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A high order uniformly convergent alternating direction scheme for time dependent reaction–diffusion singularly perturbed problems

B. Bujanda · C. Clavero · J. L. Gracia · J. C. Jorge

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Abstract In this work we design and analyze an efficient numerical method to solve two dimensional initial-boundary value reaction–diffusion problems, for which the diffusion parameter can be very small with respect to the reaction term. The method is defined by combining the Peaceman and Rachford alternating direction method to discretize in time, together with a HODIE finite difference scheme constructed on a tailored mesh. We prove that the resulting scheme is ε-uniformly convergent of second order in time and of third order in spatial variables. Some numerical examples illustrate the efficiency of the method and the orders of uniform convergence proved theoretically. We also show that it is easy to avoid the well-known order reduction phenomenon, which is usually produced in the time integration process when the boundary conditions are time dependent.

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B. Bujanda · J. C. Jorge
Departamento de Matemática e Informática,
Universidad Pública de Navarra, Pamplona, Spain
e-mail: blanca.bujanda@unavarra.es

J. C. Jorge
e-mail: jcjorge@unavarra.es

C. Clavero (✉) · J. L. Gracia
Departamento de Matemática Aplicada,
Universidad de Zaragoza, Zaragoza, Spain
e-mail: clavero@unizar.es

J. L. Gracia
e-mail: jlgracia@unizar.es

1 Introduction

We consider the two dimensional parabolic initial-boundary value problem

$$\begin{cases} u_t + L_\varepsilon u = f(\mathbf{x}, t), \ (\mathbf{x}, t) \in \Omega \times (0, T], \quad \Omega = (0, 1)^2, \\ u(\mathbf{x}, 0) = \varphi_0(\mathbf{x}), \ \mathbf{x} \in \overline{\Omega}, \quad u(\mathbf{x}, t) = 0, \ (\mathbf{x}, t) \in \partial\Omega \times (0, T], \end{cases} \tag{1}$$

where

$$L_\varepsilon u \equiv -\varepsilon \Delta u + bu. \tag{2}$$

This is a simple model of problems appearing, for instance, in chemical reaction processes. We will assume that the positive diffusion coefficient ε can be very small and that the reaction term is strictly positive ($b(\mathbf{x}) = b(x_1, x_2) \geq \beta > 0$). Also, some smoothness and compatibility conditions for functions b, f and φ_0 are going to be imposed in order to the solution of problem (1) is sufficiently smooth. Concretely, the compatibility conditions to ensure the required regularity in time of the exact solution are the following:

$$\begin{aligned} \Psi_j(\mathbf{x}) &= 0, \quad \mathbf{x} \in \partial\Omega, \quad j = 0, 1, 2, 3, \\ f(c_i, t) &= 0, \quad i = 1, 2, 3, 4, \end{aligned}$$

where $\Psi_0(\mathbf{x}) \equiv \varphi_0(\mathbf{x})$, $\Psi_j(\mathbf{x}) \equiv \dfrac{\partial^{j-1} f}{\partial t^{j-1}}(\mathbf{x}, 0) - L_\varepsilon \Psi_{j-1}(\mathbf{x})$, $j = 1, 2, 3$, and c_i, $i = 1, 2, 3, 4$, are the corners of the unit square $\overline{\Omega}$. Using similar arguments to [7] we can deduce sufficient compatibility conditions to ensure the required regularity in space for the exact solution.

It is well-known that the solution of (1) has parabolic boundary layers in $\partial\Omega \times (0, T]$ and also that there are corner layers in the neighbourhood of the four spatial corners c_i, $i = 1, 2, 3, 4$ (see [5,14]). Such behaviour causes that standard numerical methods defined on uniform meshes provide very inaccurate solutions for small values of the diffusion parameter ε, unless a high number of mesh points are used. Several alternatives to avoid this drawback can be found in the literature (see [13,18] and the references therein). One of the most successful ones is the use of simple finite difference methods defined on suitable piecewise uniform meshes introduced by Shishkin (see [6] and the references therein). Such meshes were initially designed for 1D stationary singularly perturbed problems, but they were soon extended successfully to 2D elliptic problems (see [6,10,12,13]).

To solve efficiently time dependent diffusion–reaction singularly perturbed problems, this type of spatial discretizations can be combined with suitable time integrators. Usually, the order of uniform convergence of the methods specially designed for these problems is not higher than one. For instance, in [15] a numerical scheme was developed for 1D reaction–diffusion parabolic problems, having first order in time and second order in space. For the same kind of problems, in [8,9] a defect–correction technique was proposed to increase the order of convergence in time. Nevertheless, it does not seem an easy task to prove the uniform convergence of this technique in the case of 2D parabolic problems.

In [5] a different discretization way was proposed for problem (1), where a simple (first order) alternating direction implicit method is combined with central differences on a Shishkin mesh. The deduced numerical algorithm has two main properties: on one hand, it is unconditionally and ε-uniformly convergent, i.e, its numerical solution is reliable for any value of ε and any relation between the time and spatial step sizes. On the other hand, the totally discrete method is optimal in terms of computational complexity, being capable of advancing in time with a computational cost which depends linearly on the number of mesh points and without any restriction between the sizes of the time step and the thickness of the meshes (unconditional stability).

In this paper we propose a new numerical algorithm preserving essentially the same advantages of the scheme designed in [5] and also it reaches higher orders of convergence both in space and time. Such feature permits to obtain very accurate solutions by using larger time steps and coarser spatial meshes than using lower order schemes. This scheme arises from the combination of the Peaceman and Rachford method [17], which reaches second order of convergence, and a third order HODIE finite difference scheme, defined on an appropriate piecewise uniform mesh of Shishkin type to discretize in space [2].

The paper is structured as follows. In Sect. 2, we revise the asymptotic behaviour, with respect to the parameter ε, of the solution of (1) and their partial derivatives. Also in this section we prove the uniform convergence of second order of the time semidiscretization proposed, based on the Peaceman and Rachford method. In Sect. 3 we prove the asymptotic behaviour (respect to the singular perturbation parameter ε) of the semidiscrete solutions, and their spatial derivatives, of the problems resulting of the application of the Peaceman and Rachford method. In Sect. 4 we define a classical HODIE finite difference scheme (exact for polynomials of degrees less than or equal to three), which is going to be used to discretize in space the semidiscrete problems studied in previous section. We prove that if the spatial discretization is defined on a mesh of Shishkin type, then the scheme is uniformly convergent of third order. From the previous results, we will prove the main result of this paper: the uniform convergence of the totally discrete method. In Sect. 5, we perform some numerical examples for a problem of type (1), which show a uniformly convergent behaviour, according to the theoretical results. Also, we analyze a more general problem than (1), where the boundary conditions are time dependent. It is well-known that a classical discretization of these boundary conditions causes a order reduction effect in time. We propose an alternative way to discretize the boundary conditions at each stage in order to restore the second order of uniform convergence. Some numerical examples clearly illustrate the practical improvements of this technique.

Henceforth, C and c denote any positive constant independent of the singular perturbation parameter ε and the discretization parameters N and Δt.

2 The continuous problem and the time semidiscretization

Let us assume that the data b, f and φ_0 of problem (1) are sufficiently smooth functions and also that there are sufficient compatibility conditions between them, in order to ensure that the solution $u(\mathbf{x}, t) \in C^{6,6,3}(\overline{\Omega} \times [0, T])$ (see [11]).

In [5] it was proven that the exact solution u of (1) can be decomposed in the form

$$u = u_0 + \sum_{i=1}^{4} u_i + \sum_{i=1}^{4} w_i, \tag{3}$$

where u_0 is the regular component, $u_i, i = 1, 2, 3, 4$, are the edge boundary layer functions associated at each one of the four sides of the unit square and $w_i, i = 1, 2, 3, 4$, are the corner layer functions corresponding to the corner points $c_i,\ i = 1, 2, 3, 4$, respectively. Also, in [5] the authors give the following bounds for each one of these components:

$$\left|\frac{\partial^{k_s+k_t} u_0(\mathbf{x}, t)}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_t}}\right| \le C,$$

$$\left|\frac{\partial^{k_s+k_t} u_1(\mathbf{x}, t)}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_t}}\right| \le C\varepsilon^{-k_1/2} \exp\left(-\frac{\sqrt{\beta}x_1}{\sqrt{\varepsilon}}\right),$$

$$\left|\frac{\partial^{k_s+k_t} u_2(\mathbf{x}, t)}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_t}}\right| \le C\varepsilon^{-k_1/2} \exp\left(-\frac{\sqrt{\beta}(1-x_1)}{\sqrt{\varepsilon}}\right),$$

$$\left|\frac{\partial^{k_s+k_t} u_3(\mathbf{x}, t)}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_t}}\right| \le C\varepsilon^{-k_2/2} \exp\left(-\frac{\sqrt{\beta}x_2}{\sqrt{\varepsilon}}\right),$$

$$\left|\frac{\partial^{k_s+k_t} u_4(\mathbf{x}, t)}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_t}}\right| \le C\varepsilon^{-k_2/2} \exp\left(-\frac{\sqrt{\beta}(1-x_2)}{\sqrt{\varepsilon}}\right),$$

$$\left|\frac{\partial^{k_s+k_t} w_1(\mathbf{x}, t)}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_t}}\right| \le C\varepsilon^{-k_s/2} \min\left\{\exp\left(-\frac{\sqrt{\beta}x_1}{\sqrt{\varepsilon}}\right), \exp\left(-\frac{\sqrt{\beta}x_2}{\sqrt{\varepsilon}}\right)\right\},$$

$$\left|\frac{\partial^{k_s+k_t} w_2(\mathbf{x}, t)}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_t}}\right| \le C\varepsilon^{-k_s/2} \min\left\{\exp\left(-\frac{\sqrt{\beta}(1-x_1)}{\sqrt{\varepsilon}}\right), \exp\left(-\frac{\sqrt{\beta}x_2}{\sqrt{\varepsilon}}\right)\right\},$$

$$\left|\frac{\partial^{k_s+k_t} w_3(\mathbf{x}, t)}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_t}}\right| \le C\varepsilon^{-k_s/2} \min\left\{\exp\left(-\frac{\sqrt{\beta}x_1}{\sqrt{\varepsilon}}\right), \exp\left(-\frac{\sqrt{\beta}(1-x_2)}{\sqrt{\varepsilon}}\right)\right\},$$

$$\left|\frac{\partial^{k_s+k_t} w_4(\mathbf{x}, t)}{\partial x_1^{k_1} \partial x_2^{k_2} \partial t^{k_t}}\right| \le C\varepsilon^{-k_s/2} \min\left\{\exp\left(-\frac{\sqrt{\beta}(1-x_1)}{\sqrt{\varepsilon}}\right), \exp\left(-\frac{\sqrt{\beta}(1-x_2)}{\sqrt{\varepsilon}}\right)\right\},$$

where $k_s = k_1 + k_2,\ k_s + 2k_t \le 6$, showing their asymptotic behaviour with respect to the diffusion parameter ε.

To get our totally discrete numerical method, we consider firstly a time semidiscretization of (1) by using the classical Peaceman and Rachford fractional step method, with a constant time step Δt. In order to shorten the notations, we rewrite the differential operator L_ε as $L_\varepsilon = L_{1,\varepsilon} + L_{2,\varepsilon}$, where

$$L_{i,\varepsilon} \equiv -\varepsilon \frac{\partial^2}{\partial x_i^2} + b_i(\mathbf{x}), \quad i = 1, 2, \tag{4}$$

taking $b_i(\mathbf{x})$ such that $b(\mathbf{x}) = b_1(\mathbf{x}) + b_2(\mathbf{x})$ and $b_i(\mathbf{x}) \geq \beta_i > 0, i = 1, 2$. Besides, we decompose the source term as $f = f_1 + f_2$ with

$$\begin{aligned} f_2(\mathbf{x}, t) &= f(x_1, 0, t) + x_2\big(f(x_1, 1, t) - f(x_1, 0, t)\big), \\ f_1(\mathbf{x}, t) &= f(\mathbf{x}, t) - f_2(\mathbf{x}, t), \end{aligned} \tag{5}$$

as it was suggested in [4]. Then, the Peaceman and Rachford method can be written as

$$u^0 = \varphi_0(\mathbf{x}),$$

$$\begin{cases} \big(I + \frac{\Delta t}{2} L_{1,\varepsilon}\big) u^{n+\frac{1}{2}}(\mathbf{x}) = \widehat{f}_{1,u^n}(\mathbf{x}, t_n, \Delta t), \quad x_2 \in (0, 1), \\ u^{n+\frac{1}{2}}(0, x_2) = u^{n+\frac{1}{2}}(1, x_2) = 0, \end{cases} \tag{6}$$

$$\begin{cases} \big(I + \frac{\Delta t}{2} L_{2,\varepsilon}\big) u^{n+1}(\mathbf{x}) = \widehat{f}_{2,u^{n+\frac{1}{2}}}\big(\mathbf{x}, t_{n+\frac{1}{2}}, \Delta t\big), \quad x_1 \in (0, 1), \\ u^{n+1}(x_1, 0) = u^{n+1}(x_1, 1) = 0, \end{cases} \tag{7}$$

where $t_{n+\frac{1}{2}} = (n + \frac{1}{2})\Delta t$, with

$$\widehat{f}_{1,u^n}(\mathbf{x}, t_n, \Delta t) = \left(I - \frac{\Delta t}{2} L_{2,\varepsilon}\right) u^n(\mathbf{x}) + \frac{\Delta t}{2}\left(f_1\big(\mathbf{x}, t_{n+\frac{1}{2}}\big) + f_2(\mathbf{x}, t_n)\right), \tag{8}$$

$$\widehat{f}_{2,u^{n+\frac{1}{2}}}\left(\mathbf{x}, t_{n+\frac{1}{2}}, \Delta t\right) = \left(I - \frac{\Delta t}{2} L_{1,\varepsilon}\right) u^{n+\frac{1}{2}}(\mathbf{x}) + \frac{\Delta t}{2}\left(f_1\big(\mathbf{x}, t_{n+\frac{1}{2}}\big) + f_2(\mathbf{x}, t_{n+1})\right), \tag{9}$$

and $u^n(\mathbf{x})$ denotes the approximation of the exact solution $u(\mathbf{x}, t)$ at the time level $t_n = n\Delta t$. Clearly, since the differential operators $(I + \frac{\Delta t}{2} L_{i,\varepsilon}),\ i = 1, 2$, satisfy a maximum principle, it is straightforward to prove that

$$\left\| \left(I + \frac{\Delta t}{2} L_{i,\varepsilon}\right)^{-1} \right\|_\infty \leq \frac{1}{1 + \beta_i \frac{\Delta t}{2}}, \quad i = 1, 2.$$

These bounds prove that each one of the steps of the scheme (6)–(7) has a unique solution $u^{n+1}(\mathbf{x})$, which can be bounded independently of the diffusion parameter ε.

To study the consistency of the Peaceman and Rachford method, we introduce the local error $e_{n+1} = u(\mathbf{x}, t_{n+1}) - \hat{u}^{n+1}(\mathbf{x})$, where $\hat{u}^{n+1}(\mathbf{x})$ is the approximation to

$u(\mathbf{x}, t_{n+1})$ obtained with one time step of (6)–(7), taking $u(\mathbf{x}, t_n)$ as the starting value $u^n(\mathbf{x})$, i.e., $\hat{u}^{\,n+1}(\mathbf{x})$ is the solution of the problem

$$u^n(\mathbf{x}) = u(\mathbf{x}, t_n),$$
$$\begin{cases} \left(I + \frac{\Delta t}{2} L_{1,\varepsilon}\right) \widehat{u}^{\,n+\frac{1}{2}}(\mathbf{x}) = \widehat{f}_{1,u^n}(\mathbf{x}, t_n, \Delta t), \quad x_2 \in (0,1), \\ \widehat{u}^{\,n+\frac{1}{2}}(0, x_2) = \widehat{u}^{\,n+\frac{1}{2}}(1, x_2) = 0, \end{cases} \tag{10}$$

$$\begin{cases} (I + \frac{\Delta t}{2} L_{2,\varepsilon}) \widehat{u}^{\,n+1}(\mathbf{x}) = \widehat{f}_{2,\widehat{u}^{\,n+\frac{1}{2}}}(\mathbf{x}, t_{n+\frac{1}{2}}, \Delta t), \quad x_1 \in (0,1), \\ \widehat{u}^{\,n+1}(x_1, 0) = \widehat{u}^{\,n+1}(x_1, 1) = 0, \end{cases} \tag{11}$$

where $\widehat{f_1}$, $\widehat{f_2}$ are defined in (8) and (9), respectively, but substituting $u^n(\mathbf{x})$ by $u(\mathbf{x}, t_n)$ and $u^{\,n+\frac{1}{2}}(\mathbf{x})$ by $\widehat{u}^{\,n+\frac{1}{2}}(\mathbf{x})$.

Lemma 1 *Let us suppose that*

$$\left\{u, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^3 u}{\partial t^3}\right\} \subset C^0(\bar{\Omega} \times [0, T]), \tag{12}$$

and also that they are bounded independently of ε. *Then, the local error satisfies*

$$\|e_{n+1}\|_\infty \le C(\Delta t)^3. \tag{13}$$

Proof It follows the same ideas developed in [3]. □

Using Lemma 1, we are ready to deduce the uniform convergence of this semi-discretization process. Let $E_n = u(\mathbf{x}, t_n) - u^n(\mathbf{x})$ be the global error associated to (6)–(7); then, clearly it holds $E_n = e_n + RE_{n-1}$, where

$$R \equiv \left(I + \frac{\Delta t}{2} L_{2,\varepsilon}\right)^{-1} \left(I - \frac{\Delta t}{2} L_{1,\varepsilon}\right) \left(I + \frac{\Delta t}{2} L_{1,\varepsilon}\right)^{-1} \left(I - \frac{\Delta t}{2} L_{2,\varepsilon}\right),$$

is the transition operator defined as follows: RE_{n-1} is the result obtained after one step of scheme (6)–(7) taking as starting data $u^n = E_{n-1}$, null boundary conditions and zero values for the source terms f_1 and f_2. Using this recurrence, we deduce

$$E_n = \sum_{i=1}^{n} R^{n-i} e_i.$$

Thus, if it holds that

$$\|R^i\|_\infty \le C, \quad i = 1, 2, \ldots, n, \tag{14}$$

it immediately follows that,

$$\sup_{n \le \frac{T}{\Delta t}} \|E_n\|_\infty \le C(\Delta t)^2,$$

i.e., the semidiscrete scheme (6)–(7) is a second order ε-uniform convergent method. Note that (14) is a typical stability condition; for instance, the same type of result was established in [16] for A-stable one step time discretizations of parabolic problems. In particular, in [16] the stability, in the maximum norm, of the Crank-Nicolson time semidiscretization for the heat equation was discussed. In our case, condition (14) can be obtained in a similar way when the operators $L_{1,\varepsilon}$ and $L_{2,\varepsilon}$ commute (for instance, this property is immediately satisfied if $b_1 \equiv b_1(x_1)$ and $b_2 \equiv b_2(x_2)$). In such case, the operators R^i can be rewritten in the form $R_1^i R_2^i$ where $R_j \equiv (I + \frac{\Delta t}{2} L_{j,\varepsilon})^{-1}$ $(I - \frac{\Delta t}{2} L_{j,\varepsilon})$, $j = 1, 2$, i.e., $R_j,\ j = 1, 2$, can be seen as the Crank-Nicolson transition operator corresponding to the time semidiscretization of a parabolic problem where the spatial differential operator is $L_{j,\varepsilon},\ j = 1, 2$. Then, we can use that the operators $L_{j,\varepsilon},\ j = 1, 2$, are ε-uniformly sectorial, i.e., for $0 < \theta < \pi/2$ the spectra of $L_{j,\varepsilon},\ j = 1, 2$, are contained in the set $S_\theta \equiv \{z \in \mathcal{C};\ |\arg z| \le \theta\}$ and for every $z \notin S_\theta$ it is satisfied that $\|(z - L_{j,\varepsilon})^{-1}\|_\infty \le C/|z|$ independently of ε. From [16] we can deduce that $\|R_j^i\|_\infty \le C$, $j = 1, 2$, $i = 1, 2, \ldots$, and consequently (14) follows. When operators $L_{j,\varepsilon},\ j = 1, 2$, do not commute, it is not possible to apply this argument, but in the numerical experiences performed with our method for many non commuting cases, the same stable behaviour as in the cases with commuting operators has been observed.

3 Asymptotic behaviour of the solutions of the semidiscrete problems

This section is devoted to prove that the solutions of the semidiscrete in time problems and their spatial derivatives, up to certain order, have essentially the same behaviour that the solution of a stationary 1D diffusion–reaction singularly perturbed problem. Such feature is the key that allows us to choose suitable spatial discretizations for these problems, in order to obtain a uniformly convergent numerical algorithm.

Lemma 2 *Let* $\widehat{u}^{\,n+\frac{1}{2}}(\mathbf{x})$ *and* $\widehat{u}^{\,n+1}(\mathbf{x})$ *be the solutions of* (10) *and* (11) *respectively. Then, there exists a constant C independent of* ε *and* Δt *such that, for* $0 \le i \le 6$, *it holds*

$$\left| \frac{\partial^i \widehat{u}^{\,n+\frac{1}{2}}(\mathbf{x})}{\partial x_1^i} \right| \le C \left(1 + \varepsilon^{-i/2}\Big(\exp\big(-\sqrt{\beta_1}x_1/\sqrt{\varepsilon}\big) + \exp\big(-\sqrt{\beta_1}(1 - x_1)/\sqrt{\varepsilon}\big)\Big)\right), \tag{15}$$

$$\left| \frac{\partial^i \widehat{u}^{\,n+1}(\mathbf{x})}{\partial x_2^i} \right| \le C \left(1 + \varepsilon^{-i/2}\Big(\exp\big(-\sqrt{\beta_2}x_2/\sqrt{\varepsilon}\big) + \exp\big(-\sqrt{\beta_2}(1 - x_2)/\sqrt{\varepsilon}\big)\Big)\right). \tag{16}$$

Proof Let us start by studying in detail the behaviour of $\widehat{u}^{\,n+\frac{1}{2}}(\mathbf{x})$ and their spatial derivatives. First of all, using that the operator $(I + \frac{\Delta t}{2} L_{1,\varepsilon})$ satisfies a maximum principle, the data f_1, f_2 are ε-uniformly bounded and $|u(\mathbf{x}, t_n)| \le C$, $|L_{2,\varepsilon} u(\mathbf{x}, t_n)| \le C$, it follows that $|\widehat{u}^{\,n+\frac{1}{2}}(\mathbf{x})| \le C$. Let us obtain now the corresponding bounds for the x_1-derivatives of $\widehat{u}^{\,n+\frac{1}{2}}(\mathbf{x})$. We introduce the auxiliary function

$$\omega \equiv \frac{\widehat{u}^{\,n+\frac{1}{2}}(\mathbf{x}) - u(\mathbf{x}, t_n)}{\Delta t/2}, \tag{17}$$

which is the solution of the following boundary value problem

$$\begin{cases} \left(I + \frac{\Delta t}{2} L_{1,\varepsilon}\right)\omega(\mathbf{x}) = \dfrac{\partial u}{\partial t}(\mathbf{x}, t_n) + f_1\left(\mathbf{x}, t_{n+\frac{1}{2}}\right) - f_1(\mathbf{x}, t_n), x_2 \in (0,1), \\ \omega(0, x_2) = \omega(1, x_2) = 0. \end{cases}$$

As the right-hand side of the differential equation is composed of ε-uniformly bounded terms, we can use again the maximum discrete principle of the operator $(I + \frac{\Delta t}{2} L_{1,\varepsilon})$ to deduce that $|\omega(\mathbf{x})| \le C$. Then, rewriting the boundary value problem (10) in the form

$$\begin{cases} L_{1,\varepsilon}\widehat{u}^{\,n+\frac{1}{2}}(\mathbf{x}) = -\omega - L_{2,\varepsilon}u(\mathbf{x}, t_n) + f_1\left(\mathbf{x}, t_{n+\frac{1}{2}}\right) + f_2(\mathbf{x}, t_n), x_2 \in (0,1), \\ \widehat{u}^{\,n+\frac{1}{2}}(0, x_2) = \widehat{u}^{\,n+\frac{1}{2}}(1, x_2) = 0, \end{cases}$$

similarly to [14] we can obtain

$$\left| \frac{\partial^i \widehat{u}^{\,n+\frac{1}{2}}}{\partial x_1^i}(0, x_2) \right| \le C\varepsilon^{-i/2}, \quad i = 1, 2,$$

and

$$\left| \frac{\partial^i \widehat{u}^{\,n+\frac{1}{2}}}{\partial x_1^i}(1, x_2) \right| \le C\varepsilon^{-i/2}, \quad i = 1, 2.$$

Next, differentiating with respect to x_1 the differential equation of problem (10), we see that the functions $\phi_i \equiv \dfrac{\partial^i \widehat{u}^{\,n+\frac{1}{2}}}{\partial x_1^i}$, $i = 1, 2$, are the solutions of boundary value problems of the form

$$\begin{cases} \left(I + \frac{\Delta t}{2} L_{1,\varepsilon}\right)\phi_i(\mathbf{x}) = g_i(\mathbf{x}), \\ \phi_i(0, x_2) = h_i^0(x_2), \quad \phi_i(1, x_2) = h_i^1(x_2), \end{cases}$$

where $|h_i^j(x_2)| \le C\varepsilon^{-i/2}$, $j = 0, 1$, $i = 1, 2$, and $|g_i(\mathbf{x})| \le C\left(1 + \varepsilon^{-i/2}\left(\exp(-\sqrt{\beta_1}x_1/\sqrt{\varepsilon}) + \exp(-\sqrt{\beta_1}(1-x_1)/\sqrt{\varepsilon})\right)\right)$, $\quad i = 1, 2$.
Using the same barrier function technique that in [5] we deduce the bounds (15) for $i = 1, 2$.

To find the corresponding bounds for $i = 3, 4$, we firstly observe that $\bar{\omega} \equiv L_{1,\varepsilon}\omega$ is the solution of the boundary value problem

$$\begin{cases} \left(I + \frac{\Delta t}{2} L_{1,\varepsilon}\right)\bar{\omega}(\mathbf{x}) = L_{1,\varepsilon}\left(\dfrac{\partial u}{\partial t}(\mathbf{x}, t_n) + f_1\left(\mathbf{x}, t_{n+\frac{1}{2}}\right) - f_1(\mathbf{x}, t_n)\right), \\ \qquad\qquad \bar{\omega}(0, x_2) = \frac{2}{\Delta t}\left(f_1\left(0, x_2, t_{n+\frac{1}{2}}\right) - f_1(0, x_2, t_n)\right), \\ \qquad\qquad \bar{\omega}(1, x_2) = \frac{2}{\Delta t}\left(f_1\left(1, x_2, t_{n+\frac{1}{2}}\right) - f_1(1, x_2, t_n)\right). \end{cases}$$

Using again the maximum principle and taking into account that the boundary conditions are bounded independently of Δt and ε, we deduce that $\bar{\omega}$ is bounded independently of ε and Δt. Hence, it follows $\left|\dfrac{\partial^2 \omega}{\partial x_1^2}\right| \leq C\varepsilon^{-1}$.

Now we use that ϕ_2 is the solution of the boundary value problem

$$\begin{cases} L_{1,\varepsilon}\phi_2(\mathbf{x}) = \dfrac{\partial^2}{\partial x_1^2}\left(-\omega - L_{2,\varepsilon}u(\mathbf{x}, t_n) + f_1(\mathbf{x}, t_{n+\frac{1}{2}}) + f_2(\mathbf{x}, t_n)\right) \\ \qquad\qquad -2\dfrac{\partial b_1}{\partial x_1}\dfrac{\partial \widehat{u}^{\,n+\frac{1}{2}}}{\partial x_1} - \dfrac{\partial^2 b_1}{\partial x_1^2}\widehat{u}^{\,n+\frac{1}{2}}, \\ \phi_2(0, x_2) = -f_1\left(0, x_2, t_{n+\frac{1}{2}}\right)/\varepsilon, \ \phi_2(1, x_2) = -f_1\left(1, x_2, t_{n+\frac{1}{2}}\right)/\varepsilon, \end{cases}$$

where both the right-hand side of the differential equation and the boundary conditions are bounded by $C\varepsilon^{-1}$. Again, proceeding as in [14], we can obtain

$$\left|\frac{\partial^i \widehat{u}^{\,n+\frac{1}{2}}}{\partial x_1^i}(0, x_2)\right| \leq C\varepsilon^{-i/2}, \quad \left|\frac{\partial^i \widehat{u}^{\,n+\frac{1}{2}}}{\partial x_1^i}(1, x_2)\right| \leq C\varepsilon^{-i/2}, \quad i = 3, 4.$$

Next, differentiating with respect to x_1 the differential equation of problem (10), it follows that $\phi_i = \dfrac{\partial^i \widehat{u}^{\,n+\frac{1}{2}}}{\partial x_1^i}$, $i = 3, 4$, are the solutions of boundary value problems of the form

$$\begin{cases} \left(I + \frac{\Delta t}{2} L_{1,\varepsilon}\right)\phi_i(\mathbf{x}) = g_i(\mathbf{x}), \\ \phi_i(0, x_2) = h_i^0(x_2), \quad \phi_i(1, x_2) = h_i^1(x_2), \end{cases}$$

where

$$|g_i(\mathbf{x})| \leq C\left(1 + \varepsilon^{-i/2}\left(\exp(-\sqrt{\beta_1}x_1/\sqrt{\varepsilon}) + \exp(-\sqrt{\beta_1}(1 - x_1)/\sqrt{\varepsilon})\right)\right), \quad i = 3, 4,$$
$$|h_i^j(x_2)| \leq C\varepsilon^{-i/2}, \quad j = 0, 1, \quad i = 3, 4.$$

Using again the barrier function technique of [5], the bounds (15) are obtained for $i = 3, 4$.

A similar process can be followed to prove the corresponding bounds for $i = 5, 6$. Looking for not repeating similar arguments in excess, we resume this step of the proof to the main keys. Firstly, we prove that $L_{1,\varepsilon}^2\omega$ is ε-uniformly bounded to deduce immediately that $\left|\dfrac{\partial^4\omega}{\partial x_1^4}\right| \le C\varepsilon^{-2}$. Then, we write ϕ_4 as the solution of a boundary value problem of type

$$\begin{cases} L_{1,\varepsilon}\phi_4 = g_4(\mathbf{x}), \\ \phi_4(0,x_2) = h_4^0(x_2), \quad \phi_4(1,x_2) = h_4^1(x_2), \end{cases}$$

where functions g_4, h_4^0, h_4^1 are bounded by $C\varepsilon^{-2}$ and we deduce the bounds for $\left|\dfrac{\partial^i \widehat{u}^{\,n+\frac{1}{2}}}{\partial x_1^i}\right|$, $i = 5, 6$, first in the boundaries $x_1 = 0$, $x_1 = 1$ and after for $x_1 \in (0, 1)$.

The process to prove (16) for the x_2-derivatives of $\widehat{u}^{\,n+1}(\mathbf{x})$ is very similar to the one detailed above for bounding the x_1-derivatives of $\widehat{u}^{\,n+\frac{1}{2}}(\mathbf{x})$. The main difference is that now it is required to prove previously that

$$\left|\frac{\partial^i \widehat{u}^{\,n+\frac{1}{2}}(\mathbf{x})}{\partial x_2^i}\right| \le C\left(1 + \varepsilon^{-i/2}\left(\exp(-\sqrt{\beta_2}x_2/\sqrt{\varepsilon}) + \exp(-\sqrt{\beta_2}(1-x_2)/\sqrt{\varepsilon})\right)\right),$$

holds for $1 \le i \le 6$. Fortunately, these bounds are easily deduced by using that $\psi_i \equiv \dfrac{\partial^i \widehat{u}^{\,n+\frac{1}{2}}}{\partial x_2^i}$, $1 \le i \le 6$, are solutions of boundary value problems of type

$$\begin{cases} (I + \frac{\Delta t}{2}L_{1,\varepsilon})\psi_i(\mathbf{x}) = \bar{g}_i(\mathbf{x}), \\ \psi_i(0,x_2) = 0, \quad \psi_i(1,x_2) = 0. \end{cases}$$

The maximum principle satisfied by the operator $(I + \frac{\Delta t}{2}L_{1,\varepsilon})$ joint to the available bounds for $\bar{g}_i$, for each value of x_2, permit to deduce directly the required bounds.

4 The spatial semidiscretization

In this section we discretize in space the essentially 1D elliptic singularly perturbed problems (6)–(7). Bounds (15) and (16), which show the asymptotic behaviour of the exact solutions of (10), (11), respectively, give the key to construct the appropriate mesh of Shishkin type; next, we define the HODIE finite difference scheme on this mesh.

Let $N = 4k$ where k is a positive integer; we define the transition parameters σ_l, $l = 1, 2$, in the form

$$\sigma_l = \min\left\{1/4, \sigma_{l,0}\sqrt{\varepsilon}\ln N\right\}, \quad l = 1, 2, \tag{18}$$

where $\sigma_{l,0}$, $l = 1, 2$, are positive constants to be fixed later. On each spatial direction we consider the meshes $\overline{I}^N_{l,\varepsilon} = \{x_{l,j};\ j = 0, \ldots, N\}$, where

$$x_{l,j} = \begin{cases} jh_l, & j = 0, \ldots, N/4, \\ x_{l,N/4} + (j - N/4)H_l, & j = N/4 + 1, \ldots, 3N/4, \\ x_{l,3N/4} + (j - 3N/4)h_l, & j = 3N/4 + 1, \ldots, N, \end{cases} \tag{19}$$

with $H_l = 2(1 - 2\sigma_l)/N,\ h_l = 4\sigma_l/N,\ l = 1, 2$; also, we will denote the local step sizes as $h_{l,j} = x_{l,j} - x_{l,j-1},\ j = 1, \ldots, N,\ l = 1, 2$. Then, the mesh is defined as the tensor product of the corresponding 1D Shishkin meshes, i.e., $\overline{\Omega}^N_\varepsilon = \overline{I}^N_{1,\varepsilon} \times \overline{I}^N_{2,\varepsilon}$. Note that if $\sigma_l = 1/4$, $\ell = 1, 2$, then the mesh is uniform, N^{-1} is very small respect to ε and therefore a classical analysis could be made to prove the convergence of the scheme. So, in our theoretical analysis we only consider the case $\sigma_l = \sigma_{l,0}\sqrt{\varepsilon}\ln N,\ l = 1, 2$.

To discretize (10)–(11) on this mesh, we use a classical HODIE finite difference scheme, which is exact only for polynomial functions of degrees less than or equal to three (see [2] for the details of the construction). This scheme is given by

$$\begin{cases} \left(I + \frac{\Delta t}{2} L^N_{1,\varepsilon}\right) \widehat{U}^{\,n+\frac{1}{2}}_{x_{1,j},x_2} \equiv r^{1,-}_j \widehat{U}^{\,n+\frac{1}{2}}_{x_{1,j-1},x_2} + r^{1,c}_j \widehat{U}^{\,n+\frac{1}{2}}_{x_{1,j},x_2} + r^{1,+}_j \widehat{U}^{\,n+\frac{1}{2}}_{x_{1,j+1},x_2} \\ \quad = q^{1,1}_j \widehat{F}_{1,u^n}\left(x_{1,j-1}, x_2, t_n, \Delta t\right) + q^{1,2}_j \widehat{F}_{1,u^n}\left(x_{1,j}, x_2, t_n, \Delta t\right) \\ \qquad + q^{1,3}_j \widehat{F}_{1,u^n}\left(x_{1,j+1}, x_2, t_n, \Delta t\right), \quad j = 1, \ldots, N-1,\ x_2 \in I^N_{2,\varepsilon}, \\ \widehat{U}^{\,n+\frac{1}{2}}_{0,x_2} = \widehat{U}^{\,n+\frac{1}{2}}_{N,x_2} = 0, \end{cases} \tag{20}$$

$$\begin{cases} \left(I + \frac{\Delta t}{2} L^N_{2,\varepsilon}\right) \widehat{U}^{\,n+1}_{x_1,x_{2,j}} \equiv r^{2,-}_j \widehat{U}^{\,n+1}_{x_1,x_{2,j-1}} + r^{2,c}_j \widehat{U}^{\,n+1}_{x_1,x_{2,j}} + r^{2,+}_j \widehat{U}^{\,n+1}_{x_1,x_{2,j+1}} \\ \quad = q^{2,1}_j \widehat{F}_{2,\widehat{U}^{\,n+\frac{1}{2}}}\left(x_1, x_{2,j-1}, t_{n+\frac{1}{2}}, \Delta t\right) + q^{2,2}_j \widehat{F}_{2,\widehat{U}^{\,n+\frac{1}{2}}}\left(x_1, x_{2,j}, t_{n+\frac{1}{2}}, \Delta t\right) \\ \qquad + q^{2,3}_j \widehat{F}_{2,\widehat{U}^{\,n+\frac{1}{2}}}\left(x_1, x_{2,j+1}, t_{n+\frac{1}{2}}, \Delta t\right), \quad j = 1, \ldots, N-1,\ x_1 \in I^N_{1,\varepsilon}, \\ \widehat{U}^{\,n+1}_{x_1,0} = \widehat{U}^{\,n+1}_{x_1,N} = 0, \end{cases} \tag{21}$$

where

$$\begin{aligned} \widehat{F}_{1,u^n}(x_{1,j}, x_2, t_n, \Delta t) = u(x_{1,j}, x_2, t_n) + \frac{\Delta t}{2}\Big(&- \overline{L}^N_{2,\varepsilon} u(x_{1,j}, x_2, t_n) \\ &+ f_1\left(x_{1,j}, x_2, t_{n+\frac{1}{2}}\right) + f_2(x_{1,j}, x_2, t_n)\Big), \end{aligned} \tag{22}$$

$$\begin{aligned} \widehat{F}_{2,\widehat{U}^{\,n+\frac{1}{2}}}\left(x_1, x_{2,j}, t_{n+\frac{1}{2}}, \Delta t\right) = \widehat{U}^{\,n+\frac{1}{2}}_{x_1,x_{2,j}} + \frac{\Delta t}{2}\Big(&- \overline{L}^N_{1,\varepsilon} \widehat{U}^{\,n+\frac{1}{2}}_{x_1,x_{2,j}} \\ &+ f_1\left(x_1, x_{2,j}, t_{n+\frac{1}{2}}\right) + f_2(x_1, x_{2,j}, t_{n+1})\Big), \end{aligned} \tag{23}$$

and the operators $\overline{L}^N_{i,\varepsilon},\ i = 1, 2$, are defined by

$$\overline{L}^N_{2,\varepsilon} u(x_1, x_2, t_n) = L_{2,\varepsilon}\big(u\big(x_1, x_2, t_n\big)\big), \tag{24}$$

$$\begin{aligned} \overline{L}^N_{1,\varepsilon} \widehat{U}^{n+\frac{1}{2}}_{x_1,x_2} &= -\overline{L}^N_{2,\varepsilon} u(x_1, x_2, t_n) - 2\frac{\widehat{U}^{n+\frac{1}{2}}_{x_1,x_2} - u\big(x_1, x_2, t_n\big)}{\Delta t} \\ &\quad + f_1\big(x_1, x_2, t_{n+\frac{1}{2}}\big) + f_2\big(x_1, x_2, t_{n+1}\big). \end{aligned} \tag{25}$$

The values of the coefficients $r^{l,\bullet}_j,\ j = 1, \ldots, N-1,\ l = 1, 2,\ \bullet = -, c, +$, are given by

$$\begin{aligned} r^{l,-}_j &= \frac{\Delta t}{2}\left(\frac{-2\varepsilon}{h_{l,j}(h_{l,j} + h_{l,j+1})} + q^{l,1}_j\left(\bar{b}_{l,j-1} + \frac{2}{\Delta t}\right)\right), \\ r^{l,+}_j &= \frac{\Delta t}{2}\left(\frac{-2\varepsilon}{h_{l,j+1}(h_{l,j} + h_{l,j+1})} + q^{l,3}_j\left(\bar{b}_{l,j+1} + \frac{2}{\Delta t}\right)\right), \\ r^{l,c}_j &= \frac{\Delta t}{2}\left(q^{l,1}_j \bar{b}_{l,j-1} + q^{l,2}_j \bar{b}_{l,j} + q^{l,3}_j \bar{b}_{l,j+1}\right) - r^{l,-}_j - r^{l,+}_j + 1, \end{aligned} \tag{26}$$

where $\bar{b}_l,\ l = 1, 2$, denote the restriction on the mesh $\overline{\Omega}^N_\varepsilon$ of the functions $b_l,\ l = 1, 2$ (i.e., $\bar{b}_{1,j} = b_1(x_{1,j}, x_2),\ \bar{b}_{2,j} = b_2(x_1, x_{2,j})$). Note that the values of the coefficients $r^{l,\bullet}_j,\ j = 1, \ldots, N-1,\ l = 1, 2,\ \bullet = -, c, +$, depend on the location of the mesh points and also on the relation between H_l and ε, because the coefficients $q^{l,m}_j,\ j = 1, \ldots, N-1,\ l = 1, 2,\ m = 1, 2, 3$, are defined in two different ways. In the cases $x_{l,j} \in (0, \sigma_l) \bigcup (1-\sigma_l, 1)$ the coefficients $q^{l,m}_j, l = 1, 2,\ j = 1, \ldots, N/4 - 1$ and $j = 3N/4 + 1, \ldots, N-1$ for $m = 1, 2, 3$, are given by

$$\begin{aligned} q^{l,1}_j &= \frac{1}{6}\left(1 - \frac{h^2_{l,j+1}}{h_{l,j}(h_{l,j} + h_{l,j+1})}\right), \\ q^{l,3}_j &= \frac{1}{6}\left(1 - \frac{h^2_{l,j}}{h_{l,j+1}(h_{l,j} + h_{l,j+1})}\right), \\ q^{l,2} &= 1 - q^{l,1} - q^{l,3}. \end{aligned} \tag{27}$$

For $x_{l,j} \in [\sigma_l, 1-\sigma_l]$ we must distinguish two cases depending on the relation between H_l and ε. First, when $c_1 H^2_l \|b_l\|_\infty \le \varepsilon,\ l = 1, 2$ (c_1 is a fixed positive constant independent of ε, see [2]), the coefficients $q^{l,m}_j, l = 1, 2,\ j = N/4, \ldots, 3N/4$ for $m = 1, 2, 3$ are defined again by (27). On the other hand, when $c_1 H^2_l \|b_l\|_\infty > \varepsilon,\ l = 1, 2$, then the coefficients $q^{l,m}_j, l = 1, 2,\ j = N/4, \ldots, 3N/4,\ m = 1, 2, 3$, are given by

$$q^{l,1}_j = q^{l,3}_j = 0, \quad q^{l,2}_j = 1. \tag{28}$$

Lemma 3 *Let N_0 be the smallest positive integer such that*

$$\max_{l=1,2}\left\{4\sigma_{l,0}^2(\|b_l\|_\infty + 2/\Delta t)/3\right\} < N_0^2/\ln^2 N_0. \tag{29}$$

Then, for any $N \geq N_0$, there exists a constant c independent of ε such that

$$r_j^{l,-} + r_j^{l,c} + r_j^{l,+} \geq c > 0,\ r_j^{l,-} < 0,\ r_j^{l,+} < 0, \quad l = 1, 2,\ \ 1 \leq j < N.$$

Therefore, the finite difference operators defined by (20)–(21) *are both of positive type, they satisfy a discrete maximum principle and they are uniformly stable in the maximum norm.*

Proof In the case that $x_{l,j} \in [\sigma_l, 1-\sigma_l]$ and $c_1 H_l^2\|b_l\|_\infty > \varepsilon,\ l = 1, 2$ holds, the proof is trivial. In other case, using (26), (27), the condition (29) and the definition of the step sizes of the mesh, it easily follows that $r_j^{l,-}, r_j^{l,+} < 0,\ l = 1, 2$, and $r_j^{l,-} + r_j^{l,c} + r_j^{l,+} \geq c > 0,\ l = 1, 2$.

Once checked the ε-uniform stability of the HODIE scheme, we must study the local error associated to it. Firstly, we consider the local error corresponding to the problems in the x_1 direction; to simplify the expressions we omit the dependence of $x_2 \in I_{2,\varepsilon}^N$. We know that the local error is defined by

$$\tau^1_{\widehat{u}^{n+\frac{1}{2}}}(x_{1,j}) = \left(I + \frac{\Delta t}{2} L_{1,\varepsilon}^N\right)\left[\widehat{u}_{x_{1,j}}^{n+\frac{1}{2}} - \widehat{U}_{x_{1,j}}^{n+\frac{1}{2}}\right].$$

From (10) and (22) we find that

$$\begin{aligned}\widehat{F}_{1,u^n}(x_{1,j}, x_2, t_n, \Delta t) &= \widehat{f}_{1,u^n}(x_{1,j}, x_2, t_n, \Delta t)\\ &= \left(I + \frac{\Delta t}{2} L_{1,\varepsilon}\right)\widehat{u}^{\,n+\frac{1}{2}}(x_{1,j}, x_2)\\ &\equiv \left(I + \frac{\Delta t}{2} L_{1,\varepsilon}\right)\widehat{u}_{x_{1,j}}^{n+\frac{1}{2}}.\end{aligned}$$

Then, from here, using (20) and taking Taylor expansions, it is straightforward to see that the local error satisfies

$$\tau^1_{\widehat{u}^{n+\frac{1}{2}}}(x_{1,j}) = \frac{\Delta t}{2}\,\bar{\tau}^1_{\widehat{u}^{n+\frac{1}{2}}}(x_{1,j}),$$

where $\bar{\tau}^1_{\widehat{u}^{n+\frac{1}{2}}}$ is the local error associated to the HODIE scheme defined in [2], corresponding to the discretization of a steady 1D singularly perturbed problem of reaction diffusion type, where now the reaction term is $(b_1 + 2/\Delta t)$. From [2] we know that

$$\begin{aligned}\bar{\tau}^1_{\widehat{u}^{\,n+\frac{1}{2}}}(x_{1,j}) = &\, \varepsilon \frac{\partial^3 \widehat{u}^{\,n+\frac{1}{2}}}{\partial x_1^3}(x_{1,j}) \left(\frac{h_{1,j} - h_{1,j+1}}{3} - q_j^{1,1} h_{1,j} + q_j^{1,3} h_{1,j+1} \right) \\ &+ \varepsilon \frac{\partial^4 \widehat{u}^{\,n+\frac{1}{2}}}{\partial x_1^4}(x_{1,j}) \left(-\frac{h_{1,j}^3 + h_{1,j+1}^3}{24(h_{1,j} + h_{1,j+1})} + q_j^{1,1} \frac{h_{1,j}^2}{2} + q_j^{1,3} \frac{h_{1,j+1}^2}{2} \right) \\ &- \frac{2\varepsilon R_5(x_{1,j}, x_{1,j-1}, \widehat{u}^{\,n+\frac{1}{2}})}{h_{1,j}(h_{1,j} + h_{1,j+1})} - \frac{2\varepsilon R_5(x_{1,j}, x_{1,j+1}, \widehat{u}^{\,n+\frac{1}{2}})}{h_{1,j+1}(h_{1,j} + h_{1,j+1})} \\ &+ q_j^{1,1} \varepsilon R_3 \left(x_{1,j}, x_{1,j-1}, \frac{\partial^2 \widehat{u}^{\,n+\frac{1}{2}}}{\partial x_1^2} \right) + q_j^{1,3} \varepsilon R_3 \left(x_{1,j}, x_{1,j+1}, \frac{\partial^2 \widehat{u}^{\,n+\frac{1}{2}}}{\partial x_1^2} \right),\end{aligned}$$

where $R_n(a, p, g)$ is the Taylor remainder.

Lemma 4 *Let $\widehat{u}^{\,n+\frac{1}{2}}$ be the exact solution of (10), $\widehat{U}^{\,n+\frac{1}{2}}$ the solution of the finite difference HODIE scheme (20) and let us choose N_0 satisfying (29). For any $N \geq N_0$, the global error satisfies*

$$\left| \widehat{u}^{\,n+\frac{1}{2}}_{x_{1,j}} - \widehat{U}^{\,n+\frac{1}{2}}_{x_{1,j}} \right| \leq C \Delta t \left(\max \left\{ N^{-4} \sigma_{1,0}^4 \ln^4 N, N^{-3} \right\} + N^{-\sqrt{\beta_1}\sigma_{1,0}} \right). \tag{30}$$

Proof It is a straightforward consequence from the bounds of the local error associated to the steady 1D singularly perturbed problem (see [2]) and the uniform stability proved in Lemma 3.

In order to find appropriate bounds for the global error $|\widehat{u}^{\,n+1}_{x_{2,j}} - \widehat{U}^{\,n+1}_{x_{2,j}}|$, we need the following result (now we omit the dependence of $x_1 \in I^N_{1,\varepsilon}$).

Lemma 5 *Let $\widehat{u}^{\,n+\frac{1}{2}}$ be the exact solution of (10), $\widehat{U}^{\,n+\frac{1}{2}}$ the solution of the finite difference HODIE scheme (20) and let us choose N_0 satisfying (29). For any $N \geq N_0$, it holds*

$$\left| L_{1,\varepsilon} \widehat{u}^{\,n+\frac{1}{2}}_{x_{1,j}} - \overline{L}^N_{1,\varepsilon} \widehat{U}^{\,n+\frac{1}{2}}_{x_{1,j}} \right| \leq C \left(\max \left\{ N^{-4} \sigma_{1,0}^4 \ln^4 N, N^{-3} \right\} + N^{-\sqrt{\beta_1}\sigma_{1,0}} \right).$$

Proof Using Lemma 4 and the expressions (10), (24) and (25) the result follows.

Lemma 6 *Let $\widehat{u}^{\,n+1}$ be the exact solution of (11), $\widehat{U}^{\,n+1}$ the solution of the finite difference HODIE scheme (21) and let us chose N_0 satisfying (29). If $N \geq N_0$, then the global error satisfies*

$$\begin{aligned}\left| \widehat{u}^{\,n+1}_{x_{2,j}} - \widehat{U}^{\,n+1}_{x_{2,j}} \right| \leq C \Delta t \big(&\max \left\{ N^{-4} (\sigma_{1,0}^4 + \sigma_{2,0}^4) \ln^4 N, N^{-3} \right\} \\ &+ N^{-\sqrt{\beta_1}\sigma_{1,0}} + N^{-\sqrt{\beta_2}\sigma_{2,0}} \big).\end{aligned} \tag{31}$$

Proof It is similar to the proof of the previous lemma except in the study of the local error. Now, the local error is defined by

$$\tau^2_{\widehat{u}\ n+1}(x_{2,j}) = \left(I + \frac{\Delta t}{2} L^N_{2,\varepsilon}\right)\left[\widehat{u}^{\ n+1}_{x_{2,j}} - \widehat{U}^{\ n+1}_{x_{2,j}}\right].$$

From (11) and (23) we find that

$$\begin{aligned}
&\widehat{F}_{2,\widehat{U}^{n+1/2}}(x_1, x_{2,j}, t_{n+1/2}, \Delta t)\\
&= \widehat{U}^{n+1/2}_{x_1,x_{2,j}} - \widehat{u}^{n+1/2}_{x_1,x_{2,j}}\\
&\quad + \frac{\Delta t}{2}\left(L_{1,\varepsilon}\widehat{u}^{n+1/2}_{x_1,x_{2,j}} - \overline{L}_{1,\varepsilon}\widehat{U}^{n+1/2}_{x_1,x_{2,j}}\right)\\
&\quad + \left(I - \frac{\Delta t}{2} L_{1,\varepsilon}\right)\widehat{u}^{n+1/2}_{x_1,x_{2,j}} + \frac{\Delta t}{2}\left(f_1(x_1, x_{2,j}, t_{n+1/2}) + f_2(x_1, x_{2,j}, t_{n+1})\right)\\
&\quad \equiv K^{n+1/2}_{x_{2,j}} + \widehat{f}_{2,\widehat{u}^{n+1/2}}\left(x_1, x_{2,j}, t_{n+1/2}, \Delta t\right)\\
&\quad = K^{n+1/2}_{x_{2,j}} + \left(I + \frac{\Delta t}{2} L_{2,\varepsilon}\right)\widehat{u}^{\ n+1}(x_1, x_{2,j})\\
&\quad \equiv K^{n+1/2}_{x_{2,j}} + \left(I + \frac{\Delta t}{2} L_{2,\varepsilon}\right)\widehat{u}^{\ n+1}_{x_{2,j}}.
\end{aligned}$$

Then, from here, using (21) and taking Taylor expansion, we obtain that

$$\begin{aligned}
\tau^2_{\widehat{u}\ n+1}(x_{2,j}) = &\frac{\Delta t}{2}\bar{\tau}^2_{\widehat{u}\ n+1}(x_{2,j})\\
&- \sum_{l=1}^{3} q^{2,l}_j \left[\left(\widehat{U}^{\ n+\frac{1}{2}}_{x_1,x_{2,j-2+l}} - \widehat{u}^{\ n+\frac{1}{2}}_{x_1,x_{2,j-2+l}}\right) + \frac{\Delta t}{2}\left(L_{1,\varepsilon}\widehat{u}^{\ n+\frac{1}{2}}_{x_1,x_{2,j-2+l}}\right.\right.\\
&\left.\left. - \overline{L}^N_{1,\varepsilon}\widehat{U}^{\ n+\frac{1}{2}}_{x_1,x_{2,j-2+l}}\right)\right].
\end{aligned}$$

The first term, $\bar{\tau}^2_{\widehat{u}\ n+1}(x_{2,j})$, is analyzed exactly as in previous lemma, and for the other ones it is sufficient to use Lemmas 4 and 5.

Now jointing the two discretization processes, we write the numerical algorithm that we propose here to compute the approximated solutions of (1) and, afterwards, we prove that it is ε-uniformly convergent of second order in time and of third order in space. Concretely, the numerical approaches $U^n_{x_{1,i},x_{2,j}}$ of $u(x_{1,i}, x_{2,j}, n\Delta t)$, for $i, j = 1, \ldots, N$ and $n = 1, \ldots, T/\Delta t$, are obtained by the following scheme:

$$\begin{cases} U^0_{x_{1,i},x_{2,j}} = \varphi_0(x_{1,i}, x_{2,j}), \quad 0 \le i, \ \ j \le N, \\ \bar{L}^N_{2,\varepsilon} U^0_{x_{1,i},x_{2,j}} = L_{2,\varepsilon}\varphi_0(x_{1,i}, x_{2,j}), \quad 0 \le i, \ \ j \le N, \\ \begin{cases} \left(I + \frac{\Delta t}{2} L^N_{1,\varepsilon}\right) U^{n+\frac{1}{2}}_{x_{1,i},x_2} \equiv r_i^{1,-} U^{n+\frac{1}{2}}_{x_{1,i-1},x_2} + r_i^{1,c} U^{n+\frac{1}{2}}_{x_{1,i},x_2} + r_i^{1,+} U^{n+\frac{1}{2}}_{x_{1,i+1},x_2} \\ \quad = q_i^{1,1} \widehat{F}_{1,U^n}(x_{1,i-1}, x_2, t_n, \Delta t) + q_i^{1,2} \widehat{F}_{1,U^n}(x_{1,i}, x_2, t_n, \Delta t) \\ \qquad + q_i^{1,3} \widehat{F}_{1,U^n}(x_{1,i+1}, x_2, t_n, \Delta t), \quad i = 1, \ldots, N-1, \\ \widehat{U}^{n+\frac{1}{2}}_{0,x_2} = \widehat{U}^{n+\frac{1}{2}}_{N,x_2} = 0, \quad x_2 \in I^N_{2,\varepsilon}, \end{cases} \\ \bar{L}^N_{1,\varepsilon} U^{n+\frac{1}{2}}_{x_1,x_2} = -\bar{L}^N_{2,\varepsilon} U^n_{x_1,x_2} - 2\dfrac{U^{n+\frac{1}{2}}_{x_1,x_2} - U^n_{x_1,x_2}}{\Delta t} + f_1\big(x_1, x_2, t_{n+\frac{1}{2}}\big) \\ \qquad\qquad + f_2(x_1, x_2, t_n), \quad (x_1, x_2) \in \Omega^N_\varepsilon, \\ \begin{cases} \left(I + \frac{\Delta t}{2} L^N_{2,\varepsilon}\right) U^{n+1}_{x_1,x_{2,j}} \equiv r_j^{2,-} U^{n+1}_{x_1,x_{2,j-1}} + r_j^{2,c} U^{n+1}_{x_1,x_{2,j}} + r_j^{2,+} U^{n+1}_{x_1,x_{2,j+1}} \\ \quad = q_j^{2,1} \widehat{F}_{2,\widehat{U}^{n+\frac{1}{2}}}\big(x_1, x_{2,j-1}, t_{n+\frac{1}{2}}, \Delta t\big) + q_j^{2,2} \widehat{F}_{2,\widehat{U}^{n+\frac{1}{2}}}\big(x_1, x_{2,j}, t_{n+\frac{1}{2}}, \Delta t\big) \\ \quad + q_j^{2,3} \widehat{F}_{2,\widehat{U}^{n+\frac{1}{2}}}\big(x_1, x_{2,j+1}, t_{n+\frac{1}{2}}, \Delta t\big), \quad j = 1, \ldots, N-1, \\ \widehat{U}^{n+1}_{x_1,0} = \widehat{U}^{n+1}_{x_1,N} = 0, \quad x_1 \in I^N_{1,\varepsilon}, \end{cases} \\ \bar{L}^N_{2,\varepsilon} U^{n+1}_{x_1,x_2} = -\bar{L}^N_{1,\varepsilon} U^{n+\frac{1}{2}}_{x_1,x_2} - 2\dfrac{U^{n+1}_{x_1,x_2} - U^{n+\frac{1}{2}}_{x_1,x_2}}{\Delta t} + f_1(x_1, x_2, t_{n+\frac{1}{2}}) \\ \qquad\qquad + f_2\big(x_1, x_2, t_{n+1}\big), \quad (x_1, x_2) \in \Omega^N_\varepsilon, \\ n = 0, 1, \ldots, \frac{T}{\Delta T}, \end{cases} \tag{32}$$

where $r_j^{l,\bullet}$ and $q_j^{l,m}$ are defined in (26) and (27).

Theorem 1 *Let u be the exact solution of (1), $u(t_n)|_{\overline{\Omega}^N_\varepsilon}$ the restriction of the exact solution to the mesh $\overline{\Omega}^N_\varepsilon$ and U^n the numerical solution obtained with the method (32) on the same mesh, at time level $t_n = n\Delta t$. Under the hypotheses of Lemma 3, there exists a positive constant C, independent of ε, N and Δt, such that the global errors $E^N_n = u(t_n)|_{\overline{\Omega}^N_\varepsilon} - U^n$ satisfy*

$$\begin{aligned} \|E^N_n\|_\infty \le C\big((\Delta t)^2 + \max\left\{N^{-4}\big(\sigma^4_{1,0} + \sigma^4_{2,0}\big)\ln^4 N, N^{-3}\right\} \\ + N^{-\sqrt{\beta_1}\sigma_{1,0}} + N^{-\sqrt{\beta_2}\sigma_{2,0}}\big). \end{aligned} \tag{33}$$

Proof Let $E^N_n = [u(x_{1,i}, x_{2,j}, t_n) - U^n_{i,j}]_{i,j}$ be the vector of global errors at time level t_n, $e^N_n = [u(x_{1,i}, x_{2,j}, t_n) - \widehat{u}^n(x_{1,i}, x_{2,j})]_{i,j}$ the restrictions to the mesh of the local errors considered in the time semidiscretization process and $d^N_n = [\widehat{u}^n(x_{1,i}, x_{2,j}) - \widehat{U}^n(x_{1,i}, x_{2,j})]_{i,j}$. Then,

$$E^N_n = e^N_n + d^N_n + R_N E^N_{n-1},$$

where R_N is a linear operator, called the transition operator associated to the totally discrete method (32), defined in the following way: $R_N V$ is the application of one step of scheme (32) taking $U^n = V$ and the source terms f_1, f_2 equal to zero. From this recurrence relation, it is immediately deduced that

$$E_n^N = \sum_{i=1}^{n} R_N^{n-i} e_i^N + \sum_{i=1}^{n} R_N^{n-i} d_i^N,$$

and taking into account that the powers of the transition operators of the totally discrete scheme R_N^j preserve the uniform boundedness behaviour observed for R^j, it holds that

$$\begin{aligned} \|E_n^N\|_\infty &\le \sum_{i=1}^{n} \|R_N^{n-i}\|_\infty \left(\|e_i^N\|_\infty + \|d_i^N\|_\infty\right) \\ &\le C \sum_{i=1}^{n} \Big((\Delta t)^3 + \Delta t \big(\max\left\{N^{-4}(\sigma_{1,0}^4 + \sigma_{2,0}^4)\ln^4 N, N^{-3}\right\} + N^{-\sqrt{\beta_1}\sigma_{1,0}} \\ &\quad + N^{-\sqrt{\beta_2}\sigma_{2,0}}\big)\Big) \le C \Big((\Delta t)^2 + \big(\max\left\{N^{-4}(\sigma_{1,0}^4 + \sigma_{2,0}^4)\ln^4 N, N^{-3}\right\} \\ &\quad + N^{-\sqrt{\beta_1}\sigma_{1,0}} + N^{-\sqrt{\beta_2}\sigma_{2,0}}\big)\Big). \end{aligned}$$

Remark 1 From the previous theorem it is immediate to deduce the second order of uniform convergence in time variable. Also, if we take the constants $\sigma_{l,0}, l = 1, 2$, such that $\sigma_{1,0} \ge 3/\sqrt{\beta_1}$ and $\sigma_{2,0} \ge 3/\sqrt{\beta_2}$, then we obtain third order of uniform convergence in spatial variables.

5 Numerical experiments

In this section we show the results given by the method (32) to solve some test problems. In all examples, according to Remark 1, we take the values $\sigma_{1,0} = 3/\sqrt{\min_x b_1(\mathbf{x})}$ and $\sigma_{2,0} = 3/\sqrt{\min_x b_2(\mathbf{x})}$.

The first example that we consider is given by

$$\begin{cases} u_t - \varepsilon \Delta u + \big(5 + x_1(1 - x_1) + x_2(1 - x_2)\big)u = f, & (\mathbf{x}, t) \in \Omega \times (0, 1], \\ u(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, \ t \in (0, 1], \\ u(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \overline{\Omega}, \end{cases} \tag{34}$$

with $\Omega = (0, 1)^2$, the source term is $f(\mathbf{x}, t, \varepsilon) = ((t + 1)\exp(-t) - 1)(h(x_1, 1) - 1)(h(x_2, 4) - 1)$ and

$$h(z, \alpha) = \frac{\exp(-\alpha z/\sqrt{\varepsilon}) + \exp(-\alpha(1 - z)/\sqrt{\varepsilon})}{1 + \exp(-\alpha/\sqrt{\varepsilon})}. \tag{35}$$

We decompose the reaction term in the form $b_1 = 2 + (x_1(1 - x_1) + x_2(1 - x_2))/2$, $b_2 = 3 + (x_1(1 - x_1) + x_2(1 - x_2))/2$. Note that, according to the theoretical results, the decomposition $b_1 = 2 + x_1(1 - x_1)$, $b_2 = 3 + x_2(1 - x_2)$

(commuting operators) seems more appropriate because it fulfils the requirements to prove the unconditional stability of the method. Nevertheless, we have observed that these decompositions hardly affect to the behaviour of our method. As the exact solution is unknown, we estimate the numerical errors at times $t_n = n\Delta t$ for each point $(x_{1,i}, x_{2,j})$ of the spatial mesh by

$$E_{\varepsilon,N,\Delta t}(i,j,n) = \left| U_{i,j,n}^{\varepsilon,N,\Delta t} - U_{i,j,n}^{\varepsilon,2N,\frac{\Delta t}{2}} \right|,$$

where $U_{i,j,n}^{\varepsilon,N,\Delta t}$ is the numerical solution obtained using a constant time step Δt and $(N+1)$ points in both spatial directions and $U_{i,j,n}^{\varepsilon,2N,\frac{\Delta t}{2}}$ is computed using $\frac{\Delta t}{2}$ as constant time step and $(2N+1)$ points in both spatial directions, but preserving the same transition points of the meshes used to compute $U^{\varepsilon,N,\Delta t}$. For each fixed value of ε, the maximum global errors are estimated by

$$E_{\varepsilon,N,\Delta t} = \max_{i,j,n} E_{\varepsilon,N,\Delta t}(i,j,n),$$

and, in standard way, we compute the corresponding numerical orders of convergence by

$$p = \frac{\log\left(E_{\varepsilon,N,\Delta t}/E_{\varepsilon,2N,\Delta t/2^{3/2}}\right)}{\log 2}.$$

From these values we obtain the ε-uniform errors and the ε-uniform orders of convergence by

$$E_{N,\Delta t} = \max_{\varepsilon} E_{\varepsilon,N,\Delta t}, \quad p_{uni} = \frac{\log\left(E_{N,\Delta t}/E_{2N,\Delta t/2^{3/2}}\right)}{\log 2}. \tag{36}$$

Table 1 displays the maximum errors and their corresponding numerical orders of convergence; note that we reduce in a special way the size of Δt in order to the reduction of the errors associated to the time and the spatial discretizations be similar. Clearly these results are in agreement with the theoretical result proved in Theorem 1.

The second test problem that we consider is given by

$$\begin{cases} u_t - \varepsilon\Delta u + (3 + 2\exp(x_1^2 x_2^2))u = f, & (\mathbf{x},t) \in \Omega \times (0,1], \\ u(\mathbf{x},t) = g(\mathbf{x},t), \quad \mathbf{x} \in \partial\Omega, \ t \in (0,1], \\ u(\mathbf{x},0) = 0, \quad \mathbf{x} \in \overline{\Omega}, \end{cases} \tag{37}$$

where $\Omega = (0,1)^2$, f and g are such that the exact solution is

$$u = \big((t+1)\exp(-t) - 1\big)h(x_1,1)h(x_2,2),$$

with $h(z,\alpha)$ defined in (35), for which the boundary conditions are not homogenous and also time dependent. This example belongs to a more general version of the original

Table 1 Maximum errors and numerical orders for problem (34)

N , Δt	$N = 32$ $\Delta t = 0.5$	$N = 64$ Δt=0.5/$2^{3/2}$	$N = 128$ $\Delta t = 0.5/2^3$	$N = 256$ $\Delta t = 0.5/2^{9/2}$	$N = 512$ $\Delta t = 0.5/2^6$
$\varepsilon = 2^{-6}$	4.1596E-3	5.1776E-4	6.4851E-5	8.1090E-6	1.0136E-6
	3.0061	2.9971	2.9995	3.0000	
$\varepsilon = 2^{-8}$	4.3011E-3	5.3440E-4	6.6938E-5	8.3695E-6	1.0461E-6
	3.0087	2.9970	2.9996	3.0001	
$\varepsilon = 2^{-10}$	4.4613E-3	5.5983E-4	6.8478E-5	8.3946E-6	1.0475E-6
	2.9944	3.0313	3.0281	3.0025	
$\varepsilon = 2^{-12}$	4.4590E-3	5.5965E-4	7.0497E-5	8.5911E-6	1.0577E-6
	2.9941	2.9889	3.0367	3.0220	
$\varepsilon = 2^{-14}$	4.4577E-3	5.5956E-4	7.0485E-5	8.5897E-6	1.0575E-6
	2.9940	2.9889	3.0366	3.0219	
$\varepsilon = 2^{-16}$	4.4572E-3	5.5951E-4	7.0479E-5	8.5890E-6	1.0575E-6
	2.9939	2.9889	3.0366	3.0219	
$\varepsilon = 2^{-18}$	4.4568E-3	5.5948E-4	7.0475E-5	8.5885E-6	1.0574E-6
	2.9939	2.9889	3.0366	3.0219	
$\varepsilon = 2^{-20}$	4.4567E-3	5.5948E-4	7.0474E-5	8.5884E-6	1.0574E-6
	2.9938	2.9889	3.0366	3.0219	
$\varepsilon = 2^{-22}$	4.4566E-3	5.5947E-4	7.0473E-5	8.5883E-6	1.0574E-6
	2.9938	2.9889	3.0366	3.0219	
$\varepsilon = 2^{-24}$	4.4566E-3	5.5947E-4	7.0473E-5	8.5883E-6	1.0574E-6
	2.9938	2.9889	3.0366	3.0219	
$\varepsilon = 2^{-26}$	4.4566E-3	5.5947E-4	7.0473E-5	8.5883E-6	1.0574E-6
	2.9938	2.9889	3.0366	3.0219	
$\mathrm{E}_{N,\Delta t}$	4.4613E-3	5.5983E-4	7.0497E-5	8.5911E-6	1.0577E-6
p_{uni}	2.9944	2.9894	3.0367	3.0220	

problem (1), which can be written in the form

$$\begin{cases} u_t + L_\varepsilon u = f(\mathbf{x}, t), & (\mathbf{x}, t) \in \Omega \times (0, T], \\ u(\mathbf{x}, 0) = \varphi_0(\mathbf{x}), & \mathbf{x} \in \overline{\Omega}, \\ u(\mathbf{x}, t) = g(\mathbf{x}, t), & (\mathbf{x}, t) \in \partial\Omega \times (0, T]. \end{cases} \tag{38}$$

It is well-known that the classical application of one step methods (for example Runge-Kutta methods) to semidiscretize in time problem (38), in general, causes a reduction in the order of consistency when the boundary conditions are time dependent. Such reduction can be related to the order of consistency that the internal stages possess, if they are considered as approximations of the exact solution at suitable intermediate times. This phenomenon also happens if we apply the semidiscretization scheme (6) and (7) to problem (38), obtaining only first order if we consider the corresponding classical boundary conditions:

$$\begin{cases} u^{\,n+\frac{1}{2}}(0,x_2) = g\big(0,x_2,t_{n+\frac{1}{2}}\big), \\ u^{\,n+\frac{1}{2}}(1,x_2) = g\big(1,x_2,t_{n+\frac{1}{2}}\big), \end{cases} \tag{39}$$

$$\begin{cases} u^{\,n+1}(x_1,0) = g\big(x_1,0,t_{n+1}\big), \\ u^{\,n+1}(x_1,1) = g\big(x_1,1,t_{n+1}\big). \end{cases} \tag{40}$$

Fortunately, this order reduction can be avoided if we modify the boundary conditions for $u^{\,n+\frac{1}{2}}$ and $u^{\,n+1}$ using the technique proposed in [1] for general fractional step methods. Such technique ensures that if we choose the following boundary conditions:

$$\begin{cases} u^{\,n+\frac{1}{2}}(0,x_2) = g(0,x_2,t_n) + \dfrac{\Delta t}{2}\left[\left(\dfrac{\partial g}{\partial t} + L_{2,\varepsilon}\, g\right)\big(0,x_2,t_{n+\frac{1}{2}}\big) \right. \\ \qquad\qquad \left. - L_{2,\varepsilon} g(0,x_2,t_n) - f_2\big(0,x_2,t_{n+\frac{1}{2}}\big) + f_2(0,x_2,t_n)\right], \\ u^{\,n+\frac{1}{2}}(1,x_2) = g(1,x_2,t_n) + \dfrac{\Delta t}{2}\left[\left(\dfrac{\partial g}{\partial t} + L_{2,\varepsilon}\, g\right)\big(1,x_2,t_{n+\frac{1}{2}}\big) \right. \\ \qquad\qquad \left. - L_{2,\varepsilon} g(1,x_2,t_n) - f_2\big(1,x_2,t_{n+\frac{1}{2}}\big) + f_2(1,x_2,t_n)\right], \end{cases} \tag{41}$$

$$\begin{cases} u^{\,n+1}(x_1,0) = g(x_1,0,t_n) + \dfrac{\Delta t}{2}\left[\dfrac{\partial g}{\partial t}(x_1,0,t_n) + \dfrac{\partial g}{\partial t}\big(x_1,0,t_{n+1}\big) \right. \\ \qquad\qquad + L_{1,\varepsilon}\Big(g(x_1,0,t_n) - 2g\big(x_1,0,t_{n+\frac{1}{2}}\big) + g\big(x_1,0,t_{n+1}\big)\Big) \\ \qquad\qquad \left. - \Big(f_1(x_1,0,t_n) - 2f_1\big(x_1,0,t_{n+\frac{1}{2}}\big) + f_1\big(x_1,0,t_{n+1}\big)\Big)\right], \\ u^{\,n+1}(x_1,1) = g(x_1,1,t_n) + \dfrac{\Delta t}{2}\left[\dfrac{\partial g}{\partial t}(x_1,1,t_n) + \dfrac{\partial g}{\partial t}\big(x_1,1,t_{n+1}\big) \right. \\ \qquad\qquad + L_{1,\varepsilon}\Big(g(x_1,1,t_n) - 2g\big(x_1,1,t_{n+\frac{1}{2}}\big) + g\big(x_1,1,t_{n+1}\big)\Big) \\ \qquad\qquad \left. - \Big(f_1(x_1,1,t_n) - 2f_1\big(x_1,1,t_{n+\frac{1}{2}}\big) + f_1\big(x_1,1,t_{n+1}\big)\Big)\right], \end{cases} \tag{42}$$

we recover the second order of consistency of the Peaceman and Rachford method. Moreover, without loosing the second order of consistency, in this particular case we can shorten the boundary conditions proposed in (42) for $u^{\,n+1}$ to the following ones

$$\begin{cases} u^{\,n+1}(x_1,0) = g(x_1,0,t_n) + \dfrac{\Delta t}{2}\left(\dfrac{\partial g}{\partial t}(x_1,0,t_n) + \dfrac{\partial g}{\partial t}(x_1,0,t_{n+1})\right), \\ u^{\,n+1}(x_1,1) = g(x_1,1,t_n) + \dfrac{\Delta t}{2}\left(\dfrac{\partial g}{\partial t}(x_1,1,t_n) + \dfrac{\partial g}{\partial t}(x_1,1,t_{n+1})\right), \end{cases} \tag{43}$$

since the removed terms are clearly $O((\Delta t)^3)$.

Now we decompose the reaction term in the form $b_1 = \exp(x_1^2\, x_2^2)$, $b_2 = 3 + \exp(x_1^2\, x_2^2)$. As the exact solution is known, we calculate exactly the numerical errors,

Table 2 Maximum errors and numerical orders for problem (37): classical boundary conditions

N , Δt	$N = 32$ $\Delta t = 0.5$	$N = 64$ $\Delta t = 0.5/2^{3/2}$	$N = 128$ $\Delta t = 0.5/2^3$	$N = 256$ $\Delta t = 0.5/2^{9/2}$	$N = 512$ $\Delta t = 0.5/2^6$
$\varepsilon = 2^{-6}$	3.5163E-2	7.3599E-3	1.2047E-3	1.7277E-4	2.3350 E-5
	2.2563	2.6110	2.8017	2.8874	
$\varepsilon = 2^{-8}$	2.1669E-2	5.0148E-3	8.5993E-4	1.2911E-4	1.8222 E-5
	2.1114	2.5439	2.7356	2.8249	
$\varepsilon = 2^{-10}$	8.8377E-3	2.6909E-3	5.8816E-4	1.0198E-4	1.53035-5
	1.7156	2.1938	2.5279	2.7364	
$\varepsilon = 2^{-12}$	5.3595E-3	1.2459E-3	2.5641E-4	5.1329E-5	1.0086E-6
	2.1048	2.2807	2.3206	2.3475	
$\varepsilon = 2^{-14}$	5.7532E-3	1.3194E-3	2.662E-4	4.8639E-5	8.1758E-6
	2.1246	2.3092	2.4524	2.5727	
$\varepsilon = 2^{-16}$	5.9615E-3	1.3599E-3	2.7163E-4	4.9354E-5	8.2694E-6
	2.1321	2.3238	2.4604	2.5773	
$\varepsilon = 2^{-18}$	6.0780E-3	1.3820E-3	2.7469E-4	4.9749E-5	8.3218E-6
	2.1368	2.3309	2.4651	2.5797	
$\varepsilon = 2^{-20}$	6.1373E-3	1.3938E-3	2.7637E-4	4.9967E-5	8.3507E-6
	2.1385	2.3344	2.4675	2.5810	
$\varepsilon = 2^{-22}$	6.1698E-3	1.4002E-3	2.7727E-4	5.0085E-5	8.3664E-6
	2.1396	2.3362	2.4688	2.5817	
$\varepsilon = 2^{-24}$	6.1863E-3	1.4035E-3	2.7775E-4	5.0149E-5	8.3749E-6
	2.1400	2.3372	2.4695	2.5821	
$\varepsilon = 2^{-26}$	6.1951E-3	1.4053E-3	2.7800E-4	5.0182E-5	8.3795E-6
	2.1402	2.3376	2.4698	2.5822	
$\mathrm{E}_{N,\Delta t}$	3.5163E-2	7.3599E-3	1.2047E-3	1.7277E-4	2.3350 E-5
p_{uni}	2.2563	2.6110	2.8017	2.8874	

at the times $t_n = n\Delta t$ for the mesh points $(x_{1,i}, x_{2,j})$, by

$$E_{\varepsilon,N,\Delta t}(i, j, n) = \left| u_\varepsilon(x_{1,i}, x_{2,j}, t_n) - U^{\varepsilon,N,\Delta t}_{i,j,n} \right|. \tag{44}$$

Table 2 displays the global errors obtained by using the evaluations of the classical boundary conditions (39) at the mesh points of $\partial\Omega$ and Table 3 displays the global errors computed with the evaluations of the improved boundary conditions (41) and (42) at the same points of $\partial\Omega$. In both tables we have also displayed the corresponding orders of convergence.

From Tables 2 and 3 we see that the results obtained by using the improved boundary conditions are better, because the maximum errors are smaller and also the orders of convergence are higher. Therefore, we can conclude that this technique is useful also in the discretizations of the singularly perturbed problems considered in this paper.

Table 3 Maximum errors and numerical orders for problem (37): improved boundary conditions

N, Δt	$N = 32$ $\Delta t = 0.5$	$N = 64$ $\Delta t = 0.5/2^{3/2}$	$N = 128$ $\Delta t = 0.5/2^3$	$N = 256$ $\Delta t = 0.5/2^{9/2}$	$N = 512$ $\Delta t = 0.5/2^6$
$\varepsilon = 2^{-6}$	8.9285E-3	7.8891E-4	5.3346E-5	6.1151E-6	7.4261E-7
	3.5005	3.8864	3.1249	3.0417	
$\varepsilon = 2^{-8}$	6.5266E-3	6.9612E-4	6.7693E-5	8.2635E-6	1.0237E-6
	3.2289	3.3622	3.0342	3.0130	
$\varepsilon = 2^{-10}$	5.0289E-3	7.6147E-4	8.8084E-5	1.0827E-5	1.3450E-6
	2.7234	3.1118	3.0243	3.0089	
$\varepsilon = 2^{-12}$	4.9533E-3	7.9077E-4	1.0029E-4	1.2493E-5	1.5575E-6
	2.6471	2.9791	3.0050	3.0038	
$\varepsilon = 2^{-14}$	5.3878E-3	8.5944E-4	1.0833E-4	1.3570E-5	1.6908E-6
	2.6482	2.9880	2.9970	3.0046	
$\varepsilon = 2^{-16}$	5.6399E-3	8.9839E-4	1.1307E-4	1.4181E-5	1.7686E-6
	2.6502	2.9902	2.9951	3.0033	
$\varepsilon = 2^{-18}$	5.7720E-3	9.2008E-4	1.1577E-4	1.4525E-5	1.8122E-6
	2.6492	2.9905	2.9947	3.0027	
$\varepsilon = 2^{-20}$	5.8445E-3	9.3184E-4	1.1726E-4	1.4713E-5	1.8361E-6
	2.6489	2.9904	2.9945	3.0024	
$\varepsilon = 2^{-22}$	5.8827E-3	9.3815E-4	1.1806E-4	1.4815E-5	1.8491E-6
	2.6486	2.9903	2.9944	3.0022	
$\varepsilon = 2^{-24}$	5.9030E-3	9.4153E-4	1.1849E-4	1.4870E-5	1.8561E-6
	2.6484	2.9901	2.9944	3.0021	
$\varepsilon = 2^{-26}$	5.9137E-3	9.4332E-4	1.1873E-4	1.4899E-5	1.8597E-6
	2.6483	2.9901	2.9944	3.0021	
$E_{N,\Delta t}$	8.9285E-3	9.4332E-4	1.1873E-4	1.4899E-5	1.8597E-6
p_{uni}	3.2426	2.9901	2.9944	3.0021	

The last test problem that we consider has a solution with a really small variation in time; the reason to take this example is that we wish that the error associated to the spatial discretization stage dominates in the global error. In previous examples it is observed that the contribution to the global error of the time semidiscretization dominates to the error in space for many values of ε, N and Δt. The problem is given by

$$\begin{cases} u_t - \varepsilon \Delta u + (20 + 0.2 \sin(\pi x_1) \sin(\pi x_2))u = f, & (\mathbf{x}, t) \in \Omega \times (0, 1], \\ u(\mathbf{x}, t) = g(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega,\ t \in (0, 1], \\ u(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \overline{\Omega}, \end{cases} \tag{45}$$

where $\Omega = (0, 1)^2$, $b_1 = b_2 = 10 + 0.1 \sin(\pi x_1) \sin(\pi x_2)$ and f and g are such that the exact solution is

$$u = (1 + \exp(-t/100)) \exp(-\sqrt{10}x_1/\sqrt{\varepsilon}) \exp(-\sqrt{10}x_2/\sqrt{\varepsilon}).$$

Table 4 Maximum errors for problem (45) with $\Delta t = 0.4$

ε/N	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
2^{-6}	2.7247E-4	2.2576E-5	2.1998E-6	**2.6134E-6**	**2.7995E-7**
2^{-8}	1.1954E-3	2.5976E-4	2.2565E-5	2.4317E-6	**2.8190E-6**
2^{-10}	1.1955E-3	2.5976E-4	3.8790E-5	3.8247E-6	**2.7571E-6**
2^{-12}	1.1955E-3	2.5976E-4	3.8784E-5	3.8179E-6	**2.8284E-6**
2^{-14}	1.1955E-3	2.5975E-4	3.8781E-5	3.8143E-6	**2.8680E-6**
2^{-16}	1.1955E-3	2.5975E-4	3.8778E-5	3.8125E-6	**2.8897E-6**
2^{-18}	1.1955E-3	2.5975E-4	3.8778E-5	3.8115E-6	**2.9015E-6**
2^{-20}	1.1955E-3	2.5975E-4	3.8778E-5	3.8111E-6	**2.9078E-6**
2^{-22}	1.1955E-3	2.5975E-4	3.8778E-5	3.8109E-6	**2.9111E-6**
2^{-24}	1.1955E-3	2.5975E-4	3.8778E-5	3.8107E-6	**2.9129E-6**
2^{-26}	1.1955E-3	2.5975E-4	3.8778E-5	3.8107E-6	**2.9138E-6**

Table 5 Maximum errors for problem (45) with $\Delta t = 0.2$

ε/N	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
2^{-6}	2.7108E-4	2.1970E-5	1.1311E-6	7.5213E-7	**8.1030E-7**
2^{-8}	1.1748E-3	2.5845E-4	2.1967E-5	1.1225E-6	**8.0183E-7**
2^{-10}	1.1749E-3	2.5846E-4	3.7444E-5	4.0744E-6	7.7919E-7
2^{-12}	1.1749E-3	2.5846E-4	3.7442E-5	4.0719E-6	7.9594E-7
2^{-14}	1.1749E-3	2.5846E-4	3.7441E-5	4.0719E-6	8.0484E-7
2^{-16}	1.1749E-3	2.5846E-4	3.7441E-5	4.0700E-6	8.0947E-7
2^{-18}	1.1749E-3	2.5846E-4	3.7440E-5	4.0696E-6	8.1184E-7
2^{-20}	1.1749E-3	2.5846E-4	3.7440E-5	4.0695E-6	8.1305E-7
2^{-22}	1.1749E-3	2.5846E-4	3.7440E-5	4.0694E-6	8.1366E-7
2^{-24}	1.1749E-3	2.5846E-4	3.7440E-5	4.0694E-6	8.1396E-7
2^{-26}	1.1749E-3	2.5846E-4	3.7440E-5	4.0694E-6	8.1416E-7

Again we know the exact solution and therefore we calculate exactly the maximum global errors by (44).

To illustrate the influence of the spatial discretization in the errors, we display four tables with the maximum errors obtained for different values of Δt. As the boundary conditions are time dependent and, in this test, we wish a small contribution of the time discretization to the global error, we have chosen the same improved boundary condition technique explained in the previous numerical example. We indicate in boldface the errors where the contribution of the time discretization to the global error dominates and then we cannot deduce anything about the order of convergence in space. Excluding these numbers, from Tables 4, 5, 6 and 7 it is easy to observe the uniform convergence of third order in space according to Theorem 1.

Table 6 Maximum errors for problem (45) with $\Delta t = 0.1$

ε/N	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
2^{-6}	2.5376E-4	2.0067E-5	1.1802E-6	2.2560E-7	**2.5690E-7**
2^{-8}	1.0528E-3	2.4215E-4	2.0066E-5	1.1782E-6	2.4916E-7
2^{-10}	1.0528E-3	2.4215E-4	3.3894E-5	3.7880E-6	2.9215E-7
2^{-12}	1.0529E-3	2.4215E-4	3.3894E-5	3.7873E-6	2.9178E-7
2^{-14}	1.0529E-3	2.4215E-4	3.3893E-5	3.7869E-6	2.9158E-7
2^{-16}	1.0529E-3	2.4215E-4	3.3893E-5	3.7867E-6	2.9158E-7
2^{-18}	1.0529E-3	2.4215E-4	3.3893E-5	3.7867E-6	2.9142E-7
2^{-20}	1.0529E-3	2.4215E-4	3.3893E-5	3.7866E-6	2.9139E-7
2^{-22}	1.0529E-3	2.4215E-4	3.3893E-5	3.7865E-6	2.9137E-7
2^{-24}	1.0529E-3	2.4215E-4	3.3893E-5	3.7865E-6	2.9137E-7
2^{-26}	1.0529E-3	2.4215E-4	3.3893E-5	3.7865E-6	2.9137E-7

Table 7 Maximum errors for problem (45) with $\Delta t = 0.05$

ε/N	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
2^{-6}	2.2634E-4	1.6828E-5	1.0599E-6	6.2938E-8	**7.5500E-8**
2^{-8}	1.0373E-3	2.1542E-4	1.6828E-5	1.0598E-6	7.0429E-8
2^{-10}	1.0374E-3	2.1542E-4	2.7971E-5	3.1628E-6	3.0314E-7
2^{-12}	1.0374E-3	2.1542E-4	2.7971E-5	3.1626E-6	3.0308E-7
2^{-14}	1.0374E-3	2.1542E-4	2.7971E-5	3.1625E-6	3.0305E-7
2^{-16}	1.0374E-3	2.1542E-4	2.7971E-5	3.1624E-6	3.0303E-7
2^{-18}	1.0374E-3	2.1542E-4	2.7971E-5	3.1624E-6	3.0303E-7
2^{-20}	1.0374E-3	2.1542E-4	2.7971E-5	3.1624E-6	3.0302E-7
2^{-22}	1.0374E-3	2.1542E-4	2.7971E-5	3.1624E-6	3.0302E-7
2^{-24}	1.0374E-3	2.1542E-4	2.7971E-5	3.1624E-6	3.0302E-7
2^{-26}	1.0374E-3	2.1542E-4	2.7971E-5	3.1624E-6	3.0302E-7

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