler's scheme satisfies the following estimations, for $\Delta t \le k_0 > 0$ sufficiently small,*

$$\frac{1}{2} \; \|u_h^1 - u(\Delta t)\|^2_{L^2(\Omega)} + \frac{\nu \Delta t}{2} |u_h^1 - u(\Delta t)|^2_{H^1(\Omega)} \le C\left(H^6 + h^4 + (\Delta t)^4\right), \quad (6.1)$$

and

$$(\Delta t)^{1/2} \; \|p(\Delta t) - p_h^1\|_{L^2(\Omega)} \le C\left(h^2 + H^3 + (\Delta t)^{3/2}\right). \quad (6.2)$$

Proof The error's equation is similar to (3.2).

$$\forall v_h \in V_h, \quad (u_h^1 - u(\Delta t), v_h) + \nu \Delta t (\nabla (u_h^1 - u(\Delta t)), \nabla v_h) \\ = \frac{(\Delta t)^2}{2} (u''(\theta \Delta t), v_h) - \Delta t (p(\Delta t) - r_h p(\Delta t), \operatorname{div} v_h) \\ + \Delta t \left(u(\Delta t) \cdot \nabla u(\Delta t) - u_H^1 \cdot \nabla u_h^1, v_h \right). \tag{6.3}$$

By setting $v_h = v_h^1 = P_h u(\Delta t) - u_h^1$ and $\varphi_h^1 = P_h u(\Delta t) - u(\Delta t)$, the non-linear term can be written as follows:

$$\left(u(\Delta t) \cdot \nabla u(\Delta t) - u_H^1 \cdot \nabla u_h^1, v_h \right) = \left((u(\Delta t) - u_H^1) \cdot \nabla u(\Delta t), v_h \right) \\ + \left(u_H^1 \cdot \nabla (u(\Delta t) - P_h u(\Delta t)), v_h \right) \\ + \left((u(\Delta t) - u_H^1) \cdot \nabla (u_h^1 - P_h u(\Delta t)), v_h \right) \\ + \left(u(\Delta t) \cdot \nabla (P_h u(\Delta t) - u_h^1), v_h \right) \\ = \left((u(\Delta t) - u_H^1) \cdot \nabla u(\Delta t), v_h \right) - \left(u_H^1 \cdot \nabla \varphi_h^1, v_h \right) \\ - \left((u(\Delta t) - u_H^1) \cdot \nabla v_h^1, v_h \right).$$

Then, we have three contributions of the non-linear term. For the first part, we write:

$$\Delta t \left| \left((u(\Delta t) - u_H^1) \cdot \nabla u(\Delta t), v_h^1 \right) \right| \\ \leq S_4 |v_h^1|_{H^1(\Omega)} \sup_t |u(\Delta t)|_{W^{1,4}(\Omega)} \Delta t \; \|u(\Delta t) - u_H^{n+1}\|_{L^2(\Omega)} \\ \leq \frac{1}{2} \left(\varepsilon_1 \Delta t |v_h^1|^2_{H^1(\Omega)} + \frac{1}{\varepsilon_1} C^2 \Delta t (H^6 + (\Delta t)^4 + (\Delta t) H^4) \right).$$

For the second part, we know that $\|u_H^1\|_{L^4(\Omega)}$ is bounded and we write:

$$\Delta t \left| \left(u_H^1 \cdot \nabla \varphi_h^1, v_h^1 \right) \right| \leq S_4 \Delta t |v_h^1|_{H^1(\Omega)} \; \|u_H^1\|_{L^4(\Omega)} \; |u(\Delta t) - P_H u(\Delta t)|_{H^1(\Omega)} \\ \leq \frac{1}{2} \left(\varepsilon_2 \Delta t |v_h^1|^2_{H^1(\Omega)} + \frac{C}{\varepsilon_2} (\Delta t) h^4 \right).$$

Finally, the last term can be written as:

$$\Delta t \left| \left((u(\Delta t) - u_H^1) \cdot \nabla v_h^1, v_h^1 \right) \right| \leq \Delta t \widehat{C} H^{1-\varepsilon} |v_h^1|^2_{H^1(\Omega)} |u_H^1 - u(\Delta t)|_{H^1(\Omega)},$$

with

$$\left|u_H^1 - u(\Delta t)\right|_{H^1(\Omega)} \leq C\left((\Delta t)^{3/2} + H^2 + \frac{H^3}{\sqrt{\Delta t}}\right).$$

In that case, for H (and Δt) sufficiently smooth, this term is absorbed by the left-hand side of the equation. And for the linear terms, we introduce $P_h u(\Delta t)$ in (6.3) and we obtain:

$$\begin{aligned}
&\|v_h^1\|^2_{L^2(\Omega)} + \nu\Delta t|v_h^1|^2_{H^1(\Omega)} \\
&\quad\leq \left|(\varphi_h^1, v_h^1)\right| + \nu\Delta t\left|(\nabla\varphi_h^1, \nabla v_h^1)\right| + \frac{(\Delta t)^2}{2}\sup\|u''\|_{L^2(\Omega)}\|v_h^1\|_{L^2(\Omega)} \\
&\qquad + \Delta t\ \|p(\Delta t) - r_h p(\Delta t)\|_{L^2(\Omega)}\ |v_h^1|_{H^1(\Omega)} + \text{non-linear term.}
\end{aligned} \tag{6.4}$$

For the pressure, we obtain:

$$\begin{aligned}
&\Delta t\ (r_h p(\Delta t) - p(\Delta t), \operatorname{div} v_h) + \Delta t\left(p_h^1 - r_h p(\Delta t), \operatorname{div} v_h\right) \\
&= \left(u_h^1 - u(\Delta t), v_h\right) + \nu\Delta t\left(\nabla(u_h^1 - u(\Delta t)), \nabla v_h\right) - \frac{(\Delta t)^2}{2}\left(u''(\theta\Delta t), v_h\right) \\
&\quad -\Delta t\left(u(\Delta t)\cdot\nabla u(\Delta t) - u_H^1\cdot\nabla u_h^1, v_h\right).
\end{aligned} \tag{6.5}$$

We choose $v_h \in V_h^{\perp}$ such that

$$\left(p_h^1 - r_h p(\Delta t), \operatorname{div} v_h\right) = \|p_h^1 - r_h p(\Delta t)\|^2_{L^2(\Omega)} \text{ and } |v_h|_{H^1(\Omega)} \leq \frac{1}{\beta^\star}\ \|p_h^1 - r_h p(\Delta t)\|_{L^2(\Omega)},$$

with $\beta^\star > 0$ that does not depend on h. Thus

$$\begin{aligned}
&(\Delta t)^{1/2}\ \|p_h^1 - r_h p(\Delta t)\|_{L^2(\Omega)} \\
&\leq \frac{(\Delta t)^{1/2}}{\beta^\star}\Bigg(\ \|r_h p(\Delta t) - p(\Delta t)\|_{L^2(\Omega)} + \frac{S_2}{\Delta t}\ \|u_h^1 - u(\Delta t)\|_{L^2(\Omega)} \\
&\quad + \nu|P_h u(\Delta t) - u(\Delta t)|_{H^1(\Omega)} + \frac{S_2}{2}(\Delta t)\ \|u''(\theta\Delta t)\|_{L^2(\Omega)} + S_4^2(|u(\Delta t)|_{H^1(\Omega)} \\
&\quad + |u_H^1|_{H^1(\Omega)})|u_h^1 - u(\Delta t)|_{H^1(\Omega)}\Bigg) \\
&\leq C\left(h^2 + H^3 + (\Delta t)^{3/2}\right).
\end{aligned}$$

□

The fine velocity satisfies the following error estimate:

Theorem 6.2 *Under the hypotheses of Theorem 4.7, the solution of Step 2, (u_h^{n+1}, p_h^{n+1}), satisfies the following error estimate*

$$\sup_{1\le n\le N} \|u_h^n - u(t^n)\|_{L^2(\Omega)} + \sup_{1\le n\le N} \|2(u_h^n - u(t^n)) - (u_h^{n-1} - u(t^{n-1}))\|_{L^2(\Omega)} + \left(\sum_{n=1}^{N-1} \|\delta^2(u_h^n - u(t^n))\|^2_{L^2(\Omega)}\right)^{1/2} + \sqrt{\nu}\left(\sum_{n=1}^{N} \Delta t |u_h^n - u(t^n)|^2_{H^1(\Omega)}\right)^{1/2} \le C\left(H^3 + h^2 + (\Delta t)^2 + H(\Delta t)^2\right), \tag{6.6}$$

with a constant C that does not depend on h, H and Δt.

Proof By substructing Eqs. (1.19) and (1.17), by setting $v_h^i = P_h u(t^i) - u_h^i$, $\varphi_h^i = P_h u(t^i) - u(t^i)$, by taking the test function $v_h = v_h^{n+1}$ and by summing the result from $n = 1$ to $n = m-1$, we obtain

$$\nu \sum_{n=1}^{m-1} \Delta t |v_h^{n+1}|^2_{H^1(\Omega)} + \frac{1}{4}\Bigg(\|v_h^m\|^2_{L^2(\Omega)} - \|v_h^1\|^2_{L^2(\Omega)} + \|2v_h^m - v_h^{m-1}\|^2_{L^2(\Omega)} - \|2v_h^1 - v_h^0\|^2_{L^2(\Omega)} + \sum_{n=1}^{m-1} \|\delta^2 v_h^n\|^2_{L^2(\Omega)}\Bigg)$$
$$\le \left|\sum_{n=1}^{m-1} R_1\right| + \left|\nu \sum_{n=1}^{m-1} \Delta t (\nabla\varphi_h^{n+1}, \nabla v_h^{n+1})\right| + \left|\sum_{n=1}^{m-1} \Delta t \left(p(t^{n+1}) - r_h p(t^{n+1}), \operatorname{div} v_h^{n+1}\right)\right|$$
$$+ \left|\sum_{n=1}^{m-1} \Delta t \left(\delta^1 \varphi_h^n, v_h^{n+1}\right)\right| + \left|\sum_{n=1}^{m} \Delta t \left(u_H^{n+1}\cdot\nabla u_h^{n+1} - u(t^{n+1})\cdot\nabla u(t^{n+1}), v_h^{n+1}\right)\right|. \tag{6.7}$$

Let us estimate the terms $(TG_{RH})_i$, $i = 1, \ldots, 4$ in the right-hand side of (6.7). The first term is bounded as follows:

$$|(TG_{RH})_1| \le \frac{C(\Delta t)^4}{2\varepsilon_1} \|u^{(3)}\|^2_{L^2(\Omega\times]0,T[)^2} + \frac{\varepsilon_1}{2}\sum_{n=1}^{m-1} \Delta t\ \|v_h^{n+1}\|^2_{L^2(\Omega)}.$$

The second term and third terms are bounded respectively as follows:

$$|(TG_{RH})_2| \le \frac{C\nu h^4}{2\varepsilon_2} \|u\|^2_{L^2(0,T;H^3(\Omega)^2)} + \frac{\nu\varepsilon_2}{2}\sum_{n=1}^{m-1} \Delta t |v_h^{n+1}|^2_{H^1(\Omega)}.$$

and

$$|(TG_{RH})_3| \le \frac{Ch^4}{2\varepsilon_3} \|p\|^2_{L^2(0,T;H^2(\Omega))} + \frac{\varepsilon_3}{2}\sum_{n=1}^{m-1} \Delta t |v_h^{n+1}|^2_{H^1(\Omega)},$$

and the fourth term is as follows:

$$\left|\sum_{n=1}^{m-1} \Delta t(\delta^1 \varphi_\eta^n, v_h^{n+1})\right| \leq \frac{C(\Delta t)^4}{2\varepsilon_4} \|u^{(3)}\|^2_{L^2(\Omega\times]0,T[)^2} + \frac{Ch^4}{2\varepsilon_4} \|u'\|^2_{L^\infty(0,T;H^2(\Omega)^2)}$$
$$+\frac{\varepsilon_4}{2}\sum_{n=1}^{m-1} \Delta t \; \|v_h^{n+1}\|^2_{L^2(\Omega)}.$$

The non-linear term in the right-hand side can be written as follows:

$$\begin{aligned} u_H^{n+1}\cdot\nabla u_h^{n+1} - u(t^{n+1})\cdot\nabla u(t^{n+1}) &= (u_H^{n+1} - u(t^{n+1}))\cdot\nabla u(t^{n+1}) \\ &\quad + u_H^{n+1}\cdot\nabla(P_h u(t^{n+1}) - u(t^{n+1})) \\ &\quad - (u(t^{n+1}) - u_H^{n+1})\cdot\nabla(u_h^{n+1} - P_h u(t^{n+1})) \\ &\quad - u(t^{n+1})\cdot\nabla(P_h u(t^{n+1}) - u_h^{n+1}). \end{aligned}$$

We study the four parts $(NL)_i$, $i = 1, \ldots, 4$, of the non-linear term separately. Setting $C_{\infty.1} = \sup |u|_{W^{1,4}(\Omega)}$, the first part is treated as follows:

$$\left|\sum_{n=0}^{m-1} \Delta t((NL)_1, v_h^{n+1})\right| \leq \frac{C_{\infty.1}}{2\varepsilon_{5.1}} C(H^6 + (\Delta t)^4) + \frac{S_4^2 C_{\infty.1}\varepsilon_{5.1}}{2}\sum_{n=1}^{m-1} \Delta t |v_h^{n+1}|^2_{H^1(\Omega)}.$$

Setting $C_{\infty.2} = \sup \|u_H^{n+1}\|_{L^4(\Omega)}$, the second part is treated as follows:

$$\left|\sum_{n=0}^{m-1} \Delta t((NL)_2, v_h^{n+1})\right| \leq \frac{CC_{\infty.2}h^4}{2\varepsilon_{5.2}} \|u\|^2_{L^2(0,T;H^3(\Omega)^2)}$$
$$+ \frac{S_4^2 C_{\infty.2}\varepsilon_{5.2}}{2}\sum_{n=1}^{m-1} \Delta t |v_h^{n+1}|^2_{H^1(\Omega)}.$$

For the third part, we use the following estimation (cf. [8]): there exists a constant $\widehat{C}$, that does not depend on η such that, for all $u_\eta \in V_\eta$,

$$\forall w_\eta \in X_\eta, \quad |(u_\eta \cdot \nabla w_\eta, w_\eta)| \leq \widehat{C}\eta^{1-\varepsilon} \|\mathrm{div}\, u_\eta\|_{L^2(\Omega)} |w_\eta|^2_{H^1(\Omega)}, \tag{6.8}$$

we have

$$\left|\sum_{n=0}^{m-1} \Delta t((NL)_3, v_h^{n+1})\right| \leq \widehat{C} H^{1-\varepsilon}\left(H^2 + (\Delta t)^{3/2} + \frac{H^3}{\sqrt{\Delta t}}\right)\sum_{n=1}^{m-1} \Delta t |v_h^{n+1}|^2_{H^1(\Omega)},$$

And the last part is bounded as follows:

$$\left|\sum_{n=0}^{m-1} \Delta t((NL)_4, v_h^{n+1})\right| = 0.$$

Then, collecting these inequalities, choosing suitably the parameters ε_i and δ and applying Gronwall's Lemma, we get

$$\begin{aligned}
&\sup_{1\le n\le N} \|u_h^n - P_h u(t^n)\|_{L^2(\Omega)} + \sup_{1\le n\le N} \|2(u_h^n - P_h u(t^n)) \\
&-(u_h^{n-1} - P_h u(t^{n-1}))\|_{L^2(\Omega)} + \left(\sum_{n=1}^{N-1} \|\delta^2(u_h^n - P_h u(t^n))\|^2_{L^2(\Omega)}\right)^{1/2} \\
&+\sqrt{\nu}\left(\sum_{n=1}^{N} \Delta t |u_h^n - P_h u(t^n)|^2_{H^1(\Omega)}\right)^{1/2} \le C\left(H^3 + h^2 + (\Delta t)^2\right).
\end{aligned}$$

Then, (6.6) follows readily from the above result and the P_h's properties. □

Finally, we consider the error of the pressure. As in Sect. 5, the pressure satisfies the following bound.

Lemma 6.3 *Let* $(u(t^{n+1}), p(t^{n+1}))$ *and* (u_h^{n+1}, p_h^{n+1}) *be the respective solution of* (1.1)–(1.4) *and* (1.19)–(1.20). *We have*

$$\begin{aligned}
&\left(\sum_{n=1}^{N-1} \Delta t\ \|p_h^{n+1} - r_h p(t^{n+1})\|^2_{L^2(\Omega)}\right)^{1/2} \\
&\le \frac{1}{\beta^\star}\left\{C_1 h^2\ \|p\|_{L^2(0,T;H^2(\Omega))} + C_2(\Delta t)^2\ \|u^{(3)}\|_{L^2(\Omega\times]0,T[)^2}\right. \\
&\left. + C_3(H^3 + (\Delta t)^2) + C_4 h^2 + S_2\left(\sum_{n=1}^{N-1} \Delta t\ \|\delta^1(u_h^n - u(t^n))\|^2_{L^2(\Omega)}\right)^{1/2}\right\},
\end{aligned} \tag{6.9}$$

where $\beta^\star$ *is the constant of the inf–sup condition* (1.10) *and the coefficients* $C_i, i = 1, \ldots, 4,$ *do not depend on* H, h *and* Δt.

Proof The steps of this proof are similar to those of the proof of Lemma 5.1 and the only difference between these proofs concerns the non-linear term. Here we write

$$u(t^{n+1}) \cdot \nabla u(t^{n+1}) - u_H^{n+1} \cdot \nabla u_h^{n+1} = (u(t^{n+1}) - u_H^{n+1}) \cdot \nabla u(t^{n+1})$$
$$+(u_H^{n+1} - u(t^{n+1})) \cdot \nabla(u(t^{n+1}) - u_h^{n+1})$$
$$+u(t^{n+1}) \cdot \nabla(u(t^{n+1}) - u_h^{n+1}).$$

Then, let us estimate the terms that compose the non-linear term.

$$\left|\sum_{n=1}^{N-1} \Delta t (u(t^{n+1}) \cdot \nabla u(t^{n+1}) - u_H^{n+1} \cdot \nabla u_h^{n+1}, w_h^{n+1})\right| \leq S_4 \left(\sum_{n=1}^{N-1} \Delta t |w_h^{n+1}|^2_{H^1(\Omega)}\right)^{1/2}$$
$$\times \left\{ (\sup_n |u(t^n)|_{W^{1,4}(\Omega)}) \left(\sum_{n=1}^{N-1} \Delta t \ \|u(t^{n+1}) - u_H^{n+1}\|^2_{L^2(\Omega)}\right)^{1/2} \right.$$
$$+ S_4 (\sup_n |u(t^n) - u_H^n|_{H^1(\Omega)} + \sup_n |u(t^n)|_{H^1(\Omega)})$$
$$\left. \times \left(\sum_{n=1}^{N-1} \Delta t |u(t^{n+1}) - u_h^{n+1}|^2_{H^1(\Omega)}\right)^{1/2} \right\} \leq \left(C(H^3 + (\Delta t)^2)\right)$$
$$\times \left(\sum_{n=1}^{N} \Delta t |w_h^{n+1}|^2_{H^1(\Omega)}\right)^{1/2}.$$

Then, (6.9) follows readily from these bounds and from the inf-sup condition (1.10). □

Therefore, here again, we must derive an estimate for $\left(\sum_{n=1}^{N-1} \Delta t \ \|\delta^1(u_h^{-u}(t^n))\|^2_{L^2(\Omega)}\right)^{1/2}$.

Lemma 6.4 *Under the assumptions of Theorem 4.7 and Corollary 3.4, there exists a constant C that does not depend on H, h and Δt such that:*

$$\left(\sum_{n=1}^{N-1} \Delta t \ \|\delta^1(u_h^n - u(t^n))\|^2_{L^2(\Omega)}\right)^{1/2} + \sqrt{\nu} \sup_{1\leq n\leq N} |u_h^n - u(t^n)|_{H^1(\Omega)}$$
$$+\sqrt{\nu} \sup_{1\leq n\leq N} |2(u_h^n - u(t^n)) - (u_h^{n-1} - u(t^{n-1}))|_{H^1(\Omega)}$$
$$+\sqrt{\nu}\left(\sum_{n=1}^{N-1} |\delta^2(u_h^n - u(t^n))|^2_{H^1(\Omega)}\right)^{1/2}$$
$$\leq C\left(h^2 + H^3 + (\Delta t)^2\right). \tag{6.10}$$

Proof We substruct Eqs. (1.17) and (1.19), we set $e_h^i = u_h^i - S_h u(t^i)$ and $\varphi_h^i = u(t^i) - S_h u(t^i)$ and we take the function test $w_h = \delta^1 e_h^n$. Due to the definition of the Stokes operator S_h, we have

$$\sum_{n=1}^{m-1} \Delta t \ \|\delta^1 e_h^n\|^2_{L^2(\Omega)} + \frac{\nu}{2} \sum_{n=1}^{m-1} \Delta t \left(\nabla e_h^{n+1}, \nabla \delta^1 e_h^n\right) = \sum_{n=1}^{m-1} \Delta t \left(\delta^1 \varphi_h^n, \delta^1 e_h^n\right) + \sum_{n=1}^{m-1} R_1$$
$$+ \sum_{n=1}^{m-1} \Delta t \left(u(t^{n+1}) \cdot \nabla u(t^{n+1}) - u_H^{n+1} \cdot \nabla u_h^{n+1}, \delta^1 e_h^n\right).$$

The first term of the right-hand side is bounded as follows:

$$\left|\sum_{n=1}^{m-1} \Delta t \left(\delta^1 \varphi_h^n, \delta^1 e_h^n\right)\right| \le \frac{C}{2\varepsilon_1} \Big\{ h^4 \left(\|u'\|^2_{L^\infty(0,T;H^2(\Omega)^2)} + \|p'\|^2_{L^\infty(0,T;H^1(\Omega))}\right)$$
$$+ (\Delta t)^2 h^2 \left(\|u''\|^2_{L^2(0,T;H^1(\Omega)^2)} + \|p''\|_{L^2(\Omega\times]0,T[)}\right)\Big\}$$
$$+ \frac{\varepsilon_1}{2} \sum_{n=1}^{m-1} \Delta t \ \|\delta^1 e_h^n\|^2_{L^2(\Omega)}.$$

The second term is bounded as follows:

$$\left|\sum_{n=1}^{m-1} R_1\right| \le \frac{C(\Delta t)^4}{2\varepsilon_2} \ \|u^{(3)}\|^2_{L^2(\Omega\times]0,T[)^2} + \frac{\varepsilon_2}{2} \sum_{n=1}^{m-1} \Delta t \ \|\delta^1 e_h^n\|^2_{L^2(\Omega)}.$$

Setting $C_{\infty\infty} = \sup_n \|u(t^{n+1})\|_{L^\infty(\Omega)}$, the third term is bounded as follows:

$$\left|\sum_{n=1}^{m-1} \Delta t \left((u(t^{n+1}) - u_H^{n+1}) \cdot \nabla u(t^{n+1}), \delta^1 e_h^n\right)\right|$$
$$\le \frac{C_{\infty\infty}}{2\varepsilon_3} (H^6 + (\Delta t)^4) + \frac{C_{\infty\infty}\varepsilon_3}{2} \sum_{n=1}^{m-1} \Delta t \ \|\delta^1 e_h^n\|^2_{L^2(\Omega)}.$$

Using Theorem 6.2, the fourth and fifth terms are respectively bounded as follows:

$$\left|\sum_{n=1}^{m-1} \Delta t \left((u_H^{n+1} - u(t^{n+1})) \cdot \nabla (u(t^{n+1}) - u_h^{n+1}), \delta^1 e_h^n\right)\right|$$
$$\le \frac{S_4^2}{2\varepsilon_4} \left(\sup_n \|u(t^{n+1}) - u_H^{n+1}\|_{L^\infty(\Omega)}\right)^2 \sum_{n=1}^{m-1} \Delta t \left|u(t^{n+1}) - u_h^{n+1}\right|^2_{H^1(\Omega)}$$
$$+ \frac{S_4^2 \varepsilon_4}{2} \sum_{n=1}^{m-1} \Delta t \ \|\delta^1 e_h^n\|^2_{L^2(\Omega)}$$

$$\leq \frac{S_4^2}{2\varepsilon_4} C \left(H^6 + h^4 + (\Delta t)^4 \right) + \frac{S_4^2 \varepsilon_4}{2} \sum_{n=1}^{m-1} \Delta t \ \|\delta^1 e_h^n\|_{L^2(\Omega)}^2,$$

and

$$\left| \sum_{n=1}^{m-1} \Delta t \left(u(t^{n+1}) \cdot \nabla (u(t^{n+1}) - u_h^{n+1}), \delta^1 e_h^n \right) \right|$$
$$\leq \frac{C_{\infty\infty}}{2\varepsilon_5} C \left(H^6 + h^4 + (\Delta t)^4 \right) + \frac{C_{\infty\infty}\varepsilon_5}{2} \sum_{n=1}^{m-1} \Delta t \ \|\delta^1 e_h^n\|_{L^2(\Omega)}^2 .$$

Thus, after a suitable choice of ε_i, $i = 1, \ldots, 4$ and by applying the error of the solution computed by one iteration of Euler's scheme established in Proposition 3.3, we obtain

$$\left(\sum_{n=1}^{N-1} \Delta t \ \|\delta^1 e_h^n\|_{L^2(\Omega)}^2 \right)^{1/2} + \sqrt{\nu} \sup_{1 \leq n \leq N} |e_h^n|_{H^1(\Omega)} + \sqrt{\nu} \sup_{1 \leq n \leq N} |2 e_h^n - e_h^{n-1}|_{H^1(\Omega)}$$
$$+ \sqrt{\nu} \left(\sum_{n=1}^{N-1} |\delta^2 e_h^n|_{H^1(\Omega)}^2 \right)^{1/2} \leq C(h^2 + H^3 + (\Delta t)^2).$$

□

These two lemmas yield immediately the following theorem.

Theorem 6.5 *Under the assumptions of Lemma 6.4, we have:*

$$\left(\sum_{n=1}^{N-1} \Delta t \ \|p(t^{n+1}) - p_h^{n+1}\|_{L^2(\Omega)}^2 \right)^{1/2} \leq C \left(h^2 + H^3 + (\Delta t)^2 \right), \quad (6.11)$$

with a constant C that does not depend on h, H and Δt.

Remark 6.6 As a consequence, h, H and Δt satisfy (3.16), then

$$\left(\sum_{n=1}^{N-1} \Delta t \ \|p(t^{n+1}) - p_h^{n+1}\|_{L^2(\Omega)}^2 \right)^{1/2} \leq C h^2. \quad (6.12)$$

This theoretical analysis is confirmed by numerical results cf. [1].

7 Numerical results

In order to confirm these results numerically, we did several experiments by using the FreeFem ++ software (see [12]). We use for the space discretization the Taylor–Hood

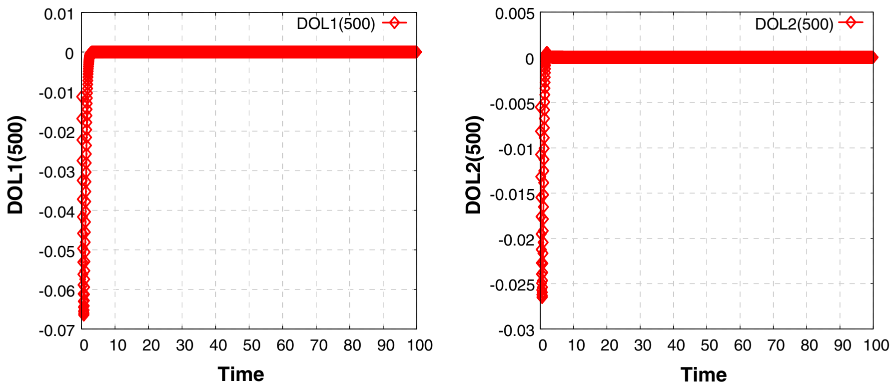


Fig. 1 The *left* figure shows the evolution of the first component of the degree of liberty 500 in time and the *right* one is related to the second component

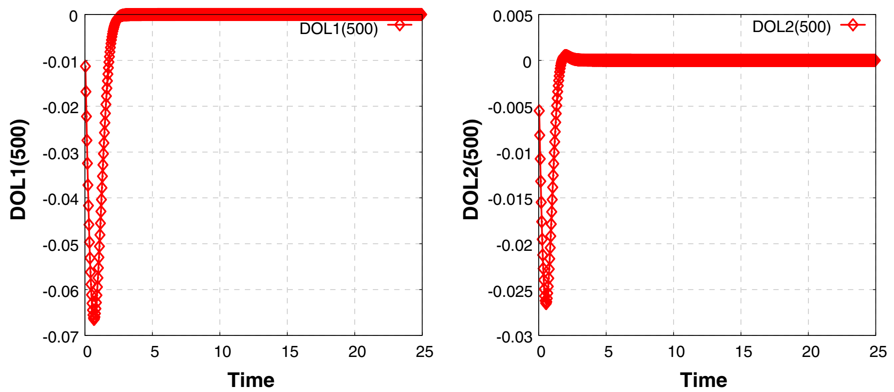


Fig. 2 The *left* figure shows a zoom on the evolution of the first component of the degree of liberty 500 in time and the *right* one is related to the second component

finite element $\mathbb{P}_2 - \mathbb{P}_1$. On the square domain $]0, 1[\times]0, 1[$, the numerical velocity and the pressure are taken as $(u, p) = (\text{curl } \psi, p)$, where:

$$\psi(t, x, y) = te^{-t^2(x+y)}y^2(1-y)^2\sin^2(\pi x) \text{ and } p(t, x, y) = te^{-t}\cos(2\pi x)\sin(2\pi y).$$

First of all, we have verified that our problem is stable in time. In fact, we have fixed the coarse grid to $N_c = 9$ points by edge of the square domain, so the fine one contains $N_f = N_c^{3/2} = 27$ points and $T = 100$ so that the number of iterations becomes $nbiter = T \times N_f$. We note that $H = \frac{1}{N_c}$ and $h = \frac{1}{N_f}$ are respectively the mesh-size of the coarse and fine grids. We present below the evolution of an arbitrary degree of liberty in time. The graphs are presented in Fig. 1.

In order to see precisely this evolution, we did a zoom on the velocity's components, in Fig. 2.

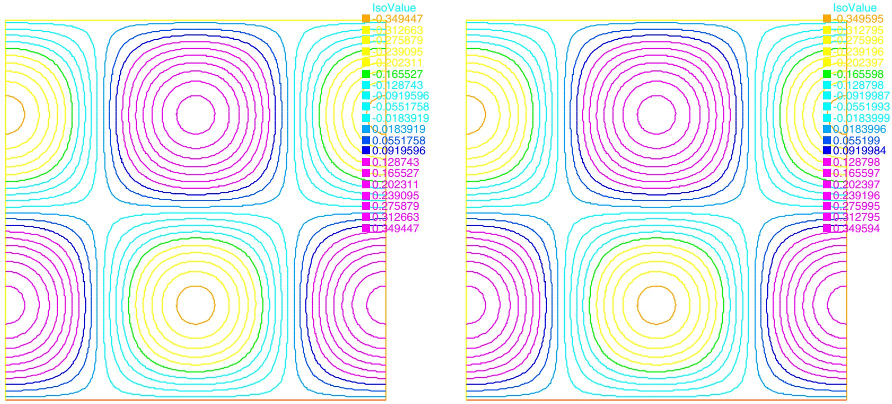


Fig. 3 *Left* and *right* figures show respectively the exact and numerical pressure's solutions

Next, we have taken $N_c = 25$, $T = 1$, $N_f^2 = N_c^3$, $\Delta t = h = \frac{1}{N_f}$ and $nbiter = T \times N_f$ and we have obtained a color comparison between the exact and numerical solutions of the pressure (Fig. 3).

The graphs related to the velocity's and pressure's error estimations have been studied. The values of these error estimations, in the logarithmic scale, are given in the following table:

Coarse and fine meshes-size	L^2 rate	H^1 rate	L^2 pressure rate
$H = 1/4, h = 1/8$	-1.9665	0.0392877	-1.91479
$H = 1/9, h = 1/27$	-3.91066	-1.78782	-2.99626
$H = 1/16, h = 1/64$	-4.91066	-2.44335	-3.74569
$H = 1/25, h = 1/125$	-5.50908	-2.75303	-4.32621

After a calculation by using these results, the $L^2(\Omega\times]0, T[)^2$ slope is of order 2.9674 and the $L^2(0, T; H^1(\Omega)^2)$ slope is of order 1.9769 and the pressure's one in norm $L^2(\Omega\times]0, T[)$ is of order 2.0199.

At the end of this paper, we show a comparison of this results with those obtained directly by the one fine grid, in Figs. 4 and 5 and we can see clearly the order of the velocity obtained by two-grid method in $L^2(\Omega\times]0, T[)^2$, $L^2(0, T; H^1(\Omega)^2)$ and the pressure in $L^2(\Omega\times]0, T[)$.

CPU comparison and advantage of the two-grid method

The goal of the two-grid strategy is to gain in time of computation of the solution. In what follows, we will show that the resolution of the Navier–Stokes problem by the two-grid method, a coarse grid $\mathcal{T}_H$ and a fine one $\mathcal{T}_h$, is less expensive than the resolution of this problem only on a fine grid $\mathcal{T}_h$.

In order to compare the time of computation between these two resolutions, we have chosen $N_c = 4, 9, 16, 25$, and $N_f^2 = N_c^3$. By noting t_{2G} and t_{1G} respectively the

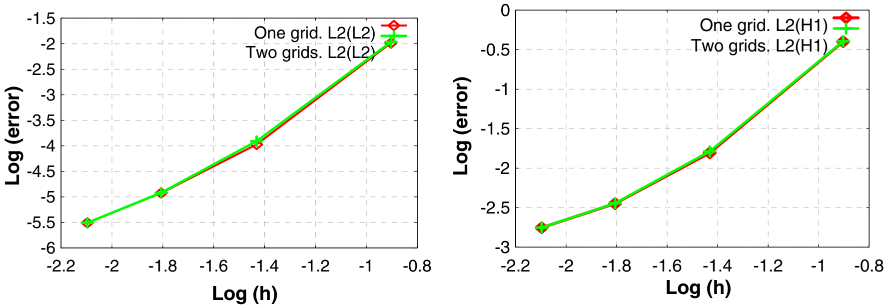
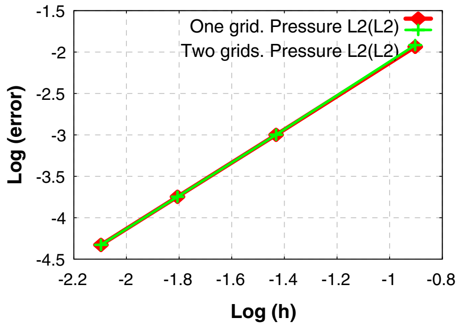


Fig. 4 Velocity's errors $L^2_{t,x}$ and $L^2_t(H^1_x)$ comparison

Fig. 5 Pressure's error $L^2_{t,x}$ comparison



computation time of the resolution of the problem by the two-grid strategy and by one grid only, we obtain Table 1.

As shown in Table 1, we notice that the resolution by the two-grid strategy is less expensive in time than the one by one fine grid. We also did a comparison in order of convergence of the velocity and the pressure. The results of the error estimates for the velocity and pressure computed by the resolution on one fine grid are as follows: the slope of the error of the velocity in $L^2(\Omega\times]0,T[)^2$ is of the order of 2.95001, the $L^2(0,T;H^1(\Omega)^2)$ slope is of order 1.97329 and the pressure's one in norm $L^2(\Omega\times]0,T[)$ is of order 2.0029, results that are similar to those of the resolution by the two-grid method. In Figs. 4 and 5, we present the comparison between the errors calculated by the two-grid strategy and by the resolution of the same problem on one fine grid. The errors are respectively in norm $L^2(\Omega\times]0,T[)^2$ and $L^2(0,T;H^1(\Omega)^2)$ for the velocity and in norm $L^2(\Omega\times]0,T[)$ for the pressure.

Table 1 CPU comparison

$N_f \times N_f$ points	8×8	27×27	64×64	125×125
t_{1G} (s)	7.25	304.953	13346.01	363402.083
t_{2G} (s)	4.89	196.25	7340.34	174433
$\frac{\lvert t_{2G}-t_{1G}\rvert}{t_{1G}}$ (%)	32.5	35.64	45	52

Remark Several numerical tests show that we have the same order of error estimations when $h = H^2$. The next step of this work will be to establish the theoretical error estimations for the velocity and pressure.

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