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Convergence analysis of a conforming adaptive finite element method for an obstacle problem

Dietrich Braess · Carsten Carstensen · Ronald H.W. Hoppe

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Abstract The adaptive algorithm for the obstacle problem presented in this paper relies on the jump residual contributions of a standard explicit residual-based a posteriori error estimator. Each cycle of the adaptive loop consists of the steps 'SOLVE', 'ESTIMATE', 'MARK', and 'REFINE'. The techniques from the unrestricted variational problem are modified for the convergence analysis to overcome the lack of Galerkin orthogonality. We establish R-linear convergence of the part of the energy above its minimal value, if there is appropriate control of the data oscillations. Surprisingly, the adaptive mesh-refinement algorithm is the same as in the unconstrained case of a linear PDE—in fact, there is no modification near the discrete free boundary necessary for R-linear convergence. The arguments are presented for a model obstacle problem with an affine obstacle χ and homogeneous Dirichlet boundary conditions. The proof of the discrete local efficiency is more involved than in the unconstrained case. Numerical results are given to illustrate the performance of the error estimator.

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D. Braess (✉)
Faculty of Mathematics, Ruhr-University, 44780 Bochum, Germany
e-mail: braess@num.ruhr-uni-bochum.de

C. Carstensen
Institute of Mathematics, Humboldt University of Berlin, 10099 Berlin, Germany
e-mail: cc@math.hu-berlin.de

Ronald H.W. Hoppe
Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA
e-mail: rohop@math.uh.edu

Ronald H.W. Hoppe
Institute of Mathematics, University of Augsburg, 86159 Augsburg, Germany

1 Introduction

Let Ω to be a bounded, polygonal domain in $\mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$. An obstacle is defined in Ω by an affine function χ on $\bar{\Omega}$ with $\chi \leq 0$ on Γ. Adopting standard notation from Sobolev space theory, we set $V := H_0^1(\Omega)$, and we denote by $K \subset V$ the non-empty, closed, convex set

$$K := \{v \in V \mid v \geq \chi \quad \text{a.e. in}\ \Omega\}.$$

Let $(\cdot,\cdot)_{0,\Omega}$ denote the L^2-inner product and introduce the bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ and, given $f \in H^1(\Omega)$, the functional $b \in V^* = H^{-1}(\Omega)$ according to

$$a(v, w) := (\nabla v, \nabla w)_{0,\Omega} \quad \text{for all}\ v, w \in V,$$
$$b(v) := (f, v)_{0,\Omega} \quad \text{for all}\ v \in V.$$

Then the energy functional

$$\Pi(v) := \frac{1}{2}a(v, v) - b(v) \tag{1}$$

is defined for $v \in V$ and minimized over K or over discrete subsets K_ℓ. The equivalent variational inequality of the elliptic obstacle problem reads: *Find $u \in K$ such that*

$$a(u, v - u) \geq b(v - u) \quad \text{for all v}\ \in K. \tag{2}$$

It is well known that (2) admits a unique solution $u \in K$; see, e.g., [23], which equals the minimizer of Π in K. We introduce $\sigma \in V^*$ as the Lagrange multiplier given by

$$\langle\sigma, v\rangle_* := a(u, v) - b(v) \quad \text{for all}\ v \in V, \tag{3}$$

where $\langle\cdot,\cdot\rangle_*$ stands for the dual pairing between V^* and V. A direct consequence of (2) reads

$$\sigma \in V_+^*, \quad \text{i.e.,}\ \langle\sigma, v\rangle_* \geq 0 \quad \text{for all}\ v \in V_+, \tag{4}$$

where $V_+ := \{v \in V \mid v \geq 0\ \text{a.e.}\}$ is the positive cone in V. Moreover, we have the following complementarity condition

$$\langle\sigma, u - \chi\rangle_* = 0. \tag{5}$$

The numerical solution of (2) by finite element discretizations has been intensively studied; see, e.g., [18]. We choose shape regular, simplicial triangulations $\{\mathcal{T}_\ell(\Omega)\}_\ell$ of Ω, and refer to $V_\ell \subset V$ as the corresponding finite element spaces of globally continuous and piecewise linear finite elements with respect to $\mathcal{T}_\ell(\Omega)$. There exists a unique finite element solution u_ℓ for the mesh $\mathcal{T}_\ell(\Omega)$ in the cone $K_\ell := K \cap V_\ell$ with

$$a(u_\ell, v_\ell - u_\ell) \geq b(v_\ell - u_\ell) \quad \text{for all}\ v_\ell \in K_\ell. \tag{6}$$

Let $\sigma_\ell \in V_\ell^*$ be the discrete Lagrange multiplier associated to (6) according to

$$\langle \sigma_\ell, v_\ell \rangle_* := a(u_\ell, v_\ell) - b(v_\ell) \quad \text{for all } v_\ell \in V_\ell.$$

Besides the efficient numerical solution of (6), adaptive refinements of the finite element mesh on the basis of appropriate a posteriori error estimators is an important issue. Adaptive finite element methods for partial differential equations and systems are well established for residual- or hierarchical-type estimators, local averaging techniques, or the so-called goal-oriented dual weighted approach; see, e.g., the monographs [1,3,4,17,26,34] and the references therein. For elliptic obstacle problems we refer to [2,5,7,15,21,22,27,28,31,32].

On the other hand, there is only little work regarding a rigorous convergence analysis. For standard conforming finite element approximations of linear elliptic boundary value problems, pioneering work has been done in [16] followed by [25] where the role of data oscillations has been clarified. A different approach with techniques from approximation theory established optimal order of convergence under mild regularity assumptions [6,30]. For nonstandard finite element methods such as mixed methods, nonconforming elements and edge elements a convergence analysis has been provided in [11–13]. The basic ingredients of the convergence proofs are the *reliability* of the estimator, its *discrete local efficiency*, and a so-called *bulk criterion* taking care of an appropriate selection of edges and elements for refinement.

In this paper, we develop an adaptive finite element algorithm for the elliptic obstacle problem (2). The analysis of the adaptive method shows that in general we can expect an energy reduction property, but not necessarily a guaranteed reduction in the energy norm. Energy reduction was also considered with adaptive algorithms for other nonlinear variational problems [33]. The analysis uses the equivalence of two well-known error estimators for the obstacle problem which depend on the jumps on the edges of the mesh.

In Sect. 2 we present the adaptive loop focusing on a residual-type a posteriori error estimator and a bulk criterion selecting edges for refinement. Section 3 is devoted to the reliability of the estimator, whereas its discrete local efficiency is shown in Sect. 4. Combined with the bulk criterion, this results in an energy reduction property which is established in Sect. 5 as the main result of the paper. The final Sect. 6 contains numerical results illustrating the performance of the error estimator.

We conclude this section with some notations. For $D \subseteq \Omega$, we denote the L^2-norm on $L^2(D)$ by $\|\cdot\|_{0,D}$ and refer to

$$|||\cdot||| := a(\cdot,\cdot)^{1/2}$$

as the energy norm. Moreover, for a simplicial triangulation $\mathcal{T}_\ell$, we denote the set of interior nodal points by $\mathcal{N}_\ell$ and the set of interior edges by $\mathcal{E}_\ell$. We set $h_T := \mathrm{diam}(T)$, $T \in \mathcal{T}_\ell$, and $h_{\mathcal{T}} := \max\{h_T | T \in \mathcal{T}_\ell\}$. We refer to h_E, $E \in \mathcal{E}_\ell$, as the length of the edge E. Further, $\Omega_E := T_+ \cup T_-$ stands for the patch formed by the triangles $T_\pm \in \mathcal{T}_\ell$ sharing $E = T_+ \cap T_-$ as a common edge. Finally, given two expressions A and B, we write $A \lesssim B$ if there exists a constant $c > 0$, depending only on the shape regularity of the triangulation such that $A \leq cB$.

2 The adaptive loop

Adaptive finite element methods consist of successive loops of a cycle involving the steps 'SOLVE', 'ESTIMATE', 'MARK', and 'REFINE'. Here, 'SOLVE' means the numerical solution of the discretized problem on the given mesh. For the numerical solution of (6), efficient iterative solvers such as multigrid methods based on active set strategies [19,20] or monotone multigrid methods [24] are available. For estimating the discretization error in the next step 'ESTIMATE', we consider the estimator

$$\eta_\ell^2 := \sum_{E\in\mathcal{E}_\ell} \eta_E^2 \tag{7}$$

based on the edge residuals

$$\eta_E := h_E^{1/2} \|\boldsymbol{\nu}_E \cdot [\nabla u_\ell]\|_{0,E} \quad \text{for } E \in \mathcal{E}_\ell. \tag{8}$$

Here $\boldsymbol{\nu}_E$ is the unit normal to the (interior) edge E and $[\nabla u_\ell]$ refers to the jump of ∇u_ℓ across E. (The product $\boldsymbol{\nu}_E \cdot [\nabla u_\ell]$ is independent of the orientation of E.) The estimator (7) is known from the unconstrained case [14]. The convergence analysis further invokes data oscillations [25]

$$\mathrm{osc}_\ell(f) := \left(\sum_{E\in\mathcal{E}_\ell(\Omega)} \mathrm{osc}_E^2(f) \right)^{1/2}, \tag{9}$$

$$\mathrm{Osc}_\ell(f) := \left(\mathrm{osc}_\ell^2(f) + \sum_{T\in\mathcal{T}_\Gamma} h_T^2 \|f\|_{0,T}^2 \right)^{1/2}, \tag{10}$$

where $\mathrm{osc}_E(f) := |\Omega_E|^{1/2} \|f - f_{\Omega_E}\|_{0,\Omega_E}$, and $f_{\Omega_E} := |\Omega_E|^{-1} \int_{\Omega_E} f dx$ is the integral mean of f on the patch Ω_E. Moreover, $\mathcal{T}_\Gamma := \{T \in \mathcal{T}_\ell |\ T \cap \Gamma \neq \emptyset\}$. The second term on the right-hand side of (10) vanishes if $\chi < 0$ holds on the boundary.

The core of the step 'MARK' is a bulk criterion [16]. Let Θ be a non-negative constant with $0 < \Theta < 1$. We select a set $\mathcal{M}_\ell$ of edges $E \in \mathcal{E}_\ell$ such that

$$\sum_{E\in\mathcal{M}_\ell} \eta_E^2 \geq \Theta \sum_{E\in\mathcal{E}_\ell} \eta_E^2. \tag{11}$$

The bulk criterion can be implemented by a greedy algorithm; see, e.g., [12,13].

Finally, in the last step 'REFINE' we generate a fine mesh $\mathcal{T}_{\ell+1}$ as follows: If $E = T_+ \cap T_- \in \mathcal{M}_\ell$, we refine $T_\pm \in \mathcal{T}_\ell$ by repeated bisection such that at least one interior nodal point in $T_\pm$ is created [25]. In order to guarantee a geometrically conforming triangulation, new nodal points are generated on edges $E \in \mathcal{E}_\ell \setminus \mathcal{M}_\ell$ such that $E = T_\pm \cap T'$ for some $T' \in \mathcal{T}_\ell$, and the element T' is bisected by joining E with the vertex of T' opposite to E.

Moreover, the refinement and the new mesh $\mathcal{T}_{\ell+1}$ shall also take care of a reduction of the data oscillation; cf. [25]. Specifically, we require that

$$\mathrm{Osc}_{\ell+1}(f) \leq \kappa\, \mathrm{Osc}_{\ell}(f) \tag{12}$$

for some $0 < \kappa < 1$. This will be achieved by additional refinements, if necessary.

3 Reliability

The reliability of the estimator will be stated in terms of the energy functional Π from (1), and *not* in terms of the energy norm $|||\cdot|||$. In this way we circumvent the lack of Galerkin orthogonality.

Theorem 1 (reliability) *Let* η_ℓ *and* $\mathrm{Osc}_\ell(f)$ *be given by* (7)–(10). *There holds*

$$\varepsilon_\ell := \Pi(u_\ell) - \Pi(u) \;\lesssim\; \eta_\ell^2 + \;\mathrm{Osc}_\ell^2(f). \tag{13}$$

Proof We start with a reliability result from [5]. Let

$$\tilde{\eta}_\ell := \min_q \|\nabla u_\ell - q\|_{0,\Omega}$$

be the a posteriori error estimator based on averaging where q is an arbitrary continuous P_1 finite element function in each of its two components. Since the function χ that defines the obstacle is assumed to be an affine function and we restrict ourselves to homogeneous Dirichlet boundary conditions, the estimate in [5, Theorem 3] reduces to

$$|||u-u_\ell||| \lesssim \tilde{\eta}_\ell + \|h_{\mathcal{T}}^2 \nabla f\|_0 + \left(\sum_{T\in\mathcal{T}_\Gamma} h_T^2 \|f\|_{0,T}^2\right)^{1/2}.$$

This estimate will be improved with respect to two items. If we look at Lemma 1 and (2.5) in [5], we see that the term $\|h_{\mathcal{T}}^2 \nabla f\|_0$ actually stems from local data oscillations $(\sum_{E\in\mathcal{E}_\ell} h_E^2 \min_z \|f-z\|_{0,\Omega_E}^2)^{1/2}$ and can thus be included in $\mathrm{osc}_\ell(f)$. Recalling (10) we have

$$|||u-u_\ell||| \lesssim \tilde{\eta}_\ell + \mathrm{Osc}_\ell(f). \tag{14}$$

Next, we revive the term $\langle\sigma, u-u_\ell\rangle_*$ that was abandoned during the proof of Theorem 2 in [5] to obtain an improvement of (14)

$$|||u-u_\ell|||^2 + \langle\sigma, u-u_\ell\rangle_* \;\lesssim\; \tilde{\eta}_\ell^2 + \mathrm{Osc}_\ell^2(f). \tag{15}$$

From (3) and the definition of Π it follows, for all $v \in K$, that

$$
\begin{aligned}
\Pi(v) - \Pi(u) &= \frac{1}{2}a(v,v) - (f,v) - \Big(\frac{1}{2}a(u,u) - (f,u)\Big) \\
&\quad + \Big(a(u,u-v) - f(u-v) - \langle \sigma, u-v\rangle_*\Big) \\
&= \frac{1}{2}|||v-u|||^2 + \langle \sigma, v-u\rangle_* \,.
\end{aligned} \tag{16}
$$

Both terms on the right-hand side are nonnegative for $v \in K$ reflecting the minimal property of the solution u. From (15) and (16) we deduce an a posteriori estimate for the energy surplus instead of the energy norm

$$
\Pi(u_\ell) - \Pi(u) \;\lesssim\; \tilde{\eta}_\ell^2 + \mathrm{Osc}_\ell^2(f).
$$

It was shown in [8–10] that the estimator $\tilde{\eta}_\ell$ is equivalent to the estimator η_ℓ with the edge residuals terms, and we obtain eventually (13). □

Remark 3.2 We like to comment on the relation to other estimators in the literature. As was shown in [7], for the obstacle problem we have

$$
|||u - u_\ell|||^2 \;\lesssim\; \hat{\eta}_\ell^2 + \langle \sigma_\ell, u_\ell - u\rangle_* - \langle \sigma, u_\ell - u\rangle_* \,, \tag{17}
$$

where $\hat{\eta}_\ell$ is an error estimator for the unconstrained problem. It may be one of the commonly used estimators. The last term on the right-hand side can be shifted to the left-hand side as done in (15). The other term in (17), namely $\langle \sigma_\ell, u_\ell - u\rangle_*$, is the difficult one. In particular, the Lagrange multiplier σ_ℓ has to be approximated (by a functional called σ_ℓ^+ in [7]) in order to achieve a monotonicity property by which the unknown solution u can be eliminated. This approximation is done, e.g. in [5] by the mapping $v \mapsto \langle \sigma_\ell, J(v)\rangle_*$ where J is a monotone interpolation operator and maps into the finite element space.

Fortunately, the difficult term under consideration can be absorbed by the estimator $\tilde{\eta}_\ell$ based on the averaging technique; cf. [5, Lemma 8]. Eventually, this process yielded (13).

4 Discrete local efficiency

For proving the discrete local efficiency, we have to establish upper bounds for the edge residual η_E, $E \in \mathcal{M}_\ell$, in terms of $u_{\ell+1} - u_\ell$. We note that the Lagrange multipliers do not enter into the estimates of this section.

Let $E = \partial T_+ \cap \partial T_-$ with $T_\pm \in \mathcal{T}_\ell$ and $\Omega_E = T_+ \cup T_-$. We set $P := \mathrm{mid}(E) \in \mathcal{N}_{\ell+1} \setminus \mathcal{N}_\ell$ and refer to $P_\pm \in \mathcal{N}_{\ell+1} \setminus \mathcal{N}_\ell$ of $T_\pm$ as interior nodes with nodal basis functions

$$
\varphi := \varphi_P \quad \text{and} \quad \varphi_\pm := \varphi_{P_\pm} \in V_{\ell+1} \quad \text{with } 0 \le \varphi,\ \varphi_\pm \in H_0^1(\Omega_E) \cap V_+.
$$

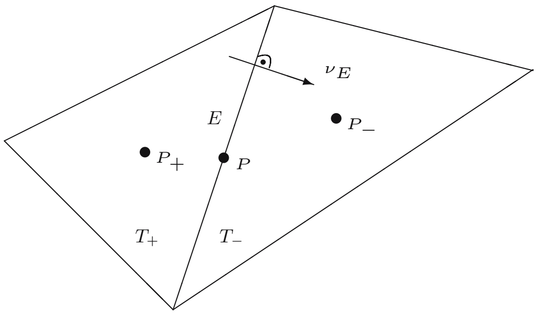

Fig. 1 Notation for $E \in \mathcal{M}_\ell$ and the adjacent elements T_+, T_-

The proof of local efficiency discusses several cases depending on

$$0 \leq w_{\ell+1} := u_{\ell+1} - \chi \in H^1(\Omega).$$

The first lemma provides a preparation for the proof of the subsequent Proposition 1 and contains arguments that are also found in proofs of the other cases (Fig. 1).

Lemma 1 *For $Q = P_+$ and $Q = P_-$ with nodal basis functions $\varphi_Q = \varphi_+$ and $\varphi_Q = \varphi_-$ supported at $K = T_+$ and $K = T_-$, respectively, and $w_{\ell+1}(Q) = 0$ there holds*

$$h_E^{1/2} \|A \cdot \nu_E\|_{0,E} \lesssim \|\nabla w_{\ell+1} - A\|_{0,K} \text{ for all } A \in \mathbb{R}2. \tag{18}$$

Proof Since $w_{\ell+1} \geq 0$ on K and $w_{\ell+1}(Q) = 0$ for an interior point Q in K, $\nabla w_{\ell+1} \cdot \nu_E$ is piecewise constant and has nonnegative as well as nonpositive values on at least one of the fine element domains in $\mathcal{T}_{\ell+1}|_K := \{T \in \mathcal{T}_{\ell+1} |\ T \subseteq K\}$. Given $A \in \mathbb{R}^2$, the products $A \cdot \nu_E$ and $\nabla w_{\ell+1} \cdot \nu_E$ have therefore opposite signs (or are zero) in at least one element $T \in \mathcal{T}_{\ell+1}|_K$, and there we have

$$|A \cdot \nu_E| \leq \left|(\nabla w_{\ell+1}|_T - A) \cdot \nu_E)\right| \leq \left|\nabla w_{\ell+1}|_T - A\right|.$$

Since $|T| \approx |K| \approx h_E\,|E|$, it follows that

$$h_E^{1/2} \|A \cdot \nu_E\|_{0,E} \lesssim \|\nabla w_{\ell+1} - A\|_{0,T} \lesssim \|\nabla w_{\ell+1} - A\|_{0,K}.$$

□

The following proposition yields already the final estimate for the special case where both points P_+ and P_- belong to the discrete coincidence set while Proposition 2 provides only a preliminary result for the case that one of them is in the discrete non-coincidence set.

Proposition 1 *If $w_{\ell+1}(P_+) = w_{\ell+1}(P_-) = 0$, then*

$$\eta_E \lesssim \|\nabla(u_{\ell+1} - u_\ell)\|_{0,\Omega_E}. \tag{19}$$

Proof Applying Lemma 1 with $A = \nabla w_\ell|_{T_\pm}$ on $T_\pm$ we obtain

$$\begin{aligned}\eta_E &\leq h_E^{1/2}\|\nabla w_\ell|_{T_+}\cdot \nu_E\|_{0,E} + h_E^{1/2}\|\nabla w_\ell|_{T_-}\cdot \nu_E\|_{0,E}\\ &\lesssim \|\nabla(w_{\ell+1}-w_\ell)\|_{0,T_+} + \|\nabla(w_{\ell+1}-w_\ell)\|_{0,T_-}\\ &= \|\nabla(u_{\ell+1}-u_\ell)\|_{0,\Omega_E}.\end{aligned}$$

□

Proposition 2 *If* $w_{\ell+1}(P_+) > 0$ *or* $w_{\ell+1}(P_-) > 0$, *then*

$$-\frac{1}{2}|\Omega_E|^{1/2}\left[\frac{\partial u_\ell}{\partial \nu_E}\right] \lesssim \|\nabla(u_{\ell+1}-u_\ell)\|_{0,\Omega_E} + \mathrm{osc}_{\Omega_E}(f). \tag{20}$$

Proof Without loss of generality, suppose that $w_{\ell+1}(P_+) > 0$. Hence,

$$b(\varphi_+) = a(u_{\ell+1}, \varphi_+).$$

Recall $\varphi = \varphi_P$ and notice $b(\varphi) \leq a(u_{\ell+1}, \varphi)$. Choose $\alpha_+ > 0$ such that

$$\varphi_E := \varphi - \alpha_+\varphi_+ \in V_{\ell+1} \cap H_0^1(\Omega_E)$$

satisfies

$$\int_{\Omega_E} \varphi_E\, dx = 0. \tag{21}$$

Notice that $\alpha_+ \approx 1$ and that $b(\varphi_E) \leq a(u_{\ell+1}, \varphi_E)$. In view of (21), this gives

$$-a(u_{\ell+1}, \varphi_E) \;\leq\; -(\varphi_E, f - f_{\Omega_E})_{L^2(\Omega_E)} \;\lesssim\; \mathrm{osc}_{\Omega_E}(f).$$

An elementwise integration by parts verifies the well-established formula

$$\frac{1}{2}\int_E \left[\frac{\partial u_\ell}{\partial \nu_E}\right] ds = \int_E \varphi_E \left[\frac{\partial u_\ell}{\partial \nu_E}\right] ds = a(u_\ell, \varphi_E).$$

The combination of the estimates above yields

$$\begin{aligned}-\frac{1}{2}\int_E \left[\frac{\partial u_\ell}{\partial \nu_E}\right] ds &= -a(u_\ell, \varphi_E) = a(u_{\ell+1}-u_\ell, \varphi_E) - a(u_{\ell+1}, \varphi_E)\\ &\lesssim \|\nabla(u_{\ell+1}-u_\ell)\|_{0,\Omega_E} + \mathrm{osc}_{\Omega_E}(f),\end{aligned}$$

and the proof is complete. □

The following two results cover the situation where $P = \mathrm{mid}(E)$ either belongs to the discrete non-coincidence set (Proposition 3) or to the discrete coincidence set (Proposition 4).

Proposition 3 *If* $w_{\ell+1}(P) > 0$, *then*

$$\eta_E \lesssim \|\nabla(u_{\ell+1} - u_\ell)\|_{0,\Omega_E} + \mathrm{osc}_{\Omega_E}(f). \tag{22}$$

Proof The assertion follows as in the unconstrained case, since the assumption $w_{\ell+1}(P) > 0$ implies

$$b(\varphi) = a(u_{\ell+1}, \varphi).$$

Hence,

$$\frac{1}{2}\eta_E = \left| \int_E \varphi_P \left[\frac{\partial u_\ell}{\partial \nu_E} \right] ds \right| = |a(u_\ell, \varphi)| = |a(u_{\ell+1} - u_\ell, \varphi) - b(\varphi)|.$$

In the case $w_{\ell+1}(P_+) = 0 = w_{\ell+1}(P_-)$, Proposition 1 proves the assertion (even without the oscillation term). Thus, we restrict our attention to the case that

$$w_{\ell+1}(P_+) > 0.$$

Since $\varphi_+ \in H_0^1(T_+)$, it follows that

$$a(u_l, \varphi_+) = \int_{T_+} \nabla u_l \nabla \varphi_+ dx = -\int_{T_+} \Delta u_l \varphi_+ dx + \int_{\partial T_+} \frac{\partial u_l}{\partial \nu} \varphi_+ ds = 0,$$

and

$$a(u_{\ell+1} - u_\ell, \varphi_+) = b(\varphi_+).$$

Defining $\varphi_E = \varphi - \alpha_+ \varphi_+$ as in the proof of Proposition 2, we obtain

$$\eta_E \le |a(u_{\ell+1} - u_\ell, \varphi_E) - b(\varphi_E)| \lesssim \|\nabla(u_{\ell+1} - u_\ell)\|_{0,\Omega_E} + \mathrm{osc}_{\Omega_E}(f).$$

□

Proposition 4 *If* $0 \le \left[\frac{\partial u_\ell}{\partial \nu_E}\right]$ *and* $w_{\ell+1}(P) = 0$, *then*

$$\eta_E \lesssim \|\nabla(u_{\ell+1} - u_\ell)\|_{0,\Omega_E}. \tag{23}$$

Proof Since $w_{\ell+1} \ge 0$ and $w_{\ell+1}(P) = 0$ at the interior point $P = \mathrm{mid}(E)$ of Ω_E, there exist $K_+ \in \mathcal{T}_{\ell+1}|_{T_+}$ and $K_- \in \mathcal{T}_{\ell+1}|_{T_-}$ with

$$\nabla w_{\ell+1}|_{K_+} \cdot \nu_E \le 0 \le \nabla w_{\ell+1}|_{K_-} \cdot \nu_E.$$

It follows that

$$0 \le \left[\frac{\partial u_\ell}{\partial \nu_E}\right] = \nabla w_\ell|_{K_+} \cdot \nu_E - \nabla w_\ell|_{K_-} \cdot \nu_E$$
$$\le \nabla(w_\ell - w_{\ell+1})|_{K_+} \cdot \nu_E - \nabla(w_\ell - w_{\ell+1})|_{K_-} \cdot \nu_E$$
$$\le \left|\nabla(w_\ell - w_{\ell+1})|_{K_+}\right| + \left|\nabla(w_\ell - w_{\ell+1})|_{K_-}\right|.$$

Since $|\Omega_E| \approx |K_\pm| \approx h_E|E|$, this leads to

$$\eta_E = |E|^{1/2} h_E^{1/2} \left[\frac{\partial u_\ell}{\partial \nu_E}\right]$$
$$\lesssim \|\nabla(w_{\ell+1} - w_\ell)\|_{0,K_+} + \|\nabla(w_{\ell+1} - w_\ell)\|_{0,K_-}$$
$$\lesssim \|\nabla(u_{\ell+1} - u_\ell)\|_{0,\Omega_E}.$$

□

The preceding results imply the announced efficiency. For completeness, we recall that the obstacle is given by an affine function.

Theorem 2 (discrete local efficiency) *For all $E \in \mathcal{M}_\ell$, there holds*

$$\eta_E \lesssim \|\nabla(u_{\ell+1} - u_\ell)\|_{0,\Omega_E} + \mathrm{osc}_{\Omega_E}(f). \tag{24}$$

Proof If $w_{\ell+1}(P_+) = w_{\ell+1}(P_-) = 0$, Proposition 1 proves the assertion and so the remaining part of the proof assumes $w_{\ell+1}(P_+) > 0$ or $w_{\ell+1}(P_-) > 0$. If $w_{\ell+1}(P) > 0$, Proposition 3 proves the assertion and the remaining part of the proof assumes $w_{\ell+1}(P) = 0$. Then, if $\left[\frac{\partial u_\ell}{\partial \nu_E}\right] \ge 0$, Proposition 4 proves the assertion. Thus, it remains to consider Proposition 2 for $\left[\frac{\partial u_\ell}{\partial \nu_E}\right] \le 0$ with

$$\eta_E \approx -|\Omega_E|^{1/2} \left[\frac{\partial u_\ell}{\partial \nu_E}\right] \lesssim \|\nabla(u_{\ell+1} - u_\ell)\|_{0,\Omega_E} + \mathrm{osc}_{\Omega_E}(f).$$

□

Remark 4.7 The fact, that χ is affine has simplified the analysis at several occasions. We point out that the error estimator does not satisfy discrete local efficiency in case of, e.g., obstacles with kinks. In such a case, the estimator needs to be modified appropriately.

5 Energy reduction

Now we are prepared to establish our main result on R-convergence of the adaptive loop. Recalling $\varepsilon_\ell := \Pi(u_\ell) - \Pi(u)$ we will show that $\limsup_{\ell\to\infty} \varepsilon_\ell^{1/\ell} < 1$; cf. [29]. In order to be more specific, we denote the (multiplicative) constant which is implicitly contained in (13) by c_r and the constant in (24) by c_{dle}.

Theorem 3 (energy reduction) *There exist constants* $0 \le \rho < 1$ *and* $C > 0$*, depending only on the constant* Θ *in the bulk criterion and on the shape regularity of the triangulations, such that*

$$\varepsilon_{\ell+1} \le \rho\, \varepsilon_\ell + C\, \mathrm{Osc}_\ell^2(f). \tag{25}$$

Proof The analogue of (16) for the discrete variational problem on the level $\ell + 1$ reads

$$\Pi(v) = \Pi(u_{\ell+1}) + \frac{1}{2}|||v - u_{\ell+1}|||^2 + \langle \sigma_{\ell+1}, v - u_{\ell+1}\rangle_* \quad \text{for all } v \in K_{\ell+1}.$$

Since the third term on the right-hand side is nonnegative, it follows that

$$\Pi(u_\ell) - \Pi(u_{\ell+1}) \ge \frac{1}{2}|||u_\ell - u_{\ell+1}|||^2.$$

Combining the bulk criterion (11) and the reliability (13) with the discrete local efficiency of Theorem 2 we obtain

$$\begin{aligned} 2(\varepsilon_\ell - \varepsilon_{\ell+1}) &\ge \frac{1}{c_{dle}} \sum_{E \in M_\ell} \eta_E^2 - \mathrm{osc}_\ell^2(f) \\ &\ge \frac{\Theta}{c_{dle}} \eta_\ell^2 - \mathrm{osc}_\ell^2(f) \\ &\ge \frac{\Theta}{c_{dle}} \left(\frac{1}{c_r} \varepsilon_\ell - \mathrm{Osc}_\ell^2(f) \right) - \mathrm{osc}_\ell^2(f). \end{aligned}$$

This proves the assertion

$$\varepsilon_{\ell+1} \le \left(1 - \frac{\Theta}{2 c_{dle} c_r}\right) \varepsilon_\ell + C \mathrm{Osc}_\ell^2(f).$$

□

Finally, the reduction of the data oscillations is guaranteed by (12), and we obtain geometrical convergence with $\varrho := \max\{\kappa^2, \rho\} < 1$ and $\rho := 1 - \Theta/(2 c_{dle} c_r)$. In fact, (5.1) and (12) combine to

$$\begin{pmatrix} \varepsilon_{\ell+1} \\ \delta_{\ell+1}^2 \end{pmatrix} \le \begin{pmatrix} \rho & C \\ 0 & \kappa^2 \end{pmatrix} \begin{pmatrix} \varepsilon_\ell \\ \delta_\ell^2 \end{pmatrix}, \tag{26}$$

where $\delta_\ell^2 := \mathrm{Osc}_\ell^2(f)$ and (26) is understood componentwise. (We have even geometric convergence of the sequence $(\varepsilon_\ell + 2C\delta_\ell^2)_\ell$ with C as above and the factor $\max\{(1+\kappa^2)/2, \rho\}$.)

This implies R-linear convergence of $(\varepsilon_\ell)_\ell$. Since (16) implies that

$$\frac{1}{2}|||u - u_\ell|||^2 \le \varepsilon_\ell,$$

there follows also R-linear convergence of the energy norm $(|||u - u_\ell|||)_\ell$.

6 Numerical results

We provide numerical results for two examples. In each case a hierarchy of simplicial triangulations is adaptively generated by the algorithm of Sect. 2. In the first example there is no data oscillation, while in the second example the reduction of the data oscillations has to be controlled. This is done with the same parameter as in the bulk criterion, i.e., we choose $\kappa = \Theta$. At each refinement level, the discretized problem has been solved by a primal-dual active set strategy.

Example 1 (Smooth rotational symmetric solution) The obstacle problem (2) is considered on the square $\Omega := (-1.5, +1.5)^2$ with a constant right-hand side $f \equiv -2$ and the obstacle fixed by $\chi \equiv 0$. The Dirichlet boundary conditions are given by the trace of the exact solution

$$u = \begin{cases} r^2/2 - \ln(r) - 1/2, & r \geq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

where $r = |x|$. The solution is visualized in Fig. 2.

Figure 3 contains the adaptively generated finite element mesh after 10 and 16 refinement steps, respectively, where $\Theta = 0.6$ has been used in the bulk criterion. We see that the refinement basically occurs in the inactive zone.

The convergence history is documented in Table 1 containing the total number N_{dof} of degrees of freedom, the square root $\sqrt{\varepsilon_\ell}$ of the energy error (error in the energy functional), the estimator, and the data oscillation $\mathrm{Osc}_\ell(f)$ per refinement level ℓ in case $\Theta = 0.6$. Although f is constant in this example, due to (10), the term $\mathrm{Osc}_\ell(f)$ contains data contributions from the boundary of the computational domain. However, as can be clearly seen, there is a rapid decrease of $\mathrm{Osc}_\ell(f)$. Figure 4 (left) provides a graphical representation of the convergence history. The experimental convergence rates for $\sqrt{\varepsilon_\ell}$ and η_ℓ are basically the same (≈ 0.5), whereas $\mathrm{Osc}_\ell(f)$ decays almost twice as fast. Figure 4 (right) compares the decrease of the square root of the energy

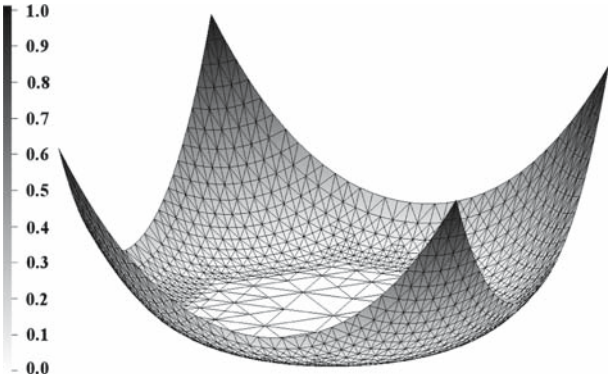

Fig. 2 Solution for Example 1

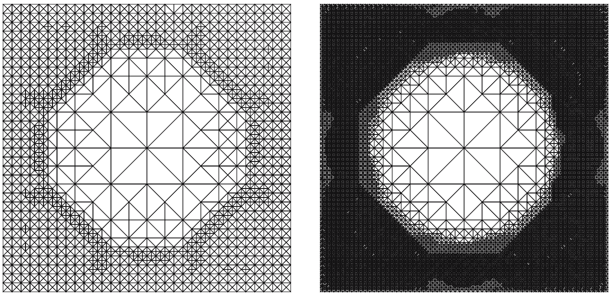

Fig. 3 Example 1: Adaptively generated mesh after 10 (*left*) and 16 (*right*) refinement steps ($\Theta = 0.6$ in the bulk criterion)

Table 1 Convergence history of the adaptive refinement process (Example 1)

ℓ	N_{dof}	sqrtε_ℓ	η_ℓ	$\mathrm{Osc}_\ell(f)$
2	41	7.90e − 01	1.55e + 00	2.25e + 00
4	143	3.68e − 01	8.93e − 01	9.49e − 01
6	430	1.95e − 01	5.12e − 01	3.84e − 01
8	1373	1.06e − 01	2.80e − 01	1.75e − 01
10	4849	5.34e − 02	1.46e − 01	6.39e − 02
12	16985	2.76e − 02	7.69e − 02	2.36e − 02
14	58739	1.44e − 02	4.17e − 02	8.60e − 03
16	190649	8.74e − 03	2.32e − 02	3.47e − 03
18	656994	4.74e − 03	1.24e − 02	1.30e − 03

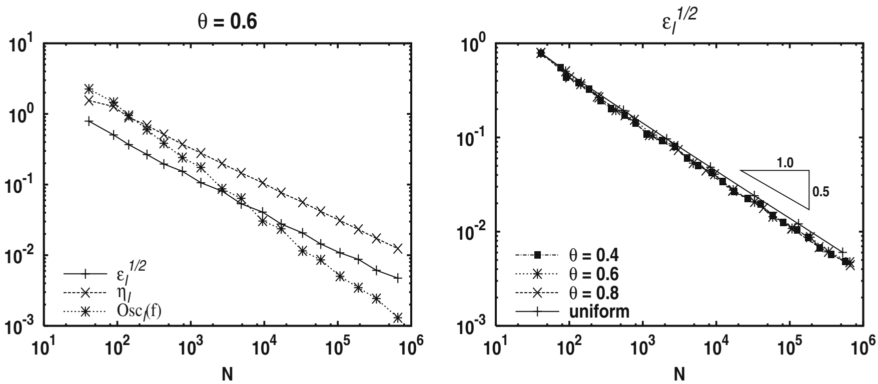

Fig. 4 Example 1: Convergence history of the adaptive refinement process for $\Theta = 0.6$ (*left*) and square root of the error in the energy functional as a function of degrees of freedom for $\Theta = 0.4, 0.6, 0.8$, and for uniform refinement (*right*)

error for adaptive refinement with the parameters $\Theta = 0.4, 0.6, 0.8$ and for uniform refinement. Due to the regularity of the solution, there are no significant benefits of adaptive refinement.

Example 2 (Corner singularity; L-shaped domain) The obstacle problem (2) is considered on the L-shaped domain $\Omega := (-2, +2)^2 \setminus [0, +2) \times (-2, 0]$ with zero obstacle, i.e., $\chi \equiv 0$, and the right-hand side

$$f(r, \varphi) := -r^{2/3} \sin(2\varphi/3)\left(\gamma_1'(r)/r + \gamma_1''(r)\right) \\ -\frac{4}{3} r^{1/3} \gamma_1'(r) \sin(2\varphi/3) - \gamma_2(r).$$

where, $\bar{r} = 2(r - 1/4)$ and

$$\gamma_1(r) = \begin{cases} 1, & \bar{r} < 0, \\ -6\bar{r}^5 + 15\bar{r}^4 - 10\bar{r}^3 + 1, & 0 \le \bar{r} < 1, \\ 0, & \bar{r} \ge 1, \end{cases}$$

$$\gamma_2(r) = \begin{cases} 0 & r \le 5/4, \\ 1 & \text{elsewhere.} \end{cases}$$

The exact solution

$$u(r, \varphi) = r^{2/3} \gamma_1(r) \sin(2\varphi/3)$$

has a corner singularity at the origin as depicted in Fig. 5. It belongs to $H^{5/3-\varepsilon}(D)$ for any $\varepsilon > 0$ and any open neighborhood D of the origin.

For $\Theta = 0.6$ in the bulk criterion, Fig. 6 displays the adaptively generated meshes after 10 and 18 refinement steps. As in the previous example, the refinement is essentially restricted to the inactive zone.

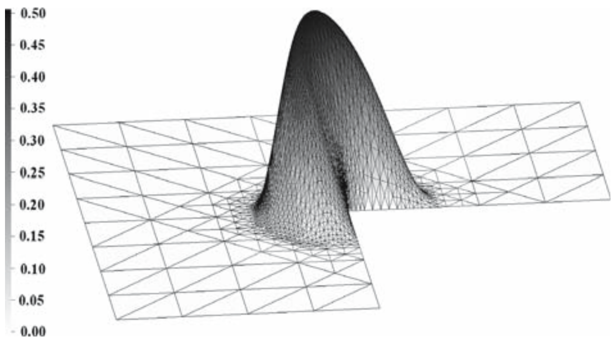

Fig. 5 Solution for Example 2

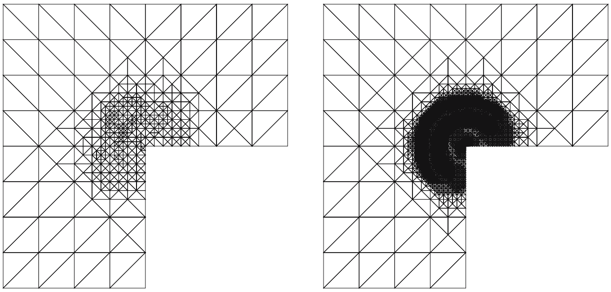

Fig. 6 Example 2: Adaptively generated mesh after 10 (*left*) and 18 (*right*) refinement steps ($\Theta = 0.6$ in the bulk criterion)

Table 2 Convergence history of the adaptive refinement process (Example 2)

ℓ	N_{dof}	$\sqrt{\varepsilon_\ell}$	η_ℓ	$\text{Osc}_\ell(f)$
2	65	7.95e − 01	9.56e − 01	7.01e + 00
4	113	3.71e − 01	1.41e + 00	1.70e + 00
6	303	1.84e − 01	7.30e − 01	4.78e − 01
8	1029	9.53e − 02	4.19e − 01	1.52e − 01
10	3248	4.85e − 02	2.20e − 01	4.72e − 02
12	11272	2.54e − 02	1.14e − 01	1.58e − 02
14	39399	1.36e − 02	5.95e − 02	4.59e − 03
16	136502	7.40e − 03	3.14e − 02	1.57e − 03
18	467972	4.67e − 03	1.69e − 02	4.72e − 04

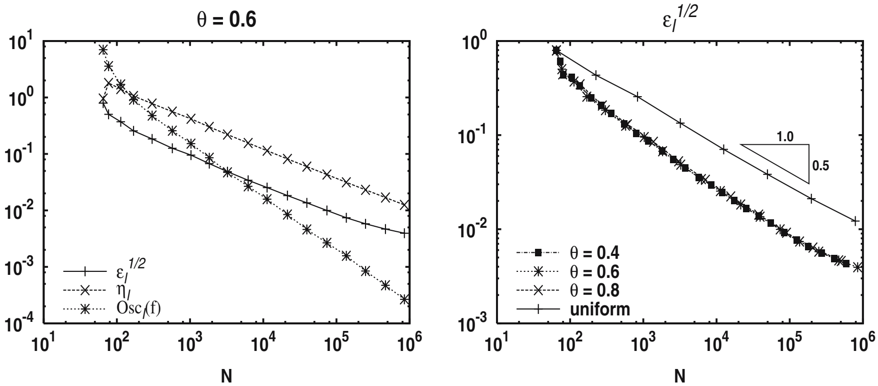

Fig. 7 Example 2: Convergence history of the adaptive refinement process for $\Theta = 0.6$ (*left*) and square root of the error in the energy functional as a function of degrees of freedom for $\Theta = 0.4, 0.6, 0.8$, and for uniform refinement (*right*)

Table 2 and Fig. 7 (left) display the convergence history with the same legends as in the previous example. The experimental convergence rates of the square root $\sqrt{\varepsilon_\ell}$ of the energy error and the estimator η_ℓ are roughly the same (≈0.5), whereas the oscillation term $\mathrm{Osc}_\ell(f)$ decays twice as fast. Figure 7 (right) is devoted to a comparison of adaptive refinement (for $\Theta = 0.4, 0.6, 0.8$) and uniform refinement. The experimental convergence rate of $\sqrt{\varepsilon_\ell}$ in case of uniform refinement is the same as for adaptive refinement. However, in order to achieve a prescribed accuracy, the adaptive refinement process needs an amount of degrees of freedom which is approximately an order of magnitude less than for uniform refinement.

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