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Hölder estimates for Green's functions on convex polyhedral domains and their applications to finite element methods

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Abstract A model second-order elliptic equation on a general convex polyhedral domain in three dimensions is considered. The aim of this paper is twofold: First sharp Hölder estimates for the corresponding Green's function are obtained. As an applications of these estimates to finite element methods, we show the best approximation property of the error in W^1_∞. In contrast to previously known results, W^2_p regularity for $p > 3$, which does not hold for general convex polyhedral domains, is not required. Furthermore, the new Green's function estimates allow us to obtain localized error estimates at a point.

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1 Introduction

In this paper we consider the model second-order elliptic problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a convex polyhedral domain in three dimensions and f is a smooth function. It is well known that for the above problem there exists a unique solution in $H^2(\Omega)$ (cf. [12]).

Let S_h be a finite dimensional subspace of $H_0^1(\Omega)$ composed of piecewise polynomials of degree k on a quasi-uniform mesh of size h and $u_h \in S_h$ be the finite element approximation to u,

$$(\nabla u_h, \nabla \chi)_\Omega = (\nabla u, \nabla \chi)_\Omega, \quad \forall \chi \in S_h,$$

where $(\nabla u, \nabla v)_S = \int_S \nabla u \cdot \nabla v$.

Our motivation is to establish the following best approximation property

$$\|\nabla(u - u_h)\|_{L^\infty(\Omega)} \leq C \inf_{\chi \in S_h} \|\nabla(u - \chi)\|_{L^\infty(\Omega)}, \tag{1.2}$$

with constant C independent of h. Such a result has many applications. For example, (1.2) is needed in order to establish the numerically observed L^2 error estimate for bi-harmonic problems (cf. [37]). By taking $\chi = 0$ in (1.2), we obtain the following stability result

$$\|\nabla u_h\|_{L^\infty(\Omega)} \leq C \|\nabla u\|_{L^\infty(\Omega)}, \tag{1.3}$$

which is essential, for example, in analyzing the finite element solution of nonlinear problems (cf. [6,7,9,11,29]).

Many important contributions have been made in order to establish (1.2) with various assumptions on the finite element spaces and geometry of Ω. Here we highlight some of the contributions. The first results valid for general quasi-uniform meshes were obtained by Natterer [23] and Scott [38]. Natterer treated piecewise linear approximation on convex polygonal domains and Scott treated the problem with Neumann boundary conditions in two dimensions. Rannacher [27] and Nitsche [24,25] considered the problem for arbitrary order approximations and on a smooth domain Ω. Schatz [30] considered non-convex polygonal domains and used a discrete maximum principle in his proof. All the above results were sub-optimal (contained a logarithmic factor) for piecewise linear elements. In 1982, Rannacher and Scott in [28] proved optimal error estimates for convex polygonal domains and smooth domains. In the book by Brenner and Scott [3], these results were extended to three dimensional polyhedral domains with certain restrictions on the geometry.

The main analytical tool used in [3,24,25,27,28] was the fact that it is possible to obtain L^∞ bounds from certain weighted L^2 estimates. However, the above technique has not been shown to give optimal error estimates for general polyhedral domains in three dimensions. For example, the estimates contained in [3] require W^2_p regularity, where p must be strictly greater than the space dimension. Although such regularity holds for general convex polygonal domains, in three dimensions such result puts strong restrictions on the geometry of polyhedral domains. More specifically, it requires that inner dihedral angles be smaller than $3\pi/4$ (cf. Theorem 7.1 in [21]). The reason for this is that the solutions both in a two-dimensional angle K and in a three-dimensional dihedral angle $K \times \mathbb{R}$ contain the singular term $r^{\pi/\theta}\sin(\pi\varphi/\theta)$, where r, φ are the polar coordinates in K and θ is the opening of the angle. This term belongs to W^2_p if $\theta < \frac{p}{2p-2}\pi$. The restriction $\theta < 3\pi/4$ in the three-dimensional case is unnatural since it is known that for any convex polyhedral domain Ω the solution u to (1.1) is in $W^1_\infty(\Omega)$. Actually, the gradient of the solution is Hölder continuous, where the modulus of the continuity depends on the geometry of the domain. Using imbedding theorems, this result can be deduced from the regularity results in weighted Sobolev spaces (cf. [19]). For the Stokes system such $C^{1+\sigma}$ result is established for the velocity field in [22].

A different technique was developed by Schatz and Wahlbin (cf. [31,34–36]). In those papers instead of using global weighted L^2 error estimates, they used local L^2 error estimates (cf. [26]), along with dyadic decompositions of Ω. The technique is independent of dimension, but relies on sharp pointwise bounds for high-order derivatives of the Green's function. These types of the Green's function estimates are known for smooth domains [18], but do not hold for general convex polyhedral domains (cf. [12,15]).

Carefully examining the arguments of Schatz and Wahlbin, one can notice that in order to establish (1.2), it is sufficient to have certain Hölder type estimates for the first order derivatives and the second order mixed derivatives of the Green's function. More precisely, one would require for some $\sigma > 0$, which may depend on the geometry of the domain Ω,

$$\begin{aligned} \frac{|\partial_{x_i} G(x,\xi) - \partial_{y_i} G(y,\xi)|}{|x-y|^\sigma} &\le C\left(|x-\xi|^{-2-\sigma} + |y-\xi|^{-2-\sigma}\right), \\ \frac{|\partial_{x_i}\partial_{\xi_j} G(x,\xi) - \partial_{y_i}\partial_{\xi_j} G(y,\xi)|}{|x-y|^\sigma} &\le C\left(|x-\xi|^{-3-\sigma} + |y-\xi|^{-3-\sigma}\right), \end{aligned} \tag{1.4}$$

for $i, j = 1, 2, 3$.

Therefore, the main contribution of this papers is twofold. First we establish Hölder estimates for the Green's function (1.4), then using these estimates we prove the best approximation property (1.2) for the finite element method for a general convex polyhedral domain.

Now we comment on (1.4). If $\Omega \subset \mathbb{R}^3$ has a smooth boundary, the following estimate holds (cf. [18])

$$|D^\alpha_x D^\beta_\xi G(x,\xi)| \le \frac{C}{|x-\xi|^{1+|\alpha|+|\beta|}}. \tag{1.5}$$

If $\partial\Omega$ is not smooth, then in general the right-hand side of the above estimate will be a function of the distance of x and ξ to the singularities of $\partial\Omega$ (cf. [19]). However, in the case Ω is convex, but not necessarily a polyhedral domain, the above estimates are known to hold for $|\alpha| \leq 1$ and $|\beta| \leq 1$ (cf. [12,15]). In the case of $C^{1+\sigma}$ boundaries, estimates of the type (1.4) were proved in [15]. However, the counter-example given by Fromm [12] indicates that estimates (1.4) can not be extended to general convex domains. In this paper we show that (1.4) holds for convex polyhedral domains Ω, where $\sigma > 0$ depends on the geometry of Ω. For example, if Ω is a cube, then the estimates (1.4) are valid for arbitrary $0 < \sigma < 1$.

Finally, we would like to present an improvement to (1.2), weighted pointwise error estimates. Until 1998, all the pointwise error estimates were global, in the sense that the error at a point $z \in \Omega$ depended equally on the smoothness of u on the whole domain Ω. However, in [31], for smooth domains Schatz proved weighted pointwise error estimates that showed that the error at a point $z \in \Omega$ depends strongly on the behavior of u in the vicinity of z and rather weakly on the behavior of u far from z. Here we prove a similar weighted result for convex polyhedral domains. More specifically, we show that for an arbitrary point $z \in \Omega$,

$$|\nabla(u - u_h)(z)| \leq C \inf_{\chi \in S_h} \|\omega^s \nabla(u - \chi)\|_{L^\infty(\Omega)}, \tag{1.6}$$

where the weight $\omega(y) = \omega_{z,h}(y) = \frac{h}{h+|z-y|}$ and the power s satisfies $0 \leq s < \sigma$. This is in contrast to smooth domains, where the power s can be taken as high as the order of the approximating polynomials (cf. [31]). Notice that if we choose $s = 0$ then the estimate reduces to (1.2). However, for $s > 0$ we have an improvement over (1.2). Finally we would like to point that the error estimate (1.6) is the first a priori weighted error estimate proved for general convex polyhedral domains which holds up to the boundary.

Such weighted results proved to be very fruitful and inspired many interesting applications, for example, asymptotic error expansion inequalities (cf. [1,31,32]), superconvergence (cf. [33]), a posteriori averaging technique (cf. [4,16]), a posteriori residual type estimators (cf. [6]), localized pointwise error estimates for quasilinear problems (cf. [8]), and Richardson Extrapolation (cf. [2]).

Remark 1 An anonymous referee pointed out to us that similar weighted result just appeared in the new edition of [3], namely Corollary 8.2.8. There the weight ω can be taken to any power $0 < s < 1 - n/p$, provided the solution $u \in W^2_p$ for $p > n$. In three dimensions, i.e. $n = 3$, it is geometrically very restrictive and do not hold for general convex polyhedral domains. In two dimensions the exponents in the above results are related through the Sobolev imbedding theorems, but in three dimensions the connection is less clear.

The rest of the paper is organized as follows. The next two sections are devoted to the proof of Green's function estimates (1.4). The proof is given for more general *polyhedral type* domains. In Sect. 4 we concentrate on the application of the Green's function estimates to finite element method. Thus, in Sect. 4.1 we state the basic assumptions on the mesh and the finite element spaces and in Sect. 4.2 we give a proof of the best approximation property (1.2). In Sect. (4.3) we state the localized pointwise estimate (1.6). Finally, in the last section we comment on possible generalizations and extension to more complicated systems.

2 Maximum modulus estimates for the Green's function in polyhedral domains

2.1 The domain

Although for our finite element error estimates we will only consider convex polyhedral domains (flat faces and straight edges) we will prove (1.4) for more general *polyhedral type* domains (curved faces and edges).

In the proof of the Green's function estimates we assume that Ω is a bounded domain of polyhedral type in $\mathbb{R}^3$. This means that

(i) the boundary $\partial\Omega$ consists of smooth (of class C^∞) open two-dimensional manifolds Γ_j (the faces of Ω), $j = 1, \ldots, N$, smooth curves M_k (the edges), $k = 1, \ldots, N'$, and corners $x^{(1)}, \ldots, x^{(d)}$,

(ii) for every $\xi \in M_k$ there exist a neighborhood $\mathcal{U}_\xi$ and a diffeomorphism (a C^∞ mapping) κ_ξ which maps $\Omega \cap \mathcal{U}_\xi$ onto $\mathcal{D}_\xi \cap B_1$, where $\mathcal{D}_\xi$ is a dihedron of the form

$$\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < r < \infty,\ 0 < \varphi < \theta_\xi,\ x_3 \in \mathbb{R}\}$$

(here r, φ are the polar coordinates in the (x_1, x_2)-plane) and B_1 is the unit ball,

(iii) for every corner $x^{(j)}$ there exist a neighborhood $\mathcal{U}_j$ and a diffeomorphism κ_j mapping $\Omega \cap \mathcal{U}_j$ onto $\mathcal{K}_j \cap B_1$, where

$$\mathcal{K}_j = \{x \in \mathbb{R}^3 : x/|x| \in X_j\} \tag{2.1}$$

is a cone with vertex at the origin.

The domains X_j in (iii) are subdomains of the unit sphere S^2 of polygonal type, i.e., the boundary of X_j consists of finitely many smooth curves meeting under nonzero angles.

2.2 Notation

In what follows, $\rho_j(x)$ denotes the distance of the point x from the vertex $x^{(j)}$, $r_k(x)$ the distance from the edge M_k and $r(x) = \min_k r_k(x)$ the distance from the set of all edge points. Let ξ be a point on the edge M_k, and let $\Gamma_{k_+}, \Gamma_{k_-}$ be the faces of Ω adjacent to ξ. Then by $\mathcal{D}_\xi$, we denote the dihedron which is bounded by the half-planes $\Gamma^\circ_{k_\pm}$ tangential to $\Gamma_{k_\pm}$ at ξ. The angle between $\Gamma^\circ_{k_+}$ and $\Gamma^\circ_{k_-}$ is denoted by $\theta(\xi)$. We set

$$\theta_k = \sup_{\xi \in M_k} \theta(\xi), \quad \mu_k = \pi/\theta_k, \text{ and } \mu(x) = \pi/\theta_{k(x)},$$

where $M_{k(x)}$ is the nearest edge to x (more precisely, $k(x)$ is the smallest k such that $r_k(x) = r(x)$). Let $x^{(j)}$ be a vertex of Ω, and let I_j be the set of all indices k such that $x^{(j)}$ is an end-point of the edge M_k. By our assumptions on Ω, there exist a neighborhood $\mathcal{U}_j$ of $x^{(j)}$ and a diffeomorphism κ_j mapping $\Omega \cap \mathcal{U}_j$ onto a subset of the cone (2.1). Without loss of generality, we may assume that the Jacobian matrix $\kappa'(x)$ coincides with the identity matrix at $x^{(j)}$. We denote by λ_j the smallest eigenvalue of the Laplace-Beltrami operator $-\delta$ on the domain X_j (with Dirichlet boundary conditions). Furthermore, let

$$\Lambda_j = -\frac{1}{2} + \sqrt{\frac{1}{4} + \lambda_j}$$

for $j = 1, \ldots, d$. This means that $\lambda_j = \Lambda_j(\Lambda_j + 1)$.

2.3 Point estimates of the Green's function

Let $G(x, \xi)$ be the Green's function of the Dirichlet problem to the Laplace equation, i.e. $G(x, \xi)$ is the solution of the problem

$$\begin{aligned} -\Delta_x G(x, \xi) &= \delta(x - \xi) \ \text{ for } x, \quad \xi \in \Omega, \\ G(x, \xi) &= 0 \ \text{ for } x \in \partial\Omega, \quad \xi \in \Omega. \end{aligned} \tag{2.2}$$

In the following, let $\mathcal{V}_j$ be a neighborhood of the vertex $x^{(j)}$ which has a positive distance to the edges M_k, $k \notin I_j$.

The following estimates of $G(x, \xi)$ were proved in [19] (cf. also [20]).

(1) If $x, \xi \in \Omega \cap \mathcal{V}_j$, and $\rho_j(\xi) < 2\rho_j(x)/3$, then

$$\begin{aligned} \left| D_x^\alpha D_\xi^\gamma G(x, \xi) \right| &\le c_{\alpha,\gamma}\, \rho_j(x)^{-1-\Lambda_j-|\alpha|+\varepsilon}\, \rho_j(\xi)^{\Lambda_j-|\gamma|-\varepsilon} \\ &\quad \times \prod_{k\in I_j} \left(\frac{r_k(x)}{\rho_j(x)}\right)^{\mu_k-|\alpha|-\varepsilon} \prod_{k\in I_j} \left(\frac{r_k(\xi)}{\rho_j(\xi)}\right)^{\mu_k-|\gamma|-\varepsilon}, \end{aligned} \tag{2.3}$$

for all multi-indices α and γ, where ε is an arbitrarily small positive number. The constant $c_{\alpha,\gamma}$ is independent of x and ξ.

(2) If $x, \xi \in \Omega \cap \mathcal{V}_j$, and $\rho_j(\xi) > 3\rho_j(x)/2$, then

$$\begin{aligned} \left| D_x^\alpha D_\xi^\gamma G(x, \xi) \right| &\le c_{\alpha,\gamma}\, \rho_j(x)^{\Lambda_j-|\alpha|-\varepsilon}\, \rho_j(\xi)^{-1-\Lambda_j-|\gamma|+\varepsilon} \\ &\quad \times \prod_{k\in I_j} \left(\frac{r_k(x)}{\rho_j(x)}\right)^{\mu_k-|\alpha|-\varepsilon} \prod_{k\in I_j} \left(\frac{r_k(\xi)}{\rho_j(\xi)}\right)^{\mu_k-|\gamma|-\varepsilon}. \end{aligned} \tag{2.4}$$

(3) If $x, \xi \in \Omega \cap \mathcal{V}_j$, $\rho_j(x)/3 < \rho_j(\xi) < 3\rho_j(x)$ and $|x-\xi| > \frac{2}{3}\min(r(x), r(\xi))$, then

$$\left|D_x^\alpha D_\xi^\gamma G(x,\xi)\right| \le c_{\alpha,\gamma}|x-\xi|^{-1-|\alpha|-|\gamma|}\left(\frac{r(x)}{|x-\xi|}\right)^{\mu(x)-|\alpha|-\varepsilon} \times \left(\frac{r(\xi)}{|x-\xi|}\right)^{\mu(\xi)-|\gamma|-\varepsilon}. \tag{2.5}$$

(4) If $x, \xi \in \Omega \cap \mathcal{V}_j$, $\rho_j(x)/3 < \rho_j(\xi) < 3\rho_j(x)$ and $|x-\xi| < \min(r(x), r(\xi))$, then

$$\left|D_x^\alpha D_\xi^\gamma G(x,\xi)\right| \le c_{\alpha,\gamma}\,|x-\xi|^{-1-|\alpha|-|\gamma|}. \tag{2.6}$$

(5) If x and ξ lie in neighborhoods $\mathcal{V}_i$ and $\mathcal{V}_j$ of different vertices $x^{(i)}$ and $x^{(j)}$ and $|x-\xi| > \delta$, where δ is a fixed positive number, then

$$\left|D_x^\alpha D_\xi^\gamma G(x,\xi)\right| \le c_{\alpha,\gamma}\,\rho_i(x)^{\Lambda_i-|\alpha|-\varepsilon}\,\rho_j(\xi)^{\Lambda_j-|\gamma|-\varepsilon} \times \prod_{k\in I_i}\left(\frac{r_k(x)}{\rho_i(x)}\right)^{\mu_k-|\alpha|-\varepsilon}\prod_{k\in I_j}\left(\frac{r_k(\xi)}{\rho_j(\xi)}\right)^{\mu_k-|\gamma|-\varepsilon}. \tag{2.7}$$

From this point on we will assume that the domain Ω satisfies the following conditions.

$$\begin{aligned}&\mathbb{C}1: \quad X_j \text{ is a proper subset of the half-sphere for all } j,\\ &\mathbb{C}2: \quad \theta_k < \pi \text{ for all } k.\end{aligned}$$

Condition $\mathbb{C}2$ implies that $\mu_k > 1$ for all k. Condition $\mathbb{C}1$ will allow us to use the following result which can be found for example in [17].

Proposition 2.1 *If X_j is a proper subset of the half-sphere then $\Lambda_j > 1$.*

It is important to note that every convex domain of polyhedral type satisfies conditions $\mathbb{C}1$ and $\mathbb{C}2$.

If Ω satisfies conditions $\mathbb{C}1$ and $\mathbb{C}2$ then using (2.3)–(2.7) one has the following bound

$$\left|\partial_{x_j} G(x,\xi)\right| \le c\,|x-\xi|^{-2} \quad \text{and} \quad \left|\partial_{x_j}\partial_{\xi_k} G(x,\xi)\right| \le c\,|x-\xi|^{-3} \tag{2.8}$$

for all $x, \xi \in \Omega$, $j, k = 1, 2, 3$. In particular, if Ω is a convex polyhedral domain the above estimate holds. In fact, by [12,15], these estimates are valid for general convex domains.

3 Hölder estimates of Green's function

Lemma 3.1 *Let* $\Omega \subset \mathbb{R}^3$ *satisfy conditions* $\mathbb{C}1$ *and* $\mathbb{C}2$ *and let* m *be an arbitrary positive number. Then the estimates* (1.4) *are satisfied with arbitrary* $\sigma \in (0, 1)$ *for* $|x - \xi| < m\,|x - y|$.

Proof If $|x - \xi| < m\,|x - y|$, then $|y - \xi| < (m + 1)\,|x - y|$ and (2.8) implies

$$\frac{\left|\partial_{x_i} G(x,\xi) - \partial_{y_i} G(y,\xi)\right|}{|x-y|^\sigma} \le \frac{\left|\partial_{x_i} G(x,\xi)\right|}{|x-y|^\sigma} + \frac{\left|\partial_{y_i} G(y,\xi)\right|}{|x-y|^\sigma}$$
$$\le c\,\frac{|x-\xi|^{-2}}{|x-y|^\sigma} + \frac{|y-\xi|^{-2}}{|x-y|^\sigma} \le c\,(m+1)^\sigma\,\left(|x-\xi|^{-2-\sigma} + |y-\xi|^{-2-\sigma}\right).$$

In the same way, the second estimate of (1.4) holds. □

In the following, we assume that m is a sufficiently large positive number.

Lemma 3.2 *Let* $\Omega \subset \mathbb{R}^3$ *satisfy conditions* $\mathbb{C}1$ *and* $\mathbb{C}2$. *Furthermore, let* σ *be a positive number,* $\sigma < 1$, $\sigma < \Lambda_j - 1$ *for all* j, *and* $\sigma < \mu_k - 1$ *for all* k. *Then there exists a constant* c *such that*

$$\frac{\left|\partial_{x_i} G(x,\xi) - \partial_{y_i} G(y,\xi)\right|}{|x-y|^\sigma} \le c\,|x-\xi|^{-2-\sigma}, \tag{3.1}$$

$$\frac{\left|\partial_{x_i}\partial_{\xi_j} G(x,\xi) - \partial_{y_i}\partial_{\xi_j} G(y,\xi)\right|}{|x-y|^\sigma} \le c\,|x-\xi|^{-3-\sigma}. \tag{3.2}$$

for all $x, y, \xi \in \Omega$, $x \ne y$, $|x - \xi| > m\,|x - y| > r(x)$.

Proof Since $|x - \xi| > m\,|x - y|$ with a sufficiently large m, we may assume that x and y lie in a neighborhood $\mathcal{V}_j$ of the same vertex $x^{(j)}$. As before we suppose that $\mathcal{V}_j$ has a positive distance to the edges M_k, $k \notin I_j$. From the condition $r(x) < m\,|x - y|$ it follows that

$$r(y) < (m+1)\,|x-y|.$$

Furthermore, the condition $|x - y| < |x - \xi|/m$ implies

$$\left(1 - \frac{1}{m}\right)\,|x-\xi| < |y-\xi| < \left(1 + \frac{1}{m}\right)\,|x-\xi|.$$

We consider the following cases

1) $\xi \in \mathcal{V}_j$ and $\rho_j(x) < \rho_j(\xi)/2$
2) $\xi \in \mathcal{V}_j$ and $\rho_j(x) > 2\rho_j(\xi)$
3) $\xi \in \mathcal{V}_j$ and $\rho_j(\xi)/2 < \rho_j(x) < 2\rho_j(\xi)$
4) ξ lies in a neighborhood of another vertex $x^{(\nu)}$ and $|x - \xi| > \delta$, where δ is a fixed positive number.

We start with *Case 1*. Then obviously $|x-\xi| < \rho_j(x)+\rho_j(\xi) < \frac{3}{2}\rho_j(\xi)$ and

$$\rho_j(y) < \rho_j(x) + |x-y| < \rho_j(x) + \frac{1}{m}|x-\xi| < \left(\frac{1}{2}+\frac{3}{2m}\right)\rho_j(\xi).$$

Consequently (2.4) yields

$$\begin{aligned}&\left|\partial_{x_i}G(x,\xi)-\partial_{y_i}G(y,\xi)\right| \le \left|\partial_{x_i}G(x,\xi)\right| + \left|\partial_{y_i}G(y,\xi)\right|\\ &\quad\le c\,\rho_j(\xi)^{-1-\Lambda_j+\varepsilon}\left(\rho_j(x)^{\Lambda_j-1-\varepsilon}\left(\frac{r(x)}{\rho_j(x)}\right)^{\sigma}+\rho_j(y)^{\Lambda_j-1-\varepsilon}\left(\frac{r(y)}{\rho_j(y)}\right)^{\sigma}\right).\end{aligned}$$

Here ε can be chosen such that $\Lambda_j-1-\varepsilon-\sigma\ge 0$. Thus,

$$\left|\partial_{x_i}G(x,\xi)-\partial_{y_i}G(y,\xi)\right| \le c\,\rho_j(\xi)^{-2-\sigma}\left(r(x)^{\sigma}+r(y)^{\sigma}\right) \le c'\,|x-\xi|^{-2-\sigma}\,|x-y|^{\sigma}.$$

Analogously, we obtain

$$\begin{aligned}&\left|\partial_{x_i}\partial_{\xi_l}G(x,\xi)-\partial_{y_i}\partial_{\xi_l}G(y,\xi)\right|\\ &\quad\le c\,\rho_j(\xi)^{-2-\Lambda_j+\varepsilon}\left(\rho_j(x)^{\Lambda_j-1-\varepsilon}\left(\frac{r(x)}{\rho_j(x)}\right)^{\sigma}+\rho_j(y)^{\Lambda_j-1-\varepsilon}\left(\frac{r(y)}{\rho_j(y)}\right)^{\sigma}\right)\\ &\quad\le c\,\rho_j(\xi)^{-3-\sigma}\left(r(x)^{\sigma}+r(y)^{\sigma}\right)\le c'\,|x-\xi|^{-3-\sigma}\,|x-y|^{\sigma}.\end{aligned}$$

Case 2 In this case $|x-\xi| < \rho_j(x)+\rho_j(\xi) < \frac{3}{2}\,\rho_j(x)$ and

$$\begin{aligned}\rho_j(y) &> \rho_j(x)-|x-y| > \rho_j(x)-\frac{1}{m}|x-\xi| > \rho_j(x)-\frac{1}{m}\left(\rho_j(x)+\rho_j(\xi)\right)\\ &> \left(1-\frac{3}{2m}\right)\rho_j(x) > \left(2-\frac{3}{m}\right)\rho_j(\xi).\end{aligned}$$

Therefore by (2.3)

$$\begin{aligned}&\left|\partial_{x_i}G(x,\xi)-\partial_{y_i}G(y,\xi)\right| \le \left|\partial_{x_i}G(x,\xi)\right| + \left|\partial_{y_i}G(y,\xi)\right|\\ &\quad\le c\,\rho_j(\xi)^{\Lambda_j-\varepsilon}\left(\rho_j(x)^{-2-\Lambda_j+\varepsilon}\left(\frac{r(x)}{\rho_j(x)}\right)^{\sigma}+\rho_j(y)^{-2-\Lambda_j+\varepsilon}\left(\frac{r(y)}{\rho_j(y)}\right)^{\sigma}\right)\\ &\quad\le c'\left(\rho_j(x)^{-2-\sigma}r(x)^{\sigma}+\rho_j(y)^{-2-\sigma}\,r(y)^{\sigma}\right)\\ &\quad\le c''\,|x-\xi|^{-2-\sigma}\,|x-y|^{\sigma}\end{aligned}$$

and analogously,

$$\begin{aligned}&\left|\partial_{x_i}\partial_{\xi_j}G(x,\xi)-\partial_{y_i}\partial_{\xi_j}G(y,\xi)\right| \le \left|\partial_{x_i}\partial_{\xi_j}G(x,\xi)\right| + \left|\partial_{y_i}\partial_{\xi_j}G(y,\xi)\right|\\ &\quad\le c\,\rho_j(\xi)^{\Lambda_j-1-\varepsilon}\left(\rho_j(x)^{-2-\Lambda_j+\varepsilon}\left(\frac{r(x)}{\rho_j(x)}\right)^{\sigma}+\rho_j(y)^{-2-\Lambda_j+\varepsilon}\left(\frac{r(y)}{\rho_j(y)}\right)^{\sigma}\right)\\ &\quad\le c'\left(\rho_j(x)^{-3-\sigma}r(x)^{\sigma}+\rho_j(y)^{-3-\sigma}\,r(y)^{\sigma}\right)\le c''\,|x-\xi|^{-3-\sigma}\,|x-y|^{\sigma}.\end{aligned}$$

Case 3 Then $|x-\xi|<3\rho_j(\xi)$ and

$$\left(\frac{1}{2}-\frac{3}{m}\right)\rho_j(\xi)<\rho_j(y)<\left(2+\frac{3}{m}\right)\rho_j(\xi)\,.$$

Since $r(x)<|x-\xi|$ and

$$r(y)<(m+1)\,|x-y|<\frac{m+1}{m}\,|x-\xi|<\frac{m+1}{m-1}\,|y-\xi|,$$

we can apply (2.5) and obtain

$$\begin{aligned}\left|\partial_{x_i}G(x,\xi)-\partial_{y_i}G(y,\xi)\right| &\le \left|\partial_{x_i}G(x,\xi)\right|+\left|\partial_{y_i}G(y,\xi)\right|\\ &\le c\left(|x-\xi|^{-2}\left(\frac{r(x)}{|x-\xi|}\right)^{\sigma}+|y-\xi|^{-2}\left(\frac{r(y)}{|y-\xi|}\right)^{\sigma}\right)\\ &\le c'\,|x-\xi|^{-2-\sigma}\,|x-y|^{\sigma}.\end{aligned}$$

Analogously,

$$\begin{aligned}\left|\partial_{x_i}\partial_{\xi_l}G(x,\xi)-\partial_{y_i}\partial_{\xi_l}G(y,\xi)\right| &\le c\left(|x-\xi|^{-3}\left(\frac{r(x)}{|x-\xi|}\right)^{\sigma}\right.\\ &\qquad\left.+\,|y-\xi|^{-3}\left(\frac{r(y)}{|y-\xi|}\right)^{\sigma}\right)\\ &\le c'\,|x-\xi|^{-3-\sigma}\,|x-y|^{\sigma}.\end{aligned}$$

Case 4 Finally, we consider the case when x and y lie in the neighborhood $\mathcal{V}_j$ of the vertex $x^{(j)}$ and ξ lies in a neighborhood of another vertex $x^{(\nu)}$ such that $|x-\xi|>\delta$, where δ is a fixed positive number. Then by (2.7)

$$\begin{aligned}&\left|\partial_{x_i}G(x,\xi)-\partial_{y_i}G(y,\xi)\right|\le\left|\partial_{x_i}G(x,\xi)\right|+\left|\partial_{y_i}G(y,\xi)\right|\\ &\quad\le c\,\rho_\nu(\xi)^{\Lambda_\nu-\varepsilon}\left(\rho_j(x)^{\Lambda_j-1-\varepsilon}\left(\frac{r(x)}{\rho_j(x)}\right)^{\sigma}+\rho_j(y)^{\Lambda_j-1-\varepsilon}\left(\frac{r(y)}{\rho_j(y)}\right)^{\sigma}\right)\\ &\quad\le c'\left(r(x)^{\sigma}+r(y)^{\sigma}\right)\le 2c'\,(m+1)^{\sigma}\,|x-y|^{\sigma}\end{aligned}$$

and analogously,

$$
\begin{aligned}
&\left|\partial_{x_i}\partial_{\xi_l}G(x,\xi)-\partial_{y_i}\partial_{\xi_l}G(y,\xi)\right| \le \left|\partial_{x_i}\partial_{\xi_l}G(x,\xi)\right| + \left|\partial_{y_i}\partial_{\xi_l}G(y,\xi)\right| \\
&\le c\,\rho_\nu(\xi)^{\Lambda_\nu-1-\varepsilon}\left(\rho_j(x)^{\Lambda_j-1-\varepsilon}\left(\frac{r(x)}{\rho_j(x)}\right)^\sigma + \rho_j(y)^{\Lambda_j-1-\varepsilon}\left(\frac{r(y)}{\rho_j(y)}\right)^\sigma\right) \\
&\le c'\,|x-y|^\sigma.
\end{aligned}
$$

This completes the proof. □

Lemma 3.3 *Let $\Omega \subset \mathbb{R}^3$ be a convex domain of polyhedral type. Furthermore, let σ be a positive number, $\sigma < \Lambda_j - 1$ for all j, $\sigma < \mu_k - 1$ for all k, and $\sigma < 1$. Then there exists a constant c such that the estimates (3.1) and (3.2) are satisfied for all $x, y, \xi \in \Omega$, $\xi \neq x \neq y$, $|x-\xi| > m\,|x-y|$, $r(x) > m\,|x-y|$.*

Proof From the mean value theorem it follows that

$$
\frac{\left|\partial_{x_i}G(x,\xi)-\partial_{y_i}G(y,\xi)\right|}{|x-y|^\sigma} \le |x-y|^{1-\sigma}\,|\nabla_z\partial_{z_i}G(z,\xi)| \tag{3.3}
$$

and

$$
\frac{\left|\partial_{x_i}\partial_{\xi_l}G(x,\xi)-\partial_{y_i}\partial_{\xi_l}G(y,\xi)\right|}{|x-y|^\sigma} \le |x-y|^{1-\sigma}\,|\nabla_z\partial_{z_i}\partial_{\xi_l}G(z,\xi)|, \tag{3.4}
$$

where $z = x + t(y-x)$, $0 < t < 1$. Since Ω is convex $z \in \Omega$. We assume again that x and y lie in the neighborhood $\mathcal{V}_j$ of the vertex $x^{(j)}$ and consider the same cases 1–4 as in the proof of Lemma 3.2.

Case 1 Since

$$
\begin{aligned}
\rho_j(z) &< \rho_j(x) + |x-y| < \rho_j(x) + \frac{1}{m}|x-\xi| < \rho_j(x) + \frac{1}{m}\left(\rho_j(x)+\rho_j(\xi)\right) \\
&< \left(\frac{1}{2}+\frac{3}{2m}\right)\rho_j(\xi),
\end{aligned}
$$

the derivatives of G at the point (z,ξ) satisfy the estimates (cf. (2.4))

$$
|\nabla_z\partial_{z_i}G(z,\xi)| \le \rho_j(z)^{\Lambda_j-2-\varepsilon}\,\rho_j(\xi)^{-1-\Lambda_j+\varepsilon}\left(\frac{r(z)}{\rho_j(z)}\right)^{\sigma-1}
$$

and

$$
|\nabla_z\partial_{z_i}\partial_{\xi_l}G(z,\xi)| \le c\,\rho_j(z)^{\Lambda_j-2-\varepsilon}\,\rho_j(\xi)^{-2-\Lambda_j+\varepsilon}\left(\frac{r(z)}{\rho_j(z)}\right)^{\sigma-1}.
$$

The number ε can be chosen such that $\Lambda_j - 1 - \varepsilon - \sigma \ge 0$. Consequently,

$$
\frac{\left|\partial_{x_i}G(x,\xi)-\partial_{y_i}G(y,\xi)\right|}{|x-y|^\sigma} \le c\,|x-y|^{1-\sigma}\,\rho_j(\xi)^{-2-\sigma}\,r(z)^{\sigma-1}
$$

and

$$\frac{\left|\partial_{x_i}\partial_{\xi_l}G(x,\xi)-\partial_{y_i}\partial_{\xi_l}G(y,\xi)\right|}{|x-y|^{\sigma}} \le |x-y|^{1-\sigma}\,\rho_j(\xi)^{-3-\sigma}\,r(z)^{\sigma-1}.$$

Using the inequalities $r(z) > (m-1)\,|x-y|$ and $\rho_j(\xi) > \frac{2}{3}|x-\xi|$, we get (3.1) and (3.2).

In *Case 2* we obtain the estimate

$$\begin{aligned}\rho_j(z) &> \rho_j(x) - |x-y| > \rho_j(x) - \frac{1}{m}|x-\xi| > \rho_j(x) - \frac{1}{m}\,\big(\rho_j(x)+\rho_j(\xi)\big)\\ &> \left(1-\frac{3}{2m}\right)\rho_j(x) > \left(2-\frac{3}{m}\right)\rho_j(\xi).\end{aligned}$$

Therefore by (2.3)

$$|\nabla_z\partial_{z_i}G(z,\xi)| \le c\,\rho_j(z)^{-3-\Lambda_j+\varepsilon}\,\rho_j(\xi)^{\Lambda_j-\varepsilon}\left(\frac{r(z)}{\rho_j(z)}\right)^{\sigma-1} \le c'\,\rho_j(z)^{-2-\sigma}\,r(z)^{\sigma-1},$$

and

$$\begin{aligned}|\nabla_z\partial_{z_i}\partial_{\xi_l}G(z,\xi)| &\le c\,\rho_j(z)^{-3-\Lambda_j+\varepsilon}\,\rho_j(\xi)^{\Lambda_j-1-\varepsilon}\left(\frac{r(z)}{\rho_j(z)}\right)^{\sigma-1}\\ &\le c'\,\rho_j(z)^{-3-\sigma}\,r(z)^{\sigma-1},\end{aligned}$$

Using the inequalities $r(z) > (m-1)\,|x-y|$ and

$$\rho_j(z) > \rho_j(x) - |x-y| > \frac{2}{3}\,|x-\xi| - \frac{1}{m}\,|x-\xi|,$$

we obtain (3.1) and (3.2).

Case 3 From the inequalities $\rho_j(\xi)/2 < \rho_j(x) < 2\rho_j(\xi)$ and $|x-\xi| > m|x-z|$ it follows that

$$\left(\frac{1}{2}-\frac{3}{m}\right)\rho_j(\xi) < \rho_j(z) < \left(2+\frac{3}{m}\right)\rho_j(\xi).$$

Furthermore, the inequalities $|x-\xi| > m\,|x-z|$ and $r(x) > m\,|x-z|$ yield

$$\begin{aligned}\left(1-\frac{1}{m}\right)|x-\xi| < |z-\xi| < \left(1+\frac{1}{m}\right)|x-\xi| \quad\text{and}\quad \left(1-\frac{1}{m}\right)r(x) < r(z)\\ < \left(1+\frac{1}{m}\right)r(x).\end{aligned} \tag{3.5}$$

If $|z-\xi| > \min(r(z), r(\xi))$, then (2.5) and (3.5) imply

$$\left|\nabla_z \partial_{z_i} G(z,\xi)\right| \le c\,|z-\xi|^{-3} \left(\frac{r(z)}{|z-\xi|}\right)^{\sigma-1} \le c'\,|x-\xi|^{-2-\sigma}\, r(x)^{\sigma-1}$$
$$\le c'\, m^{\sigma-1}\, |x-\xi|^{-2-\sigma}\, |x-y|^{\sigma-1}.$$

In the case $|z-\xi| < \min(r(z), r(\xi))$, it follows from (2.6) that

$$\left|\nabla_z \partial_{z_i} G(z,\xi)\right| \le c\,|z-\xi|^{-3} \le c'\,|x-\xi|^{-3} \le c'\, m^{\sigma-1}\, |x-\xi|^{-2-\sigma}\, |x-y|^{\sigma-1}.$$

This together with (3.3) implies (3.1). Analogously, we obtain the estimates

$$\left|\nabla_z \partial_{z_i} \partial_{\xi_l} G(z,\xi)\right| \le c\,|z-\xi|^{-4} \left(\frac{r(z)}{|z-\xi|}\right)^{\sigma-1} \le c'\,|x-\xi|^{-3-\sigma}\, |x-y|^{\sigma-1}$$

for $|z-\xi| > \min(r(z), r(\xi))$ and

$$\left|\nabla_z \partial_{z_i} \partial_{\xi_l} G(z,\xi)\right| \le c\,|z-\xi|^{-4} \le c'\, m^{\sigma-1}\, |x-\xi|^{-3-\sigma}\, |x-y|^{\sigma-1}.$$

for $|z-\xi| < \min(r(z), r(\xi))$ what together with (3.4) yields (3.2).

Case 4 Suppose that ξ lies in a neighborhood $\mathcal{V}_\nu$ of the vertex $x^{(\nu)}$ and that $|x-\xi| > \delta$, where δ is a fixed positive number. Then

$$|\nabla_z \partial_{z_i} G(z,\xi)| \le c\,\rho_j(z)^{\Lambda_j-2-\varepsilon}\, \rho_\nu(\xi)^{\Lambda_\nu-\varepsilon} \left(\frac{r(z)}{\rho_j(z)}\right)^{\sigma-1} \le c'\, r(z)^{\sigma-1}$$

and

$$|\nabla_z \partial_{z_i} \partial_{\xi_l} G(z,\xi)| \le c\,\rho_j(z)^{\Lambda_j-2-\varepsilon}\, \rho_\nu(\xi)^{\Lambda_\nu-1-\varepsilon} \left(\frac{r(z)}{\rho_j(z)}\right)^{\sigma-1} \le r(z)^{\sigma-1}.$$

Using the last two estimates together with the inequalities (3.3), (3.4), $|x-\xi| > \delta$ and $r(z) > (m-1)\,|x-y|$, we obtain (3.1) and (3.2). The proof of the lemma is complete. □

Now the following theorem holds as an immediate consequence of Lemmas 3.1–3.3.

Theorem 1 *Let $\Omega \subset \mathbb{R}^3$ be a convex domain of polyhedral type. Furthermore, let σ be a positive number, $\sigma < 1$, $\sigma < \Lambda_j - 1$ for all j, and $\sigma < \mu_k - 1$ for all k. Then there exists a constant c such that the estimates (1.4) are satisfied for all $x, y, \xi \in \Omega$, $x \ne y$.*

As an example, we consider the case when Ω is a cube. In this case, $\Lambda_j = 3$ for all j and $\mu_k = 2$ for all k. Consequently, the estimates (1.4) are valid for arbitrary $0 < \sigma < 1$.

Remark 2 In the proof of Theorem 1, we used only the estimates of Green's function in Sect. 2.3 and the fact that $\Lambda_j > 1$ and $\mu_k > 1$ for convex domains of polyhedral type. The result of the theorem is also true for other second order elliptic equations or systems provided the Green's function (matrix) satisfies the estimates (2.3)–(2.7) with exponents Λ_j and μ_k greater than 1. For example, the estimates (1.4) hold for the Green matrix of the Dirichlet problem to the Lamè system. To the best of our knowledge, whether $\Lambda_j > 1$ for general second order elliptic operator is not known.

4 Applications to finite element methods

4.1 Preliminaries and basic assumptions

For the finite element approximation of the problem, let $\{\mathcal{T}_h\}_h$, $0 < h < 1$, be a sequence of triangulations of Ω, $\overline{\Omega} = \bigcup_{\tau\in\mathcal{T}_h} \overline{\tau}$, with the elements τ mutually disjoint. The partitions are face-to-face so that simplices meet only in full lower-dimensional faces or not at all. The triangulations are assumed to be quasi-uniform, i.e. (if necessary after a renormalization of h),

$$\text{diam}\ \tau \le h \le C(\text{meas}\ \tau)^{1/3}, \quad \forall \tau \in \mathcal{T}_h.$$

Our finite element spaces are then the $\mathcal{C}^0$ simplicial Lagrange elements

$$S_h = S_h^k(\Omega) = \{\chi \in H_0^1(\Omega) : v|_\tau \in \mathbb{P}^k(\tau), \forall \tau \in \mathcal{T}_h\},$$

where $\mathbb{P}^k(\tau)$ denotes the set of polynomials of degree less than or equal to k on τ. Thus the scaling properties hold.

The particular approximation property we actually need is a standard approximation result (cf. [10]).

Lemma 4.1 *Let $Q \subset Q_d \subset \Omega$, with $d \ge \kappa h$, for some fixed κ sufficiently large and $Q_d = \{x \in \Omega : dist(x, Q) \le d\}$. Furthermore, let $I^h : C(\Omega) \to S_h$ be the Lagrange interpolant. For any $v \in C^{1+\sigma}(\Omega)$ there exists C independent of h such that*

$$\|v - I^h v\|_{W^t_\infty(Q)} \le Ch^{1-t+\sigma}\|v\|_{C^{1+\sigma}(Q_d)}, \quad t = 0, 1, \tag{4.1}$$

where

$$\|v\|_{C^{1+\sigma}(Q)} = \|v\|_{C^1(Q)} + \sup_{x_1,x_2\in Q} \frac{|\nabla v(x_1) - \nabla v(x_2)|}{|x_1 - x_2|^\sigma}. \tag{4.2}$$

Another result that we need is the local energy error estimate. First such result was proved in [26] in the interior of the domain, it was extended up to the boundary in [34], Lemma 4.4 for smooth domains. The precise result we require is Corollary 9.1 in [39], which is valid for polyhedral domains up to the boundary.

Lemma 4.2 *Let $Q \subset Q_d \subset \Omega$, with $d \geq \kappa h$, for some fixed κ sufficiently large and $Q_d = \{x \in \Omega : dist(x, Q) \leq d\}$. If $w \in H_0^1(\Omega)$ and $w_h \in S_h$ satisfy*

$$(\nabla(w - w_h), \nabla\chi)_\Omega = 0, \quad \forall\chi \in S_h,$$

then

$$\|w - w_h\|_{H^1(Q)} \leq C \min_{\chi \in S_h} \left(\|w - \chi\|_{H^1(Q_d)} + d^{-1}\|w - \chi\|_{L^2(Q_d)}\right) + Cd^{-1}\|w - w_h\|_{L^2(Q_d)}, \tag{4.3}$$

where C is independent of $Q, h, d, w,$ and w_h.

4.2 Best approximation result.

Theorem 2 *Let $\Omega \subset \mathbb{R}^3$ be a convex polyhedral domain and suppose that u satisfies (1.1) and let $u_h \in S_h$ be its finite element approximation. Then, there exists a constant C independent of h, u and u_h such that (1.2) holds.*

Proof Let $z \in \Omega$ and $z \in \tau$ for some $\tau \in \mathcal{T}_h$. We will be interested in bounding $|\nabla(u - u_h)(z)|$. There exists $\eta \in C_c^1(\tau)$, the regularized Dirac delta function, which satisfies

$$\int_\tau \chi\eta = (\chi, \eta)_\tau = \chi(z), \quad \forall\chi \in \mathbb{P}^k(\tau),$$

with the property

$$\|D^t\eta\|_{L^\infty(\Omega)} \leq Ch^{-3-t}, \quad \text{for } t = 0, 1. \tag{4.4}$$

Let $\partial_{z_l}(u - u_h)$ denote one of the partial derivatives of $u - u_h$. Now we define the function g, which satisfies the following equation,

$$(\nabla g, \nabla\phi) = (\partial_{z_l}\eta, \phi), \quad \forall\phi \in H_0^1(\Omega). \tag{4.5}$$

Then for any $\chi \in S_h$ using (4.5) and the Galerkin orthogonality,

$$\begin{aligned}
\partial_{z_l}(\chi - u_h)(z) &= (\partial_{z_l}(\chi - u_h), \eta) = -(\chi - u_h, \partial_{z_l}\eta) \\
&= -(\nabla(\chi - u_h), \nabla g) \\
&= (\nabla(u_h - u), \nabla g) + (\nabla(u - \chi), \nabla g) \\
&= (\nabla(u_h - u), \nabla(g - g_h)) + (u - \chi, \partial_{z_l}\eta) \\
&= (\nabla(\chi - u), \nabla(g - g_h)) - (\partial_{z_l}(u - \chi), \eta) \\
&\leq \|\nabla(u - \chi)\|_{L^\infty(\Omega)} \left(\|\eta\|_{L^1(\Omega)} + \|\nabla(g - g_h)\|_{L^1(\Omega)}\right) \\
&\leq \|\nabla(u - \chi)\|_{L^\infty(\Omega)} \left(C + \|\nabla(g - g_h)\|_{L^1(\Omega)}\right).
\end{aligned}$$

Hence,

$$|\partial_{z_l}(u-u_h)(z)| \leq \|\nabla(u-\chi)\|_{L^\infty(\Omega)} \left(C + \|\nabla(g-g_h)\|_{L^1(\Omega)}\right).$$

Since the above estimate is valid for arbitrary $\chi \in S_h$ and any partial derivative, taking the infimum over S_h and supremum over all partial derivatives, we obtain

$$|\nabla(u-u_h)(z)| \leq \inf_{\chi \in S_h} \|\nabla(u-\chi)\|_{L^\infty(\Omega)}(C + \|\nabla(g-g_h)\|_{L^1(\Omega)}).$$

Thus, in order to establish the estimate (1.2), we need to show

$$\|\nabla(g-g_h)\|_{L^1(\Omega)} \leq C. \tag{4.6}$$

Note, g depends on z. Thus, we shall prove the above inequality with constant C independent of z. We prove (4.6) in several steps.

Step 1: Dyadic decomposition Without loss of generality we assume that the diameter of Ω is less than 1. We use a dyadic decomposition of Ω. Let $d_j = 2^{-j}$ then we have

$$\Omega = \Omega^* \cup \bigcup_{j=0}^{J} \Omega_j,$$

where

$$\begin{aligned}\Omega^* &= \{x \in \Omega : |x-z| \leq Kh\},\\ \Omega_j &= \{x \in \Omega : d_{j+1} \leq |x-z| \leq d_j\},\end{aligned}$$

where K is a sufficiently large constant to be chosen later and J is the integer such that $2^{-J} \leq Kh \leq 2^{-J+1}$. Note that $J \approx |\log h|$. In the analysis below the generic constants will be denoted by C, but we will keep track on the explicit dependence of the constants on K. This will be important later for the double kickback argument.

Thus, by the Cauchy–Schwarz inequality, we have

$$\|\nabla(g-g_h)\|_{L^1(\Omega)} \leq CK^{3/2}h^{3/2}\|\nabla(g-g_h)\|_{L^2(\Omega^*)} + C\sum_{j=0}^{J} d_j^{3/2}\|\nabla(g-g_h)\|_{L^2(\Omega_j)}.$$

First we estimate the first term. Using the Cauchy–Schwarz inequality, the global a priori error estimates, (4.4), and H^2 regularity we have

$$\begin{aligned}(Kh)^{3/2}\|\nabla(g-g_h)\|_{L^2(\Omega^*)} &\leq CK^{3/2}h^{3/2+1}\|D^2 g\|_{L^2(\Omega)}\\ &\leq CK^{3/2}h^{5/2}\|\nabla\eta\|_{L^2(\tau)}\\ &\leq CK^{3/2}h^{4}\|\nabla\eta\|_{L^\infty(\tau)} \leq CK^{3/2}.\end{aligned}$$

Thus, we have

$$\|\nabla(g - g_h)\|_{L^1(\Omega)} \leq C K^{3/2} + \sum_{j=0}^{J} M_j, \tag{4.7}$$

where

$$M_j = d_j^{3/2} \|\nabla(g - g_h)\|_{L^2(\Omega_j)}.$$

Step 2: Initial estimate for M_j. We first need to define the following sets

$$\Omega'_j = \{x \in \Omega : d_{j+2} \leq |x - z| \leq d_{j-1}\},$$
$$\Omega''_j = \{x \in \Omega : d_{j+3} \leq |x - z| \leq d_{j-2}\}.$$

By the local energy estimate (4.3),

$$\|\nabla(g - g_h)\|_{L^2(\Omega_j)} \leq C \Big(\|\nabla(g - I^h g)\|_{L^2(\Omega'_j)} + d_j^{-1} \|g - I^h g\|_{L^2(\Omega'_j)} + d_j^{-1} \|g - g_h\|_{L^2(\Omega'_j)} \Big).$$

First we will treat the first two terms on the right hand side. By the Cauchy–Schwarz inequality and the approximation result (4.1)

$$\begin{aligned} &\|\nabla(g - I^h g)\|_{L^2(\Omega'_j)} + d_j^{-1} \|g - I^h g\|_{L^2(\Omega'_j)} \\ &\leq C d_j^{3/2} \Big(\|\nabla(g - I^h g)\|_{L^\infty(\Omega'_j)} + d_j^{-1} \|g - I^h g\|_{L^\infty(\Omega'_j)} \Big) \\ &\leq C d_j^{3/2} h^\sigma \|g\|_{C^{1+\sigma}(\Omega''_j)}. \end{aligned}$$

Now we will use the Hölder estimates (1.4) to derive a bound for $\|g\|_{C^{1+\sigma}(\Omega''_j)}$. Using the Green's function representation we have,

$$\begin{aligned} \partial_{x_i} g(x) - \partial_{y_i} g(y) &= -\int_\Omega (\partial_{x_i} G(x,\xi) - \partial_{y_i} G(y,\xi)) \partial_{\xi_l} \eta(\xi) d\xi \\ &= \int_\tau (\partial_{\xi_l} \partial_{x_i} G(x,\xi) - \partial_{\xi_l} \partial_{y_i} G(y,\xi)) \eta(\xi) d\xi, \quad i = 1, 2, 3. \end{aligned}$$

Let $x, y \in \Omega''_j$, $x \neq y$, then by (1.4),

$$\begin{aligned} \frac{|\partial_{x_i} g(x) - \partial_{y_i} g(y)|}{|x - y|^\sigma} &\leq \max_{\xi \in \tau} \frac{|\partial_{\xi_l} \partial_{x_i} G(x,\xi) - \partial_{\xi_l} \partial_{y_i} G(y,\xi)|}{|x - y|^\sigma} \|\eta\|_{L^1(\tau)} \\ &\leq C \max_{\xi \in \tau} (|x - \xi|^{-3-\sigma} + |y - \xi|^{-3-\sigma}) \|\eta\|_{L^1(\tau)} \leq d_j^{-3-\sigma}. \end{aligned}$$

In the last inequality we used that for any $\xi \in \tau$, $|x-\xi|, |y-\xi| \geq Cd_j$ and $\|\eta\|_{L^1(\tau)} \leq C$. Therefore,

$$\sup_{x,y\in\Omega_j''} \frac{|\nabla g(x)-\nabla g(y)|}{|x-y|^\sigma} \leq d_j^{-3-\sigma}.$$

Similarly, we can bound the other term of $\|g\|_{C^{1+\sigma}(\Omega_j'')}$ to obtain

$$\|g\|_{C^{1+\sigma}(\Omega_j'')} \leq Cd_j^{-3-\sigma}.$$

Thus, we have shown that

$$\|\nabla(g-I^h g)\|_{L^2(\Omega_j')} + d_j^{-1}\|g-I^h g\|_{L^2(\Omega_j')} \leq Cd_j^{-3/2-\sigma}h^\sigma.$$

Hence,

$$M_j \leq C\left((h/d_j)^\sigma + d_j^{1/2}\|g-g_h\|_{L^2(\Omega_j')}\right).$$

We still need to estimate $\|g-g_h\|_{L^2(\Omega_j')}$. We will accomplish it by a duality argument.

Step 3: Duality argument. We have the following representation

$$\|g-g_h\|_{L^2(\Omega_j')} = \sup_{v\in C_c^\infty(\Omega_j'),\|v\|_{L^2(\Omega_j')}\leq 1} (g-g_h, v).$$

Let w be the solution of the following problem

$$-\Delta w = v, \quad \text{in}\,\Omega, \tag{4.8}$$
$$w = 0, \quad \text{on}\,\partial\Omega, \tag{4.9}$$

with $v \in C_c^\infty(\Omega_j')$ and $\|v\|_{L^2(\Omega_j')} \leq 1$. Thus, if $I^h w \in S_h$ denotes the interpolant of w, we have

$$\begin{aligned}(g-g_h, v) &= (\nabla(g-g_h), \nabla w) = (\nabla(g-g_h), \nabla(w-I^h w))\\ &= (\nabla(g-g_h), \nabla(w-I^h w))_{\Omega_j''} + (\nabla(g-g_h), \nabla(w-I^h w))_{\Omega\backslash\Omega_j''}.\end{aligned}$$

First we estimate $(\nabla(g-g_h), \nabla(w-I^h w))_{\Omega_j''}$. By the Cauchy–Schwarz inequality, the global a priori error estimate, and H^2 regularity we have

$$\begin{aligned}(\nabla(g-g_h), \nabla(w-I^h w))_{\Omega_j''} &\leq \|\nabla(g-g_h)\|_{L^2(\Omega_j'')}\|\nabla(w-I^h w)\|_{L^2(\Omega)}\\ &\leq \|\nabla(g-g_h)\|_{L^2(\Omega_j'')}Ch\|D^2 w\|_{L^2(\Omega)}\\ &\leq Ch\|\nabla(g-g_h)\|_{L^2(\Omega_j'')}.\end{aligned}$$

Next we estimate the second term

$$(\nabla(g-g_h),\nabla(w-I^h w))_{\Omega\backslash\Omega_j''} \le \|\nabla(g-g_h)\|_{L^1(\Omega)}\|\nabla(w-I^h w)\|_{L^\infty(\Omega\backslash\Omega_j'')}$$
$$\le \|\nabla(g-g_h)\|_{L^1(\Omega)} Ch^\sigma \|w\|_{C^{1+\sigma}(\Omega\backslash\Omega_j'')}.$$

Since $\Omega\backslash\Omega_j''$ is separated from Ω_j' by at least d_j, we have for $x, y \in \Omega\backslash\Omega_j''$, using the first estimate of (1.4)

$$\frac{|\partial_{x_i} w(x) - \partial_{y_i} w(y)|}{|x-y|^\sigma} \le \int_{\Omega_j'} \frac{|\partial_{x_i} G(x,\xi) - \partial_{y_i} G(y,\xi)|}{|x-y|^\sigma}|v(\xi)|d\xi$$
$$\le C \max_{\xi\in\Omega_j'}(|x-\xi|+|y-\xi|)^{-2-\sigma}\int_{\Omega_j'}|v(\xi)|d\xi$$
$$\le Cd_j^{-2-\sigma} d_j^{3/2}\|v\|_{L^2(\Omega_j')} \le Cd_j^{-1/2-\sigma}.$$

Hence,

$$\|w\|_{C^{1+\sigma}(\Omega\backslash\Omega_j'')} \le Cd_j^{-1/2-\sigma},$$

which implies that

$$(\nabla(g-g_h),\nabla(w-I^h w))_{\Omega\backslash\Omega_j''} \le Ch^\sigma d_j^{-1/2-\sigma}\|\nabla(g-g_h)\|_{L^1(\Omega)}.$$

Therefore,

$$\|g-g_h\|_{L^2(\Omega_j')} \le Ch^\sigma d_j^{-1/2-\sigma}\|\nabla(g-g_h)\|_{L^1(\Omega)} + Ch\|\nabla(g-g_h)\|_{L^2(\Omega_j'')}.$$

To summarize,

$$M_j \le C(h/d_j)^\sigma + C(h/d_j)^\sigma\|\nabla(g-g_h)\|_{L^1(\Omega)} + Chd_j^{1/2}\|\nabla(g-g_h)\|_{L^2(\Omega_j'')}.$$

Step 4: Double kick-back argument. Summing over j we obtain

$$\sum_{j=0}^{J} M_j \le \frac{C}{K^\sigma} + \frac{C}{K^\sigma}\|\nabla(g-g_h)\|_{L^1(\Omega)} + \frac{Ch}{d_J}\sum_{j=0}^{J} d_j^{3/2}\|\nabla(g-g_h)\|_{L^2(\Omega_j'')},$$

where we have used that

$$\sum_{j=0}^{J}(h/d_j)^\sigma \le h^\sigma\sum_{j=0}^{J}2^{j\sigma} \le Ch^\sigma 2^{\sigma J} \le CK^{-\sigma} \quad \text{and} \quad d_j^{-1} \le d_J^{-1}.$$

Clearly,

$$\sum_{j=0}^{J} d_j^{3/2} \|\nabla(g-g_h)\|_{L^2(\Omega_j'')} \le C\sum_{j=0}^{J} M_j + C(Kh)^{3/2}\|\nabla(g-g_h)\|_{L^2(\Omega^*)}$$
$$\le C\sum_{j=0}^{J} M_j + CK^{3/2}.$$

Thus, using that $h/d_J \le K^{-1}$, and taking K large enough we have

$$\sum_{j=0}^{J} M_j \le C(K^{3/2}+1) + \frac{C}{K^\sigma}\|\nabla(g-g_h)\|_{L^1(\Omega)}.$$

Therefore, if we plug this result into (4.7) we get

$$\|\nabla(g-g_h)\|_{L^1(\Omega)} \le C(K^{3/2}+1) + \frac{C}{K^\sigma}\|\nabla(g-g_h)\|_{L^1(\Omega)}.$$

Again by choosing K large enough we can conclude

$$\|\nabla(g-g_h)\|_{L^1(\Omega)} \le C.$$

Thus the proof of (4.6) is complete and hence we have established (1.2). □

4.3 Localized pointwise error estimate.

Theorem 3 *With the assumptions of Theorem 2, the following estimate holds,*

$$|\nabla(u-u_h)(z)| \le C \inf_{\chi\in S_h} \|\omega^s \nabla(u-\chi)\|_{L^\infty(\Omega)},$$

for any $0 \le s < \sigma$, *where* $\omega = \omega_{z,h}(y) = \frac{h}{h+|z-y|}$ *is the weight function.*

The proof of this result is very similar to the proof of Theorem 2. We leave the details to the reader.

Following the presentation in [31], one can obtain the following error expansion inequality, which shows that the error is localized.

Corollary 4 *Assume that* $u \in C^{k+1+s}(\Omega)$ *for* $0 \le s < \sigma$. *With the assumptions of Theorem 2, the following estimate holds,*

$$|\nabla(u-u_h)(z)| \le Ch^k\left(\sum_{|\alpha|=k+1} |D^\alpha u(z)| + h^s\|u\|_{C^{k+1+s}(\Omega)}\right).$$

Remark 3 The proofs of Theorems 2 and 3 essentially rely on two results: the H^2 regularity of the solution u and the Green's function estimates. Adapting the proof from [31], we could treat more general uniformly elliptic operator, provided the above two results hold. Although the H^2 regularity of the solutions is well known for more general elliptic operators, the question whether the Hölder type Green's function estimates hold is more involved and depends on the operator (cf. Remark 2).

5 Concluding remarks

We proved optimal W^1_∞ error estimates for convex polyhedral domains in three dimensions. One of the main tools used in the proof are new Hölder type estimates for the Green's function on convex polyhedral domains. It is not difficult to see that if analogues Hölder estimates for the Green's function hold in higher dimensions then the same technique can be used to prove such optimal W^1_∞ error estimates in higher dimensions.

The analysis carried out here can also be applied to discontinuous Galerkin (DG) methods. Using the local error estimates found in [5] and [14] and the techniques used here, we can prove optimal W^1_∞ error estimates for various DG methods on convex polyhedral domains.

Recently Girault et al. [13] proved stability in W^1_∞ norm for certain finite element methods for Stokes problem on polygonal and polyhedral domains. In three dimensions, W^2_p regularity with $p > 3$, was required for the velocity field. This leads to strong restrictions on the inner dihedral angles of the polyhedral domain, despite the fact that the derivatives of the velocity are Hölder continuous for general convex polyhedra (cf. [22]). It would be interesting to see if the techniques used in this paper can be applied to Stokes problem in order to remove those restrictions. This is subject of ongoing work.

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