Numer. Math. (2008) 108:387–406
DOI 10.1007/s00211-007-0126-6

Numerische Mathematik

A residual based error estimator for the Mimetic Finite Difference method

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Received: 3 January 2007 / Published online: 5 December 2007

Abstract We present a local error indicator for the Mimetic Finite Difference method for diffusion-type problems on polyhedral meshes. Under essentially the same general hypotheses used in (SIAM J. Numer. Anal. 43:1872–1896, 2005) to show the convergence of the method, we prove the global reliability and local efficiency of the proposed estimator.

Mathematics Subject Classification (2000) 65N30

1 Introduction

The Mimetic Finite Difference method has been applied successfully in a large range of applications, for instance electromagnetics, gas dynamics and diffusion. The advantage of such method is twofold. First, it allows for a large choice of discrete scalar products, leading to an entire range of different (consistent and stable) numerical schemes for the same problem. This additional freedom can be used, for instance, to tackle a scheme which satisfies some additional physical or numerical properties. Second, the Mimetic Finite Difference method allows for general polyhedral meshes with degenerate and non-convex elements, which is a very useful property in many applications.

In the present paper we consider the diffusion problem written as a system of two equations

$$\begin{array}{ll} \operatorname{div} \mathbf{F} = b\ ,\ \ \mathbf{F} + \mathbb{K}\nabla p = 0 & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega\ . \end{array} \tag{1}$$

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The first equation represents the mass conservation while the second one is the constitutive equation relating the scalar variable p to the velocity field $\mathbf{F}$ through the symmetric material tensor $\mathbb{K}$. For simplicity, we consider the case of homogeneus Dirichlet boundary conditions for the scalar variable.

The diffusion problem has already been the object of a large number of papers in the literature of MFD, see for example [4,5,17–21,23,24] and references therein. In [9,11], the authors proved for the first time the convergence of the method for general polyhedral unstructured meshes with flat and curved faces, while in [10] a family of inexpensive MFD discretization schemes was introduced.

As noted above, one of the main advantages of the MFD method with respect to classical finite elements is the generality of the mesh. The elements can be general degenerate and non-convex polyhedrons, eventually of different type across the domain. Such flexibility makes the MFD method a very appealing ground for the application of adaptive strategies for error control. The present paper is a first step in this direction, presenting a local (reliable and efficient) residual-based error estimator for the MFD method applied to problem (1). In this contribution we therefore focus on the evaluation of the local error, which is a key issue in adaptivity. The aspect of developing a (optimal) converging adaptive strategy, hopefully exploiting to the best extent the flexibility of the MFD scheme, will not be considered here.

The paper is organized as follows. In Sect. 2, we briefly review the Mimetic Finite Difference method and, in particular, the assumptions on the scheme. We essentially require the same (minimal) properties on the mesh geometry and discrete scalar products introduced in [9] to prove the convergence of the method. In Sect. 3 we introduce a post-processing scheme for the scalar variable, in the spirit of [22,25]. The post-processed pressure is shown to convergence in a stronger norm and is a key ingredient in the proposed error estimator. Finally, in Sect. 4 we introduce the local error indicator, and prove the main result of the paper, i.e. global reliability and local efficiency bounds.

In the whole contribution the scalar C will indicate a general positive constant, eventually different at each occurrence, uniform in the mesh size.

Remark 1 For simplicity of exposition, in the present paper we focus the proofs and notation on the case of three dimensional problems. The error estimator for the bi-dimensional case is identic, the proofs being a simpler reformulation. Essentially, substitute "faces" with "edges" and restrict Lemma 2 to the easier case of bi-dimensional meshes.

2 The mimetic finite difference method

In the present section we give a brief description of the Mixed Finite Difference method applied to problem (1).

Let Ω be a polyhedron with Lipschitz continuous boundary. Furthermore, let Ω_h be a non-overlapping conformal partition of Ω into simply-connected polyhedral elements with flat faces. We indicate in the sequel the set of faces of Ω_h with $\mathcal{E}_h$, and with h_E the diameter of each element E. Furthermore, for every $E \in \Omega_h$ and every face $e \in \partial E$, let $\mathbf{n}_e^E$ represent the outward unit normal to e. Moreover, in the sequel we associate to every $e \in \mathcal{E}_h$ a normal unit vector $\mathbf{n}_e$, fixed once and for all.

We assume the following properties of the mesh Ω_h, introduced in [9] in order to derive the a priori error converge estimates for the scheme.

(M2) We assume that we have two positive integers N_e and N_l such that every element E has at most N_e faces, and each face e at most N_l edges.

(M3) We assume that there exist three positive constants v_*, a_* and l_* such that for every element E it holds

$$v_* h_E^3 \leq |E| \,, \quad a_* h_E^2 \leq |e| \,, \quad l_* h_E \leq |l|$$

for all faces e and edges l of E, where here and in the sequel $|E|$, $|e|$, $|l|$ represent respectively the volume of E, the area of e and the length of l.

(M4) We assume that the mesh faces are flat and that there exists a positive number γ_* such that: for each element E and for each face $e \in \partial E$ there exists a point $M_e \in e$ such that e is star-shaped with respect to every point in the disk of center M_e and radius $\gamma_* h_E$.

(M5) We further assume that for every $E \in \Omega_h$, and for every $e \in \partial E$, there exists a pyramid P_E^e contained in E such that its base equals to e, its height equals to $\gamma_* h_E$ and the projection of its vertex onto e is M_e.

(M6) We assume that there exists a positive number τ_* such that: for each element E there exists a point $M_E \in E$ such that E is star-shaped with respect to every point in the disk of center M_E and radius $\tau_* h_E$.

In the sequel, we assume for simplicity that the tensor field $\mathbb{K}$ is piecewise constant with respect to the mesh. In an adaptive framework, since subsequently refined meshes are typically obtained by subdivision, it is sufficient that such assumption holds for the initial mesh. We also assume the following standard condition.

(P1) The tensor field $\mathbb{K}$ is symmetric and uniformly strongly elliptic, implying that there exist two constants k_* and k^* such that

$$k_* ||\mathbf{v}||^2 \leq \mathbf{v}^T \mathbb{K}(\mathbf{x}) \mathbf{v} \leq k^* ||\mathbf{v}||^2 \quad \forall \mathbf{v} \in \mathbb{R}^3, \ \forall \mathbf{x} \in \Omega, \tag{2}$$

where, here and in the sequel, $|| \cdot ||$ indicates the Euclidean norm of $\mathbb{R}^3$. We are now in the position to introduce the Mimetic Finite Difference method.

The *first step* of the MFD scheme is to define the degrees of freedom for the pressure variable p and flux variable $\mathbf{F}$. We therefore introduce the space Q^d of discrete pressures that are constant on each element E. For $q \in Q^d$, we denote by q_E its value on E. For notation simplicity, for any $q \in Q^d$, in the sequel we will identify the vector of its values and the respective Ω_h-piecewise constant function.

The space of discrete velocities X^d is defined as follows. To every element $E \in \Omega_h$ and face $e \in \partial E$, we associate a number G_E^e and the respective vector field $G_E^e \mathbf{n}_e^E$. We make the continuity assumption

$$G_{E_1}^e = -G_{E_2}^e \tag{3}$$

for each face e shared by two elements E_1 and E_2. Then, the number m of the discrete velocity unknowns will be equal to the number of boundary faces plus twice the number of internal faces. We consider the space X^d as the subspace of $\mathbb{R}^m$ which satisfies (3).

We define the following corresponding *interpolation operators*. Given any function $q \in L^1(\Omega)$, we define its interpolant $q^I \in Q^d$ as

$$(q^I)_E = |E|^{-1} \int_E q \, \mathrm{d}V \quad \forall E \in \Omega_h \, . \tag{4}$$

For every function $\mathbf{G} \in [L^s(\Omega)]^3$, $s > 2$, with $\operatorname{div} \mathbf{G} \in L^2(\Omega)$, we define its interpolant $G^I \in X^d$ as

$$(G_E^I)^e = |e|^{-1} \int_e \mathbf{G} \cdot \mathbf{n}_e^E \, \mathrm{d}\Sigma \quad \forall E \in \Omega_h, \ \forall e \in \partial E. \tag{5}$$

Note that, in the sequel, there will be no confusion in the notation, because interpolant (4) is applied to scalar functions and interpolant (5) to vector functions.

The *second step* of the MFD method is to build a discrete divergence operator. For each element $\mathbf{G} \in X^d$, we define its discrete divergence $\mathcal{DIV}^d\mathbf{G}$ as the element of Q^d given by

$$(\mathcal{DIV}^d\mathbf{G})_E = |E|^{-1} \sum_{e \in \partial E} |e| \, G_E^e \quad \forall E \in \Omega_h. \tag{6}$$

It is easy to check that the following very important commuting diagram property holds. For all $\mathbf{G} \in [L^s(\Omega)]^3$, $s > 2$, with $\operatorname{div} \mathbf{G} \in L^2(\Omega)$, it holds

$$\mathcal{DIV}^d\mathbf{G}^I = (\operatorname{div} \mathbf{G})^I. \tag{7}$$

The *third step* of the MFD method is to define scalar products for the spaces Q^d and X^d. For the space Q^d, we take the only consistent choice

$$[p, q]_{Q^d} = \sum_{E \in \Omega_h} |E| \, p_E q_E \quad \forall p, q \in Q^d. \tag{8}$$

For the space X^d, the scalar product is defined as

$$[\mathbf{G}, \mathbf{Q}]_{X^d} = \sum_{E \in \Omega_h} [\mathbf{G}, \mathbf{Q}]_E \quad \forall \mathbf{G}, \mathbf{Q} \in X^d, \tag{9}$$

where $[\mathbf{G}, \mathbf{Q}]_E$ is a local scalar product on E. The choice and construction of such local scalar products is a main point in the MFD method, and it is the object of various papers in the literature, see for example [18,20]. A general procedure for building these products, such that certain fundamental assumptions are satisfied, was given in [10]. We here assume that those same (minimal) stability and consistency assumptions are satisfied.

(S1) There exist two positive constants s_* and S_* such that for every element E in the decomposition we have

$$s_*|E| \sum_{e \in \partial E} (G_E^e)^2 \leq [\mathbf{G}, \mathbf{G}]_E \leq S_*|E| \sum_{e \in \partial E} (G_E^e)^2 \quad \forall \mathbf{G} \in X^d. \tag{10}$$

(S2) For every element E, every linear function q on E and every $\mathbf{G} \in X^d$, it holds

$$[(K\nabla q)^I, \mathbf{G}]_E = -\int_E q(\mathcal{DIV}^d\mathbf{G})_E \,\mathrm{d}V + \sum_{e \in \partial E} G_E^e \int_e q \,\mathrm{d}\Sigma. \tag{11}$$

Furthermore, we introduce the local and global norms

$$|||\mathbf{G}|||_{X^d}^2 = [\mathbf{G}, \mathbf{G}]_{X^d} = \sum_{E \in \Omega_h} |||\mathbf{G}|||_E^2 \,, \quad |||\mathbf{G}|||_E^2 = [\mathbf{G}, \mathbf{G}]_E. \tag{12}$$

The *fourth step* of the MFD method is to define the discrete flux operator $\mathcal{G}^d$, as the adjoint of the discrete divergence operator with respect to the introduced scalar products. We have

$$[\mathbf{G}, \mathcal{G}^d p]_{X^d} = [p, \mathcal{DIV}^d\mathbf{G}]_{Q^d} \quad \forall p \in Q^d \,, \ \forall \mathbf{G} \in X^d. \tag{13}$$

Finally, the MFD method for problem (1) reads

$$\mathcal{DIV}^d\mathbf{F}_d = b^I \,, \quad \mathbf{F}_h = \mathcal{G}^d p_d \tag{14}$$

or, in more explicit form,

$$\begin{aligned} &[\mathbf{F}_d, \mathbf{G}]_{X^d} - [p_d, \mathcal{DIV}^d\mathbf{G}]_{Q^d} = 0 && \forall \mathbf{G} \in X^d \\ &[\mathcal{DIV}^d\mathbf{F}_d, q]_{Q^d} = [b^I, q]_{Q^d} && \forall q \in Q^d. \end{aligned} \tag{15}$$

2.1 Convergence of the method

In [9], the authors prove that properties (S1), (S2) are implied by the existence of an element lifting operator R_E with certain consistency and stability properties. We here assume the existence of such operator; as underlined in Remark 2, this is a very weak assumption. Note that this operator *never needs to be built in practice*, the knowledge of its existence being sufficient for our purposes.

(S) For every element $E \in \Omega_h$ it exists a lifting operator R_E acting on $X^d|_E$, with values in $[L^2(E)]^3$, such that

$$\begin{aligned} &R_E(\mathbf{G}_E)|_e \cdot \mathbf{n}_e = G_E^e && \forall e \in \partial E \\ &\operatorname{div} R_E(\mathbf{G}_E) = (\mathcal{DIV}^d\mathbf{G})_E && \text{in } E \end{aligned} \tag{16}$$

for all $\mathbf{G} \in X^d$,

$$R_E(\mathbf{G}_E^I) = \mathbf{G}_E \quad \forall \mathbf{G}_E \text{ constant on } E, \tag{17}$$

and the velocity scalar product can be written

$$[\mathbf{Q}, \mathbf{G}]_E = \int_E \mathbb{K}^{-1} R_E(\mathbf{Q}_E) \cdot R_E(\mathbf{G}_E)\, \mathrm{d}V \quad \forall \mathbf{Q}, \mathbf{G} \in X^d. \tag{18}$$

As shown in [11], the above properties automatically imply the following approximation property

$$||\mathbf{G} - R_E(\mathbf{G}^I)||_{L^2(E)} \leq C h_E ||\mathbf{G}||_{H^1(E)} \quad \forall \mathbf{G} \in [H^1(E)]^3,\ \forall E \in \Omega_h. \tag{19}$$

In the sequel we will indicate with R the global operator $X^d \to [L^2(\Omega)]^3$, which is obtained combining all the local lifting operators R_E element by element.

The following convergence result for the MFD method is proved in [9].

Theorem 1 *Assume that the domain Ω is convex and $\mathbb{K} \in W^{1,\infty}(\Omega)$. Let $(\mathbf{F}, p)$ be the solution of problem (1) and $(\mathbf{F}_d, p_d)$ the solution of problem (15). Then it holds*

$$\begin{aligned} &||\mathbf{F} - R\,\mathbf{F}_d||_{L^2(\Omega)} \leq Ch||p||_{H^2(\Omega)} \\ &||\operatorname{div}(\mathbf{F} - R\,\mathbf{F}_d)||_{L^2(\Omega)} = ||b - b^I||_{L^2(\Omega)} \leq Ch||b||_{H^1(\Omega)} \\ &||p^I - p_d||_{L^2(\Omega)} \leq Ch^2(||p||_{H^2(\Omega)} + ||b||_{H^1(\Omega)}). \end{aligned} \tag{20}$$

Note that the requirements Ω and $\mathbb{K}$ are needed only for $(20)_3$. In the general case, a simple modification of the proof in [9] leads to

Corollary 1 *Let $(\mathbf{F}, p)$ be the solution of problem (1) and $(\mathbf{F}_d, p_d)$ the solution of problem (15). Let $p \in H^{1+q}(\Omega),\ 0 < q \leq 1$. Then it holds*

$$\begin{aligned} &||\mathbf{F} - R\,\mathbf{F}_d||_{L^2(\Omega)} \leq Ch^q\, ||p||_{H^{1+q}(\Omega)} \\ &||p^I - p_d||_{L^2(\Omega)} \leq Ch^{s+q}\, (||p||_{H^{1+q}(\Omega)} + ||b||_{H^1(\Omega)}), \end{aligned} \tag{21}$$

where $0 \leq s \leq 1$ is a problem regularity constant, depending on K and on the shape of Ω.

Remark 2 In the recent contribution [12], the authors prove that properties (S1) and (S2), under a reasonable algebraic assumption on the discrete scalar product, imply the existence of the local lifting operator R_E. Such result confirms the general opinion that a "virtual" operator R_E essentially exists in all cases of interest.

3 A post-processing scheme for the scalar variable

In the present section we introduce a post-processing scheme, in the spirit of [22,25], for the mimetic finite difference method of Sect. 2 and show a convergence result for the improved solution. The post-processed pressure is used in the computation of the local error estimator of Sect. 4.

Let the discrete norm

$$|||q|||_{1,d}^2 = \sum_{E\in\Omega_h} ||\nabla q||^2_{L^2(E)} + \sum_{e\in\mathcal{E}_h} h_e^{-1} ||\, [\![q]\!]\, ||^2_{L^2(e)} \,, \tag{22}$$

for all q sufficiently regular, where $[\![\cdot]\!]$ represents the classical face jump operator, which is assumed to be equal to the function value on boundary faces. Moreover, let Q_1^d be the space of Ω_h-piecewise linear functions with zero average on each element.

Given $(\mathbf{F}_d, p_d)$ solution of problem (15), we define $\bar{p}_d$ as the unique function in Q_1^d that satisfies

$$|E|\, \nabla \bar{p}_d|_E \cdot \nabla q|_E = -[\mathbf{F}_d, (\nabla q)^I]_E \quad \forall E \in \Omega_h,\ \forall q \in Q_1^d \,. \tag{23}$$

We then set our post-processed pressure as

$$p_d^* = p_d + \bar{p}_d \,. \tag{24}$$

Note that, due to (18) and (17), from (23) it follows

$$\int_E \nabla p_d^* \cdot \nabla q \,\mathrm{d}V = -\int_E \mathbb{K}^{-1} R_E \mathbf{F}_d \cdot \nabla q \,\mathrm{d}V \quad \forall E \in \Omega_h,\ \forall q \in Q_1^d \,. \tag{25}$$

We then have the following result.

Proposition 1 *Assume that the domain Ω is convex, and $\mathbb{K} \in W^{1,\infty}$. Let p_d^* be computed as in* (24)*, and* $(\mathbf{F}, p)$ *be the solution of problem* (1)*. We then have*

$$|||p - p_d^*|||_{1,d} + h^{-1} ||p - p_d^*||_{L^2(\Omega)} \le Ch(||p||_{H^2(\Omega)} + ||b||_{H^1(\Omega)}) \,. \tag{26}$$

Proof The proof is based on the super convergence result (5.35) of [9], i.e. bound $(20)_3$.

Let in the sequel Π_h indicate the L^2 projection on the space of the scalar functions which are piecewise constant on the mesh Ω_h. We then have

$$||p - p_d^*||_{L^2(\Omega)} \le ||\Pi_h(p - p_d^*)||_{L^2(\Omega)} + ||(I - \Pi_h)(p - p_d^*)||_{L^2(\Omega)} \,, \tag{27}$$

where I indicates the identity operator.

The first term in the right hand side of (27) is bounded recalling that $\Pi_h p_d^* = p_d$, using the L^2 continuity of Π_h and finally applying the super convergence result $(20)_3$:

$$||\Pi_h(p - p_d^*)||_{L^2(\Omega)} = ||p^I - p_d||_{L^2(\Omega)} \le Ch^2(||p||_{H^2(\Omega)} + ||b||_{H^1(\Omega)}) \,. \tag{28}$$

Let now $\tilde{p}_d$ be the unique function in Q_1^d that satisfies

$$\int_E \nabla \tilde{p}_d \cdot \nabla q \, \mathrm{d}V = -\int_E \mathbb{K}^{-1}\mathbf{F} \cdot \nabla q \, \mathrm{d}V \quad \forall E \in \Omega_h, \ \forall q \in Q_1^d \, . \tag{29}$$

Note that, due to (1), Eq. (29) implies that $\nabla \tilde{p}_d$ is the L^2 projection of ∇p on piecewise constant vectors. Therefore a standard approximation argument, see for instance [7,8,15], immediately gives

$$||\nabla(p - \tilde{p}_d)||_{L^2(E)} \leq C h_E |p|_{H^2(E)} \, . \tag{30}$$

We now bound the second term in the right hand side of (27). Let E be a general element in Ω_h. Due to classical approximation results, see again for instance [7,8,15], we have

$$||(I - \Pi_h)(p - p_d^*)||_{L^2(E)} \leq C h_E ||\nabla(p - p_d^*)||_{L^2(E)} \tag{31}$$

which, applying a triangle inequality and bound (30), gives

$$\begin{aligned} ||(I - \Pi_h)(p - p_d^*)||_{L^2(E)} &\leq C h_E ||\nabla p - \nabla \tilde{p}_d||_{L^2(E)} + h_E ||\nabla \tilde{p}_d - \nabla p_d^*||_{L^2(E)} \\ &\leq C h_E^2 |p|_{H^2(E)} + h_E ||\nabla \tilde{p}_d - \nabla p_d^*||_{L^2(E)}. \end{aligned} \tag{32}$$

Due to definition (29) and property (25), it follows

$$||\nabla \tilde{p}_d - \nabla p_d^*||^2_{L^2(E)} = \int_E \mathbb{K}^{-1}(R_E \mathbf{F}_d - \mathbf{F}) \cdot \nabla(\tilde{p}_d - p_d^*) \, \mathrm{d}V \, , \tag{33}$$

which, using the $L^\infty(\Omega)$ bound on $\mathbb{K}^{-1}$ and a Cauchy–Schwartz inequality, gives

$$||\nabla \tilde{p}_d - \nabla p_d^*||_{L^2(E)} \leq C ||\mathbf{F} - R_E \mathbf{F}_d||_{L^2(E)} \, . \tag{34}$$

Summing the squares over all the elements and recalling the approximation result $(20)_1$, from (32) and (34) we get

$$||(I - \Pi_h)(p - p_d^*)||_{L^2(\Omega)} \leq C h^2 ||p||_{H^2(\Omega)} \, . \tag{35}$$

Combining (27), (28) and (35) gives

$$||p - p_d^*||_{L^2(\Omega)} \leq C h^2 (||p||_{H^2(\Omega)} + ||b||_{H^1(\Omega)}) \, . \tag{36}$$

The result for the $|||p - p_d^*|||_{1,d}$ norm follows easily, since the element internal contributions where already bounded as a byproduct of the steps (31)–(35). We get

$$\sum_{E \in \Omega_h} ||\nabla(p - p_d^*)||^2_{L^2(E)} \leq C h^2 ||p||^2_{H^2(\Omega)} \, . \tag{37}$$

The face contributions are bounded using an Agmon inequality—see [2,3]—and applying the previous bounds

$$\begin{aligned}\sum_{e\in\mathcal{E}_h} h_e^{-1}||[\![p-p_d^*]\!]||^2_{L^2(e)} &\leq C\sum_{E\in\Omega_h} h_e^{-2}||p-p_d^*||^2_{L^2(E)} + ||\nabla(p-p_d^*)||^2_{L^2(E)}\\ &\leq Ch^2\left(||p||_{H^2(\Omega)}+||b||_{H^1(\Omega)}\right)^2 .\end{aligned}\tag{38}$$

□

In the general case of a non-convex domain Ω and general $\mathbb{K}$, a simple modification of the proof shows that a weaker convergence result, in the spirit of Corollary 1, holds. Note that, due to property (S2), the solution $\bar{p}_d$ of (23) does not depend on the choice for the scalar product $[\cdot,\cdot]_E$.

Remark 3 An equivalent post-processing for the pressure variable is presented in [13], where the authors focus also on the computational aspects in the calculation of p_d^*.

4 A local error estimator

In the present section we introduce a local error estimator, in the spirit of [22], for the mimetic finite difference method under study, and prove reliability and efficiency results. Even for the finite element method, local error estimators for the diffusion problem in mixed form are relatively recent, see [6,14,22]. For general results regarding a posteriori error analysis, we refer for instance to [1,26].

Given $(\mathbf{F}_d, p_d)$ solution of problem (15), and p_d^* computed as in (24), the proposed error estimator is given by

$$\begin{aligned}\eta^2 &= \sum_{E\in\Omega_h}\eta_E^2\\ \eta_E^2 &= |||(\mathbb{K}\nabla p_d^*)^I+\mathbf{F}_d|||_E^2 + h_E^2||b-b^I||^2_{L^2(E)} + \frac{1}{2}\sum_{e\in\partial E} h_e^{-1}||[\![p_d^*]\!]||^2_{L^2(e)} .\end{aligned}\tag{39}$$

We have the following result, stating the global reliability and local efficiency of the estimator. The proof is shown in the following sections.

Proposition 2 *Let* $(\mathbf{F}, p)$ *be the solution of problem* (1)*. Then it holds*

$$||\mathbf{F}-R\,\mathbf{F}_d||_{L^2(\Omega)} + h||\operatorname{div}(\mathbf{F}-R\,\mathbf{F}_d)||_{L^2(\Omega)} + |||p-p_d^*|||_{1,d} \leq C\eta .\tag{40}$$

Moreover,

$$\eta_E \leq C(||\mathbf{F}-R_E\mathbf{F}_d||_{L^2(E)} + h_E||\operatorname{div}(\mathbf{F}-R_E\mathbf{F}_d)||_{L^2(E)} + |||p-p_d^*|||_{1,d,E})\tag{41}$$

where the norm $|||\cdot|||_{1,d,E}$ *stands for the discrete norm* $|||\cdot|||_{1,d}$ *with the sums restricted to the element* E *and the faces* $e\in\partial E$.

Note that the norm (40) in which we measure the error for the vector variable is essentially equivalent to the $|||\cdot|||_{\text{div}}$ norm adopted in [9], see also Remark 4 below. Simply, the assumption (S) on the existence of the lifting operator R_E allows us to write it in L^2 form. The a-priori error estimates for the norm in (40) are listed in Theorem 1 and Proposition 1.

The term $h_E||b - b^I||_{L^2(E)}$ appearing in the estimator is not a higher-order term and, in principle, it is not exactly computable. Nevertheless, it can be easily estimated up to higher order terms with a sufficiently high quadrature rule. Finally note that, due to the identity

$$b - b^I = \text{div}(\mathbf{F} - R_E\mathbf{F}_d) \tag{42}$$

shown in [9], a different scaling can be adopted for the divergence error term and the estimator η easily changed accordingly.

Remark 4 Assume that $\mathbf{F} \in W^{m,q}(\Omega)$ for some $q > 2$, $0 < m \leq 1$. Using interpolation properties, inverse inequalities and Lemma 4.1 in [9], it easily follows

$$|||\mathbf{F}^I - \mathbf{F}_d|||_{X^d} \leq C_q \left(h^m|\mathbf{F}|_{W^{m,q}(\Omega)} + h^{3(2-q)/2q}||\mathbf{F} - R\mathbf{F}_d||_{L^2(\Omega)} \right.$$
$$\left. + \left(\sum_{E\in\Omega_h} h_E^2 ||\,\text{div}\,\mathbf{F} - \text{div}\,R\mathbf{F}_d||_{L^2(E)} \right)^{1/2} \right) \tag{43}$$

where the constant C_q depends on q and blows up for $q \to 2$. Therefore, even when the lifting operator R is not explicitly known, the first two terms in the norm (40) are important nevertheless. Indeed, it is clear from the previous bound that the convergence of $R\mathbf{F}_d$ to $\mathbf{F}$ in (40) implies the convergence of $\mathbf{F}_d$ to $\mathbf{F}^I$ in (43).

4.1 Preliminary results

In the present section we introduce some preliminary result. We need the following definition. Given the mesh Ω_h, we subdivide each face e of $\mathcal{E}_h$ into triangles, connecting each vertex of each face e with the point M_e introduced in (M4). We call this set of triangles Ξ. Then, we subdivide each element E into tetrahedrons, connecting each vertex of each triangle in Ξ with the point M_E introduced in (M6). This gives a conforming mesh of tetrahedrons for Ω which we call Ω_h'.

We only sketch the proof of the following geometrical lemma.

Lemma 1 *Let $\{T_i\}_{i\in\mathbb{N}}$, be a family of shape regular triangles, with baricenter b_i and diameter h_i. Let moreover c_1, c_2 be assigned positive real constants. Assume that each T_i is embedded in $\mathbb{R}^3$ and associate to each triangle a point $\xi_i \in \mathbb{R}^3$ which satisfies the following two properties*:

(G1) $||\xi_i - b_i|| \leq c_1h_i$;
(G2) *it exists a ball of radius $r_i \geq c_2h_i$, centered in ξ_i, which does not intersect the plane containing the verticies of T_i.*

Finally, let $\{\bar{T}_i\}_{i\in\mathbb{N}}$ *be the family of tetrahedrons built connecting the verticies of each triangle* T_i *with the respective* ξ_i*. Then, the family of tetrahedrons* $\{\bar{T}_i\}_{i\in\mathbb{N}}$ *is shape-regular.*

Proof By definition, it is sufficient to show the existence of two positive constants c_3, c_4 such that every $\bar{T}_i$ is contained in a ball of radius $c_3 h_i$ and contains a ball of radius $c_4 h_i$, $i \in \mathbb{N}$. Due to property (G1), the first condition is trivial, for instance with $c_3 = \max(1, c_1)$. Since the family of the bases T_i is regular, it exists a positive constant c_5 such that every T_i contains a bi-dimensional ball B_i of radius $r_i' \geq c_5 h_i$. Let C_i indicate the (skew) cone defined by the base B_i and the vertex ξ_i; such cone is clearly contained in $\bar{T}_i$. Then, recalling properties (G1), (G2), it is easy to check that each cone C_i, $i \in \mathbb{N}$, contains a sphere of radius $r_i'' \geq c_4 h_i$ with c_4 depending only on c_1, c_2, c_5. □

We then have the following result.

Lemma 2 *The family* $\{\Omega_h'\}_h$ *is a shape-regular family of meshes.*

Proof By construction, any tetrahedron $T \in \Omega_h'$ has a face which is in the set Ξ. We start proving that all such faces are uniformly shape regular. In order to show such claim, it is sufficient to prove that it exists an $\alpha_* > 0$, independent of h, such that each face $e' \in \Xi$ satisfies the following property:
(P) the amplitude of each internal angle of the face e' is higher than or equal to α_*.

Note that, recalling properties (M3)–(M6), for all edges l_T of each tetrahedron T_E of E, $E \in \Omega_h$, it holds

$$\min\{\gamma_*, \tau_*, l_*\}\, h_E \leq |l_T| \leq h_E. \tag{44}$$

Therefore the ratios between the length of the edges of each single tetrahedron T is uniformly bounded, i.e. it exists a $0 < \beta_* \leq 1$ such that

$$\min_{l \text{ edge of } T} |l| \geq \beta_* \max_{l \text{ edge of } T} |l| \quad \forall\, T \in \Omega_h'. \tag{45}$$

Given any face $e' \in \Xi$, due to (45), basic trigonometric arguments show that it is sufficient to prove property (P) for any two angles of e'.

Let therefore e' be a generic triangular face in the set Ξ, and $e'' \in \mathcal{E}_h$ the original polyhedral face from which e' was generated. Let $E \in \Omega_h$ be a polyhedron which has e'' as a face. We call a, b, c the verticies of e', where c is the vertex corresponding to M_e. Given $\overline{ac}$, the edge connecting a with c, let x_1, x_2 indicate the two points obtained by the intersection of the circle centered in c and radius $\gamma_* h_E$—see (M4)—with the straight line perpendicular to $\overline{ac}$ at c. Call x the point, among x_1 and x_2, which is nearest to b, see Fig. 1.

Then, due to the star-shape property (M4), the segment $\overline{ax}$ is contained in e'', which easily implies

$$\widehat{cab} \geq \widehat{cax}\ . \tag{46}$$

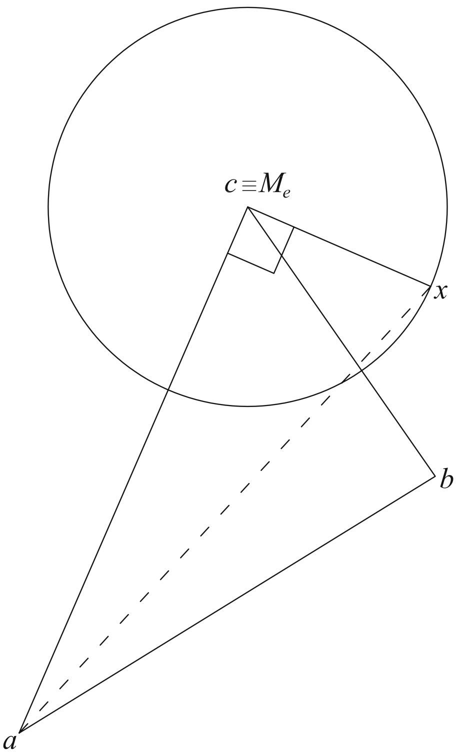

Fig. 1 Tetrahedron face in the set Ξ

Observing that the segments $|\overline{ac}| \le h_E$, $|\overline{cx}| = \gamma_* h_E$, and that the angle $\widehat{acx} = \frac{\pi}{2}$, from (46) it follows

$$\widehat{cab} \ge \widehat{cax} \ge \alpha_*, \tag{47}$$

with uniform α_*. The same argument can be repeated for the vertex b, giving

$$\widehat{abc} \ge \alpha_*. \tag{48}$$

Therefore all faces $e' \in \Xi$ satisfy property (P), i.e. are uniformly shape regular.

We now note that all tetrahedrons in Ω_h' are uniquely determined by the respective face $e' \in \Xi$ and the position of the fourth vertex (which corresponds to M_E for some E).

Due to the property (M6), it is easy to check that such tetrahedrons satisfy the assumptions of Lemma 1, where the role of the point ξ_i is played by the vertex in M_E. Therefore the proposition is proved. □

Given any function $\boldsymbol{\rho} \in H^1(\Omega)$, we define $\boldsymbol{\rho}_{II} \in H^1(\Omega)$ as the classical piecewise linear Clément interpolant of $\boldsymbol{\rho}$, based on the mesh Ω_h'. Due to the mesh shape-regularity shown in Lemma 2 it holds

$$h_T^{-1}||\boldsymbol{\rho}-\boldsymbol{\rho}_{II}||_{L^2(T)}+||\boldsymbol{\rho}_{II}||_{H^1(T)}+\sum_{e'\in\partial T} h_{e'}^{-1/2}||\boldsymbol{\rho}-\boldsymbol{\rho}_{II}||_{L^2(e')} \le C||\boldsymbol{\rho}||_{H^1(\omega_T)} \tag{49}$$

for all $T \in \Omega_h'$, where ω_T indicates the set of tetrahedrons in Ω_h' which have non-null intersection with T. For a proof of this result, see for example [7] and references therein. Note that, due to (M2), the number of tetrahedrons in ω_T, $T \in \Omega_h'$, is uniformly bounded. Therefore also the global counterpart of (49) immediately follows taking the sum of the squares

$$h||\boldsymbol{\rho} - \boldsymbol{\rho}_{II}||_{L^2(\Omega)} + ||\boldsymbol{\rho}_{II}||_{H^1(\Omega)} + \left(\sum_{e \in \mathcal{E}_h} h_e^{-1} ||\boldsymbol{\rho} - \boldsymbol{\rho}_{II}||^2_{L^2(e)} \right)^{1/2} \leq C||\boldsymbol{\rho}||_{H^1(\Omega)} . \tag{50}$$

We then have the following lemma.

Lemma 3 *Let $\mathbf{F}_d$ be the solution of problem (15). Given $\boldsymbol{\rho} \in [H^1(\Omega)]^3$, it exists $\bar{\boldsymbol{\rho}} \in [H^1(\Omega)]^3$ such that*

$$\begin{aligned}
&\int_\Omega \mathbb{K}^{-1} R\, \mathbf{F}_d \cdot \mathbf{curl}\, \bar{\boldsymbol{\rho}} \, \mathrm{d}V = 0 ;\\
&||\, \mathbf{curl}\, \bar{\boldsymbol{\rho}}||_{L^2(\Omega)} \leq C||\boldsymbol{\rho}||_{H^1(\Omega)} ; \qquad (51)\\
&\int_\Omega \nabla q^1 \cdot \mathbf{curl}(\boldsymbol{\rho}_{II} - \bar{\boldsymbol{\rho}}) \, \mathrm{d}V \leq C||\boldsymbol{\rho}||_{H^1(\Omega)} \left(\sum_{e \in \mathcal{E}_h} h_e^{-1} ||[\![q^1]\!]||^2_{L^2(e)} \right)^{1/2} \quad \forall q^1 \in Q_1^d.
\end{aligned}$$

By construction, the (distributional) divergence of the function $R(\mathbf{curl}\, \boldsymbol{\rho}_{II})^I$ is a Ω_h-piecewise constant function in $L^2(\Omega)$. Due to properties $(16)_2$ and (7), for all $E \in \Omega_h$ it holds

$$\mathrm{div}\, R_E\, (\mathbf{curl}\, \boldsymbol{\rho}_{II})^I_E = (\mathcal{DIV}^d (\mathbf{curl}\, \boldsymbol{\rho}_{II})^I)_E = (\mathrm{div}\, \mathbf{curl}\, \boldsymbol{\rho}_{II})^I_E = 0 . \tag{52}$$

Therefore the function $R(\mathbf{curl}\, \boldsymbol{\rho}_{II})^I$ in $[L^2(\Omega)]^3$ has zero divergence, which implies the existence of $\bar{\boldsymbol{\rho}} \in [H^1(\Omega)]^3$, such that

$$\mathbf{curl}\, \bar{\boldsymbol{\rho}} = R(\mathbf{curl}\, \boldsymbol{\rho}_{II})^I, \tag{53}$$

see for instance [16]. Note that $\bar{\boldsymbol{\rho}}$ is not unique; any choice is acceptable for our purposes.

From (18), $(15)_1$ and $\mathcal{DIV}^d(\mathbf{curl}\, \boldsymbol{\rho}_{II})^I = 0$, shown in (52), it follows

$$\begin{aligned}
\int_\Omega \mathbb{K}^{-1} R\mathbf{F}_d \cdot \mathbf{curl}\, \bar{\boldsymbol{\rho}} \, \mathrm{d}V &= \int_\Omega \mathbb{K}^{-1} R\mathbf{F}_d \cdot R(\mathbf{curl}\, \boldsymbol{\rho}_{II})^I \, \mathrm{d}V \\
&= [\mathbf{F}_d, (\mathbf{curl}\, \boldsymbol{\rho}_{II})^I]_{X^d} = 0,
\end{aligned} \tag{54}$$

which is $(51)_1$.

First using (2) and the definition of $\mathbf{curl}\,\bar{\boldsymbol{\rho}}$, then identity (18) and bound (S1), it follows

$$
\begin{aligned}
||\,\mathbf{curl}\,\bar{\boldsymbol{\rho}}||^2_{L^2(E)} &\le C \int_E \mathbb{K}^{-1} R_E(\mathbf{curl}\,\boldsymbol{\rho}_{II})^I \cdot R_E(\mathbf{curl}\,\boldsymbol{\rho}_{II})^I_E \,\mathrm{d}V \\
&= C|||(\mathbf{curl}\,\boldsymbol{\rho}_{II})^I_E|||^2_E \le C|E| \sum_{e\in\partial E} ((\mathbf{curl}\,\boldsymbol{\rho}_{II})^I_E)^2_e
\end{aligned}
\tag{55}
$$

for all $E \in \Omega_h$. Applying (5) and a Cauchy–Schwartz inequality, also recalling property (M3), from (55) we get

$$
\begin{aligned}
||\,\mathbf{curl}\,\bar{\boldsymbol{\rho}}||^2_{L^2(E)} &\le S_* \,|E| \sum_{e\in\partial E} |e|^{-2} \left(\int_e \mathbf{curl}\,\boldsymbol{\rho}_{II} \cdot \mathbf{n}^E_e \,\mathrm{d}\Sigma \right)^2 \\
&\le C|E| \sum_{e\in\partial E} |e|^{-1} \int_e \left(\mathbf{curl}\,\boldsymbol{\rho}_{II} \cdot \mathbf{n}^E_e\right)^2 \,\mathrm{d}\Sigma \\
&\le C h_E ||\,\mathbf{curl}\,\boldsymbol{\rho}_{II} \cdot \mathbf{n}^E_e||^2_{L^2(\partial E)}.
\end{aligned}
\tag{56}
$$

Using an Agmon and inverse inequality—tetrahedron by tetrahedron—it is easy to check that

$$
h_E||\,\mathbf{curl}\,\boldsymbol{\rho}_{II} \cdot \mathbf{n}^E_e||^2_{L^2(\partial E)} \le C||\,\mathbf{curl}\,\boldsymbol{\rho}_{II}||^2_{L^2(E)} \quad \forall E \in \Omega_h \tag{57}
$$

which, due to (56) and (50), implies $(51)_2$.

From the definition of $\bar{\boldsymbol{\rho}}$, property $(16)_1$ and definition (5), it follows

$$
\mathbf{curl}\,\bar{\boldsymbol{\rho}}|_e \cdot \mathbf{n}_e = \Pi_e(\mathbf{curl}\,\boldsymbol{\rho}_{II}|_e \cdot \mathbf{n}_e) \quad \forall e \in \mathcal{E}_h\,, \tag{58}
$$

where Π_e is the L^2 projection on constants for each face e. Therefore, $\mathbf{curl}\,\bar{\boldsymbol{\rho}}|_e \cdot \mathbf{n}_e$ is in $L^2(e)$, $e \in \mathcal{E}_h$, and integrating by parts element by element on Ω_h we get

$$
\int_\Omega \nabla q^1 \cdot \mathbf{curl}(\boldsymbol{\rho}_{II} - \bar{\boldsymbol{\rho}}) \,\mathrm{d}V = \sum_{e\in\mathcal{E}_h} \int_e [\![q^1]\!]\,\mathbf{curl}(\boldsymbol{\rho}_{II} - \bar{\boldsymbol{\rho}}) \cdot \mathbf{n}_e \,\mathrm{d}\Sigma \tag{59}
$$

for all $q^1 \in Q^d_1$. As a consequence of (58), we have

$$
||\,\mathbf{curl}(\boldsymbol{\rho}_{II} - \bar{\boldsymbol{\rho}}) \cdot \mathbf{n}_e||_{L^2(e)} \le C||\,\mathbf{curl}\,\boldsymbol{\rho}_{II} \cdot \mathbf{n}_e||_{L^2(e)} \quad \forall e \in \mathcal{E}_h\,. \tag{60}
$$

Due to identity (59), applying a Cauchy–Schwartz inequality and (60), we get

$$\int_{\Omega} \nabla q^1 \cdot \mathbf{curl}\left(\boldsymbol{\rho}_{II} - \bar{\boldsymbol{\rho}}\right) \, \mathrm{d}V = \sum_{e \in \mathcal{E}_h} \int_e [\![q^1]\!] \, \mathbf{curl}(\boldsymbol{\rho}_{II} - \bar{\boldsymbol{\rho}}) \cdot \mathbf{n}_e \, \mathrm{d}\Sigma$$
$$\leq \left(\sum_{e \in \mathcal{E}_h} h_e^{-1} ||[\![q^1]\!]||^2_{L^2(e)} \right)^{1/2} \left(\sum_{e \in \mathcal{E}_h} h_e || \mathbf{curl}\, \boldsymbol{\rho}_{II} \cdot \mathbf{n}_e ||^2_{L^2(e)} \right)^{1/2} . \quad (61)$$

Bound $(51)_3$ follows easily from (61) combined with (57) and (50). □

We will also need the following modified Helmholtz decomposition. The proof follows easily from the ellipticity of $\mathbb{K}$, using the same arguments as in the classical Helmholtz decomposition, shown for instance in [16].

Lemma 4 *Let the domain Ω and the tensor field $\mathbb{K}$ be defined as in Sect. 2. Let $\mathbf{v}$ be any function in $[L^2(\Omega)]^3$. Then there exist $\psi \in H_0^1(\Omega)$ and $\boldsymbol{\rho} \in [H^1(\Omega)]^3$, not unique, such that*

$$\begin{aligned} &\mathbf{v} = \mathbb{K}\nabla\psi + \mathbf{curl}\,\boldsymbol{\rho} \ , \\ &||\psi||_{H^1(\Omega)} + ||\boldsymbol{\rho}||_{H^1(\Omega)} \leq C_\Omega ||\mathbf{v}||_{L^2(\Omega)} \ , \end{aligned} \quad (62)$$

with the constant C_Ω depending only on Ω.

Remark 5 In the bi-dimensional case, the functions $\boldsymbol{\rho}$, $\boldsymbol{\rho}_{II}$ and $\bar{\boldsymbol{\rho}}$ appearing in the present and the following sections are scalars in $H^1(\Omega)$. As already noted, for simplicity of exposition we adopt the notation and proofs related to the three dimensional case, the bi-dimensional ones being a simpler derivation.

4.2 Proof of Proposition 2

We are now able to show the proof of Proposition (2), which we divide in four steps. We first prove separately the upper bounds for the three terms in the left hand side of (40), and finally show the lower bound (41).

Upper bound for $||\mathbf{F} - R_E\mathbf{F}_d||_{L^2(\Omega)}$.

Due to the ellipticity property (2) of $\mathbb{K}$, it is equivalent to bound $||\mathbb{K}^{-1}(\mathbf{F} - R_E\mathbf{F}_d)||_{L^2(\Omega)}$, which will be also useful in the sequel. From the definition of L^2 norm and Lemma 4, recalling the symmetry of $\mathbb{K}$, it follows

$$||\mathbb{K}^{-1}(\mathbf{F} - R\,\mathbf{F}_d)||_{L^2(\Omega)} = \sup_{v \in L^2(\Omega)} \frac{\int_\Omega \mathbb{K}^{-1}(\mathbf{F} - R\,\mathbf{F}_d) \cdot \mathbf{v}\,\mathrm{d}V}{||\mathbf{v}||_{L^2(\Omega)}}$$
$$\leq C \left(\sup_{\psi \in H_0^1(\Omega)} \frac{\int_\Omega (\mathbf{F} - R\,\mathbf{F}_d) \cdot \nabla\psi\,\mathrm{d}V}{||\psi||_{H^1(\Omega)}} + \sup_{\boldsymbol{\rho} \in [H^1(\Omega)]^3} \frac{\int_\Omega \mathbb{K}^{-1}(\mathbf{F} - R\,\mathbf{F}_d) \cdot \mathbf{curl}\,\boldsymbol{\rho}\,\mathrm{d}V}{||\boldsymbol{\rho}||_{H^1(\Omega)}} \right). \quad (63)$$

An integration by parts, identity (42) and the L^2-orthogonality of $(b - b^I)$ with respect to piecewise constants gives

$$\begin{aligned}\int_\Omega (\mathbf{F} - R\,\mathbf{F}_d) \cdot \nabla\psi \,\mathrm{d}V &= \int_\Omega \mathrm{div}(\mathbf{F} - R\,\mathbf{F}_d)\,\psi\,\mathrm{d}V = \int_\Omega (b - b^I)\,\psi\,\mathrm{d}V \\ &= \int_\Omega (b - b^I)\,(\psi - \psi^I)\,\mathrm{d}V \\ &\leq Ch||b - b^I||_{L^2(\Omega)}|\psi|_{H^1(\Omega)} \leq C\,\eta\,||\psi||_{H^1(\Omega)} \qquad (64)\end{aligned}$$

for all $\psi \in H^1_0(\Omega)$.

Given any $\boldsymbol{\rho} \in [H^1(\Omega)/\mathbb{R}]^3$, let now $\bar{\boldsymbol{\rho}}$ be the function defined in Lemma 3. Recalling definition (1) and property $(51)_1$, an integration by parts gives

$$\begin{aligned}\int_\Omega \mathbb{K}^{-1}(\mathbf{F} - R\,\mathbf{F}_d) \cdot \mathbf{curl}\,\boldsymbol{\rho}\,\mathrm{d}V &= -\int_\Omega \mathbb{K}^{-1} R\,\mathbf{F}_d \cdot \mathbf{curl}\,\boldsymbol{\rho}\,\mathrm{d}V \\ &= \int_\Omega \mathbb{K}^{-1} R\,\mathbf{F}_d \cdot \mathbf{curl}(\bar{\boldsymbol{\rho}} - \boldsymbol{\rho})\,\mathrm{d}V \\ &= \sum_{E\in\Omega_h} \int_E (\mathbb{K}^{-1} R\,\mathbf{F}_d + \nabla p_d^*) \cdot \mathbf{curl}(\bar{\boldsymbol{\rho}} - \boldsymbol{\rho})\,\mathrm{d}V \\ &\quad + \sum_{E\in\Omega_h} \int_E \nabla p_d^* \cdot \mathbf{curl}(\boldsymbol{\rho} - \bar{\boldsymbol{\rho}})\,\mathrm{d}V \\ &= I_3 + I_4. \qquad (65)\end{aligned}$$

Note that, due to (17),

$$\mathbb{K}\nabla p_d^*|_E = R_E(\mathbb{K}\nabla p_d^*)^I\ . \qquad (66)$$

Therefore, from the symmetry of $\mathbb{K}^{-1}$ and (2) , a Cauchy–Schwartz inequality and bound $(51)_2$, also recalling identity (18), it follows

$$\begin{aligned}I_3 &\leq ||\mathbb{K}^{-1/2} R\,(\mathbf{F}_d + (\mathbb{K}\nabla p_d^*)^I)||_{L^2(\Omega)}||\mathbb{K}^{-1/2}\,\mathbf{curl}(\bar{\boldsymbol{\rho}} - \boldsymbol{\rho})||_{L^2(\Omega)} \\ &\leq C||\mathbb{K}^{-1/2} R\,(\mathbf{F}_d + (\mathbb{K}\nabla p_d^*)^I)||_{L^2(\Omega)}||\boldsymbol{\rho}||_{H^1(\Omega)} \\ &= C|||\mathbf{F}_d + (\mathbb{K}\nabla p_d^*)^I|||_{X^d}||\boldsymbol{\rho}||_{H^1(\Omega)} \leq C\,\eta\,||\boldsymbol{\rho}||_{H^1(\Omega)} \qquad (67)\end{aligned}$$

Adding and subtracting the interpolant $\boldsymbol{\rho}_{II}$ introduced in (50), and recalling bound $(51)_3$, gives

$$
\begin{aligned}
I_4 &= \sum_{E\in\Omega_h}\int_E \nabla p_d^* \cdot \mathbf{curl}(\boldsymbol{\rho}_{II} - \bar{\boldsymbol{\rho}})\,\mathrm{d}V + \sum_{E\in\Omega_h}\int_E \nabla p_d^* \cdot \mathbf{curl}(\boldsymbol{\rho} - \boldsymbol{\rho}_{II})\,\mathrm{d}V \\
&\le C||\boldsymbol{\rho}||_{H^1(\Omega)} \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}||[\![p_d^*]\!]||^2_{L^2(e)}\right)^{1/2} + \left|\sum_{E\in\Omega_h}\int_E \nabla p_d^* \cdot \mathbf{curl}(\boldsymbol{\rho} - \boldsymbol{\rho}_{II})\,\mathrm{d}V\right| \\
&= I_5 + I_6
\end{aligned}
\tag{68}
$$

We have

$$
I_5 \le C\,\eta\,||\boldsymbol{\rho}||_{H^1(\Omega)}\,. \tag{69}
$$

For the second term, an integration by parts and some algebra give

$$
I_6 = \left|\sum_{e\in\mathcal{E}_h}\int_e ((\boldsymbol{\rho} - \boldsymbol{\rho}_{II}) \times \mathbf{n}_e) \cdot [\![\nabla p_d^*]\!]\,\mathrm{d}\Sigma\right|\,. \tag{70}
$$

Observing that the vector field $((\boldsymbol{\rho} - \boldsymbol{\rho}_{II}) \times \mathbf{n}_e)$ is orthogonal to $\mathbf{n}_e$, $e \in \mathcal{E}_h$, identity (70) becomes

$$
I_6 = \left|\sum_{e\in\mathcal{E}_h}\int_e ((\boldsymbol{\rho} - \boldsymbol{\rho}_{II}) \times \mathbf{n}_e) \cdot P_e[\![\nabla p_d^*]\!]\,\mathrm{d}\Sigma\right|\,, \tag{71}
$$

where P_e is the vector projection on the plane tangent to e. Moreover it is easy to check that

$$
P_e[\![\nabla p_d^*]\!] = \nabla_s[\![p_d^*]\!]\,, \tag{72}
$$

where ∇_s indicates the 2-component gradient of the function $[\![p_d^*]\!]$ living on e.

Using (71), (72), an inverse estimate and bound (50), we get

$$
\begin{aligned}
I_6 &\le \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}||\boldsymbol{\rho} - \boldsymbol{\rho}_{II}||^2_{L^2(e)}\right)^{1/2} \left(\sum_{e\in\mathcal{E}_h} h_e||\nabla_s[\![p_d^*]\!]||^2_{L^2(e)}\right)^{1/2} \\
&\le C||\boldsymbol{\rho}||_{H^1(\Omega)} \left(\sum_{e\in\mathcal{E}_h} h_e^{-1}||[\![p_d^*]\!]||^2_{L^2(e)}\right)^{1/2} \le C\,\eta\,||\boldsymbol{\rho}||_{H^1(\Omega)}.
\end{aligned}
\tag{73}
$$

Combining (65) with (67)–(69) and (73) it finally follows

$$
\int_\Omega \mathbb{K}^{-1}(\mathbf{F} - R\,\mathbf{F}_d) \cdot \mathbf{curl}\,\boldsymbol{\rho}\,\mathrm{d}V \le C\,\eta\,||\boldsymbol{\rho}||_{H^1(\Omega)}\,. \tag{74}
$$

Bound (63), using (64) and (74), gives

$$
||\mathbb{K}^{-1}(\mathbf{F} - R\,\mathbf{F}_d)||_{L^2(\Omega)} \le C\eta\,. \tag{75}
$$

Upper bound for $h|| \operatorname{div}(\mathbf{F} - R_E \mathbf{F}_d)||_{L^2(\Omega)}$.
The bound for this term is immediate due to (42). It holds

$$h|| \operatorname{div}(\mathbf{F} - R\,\mathbf{F}_d)||_{L^2(\Omega)} \leq \eta\,. \tag{76}$$

Upper bound for $|||p - p_d^*|||_{1,d}$.
Using (1) and some simple algebra, we get

$$\begin{aligned}
\sum_{E\in\Omega_h} |||\nabla(p - p_d^*)|||^2_{L^2(E)} &= \sum_{E\in\Omega_h} \int_E \mathbb{K}^{-1}(R_E\mathbf{F}_d - \mathbf{F}) \cdot \nabla(p - p_d^*)\,\mathrm{d}V \\
&\quad - \sum_{E\in\Omega_h} \int_E \mathbb{K}^{-1}(R_E\mathbf{F}_d + \mathbb{K}\nabla p_d^*) \cdot \nabla(p - p_d^*)\,\mathrm{d}V \\
&= I_7 + I_8\,.
\end{aligned} \tag{77}$$

A Cauchy–Schwartz inequality and bound (75) gives for the first term

$$\begin{aligned}
I_7 &\leq C||(\mathbb{K}^{-1}(R\,\mathbf{F}_d - \mathbf{F})||_{L^2(\Omega)} \left(\sum_{E\in\Omega_h} |||\nabla(p - p_d^*)|||^2_{L^2(E)}\right)^{1/2} \\
&\leq C\,\eta \left(\sum_{E\in\Omega_h} |||\nabla(p - p_d^*)|||^2_{L^2(E)}\right)^{1/2}
\end{aligned} \tag{78}$$

For the second term, again a Cauchy–Schwartz inequality and adopting the same steps as in (66), (67), it follows

$$\begin{aligned}
I_8 &\leq C||(\mathbb{K}^{-1/2}(R\,\mathbf{F}_d + \mathbb{K}\nabla p_d^*)||_{L^2(\Omega)} \left(\sum_{E\in\Omega_h} |||\mathbb{K}^{-1/2}\nabla(p - p_d^*)|||^2_{L^2(E)}\right)^{1/2} \\
&\leq C|||\mathbf{F}_d + (\mathbb{K}\nabla p_d^*)^I|||_{X^d} \left(\sum_{E\in\Omega_h} |||\nabla(p - p_d^*)|||^2_{L^2(E)}\right)^{1/2} \\
&\leq C\,\eta \left(\sum_{E\in\Omega_h} |||\nabla(p - p_d^*)|||^2_{L^2(E)}\right)^{1/2}.
\end{aligned} \tag{79}$$

For the face terms, noting that the face jumps of p are null, we have immediately

$$\sum_{e\in\mathcal{E}_h} h_e^{-1}||\,[\![p - p_d^*]\!]\,||^2_{L^2(e)} \leq \sum_{e\in\mathcal{E}_h} h_e^{-1}||\,[\![p_d^*]\!]\,||^2_{L^2(e)} \leq C\eta^2 \tag{80}$$

Combining bounds (77)–(80) and recalling (22), it finally follows

$$|||p - p_d^*|||_{1,d} \le C\eta \,. \tag{81}$$

Lower bounds

Using the identities (18) and (66), bound (2) and a triangle inequality coupled with (1) give

$$\begin{aligned} |||(\mathbb{K}\nabla p_d^*)^I + \mathbf{F}_d|||_E &= ||\mathbb{K}^{-1/2}(\mathbb{K}\nabla p_d^* + R_E\mathbf{F}_d)||_{L^2(E)} \le C||\mathbb{K}\nabla p_d^* + R_E\mathbf{F}_d||_{L^2(E)} \\ &\le C\left(||\mathbb{K}(\nabla p_d^* - \nabla p)||_{L^2(E)} + ||R_E\mathbf{F}_d - \mathbf{F}||_{L^2(E)}\right) \\ &\le C\left(|||p - p_d^*|||_{1,d,E} + ||R_E\mathbf{F}_d - \mathbf{F}||^2_{L^2(E)}\right). \end{aligned} \tag{82}$$

From identity (42) it follows

$$h_E^2||b - b^I||_{L^2(E)} = h_E^2||\operatorname{div}\mathbf{F} - \operatorname{div} R_E\mathbf{F}_d||_{L^2(E)} \,. \tag{83}$$

Finally, due to the L^2 trace continuity of p, we get

$$\sum_{e\in\partial E} h_e^{-1}||[\![p_d^*]\!]||^2_{L^2(e)} = \sum_{e\in\partial E} h_e^{-1}||[\![p_d^* - p]\!]||^2_{L^2(e)} \le |||p - p_d^*|||^2_{1,d,E} \,. \tag{84}$$

Bound (41) follows from (82)–(84). □

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