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Energy-conserved splitting FDTD methods for Maxwell's equations

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Abstract In this paper, two new energy-conserved splitting methods (EC-S-FDTDI and EC-S-FDTDII) for Maxwell's equations in two dimensions are proposed. Both algorithms are energy-conserved, unconditionally stable and can be computed efficiently. The convergence results are analyzed based on the energy method, which show that the EC-S-FDTDI scheme is of first order in time and of second order in space, and the EC-S-FDTDII scheme is of second order both in time and space. We also obtain two identities of the discrete divergence of electric fields for these two schemes. For the EC-S-FDTDII scheme, we prove that the discrete divergence is of first order to approximate the exact divergence condition. Numerical dispersion analysis shows that these two schemes are non-dissipative. Numerical experiments confirm well the theoretical analysis results.

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1 Introduction

Maxwell's equations are very important in the electromagnetic world and are widely used in many areas of application in modern society. For example, the design of the

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CPU in today's microelectronic field heavily depends on simulations of Maxwell's equations. Recently, it is of special importance to develop efficient numerical methods for effective and accurately simulating Maxwell's equations in large scale and long time computations.

Beginning with the pioneering works of Peaceman, Douglas and Rachford, as efficient numerical techniques to solve multidimensional parabolic problems, the alternating direction implicit methods (ADI) and the fractional step methods (FS) are very attractive and popular in solving partial differential equations for saving the memory and CPU time (see, for example, [3,5,20,21,23]; and more recent works [2,4,14,16], etc). In computations of Maxwell's equations, many works related to the ADI technique have been taken for overcoming the complexities and the huge computational costs. For example, Holland in [12] discussed the ADI method combined with Yee's scheme for the two-dimensional problems, however, the proposed scheme was difficult for obtaining the unconditional stability property for Maxwell's equations in three dimensions. Zheng et al. in [26] first proposed an unconditionally stable ADI–FDTD scheme for the three-dimensional Maxwell's equations with an isotropic and lossless medium. The further analysis of accuracy and dispersion of this scheme was studied in [10,25]. Meanwhile, Namiki [19] proposed a kind ADI–FDTD scheme for the Maxwell's equations in two dimensions, which was proved to be unconditionally stable in [19] as well as in [24].

More recently, combining the splitting technique with the staggered Yee's grid, Gao et al. in [9] proposed two splitting finite-difference time–domain methods (S-FDTDI and S-FDTDII) for Maxwell's equations in two dimensions. The two methods are both effective and easy to be implemented. By using the energy method, it is proved that the S-FDTDI scheme is unconditionally stable and convergent with first order in time and second order in space for the case with perfectly electric conducting (PEC) boundary conditions. In order to increase the accuracy in time, an improved scheme (S-FDTDII) was proposed, which is equivalent to a second-order perturbation of the Crank–Nicolson scheme. The schemes have been applied to successfully solve a scattering problem with PML boundary conditions.

On the other hand, in *lossless* medium, it is well known that the density of the electromagnetic energy of the wave is constant at different times. Then, it is natural to ask: *Is it possible for the splitting schemes to keep the conservation laws of energy?* The distinct advantages of the previous ADI or splitting schemes are unconditionally stable and effective for high dimensional problems, however, these schemes often break the property of energy conservation of Maxwell's equations. As we know, to keep the original physical features is greatly important in constructing numerical schemes for different physical problems. For example, one successful and active research is to construct structure-preserving schemes (or called symplectic schemes) for the ODE systems (see [7,11] and the references therein). Therefore, for solving Maxwell's equations, there is strong interest in developing efficient splitting methods which conserve energy.

In this paper, inspired by the above question, we propose two new energy-conserved splitting finite-difference time–domain schemes (EC-S-FDTD), which have important properties: (i) energy-conserved; (ii) Unconditionally stable; (iii) Efficient like the S-FDTDI and S-FDTDII proposed in [9]; (iv) Non-dissipative. Based on the staggered

Yee's grid, by applying the splitting technique we construct the energy-conserved splitting finite-difference time-domain scheme (called EC-S-FDTDI), which consists of two stages for each time step, and then propose an improved scheme: EC-S-FDTDII, which is a simple three stages scheme. We prove that both the EC-S-FDTDI and EC-S-FDTDII schemes satisfy two energy conversation relations in the discrete norm sense. The stability and convergence of these schemes are then analyzed rigorously by the energy method. We prove both the EC-S-FDTDI and EC-S-FDTDII schemes to be unconditionally stable. Furthermore, we prove that the EC-S-FDTDI scheme is of first order in time and of second order in space and specially, the EC-S-FDTDII is of second order both in time and space. Moreover, for analyzing the divergence-free condition of Maxwell's equations, we obtain two identities of the discrete divergences of electric fields for these two schemes. For the EC-S-FDTDI scheme, we also prove that the discrete divergence is of first order to approximate the exact divergence free condition. Through dispersion analysis, we verify both the EC-S-FDTDI and EC-S-FDTDII schemes to be non-dissipative, where we show that the numerical dispersion relations of our schemes are both of second order to approximate exact dispersion relations. Numerical experiments are given to illustrate the performance of our new schemes for the problems in the constant and piecewise constant electric permittivity medium. Numerical results confirm our theoretical analysis results.

The remaining part of the paper is organized as follows. In Sect. 2, the conservation properties of Maxwell equations are introduced and the new EC-S-FDTD schemes are proposed for the two-dimensional Maxwell equations. In Sect. 3, the energy conservations, rigorous stability and convergence analysis of the EC-S-FDTD schemes are given based on the energy method. Numerical dispersion analysis is given in Sect. 4. Numerical experiments for the constant and piecewise constant coefficients are presented in Sect. 5. Finally, some conclusions are addressed in Sect. 6.

2 Maxwell's equations and splitting schemes

Maxwell's equations in differential form are a set of four coupled partial differential equations relating the electric fields **E**, the magnetic field **H**, the electric displacement **D**, and the magnetic flux density **B** (see [1,6]):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{4}$$

where ρ is the charge density, **J** is the current density. The electric and magnetic field variables are related through the constitutive relations as

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}, \quad \mathbf{J} = \sigma \mathbf{E}, \tag{5}$$

where ϵ is the electric permittivity , μ is the magnetic permeability, and σ is the electric conductivity.

Consider a two dimensional transverse electric (TE) polarization case in a *lossless* medium and there is no source fields in this case, so we get $\rho = 0, \mathbf{J} = 0$, $\mathbf{E} = (E_x(x, y, t), E_y(x, y, t), 0)$ and $\mathbf{H} = (0, 0, H_z(x, y, t))$. Therefore, the Maxwell's equations (1)–(4) become:

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon}\frac{\partial H_z}{\partial y}, \tag{6}$$

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon}\frac{\partial H_z}{\partial x}, \tag{7}$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right), \tag{8}$$

where $\mathbf{E} = (E_x(x, y, t), E_y(x, y, t))$ and $H_z = H_z(x, y, t)$ for $(x, y) \in \Omega$ $(t \in (0, T])$ denote the electric field and the magnetic field, respectively. For simplicity, we consider the perfectly electric conducting (PEC) boundary condition on the boundary $\partial\Omega$ of the rectangle domain $\Omega = [0, a] \times [0, b]$:

$$(\mathbf{E}, 0) \times (\mathbf{n}, 0) = 0, \quad \text{on} \quad (0, T] \times \partial\Omega, \tag{9}$$

where $\mathbf{n}$ is the outward normal vector on $\partial\Omega$. The PEC condition (9) can be recast as

$$E_x(x, 0, t) = E_x(x, b, t) = E_y(0, y, t) = E_y(a, y, t) = 0, \quad \text{on} \quad (0, T] \times \partial\Omega. \tag{10}$$

To solve the system, the initial conditions are needed:

$$\mathbf{E}(x, y, 0) = \mathbf{E}_0(x, y) = (E_{x0}(x, y), E_{y0}(x, y)), \quad \text{and} \quad H_z(x, y, 0) = H_{z0}(x, y). \tag{11}$$

The problem (6)–(11) has a unique solution for suitably smooth data (see [15]).

For simplicity in notations, only constant ϵ and μ are considered. The algorithms described in this paper can be easily extended to the cases of variable coefficients. In the numerical results, one case with jump coefficients is considered which confirms all nice properties of the schemes.

2.1 Energy conservations in lossless medium

When the medium is lossless, i.e., $J = 0$ in the Maxwell's equations, then by Green's formula it gets:

$$\left\langle \frac{\partial \mathbf{D}}{\partial t}, \mathbf{E} \right\rangle + \left\langle \frac{\partial \mathbf{B}}{\partial t}, \mathbf{H} \right\rangle = \langle \nabla \times \mathbf{H}, \mathbf{E} \rangle - \langle \nabla \times \mathbf{E}, \mathbf{H} \rangle$$
$$= -\int_{\Omega} \nabla \cdot (\mathbf{E} \times \mathbf{H}) dx = -\int_{\partial\Omega} (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} dS,$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\Omega)$ inner product. The equation represents Poynting theorem which describes the conservation of electromagnetic energy in lossless medium. Using the PEC boundary condition, we have

$$\frac{1}{2} \frac{\partial \langle \epsilon \mathbf{E}, \mathbf{E} \rangle}{\partial t} + \frac{1}{2} \frac{\partial \langle \mu \mathbf{H}, \mathbf{H} \rangle}{\partial t} = 0, \tag{12}$$

which means that the electromagnetic energy conserves, i.e., the electromagnetic energy in lossless medium is constant at different times.

Theorem 1 (Energy conservation I) *If* **E** *and* **H** *are the solutions of the Maxwell's equations* (1)–(4) *in lossless medium, and satisfy the boundary conditions*:

$$\mathbf{E} \times \mathbf{n} = 0, \quad or \quad \mathbf{H} \times \mathbf{n} = 0. \tag{13}$$

Then it holds

$$\int_{\Omega} \epsilon \, |\mathbf{E}(x, t)|^2 \, dx + \int_{\Omega} \mu \, |\mathbf{H}(x, t)|^2 \, dx \equiv \text{Constant}, \tag{14}$$

which means that the density of electromagnetic energy in lossless medium keeps constant at any time.

The Poynting theorem above is well known and can be found in many classic books ([1,6]), but the following energy conservation is less obvious.

Theorem 2 (Energy conservation II) *If* **E** *and* **H** *are the solutions of the Maxwell's equations* (2)–(4) *in lossless medium, and satisfy the boundary conditions*:

$$\mathbf{E} \times \mathbf{n} = 0, \quad or \quad \mathbf{H} \times \mathbf{n} = 0. \tag{15}$$

Then, we have

$$\int_{\Omega} \left(\epsilon \left| \frac{\partial \mathbf{E}}{\partial t} \right|^2 + \mu \left| \frac{\partial \mathbf{H}}{\partial t} \right|^2 \right) dx \equiv \text{Constant}. \tag{16}$$

Proof Taking the derivative with respect to t on both sides of (1) and (2), we get

$$\nabla \times \frac{\partial \mathbf{H}}{\partial t} = \frac{\partial \mathbf{J}}{\partial t} + \frac{\partial^2 \mathbf{D}}{\partial t^2}, \quad \text{and} \quad \nabla \times \frac{\partial \mathbf{E}}{\partial t} = -\frac{\partial^2 \mathbf{B}}{\partial t^2}. \tag{17}$$

Note that $\mathbf{E}$ and $\mathbf{H}$ satisfy the boundary condition (13), then

$$\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{n} = 0, \quad \text{or} \quad \frac{\partial \mathbf{H}}{\partial t} \times \mathbf{n} = 0, \quad \text{on the boundary} \tag{18}$$

Using integration by part [18], it holds that

$$\left\langle \nabla \times \frac{\partial \mathbf{H}}{\partial t}, \frac{\partial \mathbf{E}}{\partial t} \right\rangle - \left\langle \nabla \times \frac{\partial \mathbf{E}}{\partial t}, \frac{\partial \mathbf{H}}{\partial t} \right\rangle = 0. \tag{19}$$

Thus we obtain the following relation:

$$\frac{1}{2}\frac{d}{dt}\left(\left\langle \epsilon \frac{\partial \mathbf{E}}{\partial t}, \frac{\partial \mathbf{E}}{\partial t} \right\rangle + \left\langle \mu \frac{\partial \mathbf{H}}{\partial t}, \frac{\partial \mathbf{H}}{\partial t} \right\rangle \right) = -\left\langle \frac{\partial \mathbf{J}}{\partial t}, \frac{\partial \mathbf{E}}{\partial t} \right\rangle. \tag{20}$$

For the lossless medium ($J = 0$), Eq. (16) can be directly obtained from (20). □

Remark 3 The energy conservation relation (16) can be also written as the following form:

$$\int_{\Omega} \left(\frac{1}{\epsilon} |\nabla \times \mathbf{E}|^2 + \frac{1}{\mu} |\nabla \times \mathbf{H}|^2 \right) dx \equiv \text{Constant}. \tag{21}$$

Now for the electromagnetic waves in the lossless medium, there are two conservation laws: (14) and (16). Therefore it is natural to ask whether the numerical schemes can keep these properties.

2.2 Energy-conserved splitting FDTD methods

The staggered Yee's grid is used in the FDTD methods (see Fig. 1). Let Δx and Δy be the mesh sizes along the x and y directions, respectively, and Δt the time step size. For $i = 0, 1, \ldots, I$, $j = 0, 1, \ldots, J$ and $n = 0, 1, \ldots, N$, define $(x_i, y_j, t^n) = (i\Delta x, j\Delta y, n\Delta t)$, $x_{i+\frac{1}{2}} = x_i + \frac{1}{2}\Delta x$, $y_{j+\frac{1}{2}} = y_j + \frac{1}{2}\Delta y$ and $t^{n+\frac{1}{2}} = t^n + \frac{1}{2}\Delta t$. The grid function $U^n_{\alpha,\beta}$ is defined on the staggered grid where $\alpha = i$ or $i + \frac{1}{2}$ and $\beta = j$ or $j + \frac{1}{2}$, and $\delta_x U$, $\delta_y U$ and $\delta_u \delta_v U$ are defined as follows:

$$\delta_t U^n_{\alpha,\beta} = \frac{U^{n+\frac{1}{2}}_{\alpha,\beta} - U^{n-\frac{1}{2}}_{\alpha,\beta}}{\Delta t}, \qquad \delta_x U^n_{\alpha,\beta} = \frac{U^n_{\alpha+\frac{1}{2},\beta} - U^n_{\alpha-\frac{1}{2},\beta}}{\Delta x},$$
$$\delta_y U^n_{\alpha,\beta} = \frac{U^n_{\alpha,\beta+\frac{1}{2}} - U^n_{\alpha,\beta-\frac{1}{2}}}{\Delta y}, \quad \delta_u \delta_v U^n_{\alpha,\beta} = \delta_u(\delta_v U^n_{\alpha,\beta}),$$

where u and v can be taken as x or y direction. For the grid function $U^n_{\alpha,\beta}$, we may drop the subscript if there is no confusion.

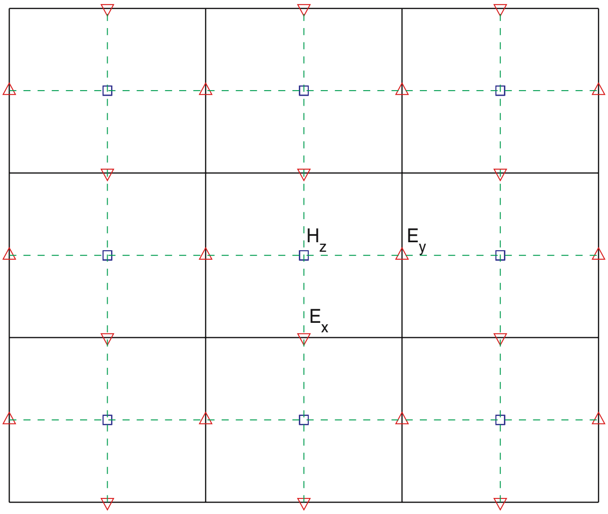

Fig. 1 Staggered grid, *square* for $H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}$, *triangle* for $E^n_{y_{i,j+\frac{1}{2}}}$, *inverted triangle* for $E^n_{x_{i+\frac{1}{2},j}}$

The Maxwell's equations (6)–(8) can be split into the following form (see [9]):

$$\begin{cases} \frac{\partial E_x}{\partial t} &= \frac{1}{\epsilon}\frac{\partial H_z}{\partial y} \\ \frac{1}{2}\frac{\partial H_z}{\partial t} &= \frac{1}{\mu}\frac{\partial E_x}{\partial y} \end{cases} \quad \text{and} \quad \begin{cases} \frac{\partial E_y}{\partial t} &= -\frac{1}{\epsilon}\frac{\partial H_z}{\partial x} \\ \frac{1}{2}\frac{\partial H_z}{\partial t} &= -\frac{1}{\mu}\frac{\partial E_y}{\partial x}. \end{cases} \tag{22}$$

Applying the spatial discretization approximation to the equations in (22) on the staggered grid, we now propose the energy-conserved splitting finite-difference time-domain method (EC-S-FDTDI) as follows.

(i) **The EC-S-FDTDI scheme**

Stage 1: Compute E_y^{n+1} and the intermediate variable H_z^* from H_z^n and E_y^n:

$$\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^n_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\epsilon}\delta_x\left\{H^*_{z_{i,j+\frac{1}{2}}} + H^n_{z_{i,j+\frac{1}{2}}}\right\}, \tag{23}$$

$$\frac{H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\mu}\delta_x\left\{E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^n_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\right\}. \tag{24}$$

Stage 2: Compute E_x^{n+1} and H_z^{n+1} from E_x^n and H_z^*:

$$\frac{E^{n+1}_{x_{i+\frac{1}{2},j}} - E^n_{x_{i+\frac{1}{2},j}}}{\Delta t} = \frac{1}{2\epsilon}\delta_y\left\{H^{n+1}_{z_{i+\frac{1}{2},j}} + H^*_{z_{i+\frac{1}{2},j}}\right\}, \tag{25}$$

$$\frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \frac{1}{2\mu}\delta_y\left\{E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^n_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\right\}. \tag{26}$$

The boundary values for the S-FDTDI scheme (23)–(26) are obtained from the PEC boundary condition (10):

$$E^n_{x\,i+\frac{1}{2},0} = E^n_{x_{i+\frac{1}{2},J}} = E^n_{y_{0,j+\frac{1}{2}}} = E^0_{y_{I,j+\frac{1}{2}}} = 0. \tag{27}$$

And the the initial values of $E^0_{\alpha,\beta}$ and $H^0_{\alpha,\beta}$ are given as:

$$E^0_{x_{\alpha,\beta}} = E_{x0}(\alpha\Delta x, \beta\Delta y);\;\; E^0_{y_{\alpha,\beta}} = E_{y0}(\alpha\Delta x, \beta\Delta y);\;\; H^0_{z_{\alpha,\beta}} = H_{z0}(\alpha\Delta x, \beta\Delta y). \tag{28}$$

Observing the EC-S-FDTDI scheme, we see that the computation of this scheme is very simple. Each stage, which contains only two equations, can be written equivalently as a tri-diagonal system of linear equations for the electric field vector E_y^{n+1} (or E_x^{n+1}) and a direct formulation of obtaining the magnetic field vector H_z^* (or H_z^{n+1}) explicitly.

After theoretical analysis and numerical experiments in the following sections, we know that the EC-S-FDTDI scheme is only of first order in time. By some modifications, the accuracy of the EC-S-FDTDI scheme can be remedied which leads us to propose a second order energy-conserved scheme: EC-S-FDTDII.

(ii) **The EC-S-FDTDII scheme:**

Stage 1: Compute the intermediate variables E_x^* and H_z^* from E_x^n and H_z^n:

$$\frac{E^*_{x_{i+\frac{1}{2},j}} - E^n_{x_{i+\frac{1}{2},j}}}{\Delta t} = \frac{1}{4\epsilon}\delta_y\left\{H^*_{z_{i+\frac{1}{2},j}} + H^n_{z_{i+\frac{1}{2},j}}\right\}, \tag{29}$$

$$\frac{H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \frac{1}{4\mu}\delta_y\left\{E^*_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^n_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\right\}. \tag{30}$$

Stage 2: Compute E_y^{n+1} and the intermediate variable H_z^{**} from E_y^n and H_z^*:

$$\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^n_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\epsilon}\delta_x\left\{H^{**}_{z_{i,j+\frac{1}{2}}} + H^*_{z_{i,j+\frac{1}{2}}}\right\}, \tag{31}$$

$$\frac{H^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\mu}\delta_x\left\{E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^n_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\right\}. \tag{32}$$

Stage 3: Compute E_x^{n+1} and H_z^{n+1} from H_z^{**} and E_x^*:

$$\frac{E^{n+1}_{x_{i+\frac{1}{2},j}} - E^*_{x_{i+\frac{1}{2},j}}}{\Delta t} = \frac{1}{4\epsilon}\delta_y\left\{H^{n+1}_{z_{i+\frac{1}{2},j}} + H^{**}_{z_{i+\frac{1}{2},j}}\right\}, \tag{33}$$

$$\frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \frac{1}{4\mu}\delta_y\left\{E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^*_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\right\}. \tag{34}$$

For this three-stages scheme, we will prove in next section that it conserves energy, unconditionally stable and has second order accuracy in both time and space.

Remark 4 Here, we would like to list the **S-FDTDI scheme** proposed by Gao et al. in [9] as follows.

Stage 1: Compute E_y^{n+1} and the intermediate variable H_z^* from H_z^n and E_y^n:

$$\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^{n}_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\epsilon}\delta_x \left\{ H^{*}_{z_{i,j+\frac{1}{2}}} + H^{n}_{z_{i,j+\frac{1}{2}}} \right\}, \tag{35}$$

$$\frac{H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\mu}\delta_x \left\{ E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} \right\}. \tag{36}$$

Stage 2: Compute E_x^{n+1} and H_z^{n+1} from E_x^n, H_x^n and H_z^*:

$$\frac{E^{n+1}_{x_{i+\frac{1}{2},j}} - E^{n}_{x_{i+\frac{1}{2},j}}}{\Delta t} = \frac{1}{2\epsilon}\delta_y \left\{ H^{n+1}_{z_{i+\frac{1}{2},j}} + H^{n}_{z_{i+\frac{1}{2},j}} \right\}, \tag{37}$$

$$\frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \frac{1}{2\mu}\delta_y \left\{ E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} \right\}. \tag{38}$$

Comparing our EC-S-FDTDI scheme (23)–(26) with the S-FDTDI scheme (35)–(38), only Eq. (37) and Eq. (25) are different, that is to say, we use H_z^* in Eq. (25) instead of H_z^n in Eq. (37). With this small modification, the new EC-S-FDTDI scheme will be proved to satisfy conservation of energy, but the S-FDTDI scheme does not. It is obvious that our EC-S-FDTDI scheme is as simple as the S-FDTDI scheme. Moreover, the EC-S-FDTDI scheme can be also rewritten as first order perturbation of Crank–Nicolson scheme for the Maxwell's equations in two dimensions.

Remark 5 Further, the **S-FDTDII scheme** was proposed by modifying Eq. (35) in the S-FDTDI scheme for improving the accuracy in [9] as bellow.

Stage 1: Compute E_y^{n+1} and the intermediate variable H_z^*:

$$\begin{aligned}\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^{n}_{y_{i,j+\frac{1}{2}}}}{\Delta t} &= -\frac{1}{2\epsilon}\delta_x \left\{ H^{*}_{z_{i,j+\frac{1}{2}}} + H^{n}_{z_{i,j+\frac{1}{2}}} \right\} \\ &\quad -\frac{\Delta t}{2\mu\epsilon}\delta_x\delta_y E^{n}_{x_{i,j+\frac{1}{2}}},\end{aligned} \tag{39}$$

$$\frac{H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\mu}\delta_x \left\{ E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} \right\}. \tag{40}$$

Stage 2: Compute E_x^{n+1} and H_z^{n+1} as stage 2 in S-FDTDI scheme.

A higher order term $-\frac{\Delta t}{2\mu\epsilon}\delta_x\delta_y E^n_{x_{i,j+\frac{1}{2}}}$ is added, and the S-FDTDII scheme is equivalent to a second order perturbation of the Crank–Nicolson scheme. The scheme has obtained efficient numerical results. On the other hand, the S-FDTDII scheme does not satisfy energy conservation. This motivates us to construct our new EC-S-FDTDII scheme (29) and (30) which satisfies energy conservation. We would like to mention that it seems difficult to find a two-stages scheme with second order accuracy in time and energy conservations property at the same time.

Comparing with two stages algorithm, the three stages algorithm (29) and (30) means one more tri-diagonal system is needed to be solved in every time step. We note that the extra tri-diagonal system also corresponds to one dimensional problem and can be solved by Thomson algorithm stably, i.e., the complexity to solve the tri-diagonal system is linear with one dimensional size N_x, so the extra cost is small. Comparing with the EC-S-FDTDI scheme, the EC-S-FDTDII scheme has second order accuracy both in time and space, and the energy conservation relations (14) and (16) can still hold well.

3 Stability and convergence analysis

In this section, we will analyze theoretically the energy conservation convergence properties for our new EC-S-FDTDI and EC-S-FDTDII schemes.

For grid functions defined on the staggered grids: $U := \{U_{i+\frac{1}{2},j}\}$, $V := \{V_{i,j+\frac{1}{2}}\}$, $W := \{W_{i+\frac{1}{2},j+\frac{1}{2}}\}$ and $\mathbf{F} := \{(U_{i+\frac{1}{2},j}, V_{i,j+\frac{1}{2}})\}$ the discrete L^2 energy norms are used:

$$\|U\|^2_{E_x} = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left|U_{i+\frac{1}{2},j}\right|^2 \Delta x\Delta y, \quad \|V\|^2_{E_y} = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left|V_{i,j+\frac{1}{2}}\right|^2 \Delta x\Delta y,$$

$$\|W\|^2_{H} = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left|W_{i+\frac{1}{2},j+\frac{1}{2}}\right|^2 \Delta x\Delta y, \quad \|\mathbf{F}\|^2_E = \|U\|^2_{E_x} + \|V\|^2_{E_y}.$$

For the analysis of the stability, the Abel transformation is introduced, which can be used to simplify the proof of the stability analysis.

Lemma 6 (Abel transformation) *Let $p \geq 1$ be any integer, and $\{a_k\}_{k=1}^p$ and $\{b_k\}_{k=1}^p$ be two sequences, then*

$$\sum_{k=1}^{p} a_k b_k = a_p B_p - \sum_{k=1}^{p-1}(a_{k+1} - a_k)B_k, \tag{41}$$

where $B_k = \sum_{i=1}^{k} b_i$.

Remark 7 The Abel transformation are essentially the discrete variant of the integral by part.

By Abel transformation, discrete Green's formula can be obtained.

Lemma 8 *Let $p \geq 1$ be any integer, and $\{a_k\}_{k=1}^{p}$ and $\{b_k\}_{k=0}^{p}$ be two sequences, then*

$$\sum_{k=1}^{p} a_k(b_k - b_{k-1}) = a_p b_p - a_1 b_0 - \sum_{k=1}^{p-1}(a_{k+1} - a_k)b_k. \tag{42}$$

The proof of this lemma can be found in [17].

Now with the PEC boundary conditions imposed, the following lemma is obtained directly. Instead of E_x, E_y and H_z, the grid functions U, V and W are used here to emphasize that the results do not depend on time.

Lemma 9 *Let the grid functions U, V and W be defined on the staggered grid. If U and V satisfy the boundary conditions*:

$$U_{i+\frac{1}{2},0} = U_{i+\frac{1}{2},J} = V_{0,j+\frac{1}{2}} = V_{I,j+\frac{1}{2}} = 0,$$

then for any integer $0 < i < I-1$ and $0 < j < J-1$, we have

$$\sum_{j=0}^{J-1} W_{i+\frac{1}{2},j+\frac{1}{2}} \delta_y U_{i+\frac{1}{2},j+\frac{1}{2}} = -\sum_{j=1}^{J-1} U_{i+\frac{1}{2},j} \delta_y W_{i+\frac{1}{2},j}, \tag{43}$$

$$\sum_{i=0}^{I-1} W_{i+\frac{1}{2},j+\frac{1}{2}} \delta_x V_{i+\frac{1}{2},j+\frac{1}{2}} = -\sum_{i=1}^{I-1} V_{i,j+\frac{1}{2}} \delta_x W_{i,j+\frac{1}{2}}, \tag{44}$$

and

$$\sum_{i=0}^{I-1}\sum_{j=1}^{J-1} U_{i+\frac{1}{2},j} \delta_x \delta_y V_{i+\frac{1}{2},j} = \sum_{i=1}^{I-1}\sum_{j=0}^{J-1} V_{i,j+\frac{1}{2}} \delta_x \delta_y U_{i,j+\frac{1}{2}}. \tag{45}$$

By this lemma, the energy conservation property of EC-S-FDTDI and II can be proven in next subsection.

3.1 Energy conservation and unconditionally stable

Theorem 10 (Discrete energy conservations) *For the integers $n \geq 0$, let $H_z^n := \{H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\}$ and $\mathbf{E}^n := \{(E^n_{x_{i+\frac{1}{2},j}}, E^n_{y_{i,j+\frac{1}{2}}})\}$ be the solutions of EC-S-FDTDI scheme* (23)–(26)*, then there exist the discrete energy conservation properties*:

$$\left\|\epsilon^{\frac{1}{2}}\mathbf{E}^{n+1}\right\|_E^2 + \left\|\mu^{\frac{1}{2}} H_z^{n+1}\right\|_H^2 = \left\|\epsilon^{\frac{1}{2}}\mathbf{E}^{n}\right\|_E^2 + \left\|\mu^{\frac{1}{2}} H_z^{n}\right\|_H^2, \tag{46}$$

and

$$\left\|\epsilon^{\frac{1}{2}}\delta_t \mathbf{E}^{n+\frac{1}{2}}\right\|_E^2 + \left\|\mu^{\frac{1}{2}}\delta_t H_z^{n+\frac{1}{2}}\right\|_H^2 = \left\|\epsilon^{\frac{1}{2}}\delta_t \mathbf{E}^{n-\frac{1}{2}}\right\|_E^2 + \left\|\mu^{\frac{1}{2}}\delta_t H_z^{n-\frac{1}{2}}\right\|_H^2. \quad (47)$$

Proof Multiplying both sides of (23) with $\epsilon \Delta t \left(E^{n+1}_{y_{i,j+\frac{1}{2}}} + E^{n}_{y_{i,j+\frac{1}{2}}}\right)$ and multiplying both sides of (24) with $\mu \Delta t \left(H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} + H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)$, we can get:

$$\epsilon\left[\left(E^{n+1}_{y_{i,j+\frac{1}{2}}}\right)^2 - \left(E^{n}_{y_{i,j+\frac{1}{2}}}\right)^2\right] = -\frac{\Delta t}{2}\delta_x\{H^{*}_{z_{i,j+\frac{1}{2}}} + H^{n}_{z_{i,j+\frac{1}{2}}}\}\left(E^{n+1}_{y_{i,j+\frac{1}{2}}} + E^{n}_{y_{i,j+\frac{1}{2}}}\right) \quad (48)$$

$$\mu\left[\left(H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2 - \left(H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right] = -\frac{\Delta t}{2}\delta_x\left\{E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\right\}\left(H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} + H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right). \quad (49)$$

Summing over all terms in the above two equations, and add them together, note that E_y satisfies the boundary condition (9), then by Lemma 9, we have

$$\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon\left(E^{n+1}_{y_{i,j+\frac{1}{2}}}\right)^2 + \mu\left(H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right) = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon\left(E^{n}_{y_{i,j+\frac{1}{2}}}\right)^2 + \mu\left(H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right). \quad (50)$$

Similarly, from (25), we have

$$\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon\left(E^{n+1}_{x_{i+\frac{1}{2},j}}\right)^2 + \mu\left(H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right) = \sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon\left(E^{n}_{x_{i+\frac{1}{2},j}}\right)^2 + \mu\left(H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right). \quad (51)$$

Combining Eq. (50) with Eq. (51), Eq. (46) is obtained.

For the discrete energy conservation II (Eq. (47)), denote by H_z^{*+1} the intermediate value H_z^* at time level $n+1$, and $\delta_t H_z^{*+\frac{1}{2}} = \frac{H_z^{*+1} - H_z^*}{\Delta t}$. Then from the EC-S-FDTDI scheme (23)–(26), $\delta_t \mathbf{E}^{n+\frac{1}{2}}$ and $\delta_t H_z^{n+\frac{1}{2}}$ satisfy the equations:

$$\delta_t E_{x_{i+\frac{1}{2},j}}^{n+\frac{3}{2}} - \delta_t E_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} = \frac{\Delta t}{2\epsilon}\delta_y \left\{\delta_t H_{z_{i+\frac{1}{2},j}}^{*+\frac{1}{2}} + \delta_t H_{z_{i+\frac{1}{2},j}}^{n+\frac{1}{2}}\right\}, \tag{52}$$

$$\delta_t H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{*+\frac{1}{2}} - \delta_t H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} = \frac{\Delta t}{2\mu}\delta_y \left\{\delta_t E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{3}{2}} + \delta_t E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}\right\}, \tag{53}$$

and

$$\delta_t E_{y_{i,j+\frac{1}{2}}}^{n+\frac{3}{2}} - \delta_t E_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} = -\frac{\Delta t}{2\epsilon}\delta_x \left\{\delta_t H_{z_{i,j+\frac{1}{2}}}^{n+\frac{3}{2}} + \delta_t H_{z_{i,j+\frac{1}{2}}}^{*+\frac{1}{2}}\right\}, \tag{54}$$

$$\delta_t H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{3}{2}} - \delta_t H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{*+\frac{1}{2}} = -\frac{\Delta t}{2\mu}\delta_x \left\{\delta_t E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{3}{2}} + \delta_t E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}\right\}. \tag{55}$$

Note that $\delta_t \mathbf{E}$ also satisfies the boundary condition, then following the proof of discrete energy conservation I (46) and using Lemma 9, we get the second discrete energy conversation (47). □

Similarly, the EC-S-FDTDII also keeps the properties of energy conservations.

Theorem 11 (Energy conservations) *For the integers* $n \geq 0$, *let* $\mathbf{E}^n := \{(E_{x_{i+\frac{1}{2},j}}^n, E_{y_{i,j+\frac{1}{2}}}^n)\}$ *and* $H_z^n := \{H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^n\}$ *be the solutions of EC-S-FDTDII scheme* (29)–(32), *then there exist the energy conservation properties*:

$$\left\|\epsilon^{\frac{1}{2}}\mathbf{E}^{n+1}\right\|_E^2 + \left\|\mu^{\frac{1}{2}} H_z^{n+1}\right\|_H^2 = \left\|\epsilon^{\frac{1}{2}}\mathbf{E}^{n}\right\|_E^2 + \left\|\mu^{\frac{1}{2}} H_z^{n}\right\|_H^2, \tag{56}$$

and

$$\left\|\epsilon^{\frac{1}{2}}\delta_t\mathbf{E}^{n+\frac{3}{2}}\right\|_E^2 + \left\|\mu^{\frac{1}{2}}\delta_t H_z^{n+\frac{3}{2}}\right\|_H^2 = \left\|\epsilon^{\frac{1}{2}}\delta_t\mathbf{E}^{n+\frac{1}{2}}\right\|_E^2 + \left\|\mu^{\frac{1}{2}}\delta_t H_z^{n+\frac{1}{2}}\right\|_H^2. \tag{57}$$

Proof Multiplying both sides of (29) with $\epsilon \Delta t \left(E_{x_{i+\frac{1}{2},j}}^{*} + E_{x_{i+\frac{1}{2},j}}^{n}\right)$ and multiplying both sides of (30) with $\mu \Delta t \left(H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{*} + H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\right)$, we can get:

$$\epsilon\left[\left(E_{x_{i+\frac{1}{2},j}}^{*}\right)^2 - \left(E_{x_{i+\frac{1}{2},j}}^{n}\right)^2\right] = \frac{\Delta t}{4}\delta_y\left\{H_{z_{i+\frac{1}{2},j}}^{*} + H_{z_{i+\frac{1}{2},j}}^{n}\right\}\left(E_{x_{i+\frac{1}{2},j}}^{*} + E_{x_{i+\frac{1}{2},j}}^{n}\right), \tag{58}$$

and

$$\mu\left[\left(H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2-\left(H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right] = \frac{\Delta t}{4}\delta_y\left\{E^{*}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\right\}\left(H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} + H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right). \tag{59}$$

Summing over all terms in the above two equations, and add them together, note that E_x satisfies the boundary condition (9), then by Lemma 9, we have

$$\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon\left(E^{*}_{x_{i+\frac{1}{2},j}}\right)^2+\mu\left(H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right)=\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon\left(E^{n}_{x_{i+\frac{1}{2},j}}\right)^2 + \mu\left(H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right). \tag{60}$$

Similarly, from (31) and (32), we have

$$\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon\left(E^{n+1}_{y_{i,j+\frac{1}{2}}}\right)^2+\mu\left(H^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right)=\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon\left(E^{n}_{y_{i,j+\frac{1}{2}}}\right)^2 + \mu\left(H^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right). \tag{61}$$

And also from (33) and (34), we have

$$\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon\left(E^{n+1}_{x_{i+\frac{1}{2},j}}\right)^2+\mu\left(H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right)=\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon\left(E^{*}_{x_{i+\frac{1}{2},j}}\right)^2 + \mu\left(H^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)^2\right). \tag{62}$$

Combining Eq. (60) with Eqs. (61) and (62), the Eq. (56) are obtained.

Denote by H_z^{*+1}, H_z^{**+1} and E_x^{*+1} the intermediate value H_z^*, H_z^{**} and E_x^* at time level $n+1$ respectively, then $\delta_t H_z^{*+\frac{1}{2}} = \frac{H_z^{*+1}-H_z^*}{\Delta t}$, $\delta_t H_z^{**+\frac{1}{2}} = \frac{H_z^{**+1}-H_z^{**}}{\Delta t}$ and $\delta_t E_x^{*+\frac{1}{2}} = \frac{E_x^{*+1}-E_x^*}{\Delta t}$. So from the EC-S-FDTDII scheme (29)–(34), $\delta_t \mathbf{E}^{n+\frac{1}{2}}$ and

$\delta_t H_z^{n+\frac{1}{2}}$ satisfy the following equations:

$$\begin{aligned} \delta_t E_{x_{i+\frac{1}{2},j}}^{*+\frac{1}{2}} - \delta_t E_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} &= \frac{\Delta t}{4\epsilon}\delta_y \left\{ \delta_t H_{z_{i+\frac{1}{2},j}}^{*+\frac{1}{2}} + \delta_t H_{z_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} \right\}, \\ \delta_t H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{*+\frac{1}{2}} - \delta_t H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} &= \frac{\Delta t}{4\mu}\delta_y \left\{ \delta_t E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{*+\frac{1}{2}} + \delta_t E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} \right\}; \end{aligned} \tag{63}$$

$$\begin{aligned} \delta_t E_{y_{i,j+\frac{1}{2}}}^{n+\frac{3}{2}} - \delta_t E_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} &= -\frac{\Delta t}{2\epsilon}\delta_x \left\{ \delta_t H_{z_{i+\frac{1}{2},j}}^{**+\frac{1}{2}} + \delta_t H_{z_{i+\frac{1}{2},j}}^{*+\frac{1}{2}} \right\}, \\ \delta_t H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{**+\frac{1}{2}} - \delta_t H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{*+\frac{1}{2}} &= -\frac{\Delta t}{2\mu}\delta_x \left\{ \delta_t E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{3}{2}} + \delta_t E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} \right\}; \end{aligned} \tag{64}$$

and

$$\begin{aligned} \delta_t E_{x_{i+\frac{1}{2},j}}^{n+\frac{3}{2}} - \delta_t E_{x_{i+\frac{1}{2},j}}^{*+\frac{1}{2}} &= \frac{\Delta t}{4\epsilon}\delta_y \left\{ \delta_t H_{z_{i+\frac{1}{2},j}}^{n+\frac{3}{2}} + \delta_t H_{z_{i+\frac{1}{2},j}}^{**+\frac{1}{2}} \right\}, \\ \delta_t H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{3}{2}} - \delta_t H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{**+\frac{1}{2}} &= \frac{\Delta t}{4\mu}\delta_y \left\{ \delta_t E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{3}{2}} + \delta_t E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{*+\frac{1}{2}} \right\}. \end{aligned} \tag{65}$$

Note that $\delta_t \mathbf{E}$ also satisfies the boundary condition, then following the proof of Theorem 10 and using Lemma 9, we get the conversation relation (57). □

Combining the above energy conservation properties, we know that the EC-S-FDTDI and EC-S-FDTDII schemes are both unconditionally stable.

Corollary 12 (Unconditionally stable) *The EC-S-FDTDI scheme and the EC-S-FDTDII scheme for Maxwell's equations in two dimensions with PEC boundary conditions are unconditionally stable.*

In this subsection, we strictly prove that our EC-S-FDTDI and EC-S-FDTDII schemes both keep the energy conservation relations (14) and (16) in the discrete sense. The stability of the algorithm is a natural conclusion from the energy conservation. Moreover the energy conservation properties are also important for the long-time computation. Just as known, the previous ADI algorithms for Maxwell's equations do not have the energy conservation properties.

3.2 Truncation errors

In fact, our new schemes can be regarded as the perturbations of Crank–Nicolson scheme for the Maxwell's equations in two dimensions. The Crank–Nicolson scheme for the Maxwell equations is one implicit scheme:

$$\frac{E^{n+1}_{x_{i+\frac{1}{2},j}} - E^{n}_{x_{i+\frac{1}{2},j}}}{\Delta t} = \frac{1}{2\epsilon}\delta_y \left\{ H^{n+1}_{z_{i+\frac{1}{2},j}} + H^{n}_{z_{i+\frac{1}{2},j}} \right\}, \tag{66}$$

$$\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^{n}_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\epsilon}\delta_x \left\{ H^{n+1}_{z_{i,j+\frac{1}{2}}} + H^{n}_{z_{i,j+\frac{1}{2}}} \right\}, \tag{67}$$

$$\begin{aligned} \frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \frac{1}{2\mu} &\left\{ \delta_y \left(E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} \right) \right. \\ &\left. -\delta_x \left(E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} \right) \right\}. \end{aligned} \tag{68}$$

Denote by $\bar{\xi}_x^{n+\frac{1}{2}}$, $\bar{\xi}_y^{n+\frac{1}{2}}$ and $\bar{\eta}_z^{n+\frac{1}{2}}$ the truncation errors of the Crank–Nicolson scheme, i.e.,

$$\begin{aligned} \bar{\xi}^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}} &= \frac{1}{\Delta t}\left(E_x\left(x_{i+\frac{1}{2}}, y_j, t^{n+1}\right) - E_x\left(x_{i+\frac{1}{2}}, y_j, t^n\right)\right) \\ &\quad -\frac{1}{2\epsilon}\delta_y \left\{ H_z\left(x_{i+\frac{1}{2}}, y_j, t^{n+1}\right) + H_z\left(x_{i+\frac{1}{2}}, y_j, t^n\right)\right\}, \\ \bar{\xi}^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}} &= \frac{1}{\Delta t}\left(E_y\left(x_i, y_{j+\frac{1}{2}}, t^{n+1}\right) - E_y\left(x_i, y_{j+\frac{1}{2}}, t^n\right)\right) \\ &\quad +\frac{1}{2\epsilon}\delta_y \left\{ H_z\left(x_i, y_{j+\frac{1}{2}}, t^{n+1}\right) + H_z\left(x_i, y_{j+\frac{1}{2}}, t^n\right)\right\}, \end{aligned}$$

and

$$\begin{aligned} \bar{\eta}^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} &= \frac{1}{\Delta t}\left(H_z\left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^{n+1}\right) - H_z\left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^n\right)\right) \\ &\quad -\frac{1}{2\mu}\left\{ \delta_y \left(E_x\left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^{n+1}\right) + E_x\left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^n\right)\right)\right\} \\ &\quad +\frac{1}{2\mu}\left\{ \delta_x \left(E_y\left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^{n+1}\right) + E_y\left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^n\right)\right)\right\}. \end{aligned}$$

Then by Taylor expansion, $\bar{\xi}^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}}$, $\bar{\xi}^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}}$ and $\bar{\eta}^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}$ are of second order both in time and space:

$$\begin{aligned} \bar{\xi}^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}} &= \Delta t^2 \left[-\frac{1}{24}\frac{\partial^3 E_x}{\partial t^3}\left(\tau_{11}, x_{i+\frac{1}{2}}, y_j\right) + \frac{1}{8\epsilon}\frac{\partial^3 H_z}{\partial t^2 \partial y}\left(\tau_{12}, x_{i+\frac{1}{2}}, y_{11}\right)\right] \\ &\quad +\frac{\Delta y^2}{24\epsilon}\frac{\partial^3 H_z}{\partial y^3}\left(t^{n+\frac{1}{2}}, x_{i+\frac{1}{2}}, y_{12}\right), \end{aligned}$$

$$\begin{aligned}
\bar{\xi}^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}} &= -\Delta t^2 \left[\frac{1}{24} \frac{\partial^3 E_y}{\partial t^3} \left(\tau_{21}, x_i, y_{j+\frac{1}{2}} \right) - \frac{1}{8\epsilon} \frac{\partial^3 H_z}{\partial t^2 \partial x} \left(\tau_{22}, x_{21}, y_{j+\frac{1}{2}} \right) \right] \\
&\quad - \frac{\Delta x^2}{24\epsilon} \frac{\partial^3 H_z}{\partial x^3} \left(t^{n+\frac{1}{2}}, x_{22}, y_{j+\frac{1}{2}} \right), \\
\bar{\eta}^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} &= \Delta t^2 \left[-\frac{1}{24} \frac{\partial^3 H_z}{\partial t^3} \left(\tau_{31}, x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}} \right) + \frac{1}{8\mu} \frac{\partial^3 E_x}{\partial t^2 \partial y} \left(\tau_{32}, x_{i+\frac{1}{2}}, y_{31} \right) \right. \\
&\quad \left. - \frac{1}{8\mu} \frac{\partial^3 E_y}{\partial t^2 \partial x} \left(\tau_{33}, x_{31}, y_{j+\frac{1}{2}} \right) \right] \\
&\quad - \frac{1}{24\mu} \left[\Delta y^2 \frac{\partial^3 E_x}{\partial y^3} \left(t^{n+\frac{1}{2}}, x_{i+\frac{1}{2}}, y_{32} \right) + \Delta x^2 \frac{\partial^3 E_y}{\partial x^3} \left(t^{n+\frac{1}{2}}, x_{32}, y_{j+\frac{1}{2}} \right) \right].
\end{aligned}$$

where $t^n \le \tau_{1\ell}, \tau_{2\ell}, \tau_{3\ell} \le t^{n+1}$, $y_{j-\frac{1}{2}} \le y_{1\ell}, y_{2\ell}, y_{3\ell} \le y_{j+\frac{1}{2}}$, $x_{i-\frac{1}{2}} \le x_{1\ell}, x_{2\ell}, x_{3\ell} \le x_{i+\frac{1}{2}}$ with $\ell = 1, 2, 3$. Then we have

$$\max_n \left\{ |\bar{\xi}_x^{n+\frac{1}{2}}|, |\bar{\xi}_y^{n+\frac{1}{2}}|, |\bar{\eta}_z^{n+\frac{1}{2}}| \right\} \le C \left\{ \Delta t^2 + \Delta x^2 + \Delta y^2 \right\}. \tag{69}$$

Now we show that the EC-S-FDTDI scheme can be rewritten as first order perturbation of Crank–Nicolson scheme. From Eq. (26), we can get the expression of $H^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}}$. And substituting it into Eq. (23), we obtain the following equivalent scheme.

The equivalent EC-S-FDTDI scheme:

$$\begin{aligned}
\frac{E^{n+1}_{x_{i+\frac{1}{2},j}} - E^n_{x_{i+\frac{1}{2},j}}}{\Delta t} &= \frac{1}{2\epsilon} \delta_y \left\{ H^{n+1}_{z_{i+\frac{1}{2},j}} + H^n_{z_{i+\frac{1}{2},j}} \right\} \\
&\quad - \frac{\Delta t}{4\mu\epsilon} \delta_x \delta_y \left\{ E^{n+1}_{y_{i+\frac{1}{2},j}} + E^n_{y_{i+\frac{1}{2},j}} \right\}, \qquad (70) \\
\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^n_{y_{i,j+\frac{1}{2}}}}{\Delta t} &= -\frac{1}{2\epsilon} \delta_x \left\{ H^{n+1}_{z_{i,j+\frac{1}{2}}} + H^n_{z_{i,j+\frac{1}{2}}} \right\} \\
&\quad + \frac{\Delta t}{4\mu\epsilon} \delta_x \delta_y \left\{ E^{n+1}_{x_{i,j+\frac{1}{2}}} + E^n_{x_{i,j+\frac{1}{2}}} \right\}, \qquad (71)
\end{aligned}$$

and

$$\begin{aligned}
\frac{H^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - H^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} &= \frac{1}{2\mu} \left\{ \delta_y \left(E^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + E^n_{x_{i+\frac{1}{2},j+\frac{1}{2}}} \right) \right. \\
&\quad \left. - \delta_x \left(E^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + E^n_{y_{i+\frac{1}{2},j+\frac{1}{2}}} \right) \right\}. \qquad (72)
\end{aligned}$$

Note that the term $\frac{\Delta t}{4\mu\epsilon}\delta_x\delta_y\{E^{n+1}_{y_{i+\frac{1}{2},j}} + E^n_{y_{i+\frac{1}{2},j}}\}$ is first order perturbation. Using the definitions of $\bar{\xi}_x$, $\bar{\xi}_y$ and $\bar{\eta}_z$, and denote by $\xi_x^{n+\frac{1}{2}}$, $\xi_y^{n+\frac{1}{2}}$ and $\eta_z^{n+\frac{1}{2}}$ the truncation errors of the EC-S-FDTDI scheme, then we can get the truncation errors:

$$\xi^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}} = \bar{\xi}^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}} + \frac{\Delta t}{4\mu\epsilon}\delta_x\delta_y\{E_{y(x_{i+\frac{1}{2}}}, y_j, t^{n+1}) + E_{y(x_{i+\frac{1}{2}}}, y_j, t^n)\}, \tag{73}$$

$$\xi^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}} = \bar{\xi}^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}} - \frac{\Delta t}{4\mu\epsilon}\delta_x\delta_y\{E_x(x_i, y_{j+\frac{1}{2}}, t^{n+1}) + E_x(x_i, y_{j+\frac{1}{2}}, t^n)\}, \tag{74}$$

$$\eta^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} = \bar{\eta}^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}. \tag{75}$$

Now the terms $\delta_x\delta_y\{E_w(t^{n+1}) + E_w(t^n)\}(w = x, y)$ are the approximations of $\frac{\partial^2 E_w(t^{n+\frac{1}{2}})}{\partial x \partial y}$, then the EC-S-FDTDI is one order approximation in time.

Lemma 13 (Truncation error) *Assume that the solutions are smooth enough, e.g, satisfy the regular condition:* $\mathbf{E} \in C^3\left([0, T]; [C^3(\bar{\Omega})]^2\right)$ *and* $H_z \in C^3\left([0, T]; C^3(\bar{\Omega})\right)$. *Let* $\xi_x^{n+\frac{1}{2}}$, $\xi_y^{n+\frac{1}{2}}$ *and* $\eta_z^{n+\frac{1}{2}}$ *be the truncation errors of EC-S-FDTDI (23)–(26). Then the truncation errors can be bounded by*:

$$\max_n\{|\xi_x^{n+\frac{1}{2}}|, |\xi_y^{n+\frac{1}{2}}|\} \leq C_1(\epsilon, \mu)\{\Delta t + \Delta x^2 + \Delta y^2\}, \tag{76}$$

$$\max_n\{|\eta_z^{n+\frac{1}{2}}|\} \leq C_1(\epsilon, \mu)\{\Delta t^2 + \Delta x^2 + \Delta y^2\}, \tag{77}$$

where $C_1(\epsilon, \mu)$ *is a constant independent of the mesh sizes* Δt, Δx *and* Δy.

Remark 14 Though the truncation errors of $\bar{\xi}_x$, $\bar{\xi}_y$ and $\bar{\eta}_z$ are second order in time and space, but the truncation errors of ξ_x and ξ_y are only first order in time since the terms

$$\frac{\Delta t}{4\mu\epsilon}\delta_x\delta_y\{E_w(x_{i+\frac{1}{2}}, y_j, t^{n+1}) + E_w(x_{i+\frac{1}{2}}, y_j, t^n)\}, \quad (w = x, y)$$

are only first order in time, therefore the convergence is expected to be first order in time. However for $\eta_z^{n+\frac{1}{2}}$, it is second order approximation, so we can propose one variant EC-S-FDTD I_v scheme.

The EC-S-FDTD I_v scheme:

Stage 1–2: Use EC-S-FDTDI scheme to obtain H_z^{n+1}.

Stage 3: Recompute E_x^{n+1} and E_y^{n+1} by explicit modification:

$$\frac{E^{n+1}_{y_{i,j+\frac{1}{2}}} - E^n_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\epsilon}\delta_x\left\{H^{n+1}_{z_{i,j+\frac{1}{2}}} + H^n_{z_{i,j+\frac{1}{2}}}\right\}, \tag{78}$$

$$\frac{E^{n+1}_{x_{i+\frac{1}{2},j}} - E^n_{x_{i+\frac{1}{2},j}}}{\Delta t} = \frac{1}{2\epsilon}\delta_y\left\{H^{n+1}_{z_{i+\frac{1}{2},j}} + H^n_{z_{i+\frac{1}{2},j}}\right\}. \tag{79}$$

The motivation of the EC-S-FDTD I_v scheme is very simple. Although EC-S-FDTDI scheme is one order perturbation of the Crank–Nicolson scheme, by careful analysis, the truncation error for H_z is still two order in time. Then substitute H_z into the Crank–Nicolson scheme, it is natural to except that the new scheme is of second order in time. Our numerical results show that the EC-S-FDTD I_v scheme has second order accuracy, but unfortunately, this scheme does not keep the energy conservation as the original scheme.

Let us turn to the EC-S-FDTDII scheme. By eliminating the intermediate variables E_x^*, H_z^* and H_z^{**}, the EC-S-FDTDII scheme also has the equivalent scheme as following.

The equivalent EC-S-FDTDII scheme:

$$\begin{aligned}\frac{E_{x_{i+\frac{1}{2},j}}^{n+1} - E_{x_{i+\frac{1}{2},j}}^{n}}{\Delta t} &= \frac{1}{2\epsilon}\delta_y\left\{H_{z_{i+\frac{1}{2},j}}^{n+1} + H_{z_{i+\frac{1}{2},j}}^{n}\right\}\\ &\quad -\frac{\Delta t}{16\mu\epsilon}\delta_y\delta_y\left\{E_{x_{i+\frac{1}{2},j}}^{n+1} - E_{x_{i+\frac{1}{2},j}}^{n}\right\}, \qquad (80)\\ \frac{E_{y_{i,j+\frac{1}{2}}}^{n+1} - E_{y_{i,j+\frac{1}{2}}}^{n}}{\Delta t} &= -\frac{1}{2\epsilon}\delta_x\left\{H_{z_{i,j+\frac{1}{2}}}^{n+1} + H_{z_{i,j+\frac{1}{2}}}^{n}\right\}\\ &\quad +\frac{\Delta t}{8\mu\epsilon}\delta_x\delta_y\left\{E_{x_{i,j+\frac{1}{2}}}^{n+1} - E_{x_{i,j+\frac{1}{2}}}^{n}\right\}, \qquad (81)\end{aligned}$$

and

$$\begin{aligned}\frac{H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1} - H_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}}{\Delta t} &= \frac{1}{2\mu}\left\{\delta_y\left(E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1} + E_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\right)\right.\\ &\quad \left. -\delta_x\left(E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1} + E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n}\right)\right\}\\ &\quad -\frac{\Delta t}{16\mu\epsilon}\delta_y\delta_y\left(H_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - H_{i+\frac{1}{2},j+\frac{1}{2}}^{n}\right)\\ &\quad +\frac{\Delta t^2}{32\mu^2\epsilon}\delta_y\delta_x\delta_y\left(E_{y^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}}} + E_{y^{n}_{i+\frac{1}{2},j+\frac{1}{2}}}\right). \qquad (82)\end{aligned}$$

Comparing with the Crank–Nicolson scheme, the terms $-\frac{\Delta t}{16\mu\epsilon}\delta_y\delta_y\{E_{x_{i+\frac{1}{2},j}}^{n+1} - E_{x_{i+\frac{1}{2},j}}^{n}\}$, $\frac{\Delta t}{8\mu\epsilon}\delta_x\delta_y\{E_{x_{i,j+\frac{1}{2}}}^{n+1} - E_{x_{i,j+\frac{1}{2}}}^{n}\}$, $-\frac{\Delta t}{16\mu\epsilon}\delta_y\delta_y(H_{i+\frac{1}{2},j+\frac{1}{2}}^{n+1} - H_{i+\frac{1}{2},j+\frac{1}{2}}^{n})$, and $\frac{\Delta t^2}{32\mu^2\epsilon}\delta_y\delta_x\delta_y(E_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1} + E_y^{n_{i+\frac{1}{2},j+\frac{1}{2}}})$ are second order terms, then the EC-S-FDTDII scheme can also be regarded as second order perturbation of Crank–Nicolson scheme. Denote by $\xi_x^{n+\frac{1}{2}}$, $\xi_y^{n+\frac{1}{2}}$ and $\eta_z^{n+\frac{1}{2}}$ the truncation errors of the EC-S-FDTDII scheme, then from the equivalent EC-S-FDTDII scheme (80)–(82), the truncation errors are:

$$\xi_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} = \bar{\xi}_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} - \frac{\Delta t}{16\mu\epsilon}\delta_y\delta_y\{E_x(x_{i+\frac{1}{2}}, y_j, t^{n+1}) - E_x(x_{i+\frac{1}{2}}, y_j, t^n)\}, \tag{83}$$

$$\xi_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} = \bar{\xi}_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} + \frac{\Delta t}{8\mu\epsilon}\delta_x\delta_y\{E_x(x_i, y_{j+\frac{1}{2}}, t^{n+1}) - E_x(x_i, y_{j+\frac{1}{2}}, t^n)\}, \tag{84}$$

$$\begin{aligned}\eta_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} &= \bar{\eta}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} - \frac{\Delta t}{16\mu\epsilon}\delta_y\delta_y\{H_z(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^{n+1}) - H_z(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^n)\} \\ &\quad + \frac{\Delta t^2}{32\mu^2\epsilon}\delta_x\delta_y\delta_y\{E_y(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^{n+1}) + E_y(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, t^n)\}.\end{aligned} \tag{85}$$

Clearly, from the above expressions, the EC-S-FDTDII is both of second order in time and space.

Lemma 15 (Truncation error) *Assume that the solutions are smooth enough, e.g, satisfy the regular condition:* $\mathbf{E} \in C^3\left([0,T]; [C^3(\bar{\Omega})]^2\right)$ *and* $H_z \in C^3\left([0,T]; C^3(\bar{\Omega})\right)$. *Let* $\xi_x^{n+\frac{1}{2}}$, $\xi_y^{n+\frac{1}{2}}$ *and* $\eta_z^{n+\frac{1}{2}}$ *be the truncation errors of EC-S-FDTDII* (29)–(32). *Then the truncation errors are second order both in space and time*:

$$\max_n\{|\xi_x^{n+\frac{1}{2}}|, |\xi_y^{n+\frac{1}{2}}|, |\eta_z^{n+\frac{1}{2}}|\} \le C_2(\epsilon,\mu)\{\Delta t^2 + \Delta x^2 + \Delta y^2\}, \tag{86}$$

where $C_2(\epsilon,\mu)$ *is a constant independent of the mesh sizes* Δt, Δx *and* Δy.

3.3 Convergence analysis for EC-S-FDTDI and EC-S-FDTDII

Based on the energy conservation and truncation error analysis, we arrive at the convergence analysis of our EC-S-FDTD schemes. Though the truncation errors are obtained from the equivalent EC-S-FDTD schemes, the energy conversation properties and the stability are based on the original EC-S-FDTD schemes, then it is not so obvious to obtain the convergence from the truncation errors and stability property.

Here we first show that the equivalent EC-S-FDTDII scheme can also be transformed back to the original scheme by introducing the intermediate variables E_x^*, H_z^* and H_z^{**}:

$$E_x^* = \frac{1}{2}(E_x^{n+1} + E_x^n) + \frac{\Delta t^2}{16\mu\epsilon}\delta_x\delta_y(E_y^{n+1} + E_y^n) - \frac{\Delta t}{8\epsilon}\delta_y(H_z^{n+1} - H_z^n), \tag{87}$$

and

$$H_z^* = H_z^n + \frac{\Delta t}{4\mu}\delta_y(E_x^* + E_x^n), \quad H_z^{**} = H_z^* - \frac{\Delta t}{2\mu}\delta_x(E_y^{n+1} + E_y^n). \tag{88}$$

By these intermediate variables, the equivalent scheme (80)–(82) will lead to the EC-S-FDTDII scheme, this fact can be verified directly. Based on this transformation, the truncation errors of the equivalent scheme will be redistributed, i.e., we obtain the truncation errors of the original EC-S-FDTDII scheme. Then by the energy method, the convergence can be obtained from the truncation error analysis and energy conservation property.

Define the error functions at the staggered grid: $\mathcal{E}^n_{w_{\alpha,\beta}} = E_w(x_\alpha, x_\beta, t^n) - E^n_{w_{\alpha,\beta}}$, and $\mathcal{H}^n_{z_{\alpha,\beta}} = H_z(x_\alpha, x_\beta, t^n) - H^n_{z_{\alpha,\beta}}$, where $E_w(x_\alpha, y_\beta, t^n)$ with $w = x, y$ and $H_z(x_\alpha, y_\beta, t^n)$ denote the values of the exact solution components E_w and H_z at the point (x_α, y_β, t^n), respectively.

Theorem 16 (Convergence) *Suppose that the exact solution components E_x, E_y and H_z are smooth enough: $\mathbf{E} \in C^3\left([0,T];[C^3(\bar{\Omega})]^2\right)$ and $H_z \in C^3\left([0,T];C^3(\bar{\Omega})\right)$. For $n \geq 0$, let E^n_x, E^n_y and $H_z{}^n$ be the solution of the EC-S-FDTDII scheme (29)–(32). Then for any fixed $T > 0$, there exists a positive constant $C_{\mu\epsilon}$ independent of Δt, Δx and Δy such that*

$$\begin{aligned}
&\max_{0\le n\le N}\{\|\epsilon^{\frac{1}{2}}[E(t^n)-E^n]\|_E^2+\|\mu^{\frac{1}{2}}[H_z(t^n)-H_z^n]\|_H^2\}\\
&\le e^T\left(\|\epsilon^{\frac{1}{2}}(\mathbf{E}(t^0)-\mathbf{E}^0)\|_E^2+\|\mu^{\frac{1}{2}}(H_z(t^0)-H_z^0)\|_H^2\right)\\
&\quad+C_{\mu\epsilon}e^T(\Delta t^2+\Delta x^2+\Delta y^2)^2.
\end{aligned}\tag{89}$$

Proof Define $\mathcal{H}^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}$, $\mathcal{H}^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}$ and $\mathcal{E}^{*}_{x_{i+\frac{1}{2},j}}$ as following:

$$\begin{aligned}
\mathcal{H}^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} &= \frac{1}{2}\left(\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{H}^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)-\frac{\Delta t}{8\mu}\delta_y\left(\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}-\mathcal{E}^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\right)\\
&\quad-\frac{\Delta t}{4\mu}\delta_x\left(\mathcal{E}^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{E}^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\right),\\
\mathcal{H}^{*}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} &= \frac{1}{2}\left(\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{H}^{n}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right)-\frac{\Delta t}{8\mu}\delta_y\left(\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}-\mathcal{E}^{n}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}\right)\\
&\quad+\frac{\Delta t}{4\mu}\delta_x\left(\mathcal{E}^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}+\mathcal{E}^{n}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}^{*}_{x_{i+\frac{1}{2},j}} &= \frac{1}{2}\left(\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j}}+\mathcal{E}^{n}_{x_{i+\frac{1}{2},j}}\right)+\frac{\Delta t^2}{16\mu\epsilon}\delta_y\delta_x\left(\mathcal{E}^{n+1}_{y_{i+\frac{1}{2},j}}+\mathcal{E}^{n}_{y_{i+\frac{1}{2},j}}\right)\\
&\quad-\frac{\Delta t}{8\epsilon}\delta_y\left(\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j}}-\mathcal{H}^{n}_{z_{i+\frac{1}{2},j}}\right).
\end{aligned}$$

And combining these variables with the other intermediate variables E^*_x, H^*_z and H^{**}_z(see (87) and (88)), we arrive at the error equations of the EC-S-FDTDII scheme:

$$\begin{cases} \dfrac{\mathcal{E}^*_{x_{i+\frac{1}{2},j}} - \mathcal{E}^n_{x_{i+\frac{1}{2},j}}}{\Delta t} = \frac{1}{4\epsilon}\delta_y(\mathcal{H}^*_{i+\frac{1}{2},j} + \mathcal{H}^n_{i+\frac{1}{2},j}) + e^{n+\frac{1}{2}}_{1_{i+\frac{1}{2},j}}, \\ \dfrac{\mathcal{H}^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - \mathcal{H}^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \frac{1}{4\mu}\delta_y(\mathcal{E}^*_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + \mathcal{E}^n_{x_{i+\frac{1}{2},j+\frac{1}{2}}}) + e^{n+\frac{1}{2}}_{2_{i+\frac{1}{2},j+\frac{1}{2}}}; \end{cases} \tag{90}$$

$$\begin{cases} \dfrac{\mathcal{E}^{n+1}_{y_{i,j+\frac{1}{2}}} - \mathcal{E}^n_{y_{i,j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\epsilon}\delta_x(\mathcal{H}^{**}_{i,j+\frac{1}{2}} + \mathcal{H}^*_{i,j+\frac{1}{2}}) + e^{n+\frac{1}{2}}_{3_{i,j+\frac{1}{2}}}, \\ \dfrac{\mathcal{H}^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - \mathcal{H}^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = -\frac{1}{2\mu}\delta_x(\mathcal{E}^{n+1}_{y_{i+\frac{1}{2},j+\frac{1}{2}}} + \mathcal{E}^n_{y_{i+\frac{1}{2},j+\frac{1}{2}}}) + e^{n+\frac{1}{2}}_{4_{i+\frac{1}{2},j+\frac{1}{2}}}; \end{cases} \tag{91}$$

$$\begin{cases} \dfrac{\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j}} - \mathcal{E}^*_{x_{i+\frac{1}{2},j}}}{\Delta t} = \frac{1}{4\epsilon}\delta_y(\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j}} + \mathcal{H}^{**}_{z_{i+\frac{1}{2},j}}) + e^{n+\frac{1}{2}}_{5_{i+\frac{1}{2},j}}, \\ \dfrac{\mathcal{H}^{n+1}_{z_{i+\frac{1}{2},j+\frac{1}{2}}} - \mathcal{H}^{**}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}}{\Delta t} = \frac{1}{4\mu}\delta_y(\mathcal{E}^{n+1}_{x_{i+\frac{1}{2},j+\frac{1}{2}}} + \mathcal{E}^*_{x_{i+\frac{1}{2},j+\frac{1}{2}}}) + e^{n+\frac{1}{2}}_{6_{i+\frac{1}{2},j+\frac{1}{2}}}. \end{cases} \tag{92}$$

Comparing $e^{n+\frac{1}{2}}_{1_{i+\frac{1}{2},j}}$, $e^{n+\frac{1}{2}}_{2_{i+\frac{1}{2},j+\frac{1}{2}}}$, $e^{n+\frac{1}{2}}_{3_{i,j+\frac{1}{2}}}$, $e^{n+\frac{1}{2}}_{4_{i+\frac{1}{2},j+\frac{1}{2}}}$, $e^{n+\frac{1}{2}}_{5_{i+\frac{1}{2},j}}$ and $e^{n+\frac{1}{2}}_{6_{i+\frac{1}{2},j+\frac{1}{2}}}$ with the truncation errors $\xi^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}}$, $\xi^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}}$ and $\eta^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}$ in the equivalent EC-S-FDTDII scheme, we can get the relations:

$$\begin{aligned} &e^{n+\frac{1}{2}}_{1_{i+\frac{1}{2},j}} = \frac{1}{2}\xi^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}}, \quad e^{n+\frac{1}{2}}_{2_{i+\frac{1}{2},j+\frac{1}{2}}} = \frac{1}{2}\eta^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}, \quad e^{n+\frac{1}{2}}_{3_{i,j+\frac{1}{2}}} = \xi^{n+\frac{1}{2}}_{y_{i,j+\frac{1}{2}}}, \\ &e^{n+\frac{1}{2}}_{4_{i+\frac{1}{2},j+\frac{1}{2}}} = 0, \quad e^{n+\frac{1}{2}}_{5_{i+\frac{1}{2},j}} = \frac{1}{2}\xi^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}}, \quad e^{n+\frac{1}{2}}_{6_{i+\frac{1}{2},j+\frac{1}{2}}} = \frac{1}{2}\eta^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}. \end{aligned}$$

Now the energy method can be used. From (90) and using Lemma 9, we have

$$\begin{aligned} &\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon(\mathcal{E}^*_{x\,i+\frac{1}{2},j})^2 + \mu(\mathcal{H}^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}})^2 - \epsilon(\mathcal{E}^n_{x\,i+\frac{1}{2},j})^2 - \mu(\mathcal{H}^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}})^2\right) \\ &\quad = \frac{1}{2}\Delta t\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\left(\epsilon\xi^{n+\frac{1}{2}}_{x_{i+\frac{1}{2},j}}(\mathcal{E}^*_{x\,i+\frac{1}{2},j} + \mathcal{E}^n_{x\,i+\frac{1}{2},j}) + \mu\eta^{n+\frac{1}{2}}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}(\mathcal{H}^*_{z_{i+\frac{1}{2},j+\frac{1}{2}}}\right. \\ &\qquad \left. + \mathcal{H}^n_{z_{i+\frac{1}{2},j+\frac{1}{2}}})\right). \end{aligned}$$

By Schwartz inequality,

$$\begin{aligned} \left(1 - \frac{\Delta t}{4}\right)\left(\epsilon\|\mathcal{E}^*_x\|^2_{E_x} + \mu\|\mathcal{H}^*_z\|^2_H\right) &\le C_1\Delta t(\Delta t^2 + \Delta x^2 + \Delta y^2)^2 \\ &\quad + \left(1 + \frac{\Delta t}{4}\right)(\epsilon\|\mathcal{E}^n_x\|^2_{E_x} + \mu\|\mathcal{H}^n_z\|^2_H). \end{aligned} \tag{93}$$

Similarly, from (91) and (92), we can obtain the inequalities as following respectively:

$$\left(1-\frac{\Delta t}{2}\right)\epsilon\|\mathcal{E}_y^{n+1}\|_{E_y}^2+\mu\|\mathcal{H}_z^{**}\|_H^2 \le C_2\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2 + \left(1+\frac{\Delta t}{2}\right)\epsilon\|\mathcal{E}_y^n\|_{E_y}^2+\mu\|\mathcal{H}_z^*\|_H^2, \quad (94)$$

and

$$(1-\frac{\Delta t}{4})\left(\epsilon\|\mathcal{E}_x^{n+1}\|_{E_x}^2+\mu\|\mathcal{H}_z^{n+1}\|_H^2\right) \le C_3\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2 + (1+\frac{\Delta t}{4})(\epsilon\|\mathcal{E}_x^*\|_{E_x}^2+\mu\|\mathcal{H}_z^{**}\|_H^2). \quad (95)$$

Let us divide both sides of (93) by $(1-\frac{\Delta t}{4})$, both sides of (95) by $(1+\frac{\Delta t}{4})$ and sum (93), (94) and (93) together. Then by eliminating the intermediate variables $\mathcal{E}_x^*$, $\mathcal{H}_z^*$ and $\mathcal{H}_z^{**}$, we have:

$$\begin{aligned}&\frac{1-\frac{\Delta t}{4}}{1+\frac{\Delta t}{4}}\epsilon\|\mathcal{E}_x^{n+1}\|_{E_x}^2+\frac{1-\frac{\Delta t}{4}}{1+\frac{\Delta t}{4}}\mu\|\mathcal{H}_z^{n+1}\|_H^2+\left(1-\frac{\Delta t}{2}\right)\epsilon\|\mathcal{E}_y^{n+1}\|_{E_y}^2\\ &\quad\le\frac{1+\frac{\Delta t}{4}}{1-\frac{\Delta t}{4}}\epsilon\|\mathcal{E}_x^n\|_{E_x}^2+\frac{1+\frac{\Delta t}{4}}{1-\frac{\Delta t}{4}}\mu\|\mathcal{H}_z^n\|_H^2+\left(1+\frac{\Delta t}{2}\right)\epsilon\|\mathcal{E}_y^n\|_{E_y}^2\\ &\quad\quad+C_4\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2. \end{aligned} \quad (96)$$

As $\frac{1-\frac{\Delta t}{4}}{1+\frac{\Delta t}{4}} \ge \left(1-\frac{\Delta t}{2}\right)$ and $\frac{1+\frac{\Delta t}{4}}{1-\frac{\Delta t}{4}} \ge \left(1+\frac{\Delta t}{2}\right)$, so from (96), we can get:

$$\begin{aligned}&\left(1-\frac{\Delta t}{2}\right)\epsilon\|\mathcal{E}_x^{n+1}\|_{E_x}^2+\left(1-\frac{\Delta t}{2}\right)\mu\|\mathcal{H}_z^{n+1}\|_H^2+\left(1-\frac{\Delta t}{2}\right)\epsilon\|\mathcal{E}_y^{n+1}\|_{E_y}^2\\ &\quad\le\frac{1+\frac{\Delta t}{4}}{1-\frac{\Delta t}{4}}\epsilon\|\mathcal{E}_x^n\|_{E_x}^2+\frac{1+\frac{\Delta t}{4}}{1-\frac{\Delta t}{4}}\mu\|\mathcal{H}_z^n\|_H^2+\frac{1+\frac{\Delta t}{4}}{1-\frac{\Delta t}{4}}\epsilon\|\mathcal{E}_y^n\|_{E_y}^2\\ &\quad\quad+C_4\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2. \end{aligned} \quad (97)$$

Divided by $1-\frac{\Delta t}{2}$ on both side of the above inequality,

$$\begin{aligned}&\epsilon\|\mathcal{E}_x^{n+1}\|_{E_x}^2+\epsilon\|\mathcal{E}_y^{n+1}\|_{E_y}^2+\mu\|\mathcal{H}_z^{n+1}\|_H^2\\ &\quad\le\frac{1+\frac{\Delta t}{4}}{1-\frac{\Delta t}{4}}\frac{1}{1+\frac{\Delta t}{2}}[\epsilon\|\mathcal{E}_x^n\|_{E_x}^2+\epsilon\|\mathcal{E}_y^n\|_{E_y}^2+\mu\|\mathcal{H}_z^n\|_H^2]\\ &\quad\quad+2C_4\Delta t(\Delta t^2+\Delta x^2+\Delta y^2)^2. \end{aligned} \quad (98)$$

Notice that when $\Delta t = \frac{T}{N}$, the following inequality is easy to be checked:

$$\left(\frac{1+\frac{\Delta t}{4}}{1-\frac{\Delta t}{4}} \quad \frac{1}{1+\frac{\Delta t}{2}} \right)^n \leq e^T.$$

Then recursively applying (98) from time level n to 0, we can obtain:

$$\begin{aligned} &\epsilon \|\mathcal{E}_x^{n+1}\|_{E_x}^2 + \epsilon \|\mathcal{E}_y^{n+1}\|_{E_y}^2 + \mu \|\mathcal{H}_z^{n+1}\|_H^2 \\ &\quad \leq e^T [\epsilon \|\mathcal{E}_x^0\|_{E_x}^2 + \epsilon \|\mathcal{E}_y^0\|_{E_y}^2 + \mu \|\mathcal{H}_z^0\|_H^2] + 4C_4 e^T \Delta t (\Delta t^2 + \Delta x^2 + \Delta y^2)^2. \end{aligned} \tag{99}$$

Taking the maximum for n from 0 to $N-1$, we complete the proof. □

We can also obtain the convergence of $\delta_t \mathbf{E}^n$ and $\delta_t H_z^n$. The proof is similar to one in Theorem 16: we first obtain the truncation estimates from the equivalent EC-S-FDTDII scheme, and then use the energy method to obtain the convergence.

Theorem 17 (Convergence II) *Suppose that the exact solution components E_x, E_y and H_z are smooth enough: $\mathbf{E} \in C^4([0,T]; [C^3(\bar{\Omega})]^2)$ and $H_z \in C^4([0,T]; C^3(\bar{\Omega}))$. For $n \geq 0$, let E_x^n, E_y^n and ${H_z}^n$ be the solution of the EC-S-FDTDII scheme (23)–(26). Then for any fixed $T > 0$, there exists a positive constant $C_{\mu\epsilon}$ independent of Δt, Δx and Δy such that*

$$\begin{aligned} &\max_{0 \leq n \leq N} \left\{ \|\epsilon^{\frac{1}{2}} [\delta_t E(t^{n+\frac{1}{2}}) - \delta_t E^{n+\frac{1}{2}}]\|_E^2 + \|\mu^{\frac{1}{2}} [\delta_t H_z(t^{n+\frac{1}{2}}) - \delta_t {H_z}^{n+\frac{1}{2}}]\|_H^2 \right\} \\ &\quad \leq e^T \left(\|\epsilon^{\frac{1}{2}} (\delta_t \mathbf{E}(t^{\frac{1}{2}}) - \delta_t \mathbf{E}^{\frac{1}{2}})\|_E^2 + \|\mu^{\frac{1}{2}} (\delta_t H_z(t^0) - \delta_t H_z^0)\|_H^2 \right) \\ &\qquad + C_{\mu\epsilon} e^T (\Delta t^2 + \Delta x^2 + \Delta y^2)^2. \end{aligned} \tag{100}$$

Proof According to the equivalent EC-S-FDTDII scheme (80)–(82), and substrating the equations at the neighbor levels n and $n-1$, we can get the error equations for $\delta_t \mathbf{E}$ and $\delta_t H_z$ as follows:

$$\begin{aligned} \frac{\delta_t \mathcal{E}_{x_{i+\frac{1}{2},j}}^{n+\frac{3}{2}} - \delta_t \mathcal{E}_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}}}{\Delta t} &= \frac{1}{2\epsilon} \delta_y \left\{ \delta_t \mathcal{H}_{z_{i+\frac{1}{2},j}}^{n+\frac{3}{2}} + \delta_t \mathcal{H}_{z_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} \right\} \\ &\quad - \frac{\Delta t}{16\mu\epsilon} \delta_y \delta_y \left\{ \delta_t \mathcal{E}_{x_{i+\frac{1}{2},j}}^{n+\frac{3}{2}} - \delta_t \mathcal{E}_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} \right\} + \zeta_{x_{i+\frac{1}{2},j}}^{n+1}, \end{aligned} \tag{101}$$

$$\begin{aligned} \frac{\delta_t \mathcal{E}_{y_{i,j+\frac{1}{2}}}^{n+\frac{3}{2}} - \delta_t \mathcal{E}_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}}}{\Delta t} &= -\frac{1}{2\epsilon} \delta_x \left\{ \delta_t \mathcal{H}_{z_{i,j+\frac{1}{2}}}^{n+\frac{3}{2}} + \delta_t \mathcal{H}_{z_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} \right\} \\ &\quad + \frac{\Delta t}{8\mu\epsilon} \delta_x \delta_y \left\{ \delta_t \mathcal{E}_{x_{i,j+\frac{1}{2}}}^{n+\frac{3}{2}} - \delta_t \mathcal{E}_{x_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} \right\} + \zeta_{y_{i,j+\frac{1}{2}}}^{n+1}, \end{aligned} \tag{102}$$

and

$$\begin{aligned}\frac{\delta_t \mathcal{H}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{3}{2}} - \delta_t \mathcal{H}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}}{\Delta t} &= \frac{1}{2\mu}\left\{\delta_y\left(\delta_t \mathcal{E}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{3}{2}} + \delta_t \mathcal{E}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}\right)\right.\\ &\left. -\delta_x\left(\delta_t \mathcal{E}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{3}{2}} + \delta_t \mathcal{E}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}\right)\right\}\\ &-\frac{\Delta t}{16\mu\epsilon}\delta_y\delta_y\left(\delta_t \mathcal{H}_{i+\frac{1}{2},j+\frac{1}{2}}^{n+\frac{3}{2}} - \delta_t \mathcal{H}_{i+\frac{1}{2},j+\frac{1}{2}}^{n+\frac{1}{2}}\right)\\ &+\frac{\Delta t^2}{32\mu^2\epsilon}\delta_y\delta_x\delta_y\left(\delta_t \mathcal{E}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{3}{2}} + \delta_t \mathcal{E}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}\right)\\ &+\zeta_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1},\end{aligned} \tag{103}$$

where $\zeta_{x_{i+\frac{1}{2},j}}^{n+1}$, $\zeta_{y_{i,j+\frac{1}{2}}}^{n+1}$ and $\zeta_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1}$ are the truncation errors. By Taylor expansion,

$$\begin{aligned}\zeta_{x_{i+\frac{1}{2},j}}^{n+1} &= \Delta t^2\left[-\frac{1}{24}\frac{\partial^4 E_x}{\partial t^4}(\tau_{11}, x_{i+\frac{1}{2}}, y_j) + \frac{1}{8\epsilon}\frac{\partial^4 H_z}{\partial t^3 \partial y}(\tau_{12}, x_{i+\frac{1}{2}}, y_{11})\right]\\ &\quad+\frac{\Delta y^2}{24\epsilon}\frac{\partial^4 H_z}{\partial t \partial y^3}(\tau_{13}, x_{i+\frac{1}{2}}, y_{12}) - \frac{\Delta t^2}{16\mu\epsilon}\frac{\partial^4 E_x}{\partial t^2 \partial y^2}(\tau_{14}, x_{i+\frac{1}{2}}, y_{13}),\\ \zeta_{y_{i,j+\frac{1}{2}}}^{n+1} &= -\,\Delta t^2\left[\frac{1}{24}\frac{\partial^4 E_y}{\partial t^4}(\tau_{21}, x_i, y_{j+\frac{1}{2}}) - \frac{1}{8\epsilon}\frac{\partial^4 H_z}{\partial t^3 \partial x}(\tau_{22}, x_{21}, y_{j+\frac{1}{2}})\right]\\ &\quad-\frac{\Delta x^2}{24\epsilon}\frac{\partial^4 H_z}{\partial t \partial x^3}(\tau_{23}, x_{22}, y_{j+\frac{1}{2}}) + \frac{\Delta t^2}{8\mu\epsilon}\frac{\partial^4 E_x}{\partial t^2 \partial x \partial y}(\tau_{24}, x_{23}, y_{21}),\\ \zeta_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1} &= \Delta t^2\left[-\frac{1}{24}\frac{\partial^4 H_z}{\partial t^4}(\tau_{31}, x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) + \frac{1}{8\mu}\frac{\partial^4 E_x}{\partial t^3 \partial y}(\tau_{32}, x_{i+\frac{1}{2}}, y_{31})\right.\\ &\quad\left.-\frac{1}{8\mu}\frac{\partial^4 E_y}{\partial t^3 \partial x}(\tau_{33}, x_{31}, y_{j+\frac{1}{2}})\right] - \frac{1}{24\mu}\left[\Delta y^2\frac{\partial^4 E_x}{\partial t \partial y^3}(\tau_{34}, x_{i+\frac{1}{2}}, y_{32})\right.\\ &\quad\left.+\Delta x^2\frac{\partial^4 E_y}{\partial t \partial x^3}(\tau_{35}, x_{32}, y_{j+\frac{1}{2}})\right] - \frac{\Delta t^2}{16\mu\epsilon}\frac{\partial^4 H_z}{\partial t^2 \partial y^2}(\tau_{33}, x_{i+\frac{1}{2}}, y_{33})\\ &\quad+\frac{\Delta t^2}{32\mu^2\epsilon}\left[\frac{\partial^4 E_y}{\partial t \partial x \partial y^2}(\tau_{34}, x_{34}, y_{34}) + \frac{\partial^4 E_y}{\partial t \partial x \partial y^2}(\tau_{35}, x_{35}, y_{35})\right].\end{aligned}$$

This means that the truncation errors of (101)–(103) are all second order in time and space. Similarly, using the same intermediate variables defined in Theorem 16, we can get:

$$\begin{cases}\delta_t \mathcal{E}_{x_{i+\frac{1}{2},j}}^{*+\frac{1}{2}} - \delta_t \mathcal{E}_{x_{i+\frac{1}{2},j}}^{n+\frac{1}{2}} = \frac{\Delta t}{4\epsilon}\delta_y\{\delta_t \mathcal{H}_{z_{i+\frac{1}{2},j}}^{*+\frac{1}{2}} + \delta_t \mathcal{H}_{z_{i+\frac{1}{2},j}}^{n+\frac{1}{2}}\} + e_{1_{i+\frac{1}{2},j}},\\ \delta_t \mathcal{H}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{*+\frac{1}{2}} - \delta_t \mathcal{H}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}} = \frac{\Delta t}{4\mu}\delta_y\{\delta_t \mathcal{E}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{*+\frac{1}{2}} + \delta_t \mathcal{E}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}\} + e_{2_{i+\frac{1}{2},j+\frac{1}{2}}},\end{cases} \tag{104}$$

$$\begin{cases}\delta_t \mathcal{E}_{y_{i,j+\frac{1}{2}}}^{n+\frac{3}{2}} - \delta_t \mathcal{E}_{y_{i,j+\frac{1}{2}}}^{n+\frac{1}{2}} = -\frac{\Delta t}{2\epsilon}\delta_x\{\delta_t \mathcal{H}_{z_{i+\frac{1}{2},j}}^{**+\frac{1}{2}} + \delta_t \mathcal{H}_{z_{i+\frac{1}{2},j}}^{*+\frac{1}{2}}\} + e_{3_{i,j+\frac{1}{2}}},\\ \delta_t \mathcal{H}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{**+\frac{1}{2}} - \delta_t \mathcal{H}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{*+\frac{1}{2}} = -\frac{\Delta t}{2\mu}\delta_x\{\delta_t \mathcal{E}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{3}{2}} + \delta_t \mathcal{E}_{y_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{1}{2}}\} + e_{4_{i+\frac{1}{2},j+\frac{1}{2}}},\end{cases} \tag{105}$$

and

$$\begin{cases}\delta_t \mathcal{E}_{x_{i+\frac{1}{2},j}}^{n+\frac{3}{2}} - \delta_t \mathcal{E}_{x_{i+\frac{1}{2},j}}^{*+\frac{1}{2}} = \frac{\Delta t}{4\epsilon}\delta_y\left\{\delta_t \mathcal{H}_{z_{i+\frac{1}{2},j}}^{n+\frac{3}{2}} + \delta_t \mathcal{H}_{z_{i+\frac{1}{2},j}}^{**+\frac{1}{2}}\right\} + e_{5_{i+\frac{1}{2},j}},\\ \delta_t \mathcal{H}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{3}{2}} - \delta_t \mathcal{H}_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{**+\frac{1}{2}} = \frac{\Delta t}{4\mu}\delta_y\left\{\delta_t \mathcal{E}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+\frac{3}{2}} + \delta_t \mathcal{E}_{x_{i+\frac{1}{2},j+\frac{1}{2}}}^{*+\frac{1}{2}}\right\} + e_{6_{i+\frac{1}{2},j+\frac{1}{2}}},\end{cases} \tag{106}$$

where

$$\begin{aligned} &e_{1_{i+\frac{1}{2},j}} = \frac{1}{2}\zeta_{x_{i+\frac{1}{2},j}}^{n+1}, \quad e_{2_{i+\frac{1}{2},j+\frac{1}{2}}} = \frac{1}{2}\zeta_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1}, \quad e_{3_{i,j+\frac{1}{2}}} = \zeta_{y_{i,j+\frac{1}{2}}}^{n+1},\\ &e_{4_{i+\frac{1}{2},j+\frac{1}{2}}} = 0, \quad e_{5_{i+\frac{1}{2},j}} = \frac{1}{2}\zeta_{x_{i+\frac{1}{2},j}}^{n+1}, \quad e_{6_{i+\frac{1}{2},j+\frac{1}{2}}} = \frac{1}{2}\zeta_{z_{i+\frac{1}{2},j+\frac{1}{2}}}^{n+1}.\end{aligned}$$

Now it is obvious that the conclusion of Theorem 17 could be obtained by the same method as in Theorem 16. □

And for the EC-S-FDTDI scheme, we have the following convergence results.

Theorem 18 (Convergence III) *Suppose that the exact solution components E_x, E_y and H_z are smooth enough: $\mathbf{E} \in C^3\left([0,T]; [C^3(\bar{\Omega})]^2\right)$, and $H_z \in C^3\left([0,T]; C^3(\bar{\Omega})\right)$. For $n \geq 0$, let E_x^n, E_y^n and $H_z{}^n$ be the solution of the EC-S-FDTDI scheme (23)–(26). Then for any fixed $T > 0$, there exists a positive constant $C_{\mu\epsilon}$ independent of Δt, Δx and Δy such that*

$$\begin{aligned}&\max_{0\le n\le N}\{\|\epsilon^{\frac{1}{2}}[E(t^n) - E^n]\|_E^2 + \|\mu^{\frac{1}{2}}[H_z(t^n) - H_z^n]\|_H^2\}\\ &\quad\le e^T\left(\|\epsilon^{\frac{1}{2}}(\mathbf{E}(t^0) - \mathbf{E}^0)\|_E^2 + \|\mu^{\frac{1}{2}}(H_z(t^0) - H_z^0)\|_H^2\right)\\ &\qquad + C_{\mu\epsilon}e^T(\Delta t + \Delta x^2 + \Delta y^2)^2.\end{aligned} \tag{107}$$

and

$$\max_{0\le n\le N}\left\{\|\epsilon^{\frac{1}{2}}[\delta_t E(t^{n+\frac{1}{2}})-\delta_t E^{n+\frac{1}{2}}]\|_E^2+\|\mu^{\frac{1}{2}}[\delta_t H_z(t^{n+\frac{1}{2}})-\delta_t H_z^{n+\frac{1}{2}}]\|_H^2\right\}$$
$$\le e^T\left(\|\epsilon^{\frac{1}{2}}(\delta_t \mathbf{E}(t^{\frac{1}{2}})-\delta_t\mathbf{E}^{\frac{1}{2}})\|_E^2+\|\mu^{\frac{1}{2}}(\delta_t H_z(t^0)-\delta_t H_z^0)\|_H^2\right)$$
$$+C_{\mu\epsilon}e^T(\Delta t+\Delta x^2+\Delta y^2)^2. \tag{108}$$

3.4 Convergence of divergence

In Subsect. 3.1 we have shown that our new schemes obey the energy conservation property. Moreover, the electronic field is divergence-free:

$$\mathrm{div}(\epsilon\mathbf{E})=0, \tag{109}$$

if the media is lossless. The divergence-free property in this case can be proved for Maxwell's equation if the initial value satisfies it. But, this property is often omitted in the FDTD methods and was not considered in the numerical algorithms in the previous papers. In this subsection, we will prove that this property holds approximately in our both schemes.

Lemma 19 *For the EC-S-FDTDI scheme, the following identity holds*:

$$\epsilon(\delta_x E_{x_{i,j}}^n+\delta_y E_{y_{i,j}}^n)-\frac{\Delta t}{2}\delta_x\delta_y H_{z_{i,j}}^n=\epsilon(\delta_x E_{x_{i,j}}^0+\delta_y E_{y_{i,j}}^0)-\frac{\Delta t}{2}\delta_x\delta_y H_{z_{i,j}}^0. \tag{110}$$

Proof Note that δ_x(or δ_y) and δ_t can be interchangeable, then from (70) to (72),

$$\begin{aligned}\delta_t(\delta_x E_{x_{i,j}}^{n+\frac{1}{2}}+\delta_y E_{y_{i,j}}^{n+\frac{1}{2}}) &= \delta_x\delta_t E_{x_{i,j}}^{n+\frac{1}{2}}+\delta_y\delta_t E_{y_{i,j}}^{n+\frac{1}{2}}\\ &=-\frac{\Delta t}{4\mu\epsilon}\delta_x\delta_x\delta_y(E_{y_{i,j}}^{n+1}+E_{y_{i,j}}^n)\\ &\quad+\frac{\Delta t}{4\mu\epsilon}\delta_y\delta_x\delta_y(E_{x_{i,j}}^{n+1}+E_{x_{i,j}}^n)\\ &=\frac{\Delta t}{2\epsilon}\delta_x\delta_y\left(-\frac{1}{2\mu}\delta_x(E_{y_{i,j}}^{n+1}\right.\\ &\quad\left.+E_{y_{i,j}}^n)+\frac{1}{2\mu}\delta_y(E_{x_{i,j}}^{n+1}+E_{x_{i,j}}^n)\right)\\ &=\frac{\Delta t}{2\epsilon}\delta_x\delta_y\delta_t H_{z_{i,j}}^{n+\frac{1}{2}}=\frac{\Delta t}{2\epsilon}\delta_t(\delta_x\delta_y H_{z_{i,j}}^{n+\frac{1}{2}}).\end{aligned} \tag{111}$$

So by summing over n,

$$\epsilon(\delta_x E_{x_{i,j}}^n+\delta_y E_{y_{i,j}}^n)-\frac{\Delta t}{2}\delta_x\delta_y H_{z_{i,j}}^n=\epsilon(\delta_x E_{x_{i,j}}^0+\delta_y E_{y_{i,j}}^0)-\frac{\Delta t}{2}\delta_x\delta_y H_{z_{i,j}}^0. \tag{112}$$

This is what we want. □

Remark 20 Roughly, we can say that for the EC-S-FDTDBEC I scheme, $\epsilon(\delta_x E^n_{x_{i,j}} + \delta_y E^n_{y_{i,j}})$ is first order approximation of the term $\text{div}(\epsilon\mathbf{E})(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$ in the sense of L^∞ norm:

$$\left|\epsilon(\delta_x E^n_{x_{i,j}} + \delta_y E^n_{y_{i,j}})\right| = O(\Delta t). \tag{113}$$

Strictly, the first order approximation is satisfied only when we can prove that the term $\delta_x\delta_y H^n$ is bounded.

We can also get similar identity for the EC-S-FDTDII scheme.

Lemma 21 *For the EC-S-FDTDII scheme, the following identity holds*:

$$\epsilon(\delta_x E^n_{x_{i,j}} + \delta_y E^n_{y_{i,j}}) - \frac{\Delta t^2}{16\mu}\delta_x\delta_y\delta_y E^n_{x_{i,j}} = \epsilon(\delta_x E^0_{x_{i,j}} + \delta_y E^0_{y_{i,j}}) - \frac{\Delta t^2}{16\mu}\delta_x\delta_y\delta_y E^0_{x_{i,j}}. \tag{114}$$

Proof Recalling what we just mentioned on the last lemma, we can obtain the next equations from (80) to (82):

$$\begin{aligned}\delta_t(\delta_x E^{n+\frac{1}{2}}_{x_{i,j}} + \delta_y E^{n+\frac{1}{2}}_{y_{i,j}}) &= \delta_x\left(\frac{1}{2\epsilon}\delta_y\left(H^{n+1}_{i,j} + H^n_{i,j}\right) - \frac{\Delta t}{16\mu\epsilon}\delta_y\delta_y\left(E^{n+1}_{x_{i,j}} + E^n_{x_{i,j}}\right)\right)\\ &\quad -\delta_y\left(\frac{1}{2\epsilon}\delta_x\left(H^{n+1}_{i,j} + H^n_{i,j}\right) + \frac{\Delta t}{8\mu\epsilon}\delta_x\delta_y\left(E^{n+1}_{x_{i,j}} + E^n_{x_{i,j}}\right)\right)\\ &= \frac{\Delta t^2}{16\mu\epsilon}\delta_x\delta_y\delta_y\frac{E^{n+1}_{x_{i,j}} - E^n_{x_{i,j}}}{\Delta t} = \frac{\Delta t^2}{16\mu\epsilon}\delta_t\left(\delta_x\delta_y\delta_y E^{n+\frac{1}{2}}_{x_{i,j}}\right).\end{aligned}$$

Also by summing over n,

$$\epsilon(\delta_x E^n_{x_{i,j}} + \delta_y E^n_{y_{i,j}}) - \frac{\Delta t^2}{16\mu}\delta_x\delta_y\delta_y E^n_{x_{i,j}} = \epsilon(\delta_x E^0_{x_{i,j}} + \delta_y E^0_{y_{i,j}}) - \frac{\Delta t^2}{16\mu}\delta_x\delta_y\delta_y E^0_{x_{i,j}}. \tag{115}$$

Which is exactly the Eq. (114) in this theorem. □

For the EC-S-FDTDII scheme, we can prove strictly that the $\epsilon(\delta_x E^n_{x_{i,j}} + \delta_y E^n_{y_{i,j}})$ is first order approximation of the divergence-free term in energy norm.

Theorem 22 *If the mesh sizes Δt, Δx and Δy are same, and $\Delta t \le 2\sqrt{\mu\epsilon}$, and the assumptions of the Theorem 16 are satisfied, then*

$$\begin{aligned}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\epsilon\left(\delta_x E^n_{x_{i,j}} + \delta_y E^n_{y_{i,j}}\right)^2\Delta x\Delta y &\le C\Delta t^2 + C\Delta t^2\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left(\delta_x\delta_y\delta_y E^0_{x_{i,j}}\right)^2\Delta x\Delta y\\ &\quad + C\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\epsilon\left(\delta_x E^0_{x_{i,j}} + \delta_y E^0_{y_{i,j}}\right)^2\Delta x\Delta y.\end{aligned} \tag{116}$$

Proof If the step sizes Δt, Δx and Δy are at same order, then from the convergence results, the terms $\delta_x\delta_y\mathcal{E}_x^n$, $\delta_y\delta_y\mathcal{E}_x^n$ and $\delta_x\delta_y\mathcal{E}_y^n$ are bounded by a constant C_1:

$$\|\delta_x\delta_y\mathcal{E}_x^n\|_{E_x} \le C_1, \quad \|\delta_y\delta_y\mathcal{E}_x^n\|_{E_x} \le C_1, \quad \text{and} \quad \|\delta_x\delta_y\mathcal{E}_y^n\|_H \le C_1. \tag{117}$$

So it is easy to show that

$$\begin{aligned} &\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\delta_x\delta_y E^n_{x_{i,j+\frac{1}{2}}})^2\Delta x\Delta y \le C_2,\\ &\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left(\delta_y\delta_y E^n_{x_{i+\frac{1}{2},j}}\delta_x\delta_y E^n_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\right)\Delta x\Delta y \le C_2. \end{aligned} \tag{118}$$

By the definition of δ_x and δ_y, if $\mathbf{E}$ satisfies the PEC boundary conditions (27), $\delta_x E^n_{x_{i,j}}$ and $\delta_y E^n_{y_{i,j}}$ still keeps the PEC boundary conditions. Therefore summing over i, j in Eq. (114), we can use the Abel transformation, and by Lemma 9,

$$\begin{aligned} &\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\epsilon\left(\delta_x E^n_{x_{i,j}}+\delta_y E^n_{y_{i,j}}\right)^2\Delta x\Delta y\\ &= -\frac{\Delta t^2}{16\mu}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left((\delta_x\delta_y E^n_{x_{i,j+\frac{1}{2}}})^2+\delta_y\delta_y E^n_{x_{i+\frac{1}{2},j}}\delta_x\delta_y E^n_{y_{i+\frac{1}{2},j+\frac{1}{2}}}\right)\Delta x\Delta y\\ &\quad -\frac{\Delta t^2}{16\mu}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\delta_x\delta_y\delta_y E^0_{x_{i,j}}\left(\delta_x E^n_{x_{i,j}}+\delta_y E^n_{y_{i,j}}\right)\Delta x\Delta y\\ &\quad +\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\epsilon\left(\delta_x E^0_{x_{i,j}}+\delta_y E^0_{y_{i,j}}\right)\left(\delta_x E^n_{x_{i,j}}+\delta_y E^n_{y_{i,j}}\right)\Delta x\Delta y. \end{aligned} \tag{119}$$

The first term in the right side of the Eq. (119) is estimated in (117), and the other two terms can be dealt with by Schwarz inequality,

$$\begin{aligned} &\left(\frac{1}{2}-\frac{\Delta t^2}{32\mu\epsilon}\right)\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\epsilon\left(\delta_x E^n_{x_{i,j}}+\delta_y E^n_{y_{i,j}}\right)^2\Delta x\Delta y\\ &\le \frac{C_2\Delta t^2}{8\mu}+\frac{\Delta t^2}{32\mu}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left(\delta_x\delta_y\delta_y E^0_{x_{i,j}}\right)^2\Delta x\Delta y\\ &\quad +\frac{1}{2}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\epsilon\left(\delta_x E^0_{x_{i,j}}+\delta_y E^0_{y_{i,j}}\right)^2\Delta x\Delta y. \end{aligned}$$

By simply requiring $\Delta t \le 2\sqrt{\mu\epsilon}$, we obtain the results. □

Remark 23 For the divergence-free condition, the finite element methods for the problems normally deal with it as a equation, in which one can construct the finite element spaces satisfying the divergence-free condition in some weak sense, (e.g., [8,18]), but it could lead to computational complexities and huge costs. On the other hand, for the finite difference methods, specially for the FDTD methods, the divergence-free condition is often omitted. The corresponding analysis of the condition can not be found for the numerical algorithms in the previous works. It is well known in physics that the divergence-free condition will always be satisfied if the initial electric wave fields are divergence-free. Here in the paper, we analyze theoretically the divergence free condition in our EC-S-FDTD schemes and strictly prove the discrete divergence term to be first order approximation to the divergence-free condition. Furthermore, numerical experiments in Sect. 5 have shown that our EC-S-FDTD schemes have excellent approximations to the condition.

4 Numerical dispersion analysis

In the previous section, we have proven that the energy conservation properties also hold. Therefore, based on this point, we can say that the both schemes are non-dissipative. In this section, the non-dissipation property can be confirmed by numerical dispersion analysis, and we also compare the dispersion relations of the different schemes at the same time.

In the dispersion analysis, we suppose that the time-harmonic solution of the Maxwell's equations are:

$$\mathbf{E}^n_{\alpha,\beta} = \mathbf{E}_0 \xi^n e^{-\mathrm{i}(k_x \alpha \Delta x + k_y \beta \Delta y)}, \quad H^n_{z\alpha,\beta} = H_{z0} \xi^n e^{-\mathrm{i}(k_x \alpha \Delta x + k_y \beta \Delta y)}, \tag{120}$$

where $\mathrm{i} = \sqrt{-1}$ is the complex number, $\mathbf{E}_0 = (E_{x0}, E_{y0})^T$, k_x and k_y are the wave numbers along the x-axis and y-axis, respectively, ξ is the stability factor.

Define a_x and b_x as

$$a_x = \frac{1}{\Delta x} \sin\left(\frac{1}{2} k_x \Delta x\right), \quad b_x = \frac{1}{\Delta y} \sin\left(\frac{1}{2} k_y \Delta y\right).$$

As a result, we obtain the equation of the stability factor ξ for the EC-S-FDTDI scheme:

$$(\xi - 1)(d_0 \xi^2 + 2 d_1 \xi + d_0) = 0, \tag{121}$$

where d_0 and d_1 are:

$$\begin{aligned} d_0 &= 1 + \frac{\Delta t^2}{\mu\epsilon}(a_x^2 + b_y^2) + \frac{\Delta t^4}{(\mu\epsilon)^2}(a_x b_y)^2, \\ d_1 &= -1 + \frac{\Delta t^2}{\mu\epsilon}(a_x^2 + b_y^2) + \frac{\Delta t^4}{(\mu\epsilon)^2}(a_x b_y)^2. \end{aligned} \tag{122}$$

So the roots of the Eq. (121) are:

$$\xi_1 = 1, \quad \xi_2 = -\frac{d_1}{d_0} + \mathrm{i}\frac{\sqrt{d_0^2 - d_1^2}}{d_0}, \quad \xi_3 = -\frac{d_1}{d_0} - \mathrm{i}\frac{\sqrt{d_0^2 - d_1^2}}{d_0}. \tag{123}$$

For the EC-S-FDTDII scheme, we still have Eq. (121) but with different d_0 and d_1:

$$d_0 = \left(1 - \frac{\Delta t^2}{4\mu} b_y^2\right)^2 + \left(\frac{\Delta t^2}{\mu\epsilon}\right)\left(a_x^2 + b_y^2\right) + \frac{\Delta t^4}{2\mu^2\epsilon^2} a_x^2 b_y^2 + \frac{\Delta t^6}{16\mu^3\epsilon^3} a_x^2 b_y^4,$$
$$d_1 = -\left(1 - \frac{\Delta t^2}{4\mu\epsilon} b_y^2\right)^2 + \frac{\Delta t^2}{\mu\epsilon}\left(a_x^2 + b_y^2\right) + \frac{\Delta t^4}{2\mu^2\epsilon^2} a_x^2 b_y^2 + \frac{\Delta t^6}{16\mu^3\epsilon^3} a_x^2 b_y^4.$$

And the roots are:

$$\xi_1 = 1, \quad \xi_2 = -\frac{d_1}{d_0} + \mathrm{i}\frac{\sqrt{d_0^2 - d_1^2}}{d_0}, \quad \xi_3 = -\frac{d_1}{d_0} - \mathrm{i}\frac{\sqrt{d_0^2 - d_1^2}}{d_0}. \tag{124}$$

Clearly, for the EC-S-FDTDI and EC-S-FDTDII schemes, the modulus of these three roots are both equal to one as well, which means that the both schemes are all non-dissipative which is consistent with the energy conservation property.

Let the wave speed $c = \frac{1}{\sqrt{\mu\epsilon}}$. Taking the stability factor $\xi = e^{\mathrm{i}\omega\Delta t}$, we obtain the numerical dispersion relations of the EC-S-FDTD schemes:

– EC-S-FDTDI:

$$\sin^2\left(\frac{1}{2}\omega\Delta t\right) = (c\Delta t)^2 \cos^2\left(\frac{1}{2}\omega\Delta t\right)\left(a_x^2 + b_y^2 + (c\Delta t)^2 a_x^2 b_y^2\right). \tag{125}$$

– EC-S-FDTDII:

$$\sin^2\left(\frac{1}{2}\omega\Delta t\right) = \cos^2\left(\frac{1}{2}\omega\Delta t\right)\frac{\frac{1}{2}(c\Delta t)^4 a_x^2 b_y^2 + (c\Delta t)^2(a_x^2 + b_y^2) + \frac{1}{16}(c\Delta t)^6 a_x^2 b_y^4}{\left(1 - \frac{(c\Delta t)^2}{4} b_y^2\right)^2}. \tag{126}$$

Note that the numerical dispersion relations (125) and (126) both converge to the analytical dispersion relation of the problem:

$$\omega^2 = c^2(k_x^2 + k_y^2), \tag{127}$$

when Δt, Δx and Δy all tend to zero. The wave number vector $\mathbf{k} = (k_x, k_y)$ can be expressed by polar coordinates: $k_x = k\cos\phi$ and $k_y = k\sin\phi$. Now the analytical dispersion relation can be rewritten as $\omega = ck$.

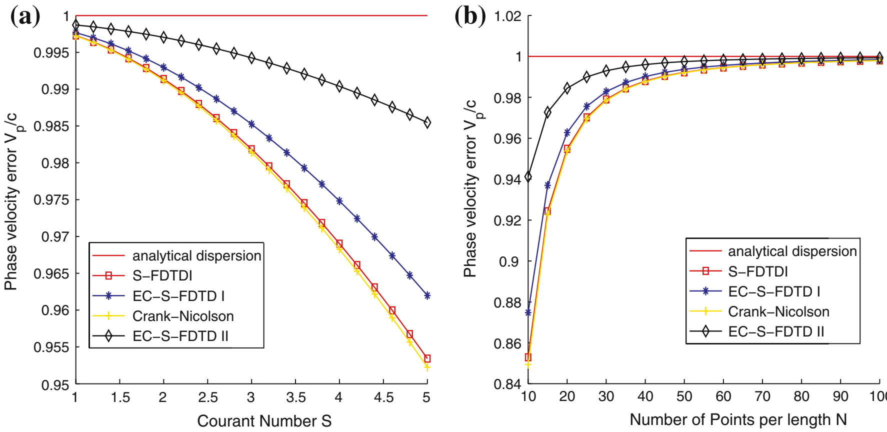

Fig. 2 **a** Numerical dispersion against the CFL number with $N_\lambda = 40$ and $\theta = 65°$, **b** numerical dispersion against the number of points per wavelength N_λ with $S = 2.4$ and $\theta = 65°$

Let λ be the wave length, $\Delta x = \Delta y = h$ the spatial step size, $N_\lambda = \frac{\lambda}{h}$ the number of points per wavelength (NPPW), and $S = \frac{c\Delta t}{h}$ the CFL number, respectively. Therefore we can regard the stability factor ξ as a function of the variables S, ϕ and N_λ: $\xi = \xi(S, \phi, N_\lambda)$. Now we present the numerical dispersion errors of the four methods (S-FDTD I, EC-S-FDTDI, EC-S-FDTDII and the Crank–Nicolson) with different gird sizes h (or N_λ), wave propagation angles ϕ and CFL numbers S.

Let $\xi = e^{\mathrm{i}\omega t}$, ω is a complex number: $\omega = \omega_R + \mathrm{i}\omega_I$, where ω_R and ω_I are the real and imaginary part, respectively, then

$$\xi = e^{-\omega_I \Delta t}(\cos(\omega_R \Delta t) + \mathrm{i}\sin(\omega_R \Delta t)). \tag{128}$$

Let $\mathrm{Im}(\xi)$ and $\mathrm{Re}(\xi)$ denote the imaginary and real parts of ξ respectively. So we can express the numerical phase velocity v_p normalized to the speed of the wave c as:

$$\frac{v_p}{c} = \frac{\omega_R/k}{c} = \frac{1}{ck\Delta t}\arctan\left(\frac{|\mathrm{Im}(\xi)|}{|\mathrm{Re}(\xi)|}\right) = \frac{N_\lambda}{2\pi S}\arctan\left(\frac{|\mathrm{Im}(\xi)|}{|\mathrm{Re}(\xi)|}\right). \tag{129}$$

Figures 2 and 3 show the normalized phase velocity v_p/c against the wave courant number S, the number of points per wavelength N_λ and the propagation angles ϕ respectively. From these figures, we can see that the numerical dispersion of EC-S-FDTDII is the closest to the analytic solution 1. Furthermore, it is interesting that in Fig. 3, the numerical dispersion of EC-S-FDTDII almost does not decline when the propagation angles ϕ is near 90 degree, until now we do not know whether this phenomena is useful.

5 Numerical experiments

Just as we have mentioned in the beginning of the paper and proven in the preceding sections, our new splitting FDTD schemes have some nice advantages. In this section,

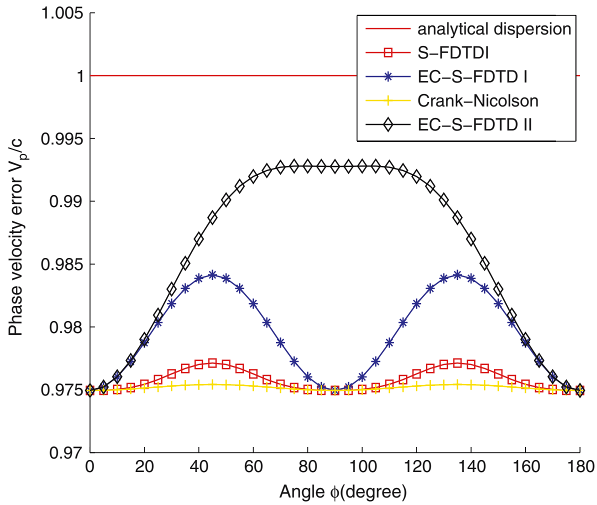

Fig. 3 Numerical dispersion against the propagation angles ϕ with $N_\lambda = 40$ and $S = 3.5$

we will demonstrate the following properties: (1) energy conservation; (2) accuracy analysis; (3) unconditionally stable even in the long time computation; (4) the convergence of the divergence free. In the numerical experiments, we will consider constant electric permittivity case and jumped coefficient case.

5.1 Constant electric permittivity case

If we assume that the magnetic field H_z is in the form of $e^{\mathrm{i}(\omega t - k_x x - k_y y)}$, then in order to satisfy the 2D Maxwell equations, the electric field $\mathbf{E}$ should be

$$\frac{1}{\epsilon}\left(-\frac{k_y}{\omega}, \frac{k_x}{\omega}\right) e^{\mathrm{i}(\omega t - k_x x - k_y y)},$$

and k_x, k_y satisfy the dispersion relation: $\omega^2 = \frac{1}{\mu\epsilon}(k_x^2 + k_y^2)$.

Now let us consider the domain $\Omega = [0, 1] \times [0, 1]$ surrounded by a perfect conductor, which means that $E_y(0, y) = E_y(1, y) = 0$ and $E_x(x, 0) = E_x(x, 1) = 0$. So we can use the following analytic solution to check the code:

$$\begin{aligned} E_x &= \frac{k_y}{\epsilon\sqrt{\mu}\omega}\cos(\omega\pi t)\cos(k_x\pi x)\sin(k_y\pi y), \\ E_y &= -\frac{k_x}{\epsilon\sqrt{\mu}\omega}\cos(\omega\pi t)\sin(k_x\pi x)\cos(k_y\pi y), \end{aligned} \tag{130}$$

and

$$H_z = \frac{1}{\sqrt{\mu}}\sin(\omega\pi t)\cos(k_x\pi x)\cos(k_y\pi y). \tag{131}$$

Table 1 Relative errors of EnergyI for the EC-S-FDTD schemes. Parameters: $T = 1$, $\Delta t = \Delta x = \Delta y = 0.01$ and different $k_x = k_y$

$k_x = k_y$	EC-S-FDTDI	EC-S-FDTDII
1	$1.18e - 14$	$3.55e - 15$
5	$9.77e - 15$	$2.66e - 15$
10	$9.55e - 15$	$2.66e - 15$

Now let us check the properties of our schemes.

1. *Energy conversations.* The energies of the solution are easily to be computed, one (Energy I) is

$$\text{EnergyI} = \left(\int_\Omega \epsilon\, |\mathbf{E}(x,t)|^2\, dx + \int_\Omega \mu\, |H_z(x,t)|^2\, dx \right)^{\frac{1}{2}} = \frac{1}{2}, \tag{132}$$

and the another (Energy II) is

$$\text{EnergyII} = \left(\int_\Omega \epsilon \left| \frac{\partial \mathbf{E}}{\partial t}(x,t) \right|^2 dx + \int_\Omega \mu \left| \frac{\partial H_z}{\partial t}(x,t) \right|^2 dx \right)^{\frac{1}{2}} = \frac{\pi \omega}{2}. \tag{133}$$

Let us fix $T = 1$, $\mu = \epsilon = 1$, $\Delta t = \Delta x = \Delta y = \frac{1}{100}$ and change $k_x = k_y$ as 1, 5 and 10. Now let us define the relative errors as

$$\text{Error of EnergyI} = \max_{0 \le n \le N} \frac{\left| \left(\|\epsilon^{\frac{1}{2}} \mathbf{E}^n\|^2 + \|\mu^{\frac{1}{2}} H_z^n\|^2 \right)^{\frac{1}{2}} - \text{EnergyI} \right|}{\text{EnergyI}}, \tag{134}$$

and

$$\text{Error of EnergyII} = \max_{0 \le n \le N-1} \frac{\left| \left(\|\epsilon^{\frac{1}{2}} \delta_t \mathbf{E}^{n+\frac{1}{2}}\|^2 + \|\mu^{\frac{1}{2}} \delta_t H_z^{n+\frac{1}{2}}\|^2 \right)^{\frac{1}{2}} - \text{EnergyII} \right|}{\text{EnergyII}}. \tag{135}$$

Table 1 shows that the energyI of the discrete solutions perfectly equal to the exact value 0.5 since the errors are near machine precision. And from results listed in the second and fourth columns of Table 2 show that the discrete energyII is also a good approximation of the continuous energyII. But comparing with the relative error of the energyI, the results seem not so confident. So we further compute the difference of the discrete energyII at two neighbor level, that is to say, we will check the value:

$$\max_{1\le n\le N-1}\left(\left[\|\epsilon^{\frac{1}{2}}\delta_t\mathbf{E}^{n+\frac{1}{2}}\|^2+\|\mu^{\frac{1}{2}}\delta_t H_z^{n+\frac{1}{2}}\|^2\right]^{\frac{1}{2}}-\left[\|\epsilon^{\frac{1}{2}}\delta_t\mathbf{E}^{n-\frac{1}{2}}\|^2+\|\mu^{\frac{1}{2}}\delta_t H_z^{n-\frac{1}{2}}\|^2\right]^{\frac{1}{2}}\right).$$

The results are listed in third column and fifth column of Table 2. From Table 2, the difference of the energyII at neighbor level is also near machine precision. Now it is clear that the relative error of energyII mainly depends the error of the first step. So from Tables 1 and 2, we can say that Theorems 10 and 11 hold, that is to say, our EC-S-FDTDI and EC-S-FDTDII both keep the energy conservation property.

2. *Accuracy analysis.* Table 3 compares the accuracy of S-FDTDII, EC-S-FDTDI and EC-S-FDTDII. First, to calculate the errors of three schemes, we use the absolute error below:

$$\text{ErrorI}=\max_{0\le n\le N}\left(\|\epsilon^{\frac{1}{2}}[\mathbf{E}(t^n)-E^n]\|_E^2+\|\mu^{\frac{1}{2}}[H_z(t^n)-H_z^n]\|_H^2\right)^{\frac{1}{2}}/\text{EnergyI},$$

and

$$\text{ErrorII}=\max_{0\le n\le N-1}\left(\|\epsilon^{\frac{1}{2}}[\delta_t\mathbf{E}(t^n)-\delta_t E^n]\|_E^2+\|\mu^{\frac{1}{2}}[\delta_t H_z(t^n)-\delta_t H_z^n]\|_H^2\right)^{\frac{1}{2}}/\text{EnergyII},$$

Table 2 Relative errors of EnergyII for the EC-S-FDTD schemes and the difference of the discrete energyII at neighbor level

$k_x = k_y$	EC-S-FDTDI		EC-S-FDTDII	
	Error of EnergyII	Difference at neighbor level	Error of EnergyII	Difference at neighbor level
1	$2.26e-4$	$1.33e-15$	$1.95e-4$	$8.88e-16$
5	$5.61e-3$	$5.33e-15$	$4.85e-3$	$5.32e-15$
10	$2.20e-2$	$1.42e-14$	$1.91e-2$	$1.07e-14$

Paramters: $T=1$, $\Delta t=\Delta x=\Delta y=0.01$ and different $k_x=k_y$

Table 3 The relative errors for different schemes

N	S-FDTD II		EC-S-FDTDI		EC-S-FDTDII	
	ErrorI	ErrorII	ErrorI	ErrorII	ErrorI	ErrorII
25	$1.08e-2$	$1.31e-3$	$4.46e-2$	$2.70e-3$	$8.00e-3$	$1.12e-3$
50	$2.806-3$	$3.31e-4$	$2.22e-2$	$9.45e-4$	$2.00e-3$	$2.86e-4$
100	$6.76e-4$	$8.28e-5$	$1.12e-2$	$3.35e-4$	$5.04e-4$	$7.14e-5$
200	$1.69e-4$	$2.07e-5$	$5.60e-3$	$1.19e-4$	$1.26e-4$	$1.79e-5$
400	$4.23e-5$	$5.17e-6$	$2.80e-3$	$4.19e-5$	$3.14e-5$	$4.47e-6$

Parameters: $T=1$, $N_x=N_y=N_t=N$, $k_x=k_y=1$ and $\mu=\epsilon=1$

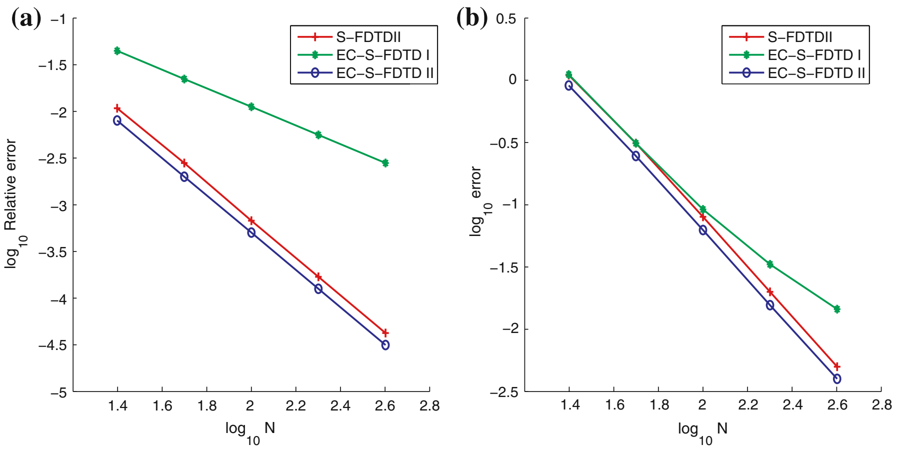

Fig. 4 **a** Error1 for different schemes when $k_x = k_y = 1$, **b** Error1 for different schemes when $k_x = k_y = 5$

where E^n, H_z^n and $E(t^n)$, $H_z(t^n)$ denote the numerical solution and the analytic solution, respectively at time level n. The results in Table 3 indicate that the accuracies of S-FDTDII and EC-S-FDTDII are both really second order in time and space for different $k_x = k_y$. For the EC-S-FDTDI scheme, the data of ErrorI indicates that this method is of first order, but the data of ErrorII seems to say that the convergence is more than first order, i.e., there is "superconvergence". Moreover, we test the code for different $k_x = k_y$: when k_x is small, Fig. 4a shows that the EC-S-FDTDI scheme is of first order in time; however, when k_x is large, Fig. 4b indicates that the EC-S-FDTDI scheme is close to second order. In other words, the ES-S-FDTDI scheme is more accurate for the high frequency components.

3. *Unconditionally stable even in the long time computation.* Here we set $T = 100$, and $\Delta x = \Delta y = \Delta t = 0.01$, i.e., the code runs $10,000$ steps. Figure 5a indicates that for the EC-S-FDTDI and EC-S-FDTDII schemes, the relative error of energyI are both controlled under 10^{-12} after $10,000$ time steps, and the error grows linearly. By contrast, the results will blow up for the S-FDTDI scheme and oscillate from 0 to 10^{-5} for the S-FDTDII scheme. And Fig. 5b suggests that when the time level increases, the error of the solutions also grows linearly. In our theoretical analysis, the errors are controlled by e^T, which means that the errors are overestimated in our theoretical analysis for long time computation.
4. *Convergence analysis of the divergence-free term.* First we want to verify the identities (110) and (114). Now for the EC-S-FDTDI scheme, we compute the "errors" of these identities, that is to say we calculate the value

$$\max_{\substack{1\le i,j\le N-1\\ 1\le n\le N}} \left| \left(\epsilon(\delta_x E^n_{x_{i,j}} + \delta_y E^n_{y_{i,j}}) - \frac{\Delta t}{2}\delta_x\delta_y H^n_{z_{i,j}} \right) - \left(\epsilon(\delta_x E^0_{x_{i,j}} + \delta_y E^0_{y_{i,j}}) - \frac{\Delta t}{2}\delta_x\delta_y H^0_{z_{i,j}} \right) \right|,$$

and for the EC-S-FDTDII scheme, we calculate the value

$$\max_{\substack{1\le i,j\le N-1\\1\le n\le N}}\left|\left(\epsilon(\delta_x E^n_{x_{i,j}}+\delta_y E^n_{y_{i,j}})-\frac{\Delta t^2}{16\mu}\delta_x\delta_y\delta_y E^n_{x_{i,j}}\right)\right.$$
$$\left.-\left(\epsilon(\delta_x E^0_{x_{i,j}}+\delta_y E^0_{y_{i,j}})-\frac{\Delta t^2}{16\mu}\delta_x\delta_y\delta_y E^0_{x_{i,j}}\right)\right|.$$

The results listed in Table 4 indicate that identities (110) and (114) hold. Now we check the solution of E_x and E_y whether they are nearly divergence-free. Here we compute the discrete divergence terms:

$$\text{Div1}=\max_{\substack{1\le i,j\le N-1\\0\le n\le N}}\left|\epsilon(\delta_x E^n_{x_{i,j}}+\delta_y E^n_{y_{i,j}})\right|,$$

and

$$\text{Div2}=\max_{0\le n\le N}\left(\sum_{1\le i\le N-1}\sum_{1\le j\le N-1}\epsilon\left(\delta_x E^n_{x_{i,j}}+\delta_y E^n_{y_{i,j}}\right)^2\Delta x\Delta y\right)^{\frac{1}{2}}.$$

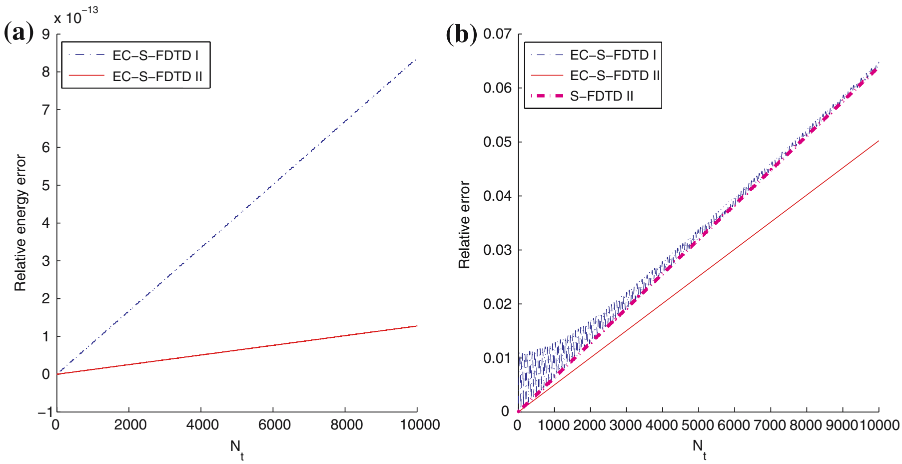

Fig. 5 **a** The energyI error in the long time computation, **b** ErrorI in the long time computation

Table 4 The errors of identities (110) and (114)

N	EC-S-FDTDI	EC-S-FDTDII
25	$2.96e-14$	$2.20e-15$
50	$9.70e-14$	$1.01e-14$
100	$3.38e-13$	$1.58e-14$
200	$1.18e-12$	$2.79e-14$
400	$4.03e-12$	$6.47e-14$

Parameters: $T=1$, $N_x=N_y=N_t=N$, $k_x=k_y=1$ and $\mu=\epsilon=1$

Table 5 The numerical divergence the EC-S-FDTD schemes

N	EC-S-FDTDI		EC-S-FDTDII	
	Div1	Div2	Div1	Div2
25	$1.96e-1$	$9.84e-2$	$4.40e-3$	$2.20e-3$
50	$9.86e-2$	$4.93e-2$	$1.10e-3$	$5.48e-4$
100	$4.93e-2$	$2.47e-2$	$2.74e-4$	$1.37e-4$
200	$2.47e-2$	$1.23e-2$	$6.85e-5$	$3.43e-5$
400	$1.23e-2$	$6.20e-3$	$1.71e-5$	$8.56e-6$

Parameters: $T = 1$, $N_x = N_y = N_t = N$, $k_x = k_y = 1$ and $\mu = \epsilon = 1$

From Table 5, we know that the numerical divergence term of EC-S-FDTDI and II is first order and second order in time, respectively. Our theoretical analysis is weaker than the numerical results. From Theorem 22, we only obtain that Div2 $= O(\Delta t)$.

5.2 Jumped electric permittivity case

Here we assume that the electric permittivity ϵ is piecewise constant on the domain $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$:

$$\epsilon = \begin{cases} 1, & \text{in } \Omega_1, \\ 4, & \text{in } \Omega_2, \end{cases}$$

where $\Omega_1 \in [0, \frac{1}{2}] \times [0, 1]$ and $\Omega_2 \in [\frac{1}{2}, 1] \times [0, 1]$ and the magnetic permeability $\mu = 1$ in Ω. Here we can still construct one exact solution to check the numerical results. Take $k_y = 8$ and let k_x be piece-wise constant on the domain Ω:

$$k_x = \begin{cases} 4, & \text{in } \Omega_1, \\ 16, & \text{in } \Omega_2. \end{cases}$$

And we take $\omega = \left(\frac{{k_x}^2+{k_y}^2}{\mu\epsilon}\right)^{\frac{1}{2}}$, which is also piece-wise constant on the domain Ω. It is easy to check that $\mathbf{E}$ defined by (130) and H_z by (131) are also the exact solutions of the Maxwell equations. Note that the exact solution may be discontinuous where the electric permittivity ϵ jumps. For example, Fig. 6 shows that the solution E_x is discontinuous when $t = \frac{1}{3}$.

The numerical results are similar as the constant coefficient case. Here we just show the results of energy conservations (see Tables 6, 7) and error behaviors (see Table 8), the results show that for the piecewise constant electric permittivity, the energy conservations still hold in our both schemes, and the convergences of the EC-S-FDTDI scheme and the EC-SFDTDII scheme are first order and second order respectively. In fact, here the wave number is some big, then for big step size ($N = 50$), the numerical methods do not converge, this is common for the high frequency wave

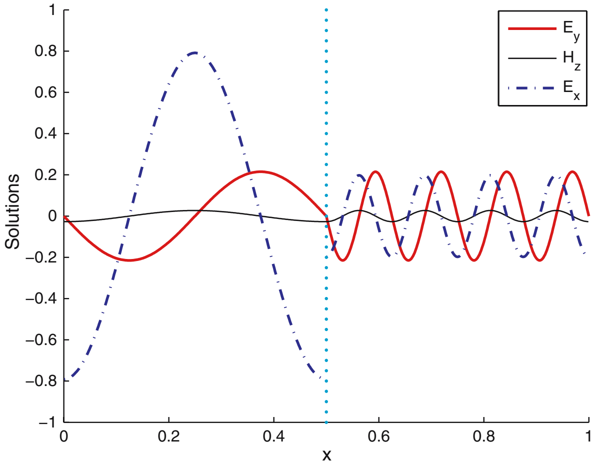

Fig. 6 The exact solutions at $t = \frac{1}{3}$ and $y = \frac{1}{3}$

Table 6 Relative errors of EnergyI by the EC-S-FDTD schemes at $T = 1$ for piecewise ϵ

$N_x = N_y = N_t$	EC-S-FDTDI	EC-S-FDTDII
50	$6.77e-15$	$6.21e-15$
100	$2.13e-14$	$1.71e-14$
200	$8.94e-14$	$8.88e-14$
400	$2.78e-13$	$2.08e-13$

Table 7 Relative errors of energyII for the EC-S-FDTD schemes and the difference of the discrete energyII at neighbor level for piecewise ϵ

N	EC-S-FDTDI		EC-S-FDTDII	
	Error of EnergyII	Difference at neighbor level	Error of EnergyII	Difference at neighbor level
50	$5.11e-2$	$1.05e-13$	$4.56e-2$	$1.31e-13$
100	$1.33e-2$	$3.61e-13$	$1.20e-2$	$3.46e-13$
200	$3.30e-3$	$2.62e-12$	$3.04e-3$	$1.65e-12$
400	$7.90e-4$	$5.66e-12$	$7.62e-4$	$4.90e-12$

Table 8 The relative errors of the solutions (ErrorI and ErrorII) for piecewise ϵ

$N_x = N_y = N_t$	EC-S-FDTDI		EC-S-FDTDII	
	ErrorI	ErrorII	ErrorI	ErrorII
50	1.11	1.06	1.01	0.965
100	$3.10e-1$	$3.10e-1$	$2.74e-1$	$2.710e-1$
200	$8.34e-2$	$8.76e-2$	$6.90e-2$	$6.89e-2$
400	$2.68e-2$	$3.01e-2$	$1.72e-2$	$1.73e-2$

(for example, see [13]). And it is interesting that for the EC-S-FDTDI scheme, the convergence is higher than first order.

6 Conclusion

In this paper, we developed two efficient energy-conserved splitting finite-difference time-domain schemes (EC-S-FDTDI and EC-S-FDTDII) for Maxwell's equations in two dimensions. From the theoretical analysis and numerical experiments, the motivations of our schemes are realized that our new EC-S-FDTD schemes are both energy conserved and have the simplicity and efficiency of computation. By the energy method, we prove that our EC-S-FDTDII scheme is of second order both in time and space. Meanwhile, the error behavior of the divergence-free condition is analyzed clearly and strictly, and the numerical results agree excellently with the theoretical analysis. Our results are being generalized to the three dimensional cases, and will be reported in another paper.

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