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Two-grid finite volume element method for linear and nonlinear elliptic problems

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Abstract Two-grid finite volume element discretization techniques, based on two linear conforming finite element spaces on one coarse and one fine grid, are presented for the two-dimensional second-order non-selfadjoint and indefinite linear elliptic problems and the two-dimensional second-order nonlinear elliptic problems. With the proposed techniques, solving the non-selfadjoint and indefinite elliptic problem on the fine space is reduced into solving a symmetric and positive definite elliptic problem on the fine space and solving the non-selfadjoint and indefinite elliptic problem on a much smaller space; solving a nonlinear elliptic problem on the fine space is reduced into solving a linear problem on the fine space and solving the nonlinear elliptic problem on a much smaller space. Convergence estimates are derived to justify the efficiency of the proposed two-grid algorithms. A set of numerical examples are presented to confirm the estimates.

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1 Introduction

The finite volume element method (FVEM) is a discretization technique for the partial differential equations, especially for those arising from physical conservation laws

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including mass, momentum, and energy. Because this method possess the crucial physical conservation properties of the original problem locally, it is popular in computational fluid mechanics. In the past several decades, many researchers have studied this method extensively and obtained some important results. We refer to the monograph [20] for the general presentation of this method, and to [2–4,7–11,14,16,18,19,23,26,28] and the reference therein for details.

For the second-order non-selfadjoint and indefinite linear elliptic problems, Mishev [23] has considered the FVEM in the linear conforming finite element space and established the error estimate in the H^1-norm; Wu and Li [28] have obtained the H^1 superconvergence and $L^p(1 < p \leq \infty)$ error estimates between the solution of the FVEM and that of the finite element method (FEM).

Li [19] have considered the finite volume element method for a nonlinear elliptic problem and obtained the error estimate in the H^1-norm. Recently, Chatzipantelidis, Ginting and Lazarov [9] have studied the finite volume element method for a nonlinear elliptic problem, established the error estimates in the H^1-, L^2- and L^∞-norms and proposed a Newton's method for the approximation of the finite volume element solution.

On the other hand, the two-grid finite element method based on two finite element spaces on one coarse and one fine grid was first introduced by Xu [29–31] for the nonsymmetric and nonlinear elliptic problems. Later on, the two-grid method was further investigated by many author, for instance, Xu and Zhou [32] for eigenvalue problems, Axelsson and Layton [1] for nonlinear elliptic problems, Dawson, Wheeler and Woodward [13] for finite difference scheme for nonlinear parabolic equations, Layton and Lenferink [17] and Utnes [27] for Navier–Stokes equations, Marion and Xu [22] for evolution equations.

In this paper, based on two linear conforming finite element spaces V_H and V_h on one coarse grid with grid size H and one fine grid with grid size h, we consider the two-grid finite volume element discretization techniques for the non-selfadjoint and indefinite linear elliptic problems and the nonlinear elliptic problems. With the proposed techniques, solving the non-selfadjoint and indefinite problem on the fine space is reduced to solving a symmetric and positive definite problems on the fine space and the non-selfadjoint and indefinite elliptic problems on a much smaller space; solving a nonlinear elliptic problem on the fine space is reduced into solving a linear problem on the fine space and solving the nonlinear elliptic problem on a much smaller space. This means that solving a nonlinear elliptic problem is not much more difficult than solving one linear problem, since $\dim V_H \ll \dim V_h$ and the work for solving the nonlinear problem is relatively negligible. If $h = O(H^2)$ is chosen, we show that the convergence rate of those two-grid methods are optimal in the H^1-norm and sub-optimal in the $W^{1,\infty}$-norm.

We shall use the standard notation for the Sobolev spaces $W^{m,p}(\Omega)$ with the norm $||\cdot||_{m,p,\Omega}$ and the seminorms $|\cdot|_{m,p,\Omega}$ [12]. In order to simplify the notations, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and skip the index $p = 2$ and Ω when possible, i.e., $||u||_{m,p,\Omega} = ||u||_{m,p}$, $||u||_{m,2,\Omega} = ||u||_m$, $||u||_0 = ||u||$. The same convention is used for the semi-norms as well.

The rest of this paper is organized as follows. In Sect. 2, we describe the FVEM for the non-selfadjoint and indefinite linear elliptic problem. Section 3 contains the two-grid finite volume element method with error analysis for the non-selfadjoint and indefinite linear elliptic problem. Section 4 is devoted to the FVEM for the nonlinear elliptic problems. The two-grid finite volume element method and error analysis for the nonlinear elliptic problem are established in Sect. 5. Finally, we give a set of numerical examples to confirm the a priori estimates.

Throughout this paper, the letter C denotes a generic positive constant independent of the mesh parameter and may be different at its different occurrences.

2 Finite volume element method for linear problem

In this section, we consider the FVEM for the following two-dimensional second-order non-selfadjoint and indefinite linear elliptic problem

$$\begin{cases} -\nabla \cdot (\mathbf{a}\nabla u) + \mathbf{b} \cdot \nabla u + cu = f, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset R^2$ is a bounded convex polygonal domain with the boundary $\partial\Omega$. We assume that $\mathbf{a}$, $\mathbf{b}$ and c are smooth functions and $\mathbf{a} = (a_{ij}(x))_{i,j=1}^2$ is a symmetric and uniformly positive definite matrix in Ω, i.e., there exists a positive constant α such that

$$\alpha|\xi|^2 \le \xi^T \mathbf{a}(x)\xi, \qquad \forall \xi \in R^2, \quad \forall x \in \overline{\Omega}.$$

As in [31], we introduce the following linear operator

$$\mathcal{L}v = -\nabla \cdot (\mathbf{a}\nabla v) + \mathbf{b} \cdot \nabla v + cv,$$

and assume that $\mathcal{L} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism, where $H^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$. A simple sufficient condition for this assumption is that $c(x) \ge 0$.

The following well-known regularity result can be found in Lemma 2.1 in [31] and [15]:

If $u \in H_0^1(\Omega)$ and $\mathcal{L}u \in L^q(\Omega)$ for $1 < q \le 2$, then $u \in W^{2,q}(\Omega)$ and

$$||u||_{2,q} \le C||\mathcal{L}u||_{0,q},$$

for some positive constant C depending on q, the coefficients of $\mathcal{L}$ and the domain Ω.

For later analysis of convergence of the proposed two-grid FVEM, we further assume that $a_{ij} \in W^{2,\infty}(\overline{\Omega})$, $1 \le i, j \le 2$, $\mathbf{b} = (b_1(x), b_2(x))$, $b_i \in W^{1,\infty}(\overline{\Omega})$, $i = 1, 2$, and $c \in W^{1,\infty}(\overline{\Omega})$.

The weak formulation of the problem (2.1) is to find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$-inner product and the bilinear form $a(\cdot,\cdot)$ is defined by

$$a(u,v)=\int_\Omega ((\mathbf{a}\nabla u)\cdot\nabla v+\mathbf{b}\cdot\nabla u v+cuv)\,\mathrm{d}x,\quad \forall u,v\in H_0^1(\Omega).$$

We assume that T_h is a quasi-uniform triangulation of Ω with $h=\max\{h_K\}$, where h_K is the diameter of the triangle $K\in T_h$ [12]. Based on this triangulation, we consider a finite element discretization of (2.2), in the standard conforming finite element space of piecewise linear functions,

$$V_h=\{v\in C(\overline{\Omega}) : v|_K \text{is linear for all} K\in T_h,\quad v=0 \text{ on } \partial\Omega\}.$$

The FEM for the problem (2.2) is to find $u_h^E\in V_h$ such that

$$a(u_h^E,v_h)=(f,v_h),\quad \forall v_h\in V_h, \tag{2.3}$$

where

$$a(u_h,v_h)=\int_\Omega ((\mathbf{a}\nabla u_h)\cdot\nabla v_h+\mathbf{b}\cdot\nabla u_h v_h+cu_h v_h)\,\mathrm{d}x,\quad \forall u_h,v_h\in V_h. \tag{2.4}$$

Schatz [25] has proved that for sufficiently small h, the problem (2.3) has a unique solution $u_h^E\in V_h$.

In order to describe the FVEM for solving problem (2.1), we construct a dual partition T_h^* based upon the original triangulation T_h whose elements are called the control volumes. We construct the control volume in the same way as in [9,14,16,20]. Let z_K be the barycenter of $K\in T_h$. We connect z_K with line segments to the midpoints of the edges of K, thus partitioning K into three quadrilaterals K_z, $z\in Z_h(K)$, where $Z_h(K)$ are the vertices of K. Then with each vertex $z\in Z_h=\cup_{K\in T_h}Z_h(K)$ we associate a control volume V_z, which consists of the union of the subregions K_z, sharing the vertex z. Thus we finally obtain a group of control volumes covering the domain Ω, which is called the dual partition $\mathcal{T}_h^*$ of the triangulation T_h. We denote the set of interior vertices of Z_h by Z_h^0.

We call the partition T_h^* *regular or quasi-uniform,* if there exists a positive constant $C>0$ such that

$$C^{-1}h^2\le \operatorname{meas}(V_z)\le Ch^2,\quad \forall V_z\in T_h^*.$$

The barycenter-type dual partition can be introduced for any finite element triangulation T_h and leads to relatively simple calculations. Besides, if the finite element triangulation T_h is quasi-uniform, then the dual partition T_h^* is also quasi-uniform.

We formulate the FVEM for the problem (2.1) as follows. Given a vertex $z \in Z_h$, integrating (2.1) over the associated control volume V_z and using Green's formula, we obtain

$$-\int_{\partial V_z} (\mathbf{a}\nabla u) \cdot \mathbf{n} \mathrm{d}s + \int_{V_z} (\mathbf{b} \cdot \nabla u + cu) \mathrm{d}x = \int_{V_z} f \mathrm{d}x, \tag{2.5}$$

where $\mathbf{n}$ denotes the unit outer-normal of the domain under consideration. It should be noted that the above formulation is a way of stating that we have an integral conservation form on the dual element.

The finite volume element approximation of (2.1) is defined as a solution $u_h \in V_h$ satisfying the equation

$$-\int_{\partial V_z} (\mathbf{a}\nabla u_h) \cdot \mathbf{n} \mathrm{d}s + \int_{V_z} (\mathbf{b} \cdot \nabla u_h + cu_h) \mathrm{d}x = \int_{V_z} f \mathrm{d}x. \tag{2.6}$$

The FVEM is viewed as a perturbation of the FEM with the help of an interpolation operator $I_h^* : V_h \to V_h^*$, defined by

$$I_h^* v_h = \sum_{z \in Z_h^0} v_h(z) \Psi_z,$$

where

$$V_h^* = \{v \in L^2(\Omega_h) : v|_{V_z} \text{is constant}, \quad \forall z \in Z_h^0; \, v|_{V_z} = 0, \forall z \in \partial\Omega_h\},$$

and Ψ_z is the characteristic function of the control volume V_z.

The finite volume element problem (2.6) can be rewritten in a variational form similar to the finite element problem. For an arbitrary $I_h^* v_h$, we multiply the integral in (2.6) by $v_h(z)$ and sum over all $z \in Z_h^0$ to obtain

$$a_h(u_h, I_h^* v_h) = (f, I_h^* v_h), \quad \forall v_h \in V_h, \tag{2.7}$$

where $a_h(\cdot, I_h^* \cdot)$ is defined by, for any $u_h, v_h \in V_h$,

$$\begin{aligned} a_h(u_h, I_h^* v_h) = &-\sum_{z \in Z_h^0} \int_{\partial V_z} (\mathbf{a}\nabla u_h) \cdot \mathbf{n} I_h^* v_h \mathrm{d}s \\ &+ \int_{\Omega} (\mathbf{b} \cdot \nabla u_h + cu_h) I_h^* v_h \mathrm{d}x. \end{aligned} \tag{2.8}$$

Mishev [23], Wu and Li [28] have proved that, for sufficiently small h, the problem (2.7) has a unique solution $u_h \in V_h$, and obtained the error estimate in the H^1-norm

$$||u - u_h||_1 \le Ch||u||_2. \tag{2.9}$$

The following Lemma 2.1, also proved in Wu and Li [28], gives the superconvergence estimate in the H^1-norm between the solution of the FVEM and that of the FEM.

Lemma 2.1 ([28]) *Assume that u_h^E and u_h are the solutions of (2.3) and (2.7) respectively. For sufficiently small h, we have*

$$||u_h - u_h^E||_1 \leq Ch^2||f||_1. \tag{2.10}$$

The following error estimate in the L^2-norm for the FEM for the problem (2.1) is obtained in [6],

$$||u - u_h^E||_0 \leq Ch^2||u||_2. \tag{2.11}$$

From (2.11) and Lemma 2.1, we get the following error estimate in the L^2-norm for the FVEM for the problem (2.1),

$$||u - u_h||_0 \leq Ch^2(||u||_2 + ||f||_1). \tag{2.12}$$

The following error estimates in the L^∞- and $W^{1,\infty}$-norms for the FEM for the problem (2.1) are also obtained in [6],

$$||u - u_h^E||_{0,\infty} \leq Ch^2|\ln h|||u||_{2,\infty}. \tag{2.13}$$
$$||u - u_h^E||_{1,\infty} \leq Ch||u||_{2,\infty}. \tag{2.14}$$

From the inverse inequality [6,12] and Lemma 2.1, we get the following error estimates in the L^∞- and $W^{1,\infty}$-norms for the FVEM for the problem (2.1),

$$||u - u_h||_{0,\infty} \leq Ch^2|\ln h|(||u||_{2,\infty} + ||f||_1). \tag{2.15}$$
$$||u - u_h||_{1,\infty} \leq Ch(||u||_{2,\infty} + ||f||_1). \tag{2.16}$$

The interpolation operator I_h^* has the following properties [8,28].

Lemma 2.2 *Let $K \in T_h$, e be the edge of K. For any $v_h \in V_h$, we have*

$$\int_K (v_h - I_h^* v_h)\mathrm{d}x = 0, \tag{2.17}$$
$$\int_e (v_h - I_h^* v_h)\mathrm{d}s = 0, \tag{2.18}$$
$$||v_h - I_h^* v_h||_{0,q,K} \leq Ch_K|v_h|_{1,q,K}, \quad 1 \leq q \leq \infty. \tag{2.19}$$

In addition in [8], the following Lemma 2.3 was derived.

Lemma 2.3 *Let e be the edge of a triangle $K \in T_h$. Then for any $u \in W^{1,p}(K)$, there exists a constant C independent of h_K such that*

$$\left| \int_e u(v_h - I_h^* v_h) \mathrm{d}s \right| \leq C h_K |u|_{1,p,K} |v_h|_{1,q,K}, \qquad \forall v_h \in V_h, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{2.20}$$

For the subsequent analysis, we introduce the following bilinear forms

$$a^{(2)}(u_h, v_h) = \int_\Omega (\mathbf{a} \nabla u_h) \cdot \nabla v_h \mathrm{d}x, \quad \forall u_h, v_h \in V_h, \tag{2.21}$$

$$a_h^{(2)}(u_h, I_h^* v_h) = -\sum_{z \in Z_h^0} \int_{\partial V_z} (\mathbf{a} \nabla u_h) \cdot \mathbf{n} I_h^* v_h \mathrm{d}s, \quad \forall u_h, v_h \in V_h. \tag{2.22}$$

Denote the bilinear forms of the lower terms of (2.4) and (2.8) by

$$l(u_h, v_h) = (a - a^{(2)})(u_h, v_h) = \int_\Omega (\mathbf{b} \cdot \nabla u_h v_h + c u_h v_h) \mathrm{d}x, \tag{2.23}$$

$$l_h(u_h, I_h^* v_h) = (a_h - a_h^{(2)})(u_h, I_h^* v_h) = \int_\Omega (\mathbf{b} \cdot \nabla u_h + c u_h) I_h^* v_h \mathrm{d}x. \tag{2.24}$$

The following Lemma 2.4 is proved in [14,28].

Lemma 2.4 *For any $u_h, v_h \in V_h$, we have*

$$\begin{aligned} a^{(2)}(u_h, v_h) - a_h^{(2)}(u_h, I_h^* v_h) = & \sum_{K \in T_h} \int_{\partial K} (\mathbf{a} \nabla u_h) \cdot \mathbf{n} (v_h - I_h^* v_h) \mathrm{d}s \\ & - \sum_{K \in T_h} \int_K (\nabla \mathbf{a} \cdot \nabla u_h)(v_h - I_h^* v_h) \mathrm{d}x. \end{aligned} \tag{2.25}$$

The following Lemma 2.5 characterizes the difference between the bilinear form of the FVEM and that of the FEM which plays the key role in the subsequent analysis.

Lemma 2.5 *For any $u_h, v_h \in V_h, u \in H^2(\Omega)$, $p \geq 1$, $1/p + 1/q = 1$, we have*

$$|a^{(2)}(u_h, v_h) - a_h^{(2)}(u_h, I_h^* v_h)| \leq C h^2 (h^{-1} |u - u_h|_{1,p} + ||u||_{2,p}) |v_h|_{1,q}, \tag{2.26}$$

$$|l(u_h, v_h) - l_h(u_h, I_h^* v_h)| \leq C h^2 ||u_h||_{1,p} |v_h|_{1,q}. \tag{2.27}$$

Proof By observing that any interior edge e is a common side of two element with opposite outer unit normal vectors on the edge, we deduce that

$$\sum_{K\in T_h}\sum_{e\subset\partial K}\int_e ((\mathbf{a}-\mathbf{a}(m_e))\nabla u)\cdot\mathbf{n}(v_h-I_h^* v_h)\mathrm{d}s=0, \tag{2.28}$$

where m_e is the midpoint of the edge e.

Noting that ∇u_h is a constant vector on K, from (2.28), Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned} &a^{(2)}(u_h, v_h)-a_h^{(2)}(u_h, I_h^* v_h)\\ &\quad=\sum_{K\in T_h}\sum_{e\subset\partial K}\int_e ((\mathbf{a}-\mathbf{a}(m_e))\nabla(u_h-u))\cdot\mathbf{n}(v_h-I_h^* v_h)\mathrm{d}s\\ &\qquad+\sum_{K\in T_h}\int_K (\nabla\mathbf{a}(z_K)-\nabla\mathbf{a})\cdot\nabla u_h(v_h-I_h^* v_h)\mathrm{d}x\\ &\quad=\sum_{K\in T_h}(I_K+II_K). \end{aligned} \tag{2.29}$$

It follows from Lemma 2.3 that

$$|I_K|\le Ch\sum_{e\subset\partial K}|(\mathbf{a}-\mathbf{a}(m_e))\nabla(u_h-u)|_{1,p,K}|v_h|_{1,q,K}. \tag{2.30}$$

Further, a simple calculations gives

$$|(\mathbf{a}-\mathbf{a}(m_e))\nabla(u_h-u)|_{1,p,K}\le C|u-u_h|_{1,p,K}+Ch||u||_{2,p,K}. \tag{2.31}$$

From (2.30) and (2.31), we get

$$\left|\sum_{K\in T_h} I_K\right|\le Ch^2(h^{-1}|u-u_h|_{1,p}+||u||_{2,p})|v_h|_{1,q}. \tag{2.32}$$

By Hölder inequality, Lemma 2.2 and the triangle inequality,

$$\left|\sum_{K\in T_h} II_K\right|\le Ch^2|u_h|_{1,p}|v_h|_{1,q}\le Ch^2(|u-u_h|_{1,p}+|u|_{1,p})|v_h|_{1,q}. \tag{2.33}$$

The desired result (2.26) is obtained from (2.29), (2.32) and (2.33). The proof of (2.27) is easy. Noting that u_h is a linear function over K and from Lemma 2.2, we have

$$\begin{aligned}|l(u_h, v_h) - l_h(u_h, I_h^* v_h)| &= \Bigg| \sum_{K\in T_h} \int_K [(\mathbf{b} - \mathbf{b}(z_K))\nabla u_h + (c - c(z_K))u_h \\ &\qquad + c(z_K)(u_h - u_h(z_K))](v_h - I_h^* v_h)\mathrm{d}x \Bigg| \\ &\le Ch^2 ||u_h||_{1,p} |v_h|_{1,q}.\end{aligned}$$

3 Two-grid finite volume element method for linear problem

In this section, we shall present the two-grid finite volume element algorithm for the non-selfadjoint and indefinite linear elliptic problems based on two finite element spaces. The idea of the two-grid method is to reduce the non-selfadjoint and indefinite elliptic problem on a fine grid into a symmetric and positive definite elliptic problem on a fine grid by solving a non-selfadjoint and indefinite elliptic problem on a coarse grid.

The basic mechanisms are two quasi-uniform triangulations of Ω, T_H and T_h, with two different mesh sizes H and h ($H > h$), and the corresponding finite element spaces V_H and V_h which satisfies $V_H \subset V_h$ and will be called the coarse and the fine spaces, respectively.

In order to present the two-grid finite volume element method, we introduce the following bilinear forms

$$a_c^{(2)}(u_h, v_h) = \int_\Omega (\bar{\mathbf{a}}\nabla u_h) \cdot \nabla v_h \mathrm{d}x, \quad \forall u_h, v_h \in V_h, \tag{3.1}$$

$$a_{h,c}^{(2)}(u_h, I_h^* v_h) = -\sum_{z\in Z_h^0} \int_{\partial V_z} (\bar{\mathbf{a}}\nabla u_h) \cdot \mathbf{n} I_h^* v_h \mathrm{d}s, \quad \forall u_h, v_h \in V_h, \tag{3.2}$$

where $\bar{\mathbf{a}}|_K = \mathbf{a}_K$ and

$$\mathbf{a}_K = \frac{1}{\mathrm{meas}(K)} \int_K \mathbf{a}(x)\mathrm{d}x, \quad \forall K \in T_h.$$

Let us now present the two-grid finite volume element algorithm.

Algorithm 3.1

1. Find $u_H \in V_H$ such that $a_H(u_H, I_H^* v_H) = (f, I_H^* v_H), \forall v_H \in V_H$.
2. Find $u^h \in V_h$ such that $a_{h,c}^{(2)}(u^h, I_h^* v_h) + l_h(u_H, I_h^* v_h) = (f, I_h^* v_h), \forall v_h \in V_h$.

The following lemma is proved in [10,14,16].

Lemma 3.1 *For any $u_h, v_h \in V_h$, we have*

$$a_{h,c}^{(2)}(u_h, I_h^* v_h) = a_c^{(2)}(u_h, v_h).$$

In general case, the matrix obtained from $a_h^{(2)}(u_h, I_h^* v_h)$ is not symmetric. This introduce some difficulties in real implementations, the method suitable to symmetric linear system can not be used in this case. From Lemma 3.1, we know that the coefficient matrix of the linear system in the second step of Algorithm 3.1 is symmetric and positive definite and is in general much easier to solve (e.g. conjugate gradient like methods can be applied effectively).

Theorem 3.1 *Assume that u_h and u^h are the solutions obtained by (2.7) and Algorithm 3.1 for $H \ll 1$, then*

$$||u_h - u^h||_1 \le CH^2(||u||_2 + ||f||_1); \tag{3.3}$$

$$||u - u^h||_1 \le C(h + H^2)(||u||_2 + ||f||_1). \tag{3.4}$$

Proof It follows from Algorithm 3.1 and (2.7) that

$$\begin{aligned} &a_{h,c}^{(2)}(u_h - u^h, I_h^* v_h) \\ &= a_{h,c}^{(2)}(u_h, I_h^* v_h) - a_{h,c}^{(2)}(u^h, I_h^* v_h) \\ &= a_{h,c}^{(2)}(u_h, I_h^* v_h) - (f, I_h^* v_h) + l_h(u_H, I_h^* v_h) \\ &= (a_{h,c}^{(2)}(u_h, I_h^* v_h) - a_h^{(2)}(u_h, I_h^* v_h)) + (l_h(u_H, I_h^* v_h) - l_h(u_h, I_h^* v_h)) \\ &= R_1 + R_2. \end{aligned} \tag{3.5}$$

We first estimate R_1. For this purpose, we rewrite R_1 as follows.

$$R_1 = \left[a_{h,c}^{(2)}(u_h, I_h^* v_h) - a^{(2)}(u_h, v_h)\right] + \left[a^{(2)}(u_h, v_h) - a_h^{(2)}(u_h, I_h^* v_h)\right]. \tag{3.6}$$

Using Lemma 3.1, noting that u_h and v_h are piecewise linear functions and from the definition of $\bar{\mathbf{a}}$, we have

$$\begin{aligned} a_{h,c}^{(2)}(u_h, I_h^* v_h) - a^{(2)}(u_h, v_h) &= a_c^{(2)}(u_h, v_h) - a^{(2)}(u_h, v_h) \\ &= \sum_{K \in T_h} \int_K (\bar{\mathbf{a}} - \mathbf{a}) \nabla u_h \cdot \nabla v_h \mathrm{d}x \\ &= 0. \end{aligned} \tag{3.7}$$

From Lemma 2.5 and (2.9), we get

$$\begin{aligned} \left| a^{(2)}(u_h, v_h) - a_h^{(2)}(u_h, I_h^* v_h) \right| &\le Ch^2 \left(h^{-1} |u - u_h|_1 + ||u||_2 \right) |v_h|_1 \\ &\le Ch^2 ||u||_2 |v_h|_1. \end{aligned} \tag{3.8}$$

It follows from (3.6), (3.7) and (3.8) that

$$|R_1| \leq Ch^2||u||_2|v_h|_1. \tag{3.9}$$

In order to estimate R_2, we rewrite R_2 as follows.

$$\begin{aligned} R_2 &= [l_h(u_H, I_h^* v_h) - l(u_H, v_h)] + [l(u_H, v_h) - l(u_h, v_h)] \\ &\quad + [l(u_h, v_h) - l_h(u_h, I_h^* v_h)] \\ &= S_1 + S_2 + S_3. \end{aligned} \tag{3.10}$$

By Lemma 2.5, the triangle inequality and (2.9),

$$|S_1| + |S_3| \leq Ch^2(||u_H||_1 + ||u_h||_1)|v_h|_1 \leq Ch^2||u||_2|v_h|_1. \tag{3.11}$$

Using Green's formula, the Cauchy–Schwarz inequality and (2.12), we have

$$\begin{aligned} |S_2| &= \left| \int_\Omega \mathbf{b} \cdot \nabla(u_H - u_h) v_h \mathrm{d}x + \int_\Omega c(u_H - u_h) v_h \mathrm{d}x \right| \\ &\leq C||u_H - u_h||\,||v_h||_1 \\ &\leq CH^2(||u||_2 + ||f||_1)||v_h||_1. \end{aligned} \tag{3.12}$$

From (3.10)–(3.12), we get the estimation for R_2,

$$|R_2| \leq CH^2(||u||_2 + ||f||_1)||v_h||_1. \tag{3.13}$$

Combining (3.5), (3.9) with (3.13) yields

$$a_{h,c}^{(2)}(u_h - u^h, I_h^* v_h) \leq CH^2(||u||_2 + ||f||_1)||v_h||_1. \tag{3.14}$$

By means of the discrete Poincare inequality, we get the coerciveness of the bilinear form $a_c^{(2)}(\cdot,\cdot)$,

$$c_0||v_h||_1^2 \leq a_c^{(2)}(v_h, v_h), \quad \forall v_h \in V_h. \tag{3.15}$$

Then, by setting $v_h = u_h - u^h$ in (3.14), from Lemma 3.1 and (3.15), we get the desired result (3.3). The desired result (3.4) follows from (3.3) and (2.9).

Remark From Theorem 3.1, we know that if $h = O(H^2)$ is chosen, the solution obtained by the two-grid finite volume element method is convergent to the solution of the original problem with the optimal order in the H^1-norm.

In the following, we introduce the definition of the discrete Green's function [14,21,24], which will be used in the error estimate in $W^{1,\infty}$-norm between the solution of (2.7) and that of the two-grid finite volume element method.

Given any point $x \in \Omega$, we define the regularized Green's function $G_x \in H_0^1(\Omega) \cap H^2(\Omega)$ to be the solution of the equation

$$-\nabla \cdot (\bar{\mathbf{a}} \nabla G_x) = \delta_x^h, \quad \forall x \in \Omega,$$

where $\delta_x^h \in V_h$ is a smoothed δ-function associated with the point x, which is defined by

$$(\delta_x^h, v_h) = v_h(x), \quad \forall v_h \in V_h.$$

Let $G_x^h \in V_h$ be the finite element approximation of the regularized Green's function, i.e.,

$$a_c^{(2)}(G_x - G_x^h, v_h) = 0, \quad \forall v_h \in V_h.$$

Moreover, following [14,21], for a given point $x \in \Omega$, we define $\partial_x G_x$ by

$$\partial_x G_x = \lim_{\triangle x \to 0, \triangle x || l} \frac{G_{x+\triangle x} - G_x}{|\triangle x|},$$

where l is any fixed direction, $\triangle x || l$ means that $\triangle x$ is parallel to l. Clearly $\partial_x G_x$ satisfies

$$a_c^{(2)}(\partial_x G_x, v_h) = -(\partial_x \delta_x^h, v_h) = \partial_x v_h(x), \quad \forall v_h \in V_h. \tag{3.16}$$

The finite element approximation $\partial_x G_x^h \in V_h$ of $\partial_x G_x$ is then defined by

$$a_c^{(2)}(\partial_x G_x - \partial_x G_x^h, v_h) = 0, \quad \forall v_h \in V_h. \tag{3.17}$$

The following estimates have been established in the literatures [14,21,24,31].

$$||\partial_x G_x^h||_{1,1} \leq C|\ln h|. \tag{3.18}$$
$$||G_x^h||_{1,1} \leq C|\ln h|. \tag{3.19}$$

Theorem 3.2 *Assume that u_h and u^h are the solutions obtained by* (2.7) *and Algorithm* 3.1 *for $H \ll 1$, then*

$$||u_h - u^h||_{1,\infty} \leq CH^2|\ln h|^2(||u||_{2,\infty} + ||f||_1), \quad u \in W^{2,\infty}(\Omega). \tag{3.20}$$

Proof It follows from the definition of $\partial_x G_x^h$, Lemma 3.1, (2.7) and Algorithm 3.1 that

$$\begin{aligned}\partial_x(u_h - u^h)(x) &= a_c^{(2)}(u_h - u^h, \partial_x G_x^h)\\ &= a_{h,c}^{(2)}(u_h - u^h, I_h^* \partial_x G_x^h)\\ &= a_{h,c}^{(2)}(u_h, I_h^* \partial_x G_x^h) - (f, I_h^* \partial_x G_x^h) + l_h(u_H, I_h^* \partial_x G_x^h)\\ &= \left(a_{h,c}^{(2)}(u_h, I_h^* \partial_x G_x^h) - a_h^{(2)}(u_h, I_h^* \partial_x G_x^h)\right)\\ &\quad + \left(l_h(u_H, I_h^* \partial_x G_x^h) - l_h(u_h, I_h^* \partial_x G_x^h)\right)\\ &= I_1 + I_2. \end{aligned} \tag{3.21}$$

In order to estimate I_1, we rewrite I_1 as follows

$$\begin{aligned} I_1 &= \left(a_{h,c}^{(2)}(u_h, I_h^* \partial_x G_x^h) - a^{(2)}(u_h, \partial_x G_x^h)\right)\\ &\quad + \left(a^{(2)}(u_h, \partial_x G_x^h) - a_h^{(2)}(u_h, I_h^* \partial_x G_x^h)\right). \end{aligned} \tag{3.22}$$

Similar to the derivation of (3.7), we have

$$a_{h,c}^{(2)}(u_h, I_h^* \partial_x G_x^h) - a^{(2)}(u_h, \partial_x G_x^h) = 0. \tag{3.23}$$

From Lemma 2.5, (2.16) and (3.18), we get

$$\begin{aligned} a^{(2)}(u_h, \partial_x G_x^h) - a_h^{(2)}(u_h, I_h^* \partial_x G_x^h) &\le Ch^2(h^{-1}|u - u_h|_{1,\infty} + |u|_{2,\infty})|\partial_x G_x^h|_{1,1}\\ &\le Ch^2|\ln h|(||u||_{2,\infty} + ||f||_1). \end{aligned} \tag{3.24}$$

The estimation of I_1 is obtained from (3.22), (3.23) and (3.24),

$$|I_1| \le Ch^2|\ln h|(||u||_{2,\infty} + ||f||_1). \tag{3.25}$$

In order to estimate I_2, we rewrite I_2 as follows.

$$\begin{aligned} I_2 &= \left[l_h(u_H, I_h^* \partial_x G_x^h) - l(u_H, \partial_x G_x^h)\right] + \left[l(u_H, \partial_x G_x^h) - l(u_h, \partial_x G_x^h)\right]\\ &\quad + \left[l(u_h, \partial_x G_x^h) - l_h(u_h, I_h^* \partial_x G_x^h)\right]\\ &= T_1 + T_2 + T_3. \end{aligned} \tag{3.26}$$

It follows from Lemma 2.5 and (3.18) that

$$|T_1| + |T_3| \le Ch^2(||u_H||_{1,\infty} + ||u_h||_{1,\infty})|\partial_x G_x^h|_{1,1} \le Ch^2|\ln h|||u||_{2,\infty}. \tag{3.27}$$

Using Green's formula, Hölder inequality, the triangle inequality and (2.15),

$$\begin{aligned}|T_2| &= \left|\int_\Omega \mathbf{b}\cdot\nabla(u_H-u_h)\partial_x G_x^h \mathrm{d}x + \int_\Omega c(u_H-u_h)\partial_x G_x^h \mathrm{d}x\right| \\ &\le C||u_H-u_h||_{0,\infty}||\partial_x G_x^h||_{1,1} \\ &\le CH^2|\ln h|^2(||u||_{2,\infty}+||f||_1). \end{aligned} \tag{3.28}$$

From (3.26), (3.27) and (3.28), we get

$$|I_2| \le CH^2|\ln h|^2(||u||_{2,\infty}+||f||_1). \tag{3.29}$$

Combining (3.21), (3.25) with (3.29) yields

$$|u_h-u^h|_{1,\infty} \le CH^2|\ln h|^2(||u||_{2,\infty}+||f||_1). \tag{3.30}$$

Using the definition of G_x^h, (3.19) and the method above to estimate $|u_h-u^h|_{1,\infty}$, we obtain the estimation of $|u_h-u^h|_{0,\infty}$ easily

$$|u_h-u^h|_{0,\infty} \le CH^2|\ln h|^2(||u||_{2,\infty}+||f||_1). \tag{3.31}$$

We get the desired result (3.20) from (3.30) and (3.31).

Remark From Theorem 3.2 and the error estimate (2.16), we get

$$||u-u^h||_{1,\infty} \le C(h+H^2|\ln h|^2)(||u||_{2,\infty}+||f||_1). \tag{3.32}$$

4 Finite volume element method for nonlinear problem

In this section, we consider the FVEM for the two-dimensional second-order nonlinear elliptic problem

$$\begin{cases} L(u)u \equiv -\nabla\cdot(A(u)\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded convex polygonal domain in R^2 with the boundary $\partial\Omega$, $A: R\to R$ is sufficiently smooth and there exist constants $\beta_i, i=1,2,3,$ satisfying

$$0<\beta_1\le A(x)\le\beta_2, |A'(x)|\le\beta_3, \quad \forall x\in R. \tag{4.2}$$

Moreover, as in [9], we assume that A' is Lipschitz continuous with constant L, i.e.,

$$|A'(x)-A'(y)|\le L|x-y|, \quad \forall x,y\in R. \tag{4.3}$$

As in [9], in order to get the error estimates in the L^2-norm for the FVEM for the problem (4.1), we also assume that u is Lipschitz continuous and $A'' \in L^1(R)$.

In this paper, we assume that the function $A(x)$ and $f(x)$ is sufficiently smooth so that the problem (4.1) has a unique solution in a certain Sobolev space.

In this section, the dual partition T_h^* and the interpolation operator I_h^* are the same as in Sect. 2.

We formulate the FVEM for the problem (4.1) as follows. Given a vertex $z \in Z_h$, integrating (4.1) over the associated control volume V_z and using Green's formula, we obtain

$$-\int_{\partial V_z} (A(u)\nabla u) \cdot \mathbf{n} \mathrm{d}s = \int_{V_z} f \mathrm{d}x.$$

The FVEM is to find $u_h \in V_h$ such that

$$-\int_{\partial V_z} (A(u_h)\nabla u_h) \cdot \mathbf{n} \mathrm{d}s = \int_{V_z} f \mathrm{d}x, \quad \forall V_z \in T_h^*. \tag{4.4}$$

The finite volume element problem (4.4) can be rewritten in a variational form. Find $u_h \in V_h$ such that

$$a_h(u_h; u_h, I_h^* v_h) = (f, I_h^* v_h), \quad \forall v_h \in V_h, \tag{4.5}$$

where $a_h(\cdot; \cdot, I_h^* \cdot)$ is defined by

$$a_h(\omega_h; u_h, I_h^* v_h) = -\sum_{z \in Z_h^0} \int_{\partial V_z} (A(\omega_h)\nabla u_h) \cdot \mathbf{n} I_h^* v_h \mathrm{d}s, \quad \forall \omega_h, u_h, v_h \in V_h. \tag{4.6}$$

By means of a fixed point iteration, Chatzipantelidis, Ginting and Lazarov [9] have proved the existence and uniqueness of the solution u_h of the problem (4.5):

1. Choose $M > 0$ such that $||f||_0 \le M\alpha^{-1}$, then there exists a solution of (4.5) in a ball

$$B_M = \{v_h \in V_h : ||\nabla v_h||_{0,p} \le M\}, \quad p > 2, \tag{4.7}$$

where α appears in the inf-sup condition employed in [9]: there exist constants $\alpha = \alpha(A, \Omega) > 0$, $h_\alpha > 0$ and $\delta = \delta(A, \Omega) > 0$ such that for all $0 < h \le h_\alpha$ and $\chi_h \in V_h, \omega_h \in B_M$,

$$||\nabla \chi_h||_{0,p} \le \alpha \sup_{0 \ne v_h \in V_h} \frac{a(\omega_h; u_h, I_h^* v_h)}{||\nabla v_h||_{0,q}}, \quad 2 < p \le 2 + \delta, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{4.8}$$

2. For a sufficiently small data, f, and for sufficiently small h, the solution u_h is unique.

Chatzipantelidis, Ginting and Lazarov [9] have obtained the error estimates in the H^1- and L^2-norms for the FVEM for the problem (4.1) under the condition $\gamma = \alpha\beta_3 M < 1$, where β_3, M and α appeared in (4.2), (4.7) and (4.8).

Lemma 4.1 ([9]) *Let u and u_h be the solutions of (4.1) and (4.5), respectively. Then, if $\gamma = \alpha\beta_3 M < 1$, there exists a constant $h_0 > 0$ such that for $0 < h \le h_0$,*

$$|u - u_h|_1 \le C(u, f)h, \quad u \in H^2(\Omega), \quad f \in L^2(\Omega); \tag{4.9}$$

$$||u - u_h||_0 \le C(u, f)h^2, \quad u \in H^2(\Omega) \cap W^{1,\infty}(\Omega), \quad f \in H^1(\Omega). \tag{4.10}$$

In order to simplify the notations and overall exposition of the material, we shall use the notation $C(u, f)$ to denote the constant which only depends on u and f and may stand for the different dependence of u and f at its different appearance.

The bound of $|u_h|_{1,\infty}$ has been shown in Theorem 4.4 in [9].

Lemma 4.2 ([9]) *Let $u \in W^{1,\infty}(\Omega) \cap H^2(\Omega) \cap H_0^1(\Omega)$ and u_h be the solutions of (4.1) and (4.5), respectively. Then, if $f \in L^2(\Omega)$ and $\gamma = \alpha\beta_3 M < 1$, there exists a constant $h_0 > 0$ such that for $0 < h \le h_0$,*

$$|u_h|_{1,\infty} \le C(u, f). \tag{4.11}$$

As an auxiliary tool, we introduce for any $\omega_h, u_h, v_h \in V_h$, the bilinear form associated with the finite element method,

$$a(\omega_h; u_h, v_h) = \int_\Omega (A(\omega_h)\nabla u_h) \cdot \nabla v_h \mathrm{d}x. \tag{4.12}$$

For the sake of the later analysis, we introduce the error functional

$$\varepsilon_a(\omega_h; u_h, v_h) = a(\omega_h; u_h, v_h) - a_h(\omega_h; u_h, I_h^* v_h), \quad \forall \omega_h, u_h, v_h \in V_h.$$

The following Lemma 4.3 characterizes the difference between $a(\omega_h; u_h, v_h)$ and $a_h(\omega_h; u_h, I_h^* v_h)$, which plays the key role in later error analysis of the two-grid finite volume element method for the nonlinear elliptic problem. It should be pointed out that the following Lemma 4.3 is a modification of Lemma 2.4 in [9] and has been proved in Lemma 3.2 in [5].

Lemma 4.3 ([5]) *Assume that $u \in W^{2,p}(\Omega)$, $p \ge 1$. Then there exists a constant C independent of h such that for $\omega_h, u_h, v_h \in V_h$,*

$$|\varepsilon_a(\omega_h; u_h, v_h)| \le Ch^2 |\omega_h|_{1,\infty} (|\omega_h|_{1,\infty} |u_h|_{1,p} + h^{-1}|u - u_h|_{1,p} + ||u||_{2,p}) |v_h|_{1,q},$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

The error estimate in the L^∞-norm for the FVEM for the problem (4.1) has been obtained in [9]. In [5], we reproduce the error estimate in the L^∞-norm under the condition $\gamma = \alpha\beta_3 M < 1$ by means of the superconvergence estimate between the solution of the FVEM and that of the FEM, and also get the error estimate in the $W^{1,\infty}$-norm for the FVEM for the problem (4.1) which will be used in the two-grid finite volume element method for the problem (4.1).

Lemma 4.4 ([5]) *Let u and u_h be the solutions of* (4.1) *and* (4.5)*, respectively, with $u \in W^{2,\infty}(\Omega)$, $f \in H^1(\Omega)$. Then, if $\gamma = \alpha\beta_3 M < 1$, there exists a constant $h_0 > 0$ such that for $0 < h \le h_0$,*

$$||u - u_h||_{0,\infty} \le C(u, f)h^2|\ln h|; \tag{4.13}$$

$$||u - u_h||_{1,\infty} \le C(u, f)h. \tag{4.14}$$

5 Two-grid finite volume element method for nonlinear problem

In this section, we shall present the two-grid finite volume element algorithm for the second-order nonlinear elliptic problems.

In the two-grid finite volume element method proposed in this section, on the coarser space V_H, we use the FVEM to obtain a rough approximation $u_H \in V_H$, and on the fine space V_h, solve a linearized problem based on u_H to produce a corrected solution $u^h \in V_h$. This means that solving a nonlinear equation is not much more difficult than solving one linear equation, since $\dim V_H \ll \dim V_h$ and the work for solving u_H is relatively negligible.

In order to present the two-grid finite volume element algorithm, we first introduce some notations.

$$a_{h,c}(\omega_h; u_h, I_h^* v_h) = -\sum_{z\in Z_h^0} \int_{\partial V_z} (\overline{A(\omega_h)}\nabla u_h) \cdot \mathbf{n} I_h^* v_h \mathrm{d}s, \tag{5.1}$$

where

$$\overline{A(\omega_h)}|_K = \frac{1}{\mathrm{meas}(K)} \int_K A(\omega_h)\mathrm{d}x.$$

Since $\overline{A(\omega_h)}$ is piecewise constant over all element $K \in T_h$, from Lemma 4.1, we have

$$a_{h,c}(\omega_h; u_h, I_h^* v_h) = a_c(\omega_h; u_h, v_h), \quad \forall \omega_h, u_h, v_h \in V_h, \tag{5.2}$$

where $a_c(\omega_h; u_h, v_h)$ is defined by

$$a_c(\omega_h; u_h, v_h) = \int_\Omega \overline{A(\omega_h)}\nabla u_h \cdot \nabla v_h \mathrm{d}x.$$

Noting that u_h and v_h are piecewise linear functions, by the definition of $\overline{A(\omega_h)}$, we have

$$a_c(\omega_h; u_h, v_h) = a(\omega_h; u_h, v_h), \quad \forall \omega_h, u_h, v_h \in V_h. \tag{5.3}$$

Now, let us present the two-grid finite volume element algorithm for the nonlinear elliptic problem.

Algorithm 5.1

1. Find $u_H \in V_H$ such that $a_H(u_H; u_H, I_H^* v_H) = (f, I_H^* v_H), \forall v_H \in V_H$.
2. Find $u^h \in V_h$ such that $a_{h,c}(u_H; u^h, I_h^* v_h) = (f, I_h^* v_h), \forall v_h \in V_h$.

From (5.2), we know that the coefficient matrix of the linearized problem in the second step of Algorithm 5.1 is symmetric and positive definite and is in general much easier to solve (e.g. conjugate gradient like methods can be applied effectively).

Theorem 5.1 *Assume that u_h and u^h are the solutions obtained by* (4.5) *and Algorithm* 5.1 *for $H \ll 1$, then*

$$\|u_h - u^h\|_1 \le C(u, f)H^2; \tag{5.4}$$

$$\|u - u^h\|_1 \le C(u, f)(h + H^2). \tag{5.5}$$

Proof It follows from the Algorithm 5.1, (4.5), (5.2) and (5.3) that

$$\begin{aligned} a_{h,c}(u_H; u_h - u^h, I_h^* v_h) &= a_{h,c}(u_H; u_h, I_h^* v_h) - (f, I_h^* v_h) \\ &= a(u_H; u_h, v_h) - a_h(u_h; u_h, I_h^* v_h) \\ &= [a(u_H; u_h, v_h) - a(u_h; u_h, v_h)] \\ &\quad + [a(u_h; u_h, v_h) - a_h(u_h; u_h, I_h^* v_h)] \\ &= Q_1 + Q_2. \end{aligned} \tag{5.6}$$

Using the Cauchy–Schwarz inequality, (4.2), Lemma 4.1 and Lemma 4.2, we obtain

$$\begin{aligned} |Q_1| &= \left| \int_\Omega (A(u_H) - A(u_h)) \nabla u_h \cdot \nabla v_h \mathrm{d}x \right| \\ &\le \beta_3 \int_\Omega |u_H - u_h| |\nabla u_h \cdot \nabla v_h| \mathrm{d}x \\ &\le C |u_h|_{1,\infty} (\|u_H - u\| + \|u - u_h\|) |v_h|_1 \\ &\le C(u, f) H^2 |v_h|_1. \end{aligned} \tag{5.7}$$

The estimation of Q_2 is obtained from Lemma 4.3, Lemma 4.1 and Lemma 4.2,

$$|Q_2| \le C(u, f) h^2 |v_h|_1. \tag{5.8}$$

By means of the discrete Poincare inequality, we get the coercivity of the bilinear form $a(u_H; \cdot, \cdot)$,

$$c_1||v_h||_1^2 \leq a(u_H; v_h, v_h), \quad \forall v_h \in V_h. \tag{5.9}$$

It follows from (5.2), (5.3) and (5.9) that

$$c_1||v_h||_1^2 \leq a_{h,c}(u_H; v_h, I_h^* v_h), \quad \forall v_h \in V_h. \tag{5.10}$$

By taking $v_h = u_h - u^h$ in (5.6), from (5.7), (5.8) and (5.10), we get the desired result (5.4). The desired result (5.5) follows from (5.4) and Lemma 4.1.

In the following, in order to get the error estimate in the $W^{1,\infty}$-norm between the solution of (4.5) and Algorithm 5.1, we introduce the regularized Green's function $G_x \in H_0^1$ defined by

$$-\nabla \cdot (\overline{A(u_H)} \nabla G_x) = \delta_x^h, \quad \forall x \in \Omega, \tag{5.11}$$

where the function δ_x^h is given in Sect. 3.

Similar to that in Sect. 3, we assume $G_x^h \in V_h$ is the finite element approximation of (5.11) and define $\partial_x G_x^h$. We also have the following estimates

$$||G_x^h||_{1,1} \leq C|\ln h|, \quad ||\partial_x G_x^h||_{1,1} \leq C|\ln h|. \tag{5.12}$$

Theorem 5.2 *Assume that u_h and u^h are the solutions obtained by* (4.5) *and Algorithm* 5.1 *for $H \ll 1$, then*

$$||u_h - u^h||_{1,\infty} \leq C(u, f) H^2 |\ln h|^2, \quad u \in W^{2,\infty}(\Omega). \tag{5.13}$$

Proof From the definition of $\partial_x G_x^h$ and (5.2), similar to the derivation of (5.6), we have

$$\begin{aligned} \partial_x (u_h - u^h)(x) &= a_c(u_H, u_h - u^h, \partial_x G_x^h) \\ &= a_{h,c}(u_H; u_h - u^h, I_h^* \partial_x G_x^h) \\ &= Q_1' + Q_2', \end{aligned} \tag{5.14}$$

where

$$\begin{aligned} Q_1' &= a(u_H; u_h, \partial_x G_x^h) - a(u_h; u_h, \partial_x G_x^h), \\ Q_2' &= a(u_h; u_h, \partial_x G_x^h) - a_h(u_h; u_h, I_h^* \partial_x G_x^h). \end{aligned}$$

Similar to the estimation of Q_1 in (5.7), from Lemma 4.4, Lemma 4.2 and (5.12), we get

$$\begin{aligned} |Q_1'| &\leq C|u_h|_{1,\infty}(||u_H - u||_{0,\infty} + ||u - u_h||_{0,\infty})|\partial_x G_x^h|_{1,1} \\ &\leq C(u, f) H^2 |\ln h|^2. \end{aligned} \tag{5.15}$$

It follows from Lemma 4.3 in the case $p = \infty, q = 1$, Lemma 4.4 and (5.12) that

$$|Q_2'| \leq C(u, f)h^2|\ln h|. \tag{5.16}$$

From (5.14), (5.15) and (5.16), we obtain

$$|u_h - u^h|_{1,\infty} \leq C(u, f)H^2|\ln h|^2. \tag{5.17}$$

From the definition of G_x^h, (5.12) and using the method above to estimate $|u_h - u^h|_{1,\infty}$, we get the estimation of $|u_h - u^h|_{0,\infty}$ easily

$$|u_h - u^h|_{0,\infty} \leq C(u, f)H^2|\ln h|^2. \tag{5.18}$$

Combining (5.17) with (5.18) yields the desired result (5.13).

Remark From Theorem 5.2 and Lemma 4.4, we get

$$||u - u^h||_{1,\infty} \leq C(u, f)(h + H^2|\ln h|^2). \tag{5.19}$$

6 Numerical examples

In this section we present several numerical experiments to gain insights on the theoretical findings presented earlier. In particular, our main interest is to verify Theorems 3.1 and 3.2 for the indefinite problems, and Theorems 5.1 and 5.2 for the nonlinear problems. As described in the previous sections, all these theorems guarantee the superconvergence of the two-grid FVEM solution to the standard FVEM solution on the fine grid. In all examples, the problems are posed in domain $\Omega = [0, 1] \times [0, 1]$. The domain is discretized into N numbers of rectangle in each direction and then each rectangle is divided into two triangles, resulting in a mesh with size $1/N$. For the computation, the finite element space V_h is built on the fine grid whose mesh size is $h = 1/512$, and thus the two-grid solution is compared against the solution in this fine grid.

The first example is the nondefinite problem $-\epsilon\Delta u + b\cdot\nabla u = f$ with homogeneous Dirichlet boundary. We set $\epsilon = 0.5$, $b = (1, 1)^\top$, and $f = 1$. The finite element space V_H is built on the coarse grid whose mesh size is $H = 1/N$, with $N = 4, 8, 16, 32, 64$. It is clear that with this construction, V_H and V_h are conforming. Figure 1 shows the log–log plot of the deviation $u^h - u_h$ against the coarse mesh size H. This deviation is computed in H^1- and $W^{1,\infty}$-norm. A least-squares fit of each of these pair of data indicates that the rate of convergence of the deviation is approximately 1.96 and 1.81, respectively, for each norm. These results are in accordance with the estimates in Theorems 3.1 and 3.2.

For the nonlinear problem, we solve a homogeneous Dirichlet boundary value problem $-\nabla\cdot(A(u)\nabla u) = f$ in Ω, where the function f is chosen such that the known solution is $u(x, y) = \sin(3\pi x)\sin(3\pi y)$. Here we choose the nonlinear coefficient $A(u) = 1 + 1/(1 + u^2)$. The finite element space V_H is built on the coarse grid whose

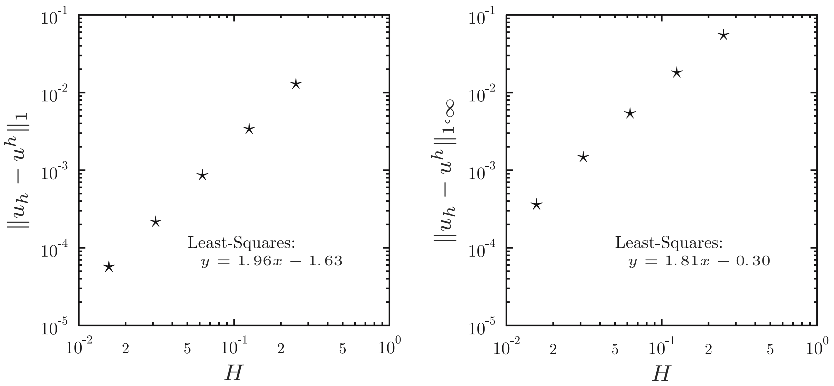

Fig. 1 Plot of errors against the coarse grid mesh size for nondefinite problem: *left* error in H^1-norm, *right* error in $W^{1,\infty}$-norm. All plots are in log–log axis

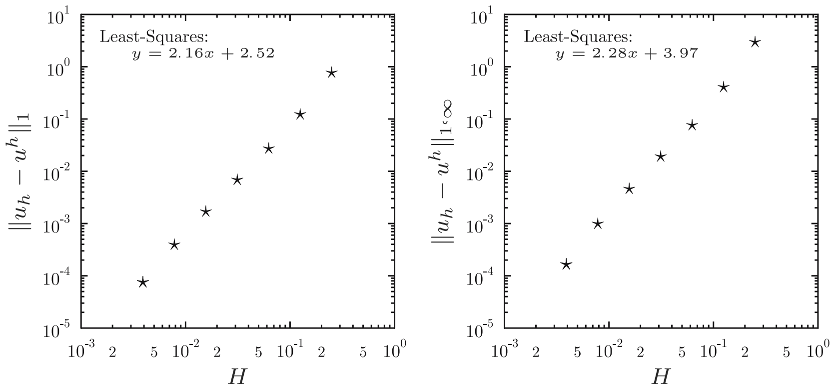

Fig. 2 Plot of errors against the coarse grid mesh size for nonlinear problem: *left* error in H^1-norm, *right* error in $W^{1,\infty}$-norm. All plots are in log–log axis

mesh size is $H = 1/N$, with $N = 4, 8, 16, 32, 64, 128, 256$. Figure 2 shows the log–log plot of the deviation $u^h - u_h$ against the coarse mesh size H. A least-squares fit of each of these pair of data give the rate of convergence of approximately 2 for both norms, which confirms Theorems 5.1 and 5.2.

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