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Smoothness equivalence properties of univariate subdivision schemes and their projection analogues

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Abstract We study the following modification of a linear subdivision scheme S: let M be a surface embedded in Euclidean space, and P a smooth projection mapping onto M. Then the P-projection analogue of S is defined as $T := P \circ S$. As it turns out, the smoothness of the scheme T is always at least as high as the smoothness of the underlying scheme S or the smoothness of P minus 1, whichever is lower. To prove this we use the method of proximity as introduced by Wallner et al. (Constr Approx 24(3):289–318, 2006; Comput Aided Geom Design 22(7):593–622, 2005). While smoothness equivalence results are already available for interpolatory schemes S, this is the first result that confirms smoothness equivalence properties of arbitrary order for general non-interpolatory schemes.

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1 Introduction

The analysis of nonlinear subdivision schemes derived from linear schemes has been an active topic in the past few years. The main idea is to modify a linear subdivision scheme which is defined in Euclidean space so as to operate in nonlinear geometries. In [9,10] it is shown that if the difference between the linear scheme and its nonlinear modification is small enough, then the nonlinear scheme enjoys the same smoothness as the linear scheme. The condition for the nonlinear scheme to be close enough to the linear scheme is referred to as *proximity condition*. It is also known that for some natural nonlinear analogues of linear schemes that operate in nonlinear geometries, the nonlinear schemes satisfy proximity conditions with the linear schemes that ensure

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C^2 smoothness of the nonlinear scheme, provided the linear scheme is C^2 and some other technical conditions are met [5,9,11].

Up to now this was the strongest result concerning smoothness equivalence properties of general nonlinear schemes.

If the underlying linear subdivision scheme is interpolatory, then it can be shown that for a very general family of nonlinear analogues, the corresponding nonlinear scheme enjoys the same smoothness as the linear scheme [4,5,14,17].

In the present work we consider the following nonlinear perturbation of a linear subdivision scheme S that operates on a surface M embedded in some Euclidean space: first perform linear subdivision in Euclidean space, and then map the obtained data back onto M via a suitable projection mapping P. This yields the so called P-projection analogue $T := P \circ S$ of S.

We show that if S is of C^k smoothness, then so is T.

Our results for the first time confirm a general smoothness equivalence conjecture [2] for noninterpolatory subdivision schemes.

1.1 Contents

The outline of this paper is as follows: after briefly introducing some well-known definitions and results on linear subdivision schemes in Sect. 2, in Sect. 3 we define the projection analogue of a linear subdivision scheme and discuss some applications. Section 4 contains the main result, namely the (rather long) proof of a general proximity condition that is satisfied between a linear scheme and its projection analogue. Finally in Sect. 5 we use a result from [9] to show the implication "proximity ⇒ smoothness of the nonlinear scheme".

2 Linear subdivision

We briefly review some well-known facts from the theory of linear subdivision schemes. All the material in this section can be found in [8,3], where only the case of dilation factor 2 is considered. The extension to arbitrary dilation factors is easy.

Basically a convergent subdivision scheme takes a sequence of points as input and produces another, more dense, sequence of points. Iterative application of the subdivision operator yields a continuous curve in the limit. If this subdivision operator is a linear mapping, we speak of linear subdivision. We make this more precise: let $(a_i)_{i\in\mathbb{Z}}$ be a sequence of real numbers, only finitely many of them nonzero. Then we define the linear subdivision operator S associated with this sequence by

$$(Sp)_i = \sum_{j\in\mathbb{Z}} a_{i-Nj}\, p_j, \tag{1}$$

where $p_i \in \mathbb{R}^m$ are the input data. The sequence $(a_i)_{i\in\mathbb{Z}}$ is called the mask of S, the number $N > 1$ is called dilation factor. S is called of finite mask, since the support of the sequence (a_i) is finite.

A (linear or nonlinear) subdivision scheme is called convergent, if the sequences $p^l := S^l p$ converge to a continuous curve. This can be made precise as follows:

denote by $\mathcal{F}_l(p^l)$ the piecewise linear interpolation of the sequence p^l on the grid $\frac{1}{N^l}\mathbb{Z}$. Then S is said to be convergent, if the sequence $\mathcal{F}_l(p^l)$ converges uniformly to a curve $\mathcal{F}_\infty(p)(t)$. $\mathcal{F}_\infty(p)$ is called the limit function of S corresponding to the input data p. If the limit function is C^n for all p, we say that S is C^n.

A natural estimate for the nth derivative of the function $\mathcal{F}_\infty(p)$ sampled on the grid $\frac{1}{N^l}\mathbb{Z}$ is the left hand derivative of $\mathcal{F}_l(p^l)$. It is given by $N^{ln}\Delta^n S^l p$, where

$$(\Delta p)_i := p_{i+1} - p_i.$$

Note that

$$\Delta^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j E^{n-j} = (E - I)^n, \tag{2}$$

with E as the shift operator that maps a sequence $(p_i)_{i\in\mathbb{Z}}$ to the sequence $(p_{i+1})_{i\in\mathbb{Z}}$.

The derived schemes (provided they exist) satisfy the following relations:

$$S^{[0]} := S, \quad S^{[n]}\Delta := N\Delta S^{[n-1]},\ n \geq 1.$$

Consider the generating function $a(x) := \sum_{j\in\mathbb{Z}} a_i x^i$. The Laurent polynomial $a(x)$ is called the symbol of S. It is not hard to see that the existence of derived schemes up to order $n+1$ is equivalent to the fact that

$$a(1) = N \quad \text{and} \quad a^{(l)}(\zeta) = 0, \quad l = 0, \ldots, n, \quad \text{for all } \zeta^N = 1 \neq \zeta \tag{3}$$

These conditions also imply that

$$\sum_{j\in\mathbb{Z}} a_{i-Nj} = 1 \quad \text{for all } i.$$

The above property is called *reproduction of constants*. In what follows we will always assume that (3) holds and thus the derived schemes up to order $n+1$ exist. This assumption is actually not a big restriction. The following theorem is well known, see e.g. [8]:

Theorem 1 *If a linear subdivision scheme S produces C^k limit functions, then the derived schemes $S^{[1]}, \ldots, S^{[k+1]}$ exist, if S is nonsingular, i.e. nonzero sequences get mapped to nonzero functions.*

A (linear or nonlinear) subdivision operator operates on the space of sequences $p = (p_i \in \mathbb{R}^m)_{i\in\mathbb{Z}}$. We define a norm on this space via

$$\|p\| := \sup_{i\in\mathbb{Z}} \|p_i\|',$$

where $\|\cdot\|'$ refers to any norm on the finite dimensional vector space $\mathbb{R}^m$.

Note that this norm is finite only for a bounded sequence of points. However, we are only interested in smoothness properties of $\mathcal{F}_\infty(p)(t)$, which, near a given parameter value, depend only on a finite number of points of the initial sequence p. This is well known and follows from the fact that S is of finite mask. Therefore we can without loss of generality assume that p is bounded. We can even assume that p is only a finite sequence.

The following theorem is well known.

Theorem 2 *Let S be a linear subdivision scheme such that the derived schemes up to order $n+1$ exist. S produces C^n functions if and only if*

$$\mu_j := \frac{1}{N}\rho\left(S^{[j+1]}\right) < 1 \quad \textit{for all } j = 0, \ldots, n. \tag{4}$$

Here $\rho(S^{[j+1]})$ means the spectral radius of the operator $S^{[j+1]}$ with respect to the norm $\|\cdot\|$.

Actually for Theorem 2 to be valid it is necessary and sufficient that (4) holds only for $j = n$. If the subdivision scheme produces smooth limit functions, there is a better estimate for the constants μ_j:

Theorem 3 *Under the assumptions of Theorem 2*

$$\mu_j = \frac{1}{N} \quad \textit{for } j = 0, \ldots, n-1.$$

Proof It is easy to show that under the assumptions of Theorem 2, the derived schemes $S^{[j]}$ converge to the jth derivative of the limit function of S for $j = 1, \ldots, n$ (cf. [3]). Therefore there exists a constant A such that $\|(S^{[j]})^l\| \leq A$ for $l \in \mathbb{N}$ and all $j = 1, \ldots, n$. It follows that the spectral radius $\rho(S^{[j+1]}) \leq 1$ for $j = 1, \ldots, n$. Since $Sp = p$ for all constant sequences p, we get $\rho(S^{[j+1]}) = 1$.

We give an example: the Lane-Riesenfeld subdivision scheme of order n, which produces B-spline curves of degree n, is given by the symbol

$$a(x) = \frac{(1 + x + \cdots + x^{N-1})^{n+1}}{(Nx)^n}.$$

It satisfies (4) with $\mu_j = \frac{1}{N}$, $j = 0, \ldots, n$. By Theorem 2, the Lane-Riesenfeld scheme is C^n.

3 The projection analogue

The projection analogue of a linear subdivision scheme S is defined by first applying a linear subdivision scheme S followed by a projection mapping P. If linear subdivision is applied to data contained in a surface in some vector space $\mathbb{R}^m$, the result is in general not contained in that surface. If we combine linear subdivision with a projection-type mapping onto that surface, we obtain a refinement operator which creates data contained in the surface again. Such a projection mapping does not have to be globally defined, because we can reasonably assume that linear subdivision will create data which lie not too far away from the surface.

Definition 1 Let $P : O \subseteq \mathbb{R}^m \to \mathbb{R}^m$, O open in $\mathbb{R}^m$, be a smooth mapping with the property that $P|_{\mathrm{ran}(P)} = \mathrm{id}$ and there exists an ε-neighborhood of $\mathrm{ran}(P)$ contained in P's domain of definition, O. Then P is called a *Projection mapping*.

The P-projection analogue is defined by first applying the subdivision operator S to the points $(p_i \in \mathrm{ran}(P))_{i\in\mathbb{Z}}$ and then projecting onto the range of P via a projection mapping P. In other words, the P-projection analogue T of S is the subdivision scheme defined by

$$P(Sp), \quad p = (p_i \in \mathrm{ran}(P))_{i\in\mathbb{Z}}. \tag{5}$$

We would like to give some examples of what the mapping P might look like (compare also [12,10]):

3.1 Subdivision on surfaces

If $\mathcal{S} \subseteq \mathbb{R}^m$ is a surface embedded into Euclidean space $\mathbb{R}^m$, then it is possible to define a subdivision scheme that operates in $\mathcal{S}$: first we need a projection mapping $P : O \subseteq \mathbb{R}^m \to \mathcal{S}$, where O is an open set with $\mathcal{S} \subseteq O$. This mapping can be realized e.g. as a closest point projection, or if the surface is given as a level set $f(x_1, \ldots, x_n) = c$, we can let P equal the gradient flow. Then the P-projection analogue of any linear subdivision scheme operates in $\mathcal{S}$.

3.2 Avoiding obstacles

Another application is the following: assume that we are subdividing in Euclidean space $\mathbb{R}^m$. For certain applications it may be necessary for the subdivided curve to avoid an obstacle $\mathcal{O} \subseteq \mathbb{R}^m$. We define P as follows: if a point $p \in \mathbb{R}^m$ is not in $\mathcal{O}$, then P is just the identity mapping. If $p \in \mathcal{O}$, then P projects p onto the boundary of $\mathcal{O}$. $\mathcal{O}$ can for example be the interior of a closed surface $\mathcal{S} \subseteq \mathbb{R}^m$. Then the mapping P acts as the projection on $\mathcal{S}$ on the points in the interior of $\mathcal{S}$ and as the identity mapping on all the other points. Note that the mapping P is only continuous. Nevertheless the P-projection analogue T fulfills the lowest order proximity condition and limit curves enjoy C^1 smoothness [12].

3.3 Lie groups

We show how to define a projection mapping onto a compact subgroup G of the orthogonal group O_m. The scalar product

$$\langle x, y\rangle := \mathrm{trace}\big(x^T y\big)$$

corresponding to the Frobenius norm is both right- and left-invariant [1]. We can thus define P as the closest point projection onto G with respect to the Frobenius norm. We can extend this construction to the case that G is any compact matrix group, because after a suitable change of coordinates, we can think of G as a compact subgroup of O_m [1].

Now let us define a semidirect product $G \ltimes \mathbb{R}^m$, where G is a compact matrix group. After choosing suitable coordinates, we can assume that G is a subgroup of O_m. We can denote elements of $G \ltimes \mathbb{R}^m$ by $(m+1) \times (m+1)$-matrices of the form

$$\begin{pmatrix} 1 & 0 \\ t & g \end{pmatrix},$$

with $t \in \mathbb{R}^m$ and $g \in G$. We consider the group operation

$$\begin{pmatrix} 1 & 0 \\ t_1 & g_1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ t_2 & g_2 \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ t_1 + g_1 t_2 & g_1 g_2 \end{pmatrix}.$$

If G is equal to SO_m, then $G \ltimes \mathbb{R}^m$ is just the Euclidean motion group SE_m. A short computation shows that the Frobenius norm on $G \ltimes \mathbb{R}^m$ induced by the Frobenius norm on $\mathbb{R}^{(m+1)\times(m+1)}$ is right invariant, but not left invariant. However, this does not prevent us from defining P as the closest point projection onto $G \ltimes \mathbb{R}^m$ with respect to the Frobenius norm. If a matrix of the form

$$\begin{pmatrix} 1 & 0 \\ t & h \end{pmatrix}, \quad h \in \mathrm{GL}_m$$

is given, then its closest point projection onto $G \ltimes \mathbb{R}^m$ is given by

$$\begin{pmatrix} 1 & 0 \\ t & g \end{pmatrix}, \quad g = P(h),$$

where P is now the closest point projection onto G.

If $G = \mathrm{O}_m$, we can compute the closest point projection explicitly: let $h \in \mathrm{GL}_m$, with the singular value decomposition $h = v^T \Sigma w$, where $v, w \in \mathrm{O}_m$. The closest point projection onto O_m of h is now given by $P(h) = v^T w$ [7].

A different kind of projection analogue has been studied in [9,10,12], where smoothness equivalence results up to C^2 have been obtained. In [14] smoothness equivalence has been proven for interpolatory subdivision schemes if P is the closest point projection onto the sphere and related manifolds. In [5] this result is extended to arbitrary mappings P and multivariate subdivision schemes. For non-interpolatory subdivision schemes, however, the result in the present paper is the first one that establishes smoothness equivalence for arbitrary smoothness order.

4 Establishing a proximity condition

Our analysis considers the nonlinear scheme T as a perturbation of the linear scheme S. In this spirit we define

Definition 2 A linear scheme S and its P-projection analogue T satisfy a *local proximity condition of order* n if for every compact set $K \subseteq \operatorname{ran}(P)$ there exist constants $\delta > 0$, C such that for all initial data $p \in K^{\mathbb{Z}}$ with $\|\Delta p\| < \delta$ we have the estimate

$$\|\Delta^{n-1}(Tp - Sp)\| \leq C \sum_{\substack{\alpha_1,\ldots,\alpha_n \in \mathbb{N}_0 \\ \alpha_1+2\alpha_2+\cdots+n\alpha_n=n+1}} \|\Delta p\|^{\alpha_1} \cdots \|\Delta^n p\|^{\alpha_n}. \tag{6}$$

The goal of the present section is to prove the following theorem:

Theorem 4 *Let S be a linear subdivision scheme such that the derived schemes $S^{[j]}$, $j = 1, \ldots, n+1$ exist, and let $P : O \to \mathbb{R}^m$ be a C^{n+1} projection mapping. Then S satisfies a local proximity condition of order n with its P-projection analogue T*

The nonlinear scheme T is only defined if the subdivided data Sp lies within O. This can always be achieved by choosing initial data p with $\|\Delta p\|$ small:

Lemma 1 *Let O be such that an ε neighborhood of* $\operatorname{ran}(P)$ *is still contained in O. Then there exists $\delta > 0$ such that for initial data p with $\|\Delta p\| \leq \delta$, the subdivided data Sp remains within O. The constant δ depends on S and on ε but not on p.*

Proof It is well known (compare [3]) that for a linear subdivision scheme S there exists a constant $C > 0$ such that

$$\sup_{i\in\mathbb{Z}} \inf_{j\in\mathbb{Z}} \|Sp_i - p_j\|' \leq C\|\Delta p\|.$$

This implies our statement.

The proof of Theorem 4 takes up the whole section. In the first subsection, we derive a tractable condition for a proximity condition to hold. In the second subsection, we prove that this condition is always fulfilled if the derived schemes up to order $n+1$ exist.

4.1 Expressing the differences through a generating function

In order to obtain a proximity condition as required by Theorem 4, we first look for a way to rewrite the difference $\Delta^{n-1}(Sp - Tp)$. At the end of this section, we will see that a proximity condition holds if a certain generating function has many vanishing derivatives (compare also [4–6]). The checking of this second fact then is the topic of Sect. 4.3.

We start with the following lemma:

Lemma 2 *Let S be a linear subdivision scheme with mask $(a_i)_{i\in\mathbb{Z}}$ and T its P-projection analogue. Let $K \subseteq \operatorname{ran}(P)$ be a compact set. Then there exists $\delta > 0$ (depending only on K) such that for all initial data $(p_i)_{i\in\mathbb{Z}} \in K^{\mathbb{Z}}$ with $\|\Delta p\| < \delta$ and $v_j := p_j - p_i$*

$$\Delta^{n-1}(Tp - Sp)_i = \sum_{l=2}^{n} \frac{1}{l!} B_{l,i} + O\big(\|\Delta p\|^{n+1}\big), \tag{7}$$

where

$$B_{l,i} = \sum_{\mathbf{j}=(j_1,\ldots,j_l)\in\mathbb{Z}^l} \big(\Delta^{n-1} b_{l,\mathbf{j}}\big)_i d^l P|_{p_i} \left(v_{j_1}, \ldots, v_{j_l}\right) \tag{8}$$

and

$$b_{l,\mathbf{j}} = \begin{cases} (a^l_{i-Nj_1} - a_{i-Nj_1})_{i\in\mathbb{Z}} & \text{if } j_1 = \cdots = j_l, \\ (a_{i-Nj_1} \cdots a_{i-Nj_l})_{i\in\mathbb{Z}} & \text{else.} \end{cases} \tag{9}$$

The Δ^{n-1} operator acts on the index i and d^l denotes the lth order differential.

Proof The proof is done by Taylor expansion. We have $Tp = P(Sp)$ with $p \in \operatorname{ran}(P)$ and clearly, $Sp = S(P(p))$. Therefore, we can write

$$(Tp - Sp)_{i+h} = (P(Sp) - S(P(p)))_{i+h} = P\left(\sum_{j\in\mathbb{Z}} a_{i+h-Nj} p_j\right) - \sum_{j\in\mathbb{Z}} a_{i+h-Nj} P(p_j), \;\; h=0,\ldots,N-1. \tag{10}$$

By compactness of K, there exists $\delta' > 0$ such that for all v with $\|v\| \le \delta'$ and all $p \in K$ we can write P in its Taylor expansion (recall that we assumed that $P \in C^{n+1}$) as

$$P(p+v) = \sum_{l=0}^{n} \frac{1}{l!} d^l P|_p(v, \ldots, v) + O\big(\|v\|'^{n+1}\big). \tag{11}$$

Since the mask of S is finite, only a finite number (depending only on the support of the mask) of initial data points contributes to the computation of $(Sp)_{i+h}$ and $(Tp)_{i+h}$, $h = 0, \ldots, N-1$, and hence $(\Delta^{n-1}(S - T)p)_i$. Therefore we may assume that p is only a finite sequence and thus

$$\sup_j \left\| p_j - p_i \right\|' = O\left(\|\Delta p\|\right)$$

and

$$\sup_i \left\| \sum_j a_{i-Nj} \left(p_j - p_i\right) \right\|' = O\left(\|\Delta p\|\right).$$

The implicit constants only depend on the support size of the mask of S and the operator norm of S. It follows that there exists $\delta > 0$ such that for all sequences $p \in K^{\mathbb{Z}}$ with $\|\Delta p\| < \delta$ we can write

$$p_j = P(p_j) = \sum_{l=0}^{n} \frac{1}{l!} d^l P|_{p_i} \left(p_j - p_i, \ldots, p_j - p_i\right) + O\left(\|p_j - p_i\|'^{n+1}\right),$$

and thus

$$\begin{aligned}(Sp)_{i+h} &= \sum_{j\in\mathbb{Z}} a_{i+h-Nj} p_j \\ &= \sum_{j\in\mathbb{Z}} a_{i+h-Nj} \sum_{l=0}^{n} \frac{1}{l!} d^l P|_{p_i} \left(p_j - p_i, \ldots, p_j - p_i\right) + O\left(\|v\|^{n+1}\right) \\ &= \sum_{l=0}^{n} \sum_{j\in\mathbb{Z}} a_{i+h-Nj} \frac{1}{l!} d^l P|_{p_i} \left(v_j, \ldots, v_j\right) + O\left(\|v\|^{n+1}\right), \end{aligned} \tag{12}$$

with $v_j := p_j - p_i$ and $v = (v_j)_{j\in\mathbb{Z}}$.

The nonlinear scheme can be written as

$$\begin{aligned}(Tp)_{i+h} &= P\left(\sum_{j\in\mathbb{Z}} a_{i+h-Nj} p_j\right) = P\left(p_i + \sum_{j\in\mathbb{Z}} a_{i+h-Nj} v_j\right) \\ &= \sum_{l=0}^{n} \frac{1}{l!} d^l P|_{p_i} \left(\sum_{j\in\mathbb{Z}} a_{i+h-Nj} v_j, \ldots, \sum_{j\in\mathbb{Z}} a_{i+h-Nj} v_j\right) \\ &\quad + O\left(\|\Delta p\|^{n+1}\right). \end{aligned} \tag{13}$$

(recall that $\sum_{j\in\mathbb{Z}} a_{i+h-Nj} = 1$ and use (11)). For $l = 0, \ldots, n$ we consider the expression

$$\begin{aligned} & d^l P|_{p_i} \left(\sum_{j\in\mathbb{Z}} a_{i+h-Nj} v_j, \ldots, \sum_{j\in\mathbb{Z}} a_{i+h-Nj} v_j\right) \\ & - \sum_{j\in\mathbb{Z}} a_{i+h-Nj} d^l P|_{p_i} \left(v_j, \ldots, v_j\right). \end{aligned} \tag{14}$$

Observe that by the multilinearity of $d^l P|_{p_i}$ the expression (14) equals

$$\sum_{\mathbf{j}\in\mathbb{Z}^l} \left(b_{l,\mathbf{j}}\right)_{i+h} d^l P|_{p_i} \left(v_{j_1}, \dots, v_{j_l}\right).$$

With (12) and (13), (10) becomes

$$(Tp - Sp)_{i+h} = \sum_{l=0}^{n} \frac{1}{l!} \sum_{\mathbf{j}\in\mathbb{Z}^l} \left(b_{l,\mathbf{j}}\right)_{i+h} d^l P|_{p_i} \left(v_{j_1}, \dots, v_{j_l}\right) + O\left(\|\Delta p\|^{n+1}\right).$$

It is easily seen from the reproduction of constants that the terms corresponding to $l = 0, 1$ vanish. We therefore have

$$(Tp - Sp)_{i+h} = \sum_{l=2}^{n} \frac{1}{l!} \sum_{\mathbf{j}\in\mathbb{Z}^l} \left(b_{l,\mathbf{j}}\right)_{i+h} d^l P|_{p_i} \left(v_{j_1}, \dots, v_{j_l}\right) + O\left(\|\Delta p\|^{n+1}\right).$$

It remains to apply the linear operator Δ^{n-1} to arrive at the desired result.

We show how to establish a proximity condition by rewriting the expressions $B_{l,i}$.

Lemma 3 *Let the assumptions be as in Lemma 2. If for any $l = 2, \dots, n,\ i \in \mathbb{Z}$ it is possible to express $B_{l,i}$ in the form*

$$\sum_{\beta\in\Gamma_l} \sum_{\mathbf{j}=(j_1,\dots,j_l)\in\mathbb{Z}^l} c_{\mathbf{j},\beta} d^l P|_{p_i} \left(\Delta^{\beta_1} v_{j_1}, \dots, \Delta^{\beta_l} v_{j_l}\right) \tag{15}$$

with $\Gamma_l := \{(\beta_1, \dots, \beta_l) \in \mathbb{N}_0^l : \sum_{i=1}^l \beta_i = n + 1\}$ and multivariate, finite sequences $c_{\mathbf{j},\beta}$, then the P-projection analogue T of S and S satisfy a local proximity condition of order n in the sense of Definition 2.

Proof Since the sequence p is K-valued and K is compact, there exists a uniform constant C_1 such that

$$\|d^l P|_{p_i}(v_1, \dots, v_l)\|' \leq C_1 \|v_1\|' \cdot \ldots \cdot \|v_l\|'$$

for all vectors $v_1, \dots, v_l$ and $i \in \mathbb{Z}$. With

$$C_{2,\beta} := |\{\mathbf{j} : c_{\mathbf{j},\beta} \neq 0\}| \max_{\mathbf{j}} |c_{\mathbf{j},\beta}|$$

it follows that

$$\|(15)\|' \leq \sum_{\beta\in\Gamma_l} C_1 C_{2,\beta} \left\|\Delta^{\beta_1} v\right\| \cdot \ldots \cdot \left\|\Delta^{\beta_l} v\right\|,$$

v being the sequence $(p_j - p_i)_{j\in\mathbb{Z}}$. Since $l \geq 2$, $\|v\| = O(\|\Delta p\|)$, $\|\Delta^r v\| = O(\|\Delta^r p\|)$ and $\|\Delta^{n+1} p\| \leq 2\|\Delta^n p\|$, there exists a constant C_3 such that

$$\|(15)\|' \leq C_3 \sum_{\beta\in\Gamma_l,\beta_k\leq n\forall k} \left\|\Delta^{\beta_1} p\right\| \cdot \ldots \cdot \left\|\Delta^{\beta_l} p\right\|.$$

By setting $\alpha_r := |\{j : \beta_j = r\}|$, we obtain an estimate of the form

$$\|(15)\|' \leq C_4 \sum_{\sum_{r=1}^{n+1} r\alpha_r = n+1} \|\Delta p\|^{\alpha_1} \cdot \ldots \cdot \left\|\Delta^n p\right\|^{\alpha_n}.$$

By Lemma 11 we get for any i a constant such that

$$\left\|\Delta^{n-1} p_i\right\|' \leq C_i \sum_{\sum_{r=1}^{n+1} r\alpha_r = n+1} \|\Delta p\|^{\alpha_1} \ldots \left\|\Delta^n p\right\|^{\alpha_n}.$$

By shift-invariance it suffices to only consider the values $i = 0, \ldots, N-1$. This proves the statement.

Our goal is to rewrite $B_{l,i}$ in the form (15). Note that this is a purely algebraic problem. We first derive a necessary and sufficient condition for such a rewriting rule to exist. To do this we make some definitions. With $v_j := p_j - p_i$ let

$$\mathcal{P}_l := \left\{ \sum_{\mathbf{j}\in\mathbb{Z}^l} e_{\mathbf{j}} d^l P|_{p_i} \left(v_{j_1}, \ldots, v_{j_l}\right) : \ e_{\mathbf{j}} = 0 \text{ for almost all } \mathbf{j} \right\}.$$

$\mathcal{P}_l$ is the set of all formal expressions of the form (8). Of course $B_{l,i}$ is in $\mathcal{P}_l$. Define a mapping

$$\Phi : \begin{cases} \mathcal{P}_l & \rightarrow \ \mathbb{R}\left[x_1, x_1^{-1}, \ldots, x_l, x_l^{-1}\right] \\ \sum_{\mathbf{j}\in\mathbb{Z}^l} e_{\mathbf{j}} d^l P|_{p_i} \left(v_{j_1}, \ldots, v_{j_l}\right) & \mapsto \ \sum_{\mathbf{j}\in\mathbb{Z}^l} e_{\mathbf{j}} x_1^{j_1} \cdots x_l^{j_l}. \end{cases}$$

We impose a ring structure on $\mathcal{P}_l$ such that Φ is a ring isomorphism. Multiplication by $(1 - x_r)^\alpha$ in the space $\mathbb{R}\left[x_1, x_1^{-1} \cdots, x_l, x_l^{-1}\right]$ corresponds to the mapping

$$\sum_{\mathbf{j}\in\mathbb{Z}^l} e_{\mathbf{j}} d^l P|_{p_i}(v_{j_1}, \ldots, v_{j_l}) \mapsto \sum_{\mathbf{j}\in\mathbb{Z}^l} e_{\mathbf{j}} d^l P|_{p_i}(v_{j_1}, \ldots, \Delta^\alpha v_{j_r}, \ldots, v_{j_l}),$$

as seen from (2). It follows that a rewriting rule of the form (15) exists if and only if the generating function $\tilde{B}_{l,i}(x_1, \ldots, x_l) = \Phi(B_{l,i})$ can be written in the form

$$\sum_{\beta_1+\cdots+\beta_l=n+1} c_\beta(x_1, \ldots, x_l)(1 - x_1)^{\beta_1} \cdots (1 - x_l)^{\beta_l}, \tag{16}$$

for some Laurent polynomials $c_\beta \in \mathbb{R}\left[x_1, x_1^{-1}, \dots, x_l, x_l^{-1}\right]$. Applying Taylor's formula yields a necessary and sufficient condition for (15), namely that all derivatives of $\tilde{B}_{l,i}(x_1, \dots, x_l)$ of order $\le n$ are equal to zero for $(x_1, \dots, x_l) = (1, \dots, 1)$. We summarize:

Lemma 4 *If all derivatives of order* $\le n$ *of the Laurent polynomials* $\tilde{B}_{l,i}(x_1, \dots, x_l) = \Phi(B_{l,i})$, $l = 2, \dots, n$ *vanish at the point* $(1, \dots, 1)$*, then* S *and* T *satisfy a local proximity condition of order* n*. We have*

$$\tilde{B}_{l,i}(x_1, \dots, x_l) = \Delta^{n-1}\left(\left(\sum_{j\in\mathbb{Z}} a_{i-Nj}x_1^j\right)\cdots\left(\sum_{j\in\mathbb{Z}} a_{i-Nj}x_l^j\right) - \left(\sum_{j\in\mathbb{Z}} a_{i-Nj}x_1^j\cdots x_l^j\right)\right)_i. \tag{17}$$

Proof By the previous discussion it follows that the vanishing of derivatives of the Laurent polynomials $\tilde{B}_{l,i}(x_1, \dots, x_l)$ implies that a rewriting rule of the form (15) for the expressions $B_{l,i}$ exists. If we apply Lemma 3 we get the first claim that a local proximity condition of order n holds between S and T. What remains to show is that (17) holds. Thus, we have to verify that

$$\Phi(B_{l,i}) = \Delta^{n-1}\left(\left(\sum_{j\in\mathbb{Z}} a_{i-Nj}x_1^j\right)\cdots\left(\sum_{j\in\mathbb{Z}} a_{i-Nj}x_l^j\right) - \left(\sum_{j\in\mathbb{Z}} a_{i-Nj}x_1^j\cdots x_l^j\right)\right)_i.$$

This follows from (9) and the fact that $\Phi(\Delta E_i) = \Delta\Phi(E_i)$ for every $E_i \in \mathcal{P}_l$. Again, we let Δ act on the index i.

4.2 Combinatorics of generating functions

The goal of this section is to verify the condition given in Lemma 4. This will be done by a series of lemmas which are subsequently used in the proof of Theorem 4 in Sect. 4.3. We would like to stress that this section may be read without any connection to nonlinear subdivision. It contains purely algebraic results for the symbols of linear schemes which are in our opinion interesting in their own right. It is easy to derive the following identity (ζ is a primitive Nth root of the unit):

$$q_k(x) := \sum_{j\in\mathbb{Z}} a_{k-Nj}x^{k-Nj} = \frac{1}{N}\left(a(x) + \sum_{r=1}^{N-1} \zeta^{-kr} a\left(\zeta^r x\right)\right). \tag{18}$$

It will turn out that the crux of the proof is to show that all derivatives of $\tilde{B}_{l,i}(x_1, \dots, x_l)$ vanish. First we look at the moments of the sequences $(a_{k+Nj})_{j\in\mathbb{Z}}$. To do that, we use a well-known fact from the theory of generating functions [13]:

Lemma 5 *Let $q(x) := \sum_{j\in\mathbb{Z}} q_j x^j$ be the generating function of the sequence $(q_j)_{j\in\mathbb{Z}}$. Then the generating function of the sequence $(j^l q_j)_{j\in\mathbb{Z}}$ is given by*

$$\left(x\frac{\partial}{\partial x}\right)^l q(x). \tag{19}$$

From (18) and Lemma 5 we can deduce the following identities for the first two moments of $(a_{k+Nj})_{j\in\mathbb{Z}}$:

$$N^2 \sum_{j\in\mathbb{Z}} j a_{k-Nj} = kN - a^{(1)}(1) \tag{20}$$

$$N^3 \sum_{j\in\mathbb{Z}} j^2 a_{k-Nj} = k^2 N - (2k-1)a^{(1)}(1) + a^{(2)}(1). \tag{21}$$

Lemma 6 *Let $a(x)$ be a Laurent polynomial satisfying (3) and define $p_r(k) := \sum_{j\in\mathbb{Z}} j^r a_{k-Nj}$. Then the following holds: $p_r(k)$ is a polynomial of degree r in k for $0 \le r \le n$. There exist constants C_r with*

$$(-N)^n p_r(k) = -\sum_{l=1}^{r} \binom{r}{l} i^l (-N)^{r-l} p_{r-l}(k) + C_r. \tag{22}$$

The leading terms of $p_r(k)$ are given by

$$\frac{1}{N^r}k^r \quad \text{and} \quad -\frac{r}{N^{r+1}}a^{(1)}(1)k^{r-1}. \tag{23}$$

Proof In view of (18) and Lemma 5, the following holds:

$$\sum_{j\in\mathbb{Z}} (k-Nj)^r a_{k-Nj} = \left(x\frac{\partial}{\partial x}\right)^r \left(\frac{1}{N}\left(a(x) + \sum_{r=1}^{N} \zeta^{-kr} a(\zeta^r x)\right)\right)\Bigg|_{x=1}. \tag{24}$$

From the assumption that the derived schemes of orders up to $n+1$ exist, it follows by (3) that the right hand side of (24) is a constant, i.e., does not depend on k. We denote it by C_r. The binomial formula shows that

$$(-N)^r p_r(k) = -\sum_{l=1}^{r} \binom{r}{l} k^l (-N)^{r-l} p_{r-l}(k) + C_r. \tag{25}$$

It remains to show the statement about the two leading terms of p_r. We use induction on r, the cases $r = 1, 2$ following from (20) and (21). Let us first show that the leading coefficient of $p_r(k)$ equals $\frac{1}{N^r}$. By the induction hypothesis, this holds for all $p_{r-l}(k)$,

$l \geq 1$. Thus the leading coefficient of $p_r(k)$ is given by

$$-(-N)^{-n}\sum_{l=1}^{r}\binom{r}{l}(-N)^{r-l}N^{-(r-l)} = \frac{1}{N^r}(-1)^{r+1}\sum_{l=1}^{r}\binom{r}{l}(-1)^{r-l}. \quad (26)$$

It is not hard to see that (26) equals $\frac{1}{N^r}$: use the formula

$$x^r = (x-1+1)^r = \sum_{l=0}^{r}\binom{r}{l}(x-1)^{r-l}, \quad (27)$$

and substitute $x = 0$. This gives $\sum_{l=1}^{r}\binom{r}{l}(-1)^{r-l} = -(-1)^r = (-1)^{r+1}$. Putting this into (26) gives the desired result.

It remains to prove that the term of order $r-1$ in $p_r(k)$ is given by $\frac{-r}{N^{r+1}}a^{(1)}(1)k^{r-1}$:

We start with (25) and extract the terms of order $r-1$. Using our induction assumption we come up with

$$(-N)^{-r}a^{(1)}(1)\sum_{l=1}^{r}\binom{r}{l}(-N)^{r-l}\frac{(r-l)}{N^{r-l+1}}.$$

If we differentiate (27) and let $x = 0$ (assuming that $r \geq 2$), we see that

$$\sum_{l=1}^{r}\binom{r}{l}(r-l)(-1)^{r-l} = r(-1)^{r+1}.$$

Now the proof is complete.

Remark 1 The left hand side of (23) is actually well known: if the derived schemes up to order $n+1$ exist, then the subdivision scheme S generates polynomials up to degree n. $p_r(k)$ can be viewed as taking the integer samplings of the polynomial k^r as input and applying the linear subdivision scheme S. From the polynomial generation property of S it immediately follows that $p_r(k)$ is a polynomial, and it is also well known that the term of degree r is also reproduced [8].

Remark 2 It would not be difficult to make the constants C_r explicit using Stirling numbers (which describe the base change from the monomial basis to the basis consisting of polynomials $Q_u(x) := x(x-1)\cdots(x-u+1)$ c.f. [13]). We do not need this for our analysis.

The next well-known lemma describes some elementary properties of the Stirling numbers S_i^j. We will use this result later. The proof is an easy exercise.

Lemma 7 *Let* $Q_u(x) = x(x-1)\cdots(x-u+1)$. *Then*

$$Q_u(x) = S_u^u x^l + S_{u-1}^u x^{u-1} + \cdots + S_0^u,$$

with $S_u^u = 1$ *and* $S_{u-1}^u = -\frac{u(u-1)}{2}$.

We continue to collect lemmas which contribute to our final proximity result.

Lemma 8 *Let $a(x)$ be a Laurent polynomial with (3) and $u_1 + \cdots + u_l = n$. Define*

$$E(k) := \left(\sum_{j\in\mathbb{Z}} Q_{u_1}(j) a_{k-Nj} \right) \cdots \left(\sum_{j\in\mathbb{Z}} Q_{u_l}(j) a_{k-Nj} \right) \quad \textit{and} \tag{28}$$

$$F(k) := \sum_{j\in\mathbb{Z}} Q_{u_1}(j) \cdots Q_{u_l}(j) a_{k-Nj}. \tag{29}$$

Then $E(k)$ and $F(k)$ are polynomials of degree n in k. The terms of degree n and $n-1$ of $E(k)$ and $F(k)$ agree.

Proof The fact that $E(k)$ and $F(k)$ are polynomials of degree n is a direct consequence of Lemma 6. From Lemma 7, we obtain

$$F(k) = \sum_{j\in\mathbb{Z}} \left(j^{u_1} - \frac{u_1(u_1-1)}{2} j^{u_1-1} + \cdots \right) \cdots \left(j^{u_l} - \frac{u_l(u_l-1)}{2} j^{u_l-v1} + \cdots \right) a_{k-Nj}, \tag{30}$$

where the dots indicate terms of lower order which do not contribute to the two leading terms of $F(k)$. Multiplying the polynomials in (30) yields

$$F(k) = \sum_{j\in\mathbb{Z}} \left(j^n - \left(\frac{u_1(u_1-1)}{2} + \cdots + \frac{u_l(u_l-1)}{2} \right) j^{n-1} + \cdots \right) a_{k-Nj}.$$

It follows that

$$F(k) = p_n(k) - \left(\frac{u_1(u_1-1)}{2} + \cdots + \frac{u_l(u_l-1)}{2} \right) p_{n-1}(k) + s(k),$$

where $s(k)$ is a polynomial of degree $\leq n-2$ in k. Now we use Lemma 6 to conclude that

$$F(k) = \frac{1}{N^n} k^n - \left(\left(\frac{u_1(u_1-1)}{2} + \cdots + \frac{u_l(u_l-1)}{2} \right) \frac{1}{N^{n-1}} + \frac{n}{N^{n+1}} a^{(1)}(1) \right) k^{n-1} + \cdots, \tag{31}$$

where the dots indicate terms of order $\leq n-2$. We have to check that $E(k)$ is of the same form: using Lemma 7, $E(k)$ can be written as

$$\prod_{r=1}^{l} \sum_{j\in\mathbb{Z}} \left(j^{u_r} - \frac{u_r(u_r-1)}{2} j^{u_r-1} + \cdots \right) a_{k-Nj}.$$

It follows that

$$E(k) = \prod_{r=1}^{l} \left(p_{u_r}(k) - \frac{u_r(u_r-1)}{2} p_{u_r-1}(k) + \cdots \right).$$

If we use (23), we see that also $E(k)$ has the form (31) and the proof is complete.

4.3 Proof of the proximity condition

Now we are ready to sum up the proof of Theorem 4.

Proof of Theorem 4 In view of Lemma 4 we need to verify that if we apply a differential operator $D := \frac{\partial^n}{(\partial x_1)^{u_1} \cdots (\partial x_l)^{u_l}}$, $u_1 + \cdots + u_l = n$ to

$$\left(\sum_{j \in \mathbb{Z}} a_{i-Nj} x_1^j \right) \cdots \left(\sum_{j \in \mathbb{Z}} a_{i-Nj} x_l^j \right) - \left(\sum_{j \in \mathbb{Z}} a_{i-Nj} x_1^j \cdots x_l^j \right), \tag{32}$$

then applying the operator Δ^{n-1} in i and evaluating at $(x_1, \ldots, x_l) = (1, \ldots, 1)$ yields zero for $l = 1, \ldots, n$. In the notation of Lemma 8, the expression we get if we apply the operator D to (32), and evaluate at $(x_1, \ldots, x_l) = (1, \ldots, 1)$ is

$$T(i) := F(i) - E(i).$$

From Lemma 8 and the fact that the derived schemes of S of order $\leq n+1$ exist we know that $T(i)$ is a polynomial of degree $\leq n-2$ in i. The operator Δ^{n-1} annihilates all polynomials of degree less than $n-1$, and thus $\Delta^{n-1} T(i) = 0$. Therefore the condition (17) is satisfied and we can use Lemma 4. This completes the proof.

5 Smoothness equivalence

Having established a general proximity condition in the previous section, we now draw on results from [9] to show smoothness equivalence between a linear scheme and its projection analogue.

Theorem 5 *Let S be a linear subdivision scheme with derived schemes $S^{[1]}, \ldots, S^{[n+1]}$ and P a C^{n+1} projection mapping. Then the P-projection analogue T of S produces C^n limit curves for all initial data $p = (p_i)_{i \in \mathbb{Z}}$ such that $T^l p$ converges.*

Proof Since the mask of S is finite, the limit function $\mathcal{F}_\infty(p)$ locally depends only on a finite number of initial data points p_i. It is therefore no restriction to assume that p takes its values in a compact set K. By the assumption that T is convergent, we may also without loss of generality assume that $\|\Delta p\| < \delta$ for any $\delta > 0$. From Theorem 4 we get a proximity condition of the form

$$\left\| \Delta^{n-1}(S-T)p \right\| \leq C \sum_{\alpha_1+\cdots+n\alpha_n=n+1} \|\Delta p\|^{\alpha_1} \cdots \left\| \Delta^n p \right\|^{\alpha_n}$$

which holds on compact sets and initial data p with $\|\Delta p\|$ small enough. Define $\mu_i := \frac{1}{N}\rho(S^{[i]}) < 1$, $i = 1, \ldots, n+1$. The results of [9] say that if the contractivity constants $\mu_1, \ldots, \mu_n < 1$ satisfy

$$(\mu_1)^{\alpha_1} \left(\frac{\mu_2}{N}\right)^{\alpha_2} \cdots \left(\frac{\mu_n}{N^{n-1}}\right)^{\alpha_n} < \frac{1}{N^n}$$

for all n-tuples $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 + \cdots + n\alpha_n = n+1$, then T is C^n. By Theorem 3 this is always satisfied and that concludes the proof.

5.1 Discussion

The statement "such that $T^l p$ converges" in Theorem 5 means that in general we have to start with a sequence p with $\delta := \|\Delta p\|$ small. From the results in [10] it follows that we can choose δ so that $1 + 2C\delta < 1$, where C is chosen such that $\|Sp - Tp\| \leq C\|\Delta p\|^2$. The constant δ is typically rather small (compare also the discussion in [10,11]).

Only for special schemes and certain nonlinear analogues (geodesic averaging) it is possible to show that the nonlinear scheme converges for all initial data [15].

It has been reported [17] that it is surprising that the smoothness equivalence can be derived solely by algebraic manipulations of the mask coefficients, i.e. the smoothness of the nonlinear subdivision scheme does not depend on the "amount" of nonlinearity. For the present paper, this is not entirely true in the following sense: for any nonlinear analogue smoothness equivalence is true for sequences p with $\delta := \|\Delta p\|$ such that $1 + 2C\delta < 1$. The constant C comes from the proximity condition and is directly related to the nonlinearity of the subdivision scheme. Even if this bound is in all probability far from sharp, it indicates that the more nonlinear our subdivision scheme is, the more dense our initial point sequence has to be.

There exist results where the smoothness of a nonlinear subdivision scheme depends on the initial data [16]. These nonlinear schemes are, however, quite different from the ones considered here, in that they are not constructed from linear schemes.

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