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A unifying theory of a posteriori error control for nonconforming finite element methods*

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Abstract Residual-based a posteriori error estimates were derived within one unifying framework for lowest-order conforming, nonconforming, and mixed finite element schemes in Carstensen [Numer Math 100:617–637, 2005]. Therein, the key assumption is that the conforming first-order finite element space V_h^c annulates the linear and bounded residual ℓ written $V_h^c \subseteq \ker \ell$. That excludes particular nonconforming finite element methods (NCFEMs) on parallelograms in that $V_h^c \not\subset \ker \ell$. The present paper generalises the aforementioned theory to more general situations to deduce new a posteriori error estimates, also for mortar and discontinuous Galerkin methods. The key assumption is the existence of some bounded linear operator $\Pi : V_h^c \to V_h^{nc}$ with some elementary properties. It is conjectured that the more general hypothesis (H1)–(H3) can be established for all known NCFEMs. Applications on various nonstandard finite element schemes for the Laplace, Stokes, and Navier–Lamé equations illustrate the presented unifying theory of a posteriori error control for NCFEM.

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1 Unified mixed approach to error control

Suppose that the primal variable $u \in V$ (e.g., the displacement field) is accompanied by a dual variable $p \in L$ (e.g., the flux or stress field). Typically L is some Lebesgue and V is some Sobolev space; suppose throughout this paper that L and V are Hilbert spaces and $X := L \times V$. Given bounded bilinear forms

$$a : L \times L \to \mathbb{R} \quad \text{and} \quad b : L \times V \to \mathbb{R} \tag{1.1}$$

and well established conditions on a and b [13,17], the linear and bounded operator $A : X \to X^*$, defined by

$$(A(p,u))(q,v) := a(p,q) + b(p,v) + b(q,u), \tag{1.2}$$

is bijective. Then, given right-hand sides $f \in L^*$ and $g \in V^*$, there exists some unique $(p,u) \in X$ with

$$a(p,q) + b(q,u) = f(q) \quad \text{for all } q \in L, \tag{1.3}$$

$$b(p,v) = g(v) \quad \text{for all } v \in V. \tag{1.4}$$

Suppose $(p_h, \tilde{u}_h) \in L \times V$ is some approximation to (p,u) and define

$$\mathcal{R}es_L(q) := f(q) - a(p_h,q) - b(q,\tilde{u}_h) \quad \text{for all } q \in L, \tag{1.5}$$

$$\mathcal{R}es_V(v) := g(v) - b(p_h,v) \quad \text{for all } v \in V. \tag{1.6}$$

Here and throughout, $\tilde{u}_h$ is some continuous and *not* necessarily discrete function established as the key ingredient in [20]; however, the subindex in $\tilde{u}_h$ refers to the fact that $\tilde{u}_h$ might be closely related (or designed with some post-processing) to some discrete function u_h and hence that $\tilde{u}_h$ is on our disposal. Since $A : X \to X^*$ is an isomorphism, there holds

$$\|p - p_h\|_L + \|u - \tilde{u}_h\|_V \approx \|\mathcal{R}es_L\|_{L^*} + \|\mathcal{R}es_V\|_{V^*}. \tag{1.7}$$

Here and throughout, an inequality $a \lesssim b$ replaces $a \le C\,b$ with some multiplicative mesh-size independent constant $C > 0$ that depends only on the domain Ω and the shape (e.g., through the aspect ratio) of elements ($C > 0$ is also independent of crucial parameters as the Lamè parameter λ below). Finally, $a \approx b$ abbreviates $a \lesssim b \lesssim a$.

Remark 1.1 Note that (1.3) and (1.4) are a primal mixed formulation with $L := L^2(\Omega)^{m\times n}$ for the Laplace, Stokes, and Navier–Lamé equations under consideration. Throughout this paper, the discrete component p_h is derived from u_h, e.g, $p_h = \nabla_{\mathcal{T}} u_h$ in case $p = \nabla u$ for the Laplace equation; while u_h is solved from the discrete problem in the displacement-oriented formulation (Sects. 4–6 below).

The examples in [20] include conforming, nonconforming and mixed finite element schemes for the Laplace, Stokes, and Navier–Lamé equations. This paper will consider such applications in Sects. 4, 5, and 6 below for with focus on nonconforming finite element methods (NCFEMs) displayed in Tables 1, 2, 3, and 4. For conforming finite element schemes and the setting of a posteriori error control we refer to [1,55]. The applications of the present theory to mortar and discontinuous Galerkin methods are also condidered in Sect. 4 for the Poisson problem. Therein, the norms of $\mathcal{R}es_L$ and $\mathcal{R}es_V$ are estimated under the general hypothesis that each of those has the form

$$\mathcal{R}es(v) := \int_\Omega g \cdot v\, dx + \int_{\cup\mathcal{E}} g_\mathcal{E} \cdot v\, ds \quad \text{for } v \in V. \tag{1.8}$$

Here and below, V belongs to some Sobolev space $V = H_0^1(\Omega)^m$ and $g \in L^2(\Omega)^m$, while $g_\mathcal{E} \in L^2(\cup\mathcal{E})^m$ with some domain $\Omega \subset \mathbb{R}^n$ and the union $\cup\mathcal{E}$ of edges (if $n = 2$) or faces (if $n = 3$) related to a regular triangulation of Ω. Some required key property in [20] on both $\mathcal{R}es = \mathcal{R}es_L$ and $\mathcal{R}es = \mathcal{R}es_V$ reads

$$V_h^c \subset \ker \mathcal{R}es \subset V. \tag{1.9}$$

In this situation, a typical result of an explicit residual-based error estimation reads

$$\|\mathcal{R}es\|_{V^*}^2 \lesssim \|h_\mathcal{T} g\|_{L^2(\Omega)}^2 + \sum_{E\in\mathcal{E}} h_E \|g_\mathcal{E}\|_{L^2(E)}^2 =: \eta^2. \tag{1.10}$$

Here and throughout, $h_\mathcal{T}$ and h_E denote local mesh-sizes in the underlying triangulation, i.e.,

$$h_\mathcal{T}|_T = \operatorname{diam}(T) \quad \text{for any } T \in \mathcal{T}, \quad \text{and} \quad h_E = \operatorname{diam}(E) \quad \text{for any } E \in \mathcal{E}.$$

V_h^c includes the first-order finite element functions to ensure (1.10). Details on the notation and the concrete examples will be given below. The terms in (1.8) often result from some discretisation of the equilibration condition (1.4), e.g., via an integration by parts, and hence the term $\mathcal{R}es_V$ is referred to as the equilibration residual.

The *first aim* of this paper is the generalisation of (1.10) for $\mathcal{R}es = \mathcal{R}es_V$ in Theorem 2.1 of Sect. 2 to allow the control of certain nonstandard finite element schemes *without* the condition (1.9) in Sects. 4–6. Here, one key theory is to replace (1.9) by assumptions (H1)–(H3) on some Clément-type operator J, [8,21,22] and some linear bounded operator Π between the conforming and nonconforming finite element spaces.

For the Laplace, Stokes, and Navier–Lamé equations considered herein, one can observe from the definitions of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ in Sects. 4–6 below that the consistency residuum $\mathcal{R}es_L$ from (1.5) can also be written in the form (1.8). With some bounded linear operator $\mathcal{A} : L := L^2(\Omega)^{m\times n} \to L$, the norm of

$\mathcal{R}es_L$ allows the form

$$\min_{\tilde{u}_h \in V} \|\mathcal{R}es_L\|_{L^*} \approx \min_{\tilde{u}_h \in V} \|\mathcal{A}(p_h) - D\tilde{u}_h\|_{L^2(\Omega)}. \tag{1.11}$$

Therein $D\tilde{u}_h$ denotes the functional matrix of all first-order partial derivatives (e.g., the gradient and possibly also the Green strain of linear elasticity) of the Sobolev function $\tilde{u}_h$ in Sects. 4–6.

Remark 1.2 This observation can also be found in [20, Theorem 2.2] for the Laplace equation with $\mathcal{A} = \text{id}$ the identity operator. For the Stokes and Navier–Lamé equations, the operators $\mathcal{A}$ are $\frac{\text{dev}}{\mu}$ and $\mathbb{C}^{-1}$ with the operators dev (and μ) and $\mathbb{C}^{-1}$ of Sects. 5 and 6.

Since $\|\mathcal{A}(p_h) - D\tilde{u}_h\|_{L^2(\Omega)} \lesssim \|D_{\mathcal{T}} u_h - D\tilde{u}_h\|_{L^2(\Omega)}$ (plus some computable term in the case of the Stokes problem) for the aforementioned problems, the second aim of this paper reads

$$\min_{\tilde{u}_h \in V} \|D_{\mathcal{T}} u_h - D\tilde{u}_h\|^2_{L^2(\Omega)} \lesssim \sum_{E \in \mathcal{E}} \sum_{z \in \mathcal{K}(E)} h_E \|\gamma_{\tau_E}([D_{\mathcal{T}}(\psi_z\, u_h)])\|^2_{L^2(E)} =: \mu^2$$

for the jumps $[D_{\mathcal{T}}(u_h\, \psi_z)]$ of a discrete nonconforming finite element function u_h times a weight-function ψ_z across some side E with vertex z; details on the notation can be found in Sect. 3. The second main result (Theorem 3.1) holds for all piecewise gradients and employs a localisation argument with the (modified) hat functions $(\psi_z : z \in \mathcal{K})$ of the free nodes $\mathcal{K}$.

Then, a summary of these two aims (See, Theorems 2.1 and 3.1) and (1.7) concludes the main result of this paper

$$\|p - p_h\|_L \lesssim \eta + \mu + \text{osc}(g) \tag{1.12}$$

for the unified a posterior error estimate of the nonconforming finite element methods with (H1)–(H3) of Sect. 2 and for all aforemented problems. This conclusion will be exhibited for each problem in Sects. 4–6, what is left is to check the well-posdeness of (1.3)-(1.4) [or (1.2)] for each problem and (H1)–(H3) for each nonconforming finite element scheme; see Sects. 4–6 for further details.

The rest of this paper is organized as follows. While Sects. 2-3 treat general assertions on (1.10) and (1.11) where condition (1.9) is substituted by (H1)–(H3), Sects. 4–6 conclude this paper with particular model examples in $2D$ (and some in $3D$) with first reliability proofs for many nonstandard finite element error estimates.

Throughout this paper, V_h^c and V_h^{nc} denote conforming and nonconforming finite element spaces based on a regular triangulation $\mathcal{T}$ of Ω; ν denotes the normal unit vector along the boundary $\partial\Omega$; τ denotes the tangent vector along the boundary for 2D. Colon ":" denotes the scalar product in $\mathbb{R}^{m\times n}$, i.e., $A : B := \sum_{j=1}^{m} \sum_{k=1}^{n} A_{jk} B_{jk}$.

2 Reliability control of the equilibrium residual

This section establishes an explicit residual-based error estimate (1.10) for a class of nonstandard finite element schemes.

Let $V = H^1_0(\Omega)^m$ and $L = L^2(\Omega;\mathbb{R}^{m\times n})$ denote standard Sobolev and Lebesgue spaces on some bounded Lipschitz domain Ω in $\mathbb{R}^n$ with a piecewise flat boundary Γ. Suppose that the closure $\overline{\Omega}$ is covered exactly by a regular triangulation $\mathcal{T}$ of $\overline{\Omega}$ into (closed) triangles or parallelograms in $2D$, tetrahedrons or parallelepipeds in $3D$ (or other unions of simplices). It is assumed, that

$$\overline{\Omega} = \cup\mathcal{T} \quad \text{and} \quad |T_1 \cap T_2| = 0 \quad \text{for } T_1, T_2 \in \mathcal{T} \quad \text{with } T_1 \neq T_2, \tag{2.1}$$

where $|\cdot|$ denotes the volume (as well as the modulus of a vector etc. where there is no real risk of confusion). The remaining assumptions on the shape regularity of $\mathcal{T}$ are hidden in the following abstract conditions.

(H1) There exists a Clément-type operator $J : V \to V^c_h$ into some (conforming) subspace $V^c_h \subseteq V$ of $\mathcal{T}$-piecewise smooth functions such that, for all $v \in V$ and $T \in \mathcal{T}$

$$h_T^{-1}\|v - Jv\|_{L^2(T)} + h_T^{-1/2}\|v - Jv\|_{L^2(\partial T)} + \|D(v - Jv)\|_{L^2(T)} \lesssim \|Dv\|_{L^2(\omega_T)}, \tag{2.2}$$

with some neighbourhood ω_T of T such that $(\omega_T : T \in \mathcal{T})$ has finite overlap

$$\max_{x\in\overline{\Omega}} \operatorname{card}\{T \in \mathcal{T} : x \in \omega_T\} \lesssim 1. \tag{2.3}$$

(H2) There exists a nonconforming space $V^{nc}_h \subseteq L^2(\Omega)^m$ of $\mathcal{T}$-piecewise smooth and, in general, discontinuous functions $V^{nc}_h \subseteq H^1(\mathcal{T})^m \not\subset V$. Given distinct $T_1, T_2 \in \mathcal{T}$, their intersection $T_1 \cap T_2$ has zero volume measure by (2.1) but possibly a positive surface measure h_E. The set of all interior (edges or faces etc.) $T_1 \cap T_2 = E$ is denoted by $\mathcal{E}$. For any $v_h \in V^{nc}_h$, the jump

$$[v_h]_E(x) := (v_h|_{T_2})(x) - (v_h|_{T_1})(x) \quad \text{for } x \in E. \tag{2.4}$$

across $E \in \mathcal{E}$ with $E = T_1 \cap T_2$ is fixed up to the sign which results from the orientation of the unit vector ν_E on E (e.g. ν_E points outward of T_2). The shape regularity of $\mathcal{T}$ and $\mathcal{E}$ is described by the assumption

$$h_E \approx h_T \approx \operatorname{diam}(\omega_T) \quad \text{for all } E \in \mathcal{E}, T \in \mathcal{T} \text{ with } E \cap T \neq \emptyset. \tag{2.5}$$

Remark 2.1 The trace inequality yields, for $v \in V$ and $T \in \mathcal{T}$ [15,35],

$$\|v\|_{L^2(\partial T)} \lesssim h_T^{-1/2}\|v\|_{L^2(T)} + h_T^{1/2}\|Dv\|_{L^2(T)}. \tag{2.6}$$

Hence the trace term with $L^2(\partial T)$ in (2.2) is estimated by the other two $L^2(T)$ norms. More over, if $\mathcal{E}(T)$ denotes the set of all E with $E \subseteq \partial T$, the shape regularity (2.5) shows that

$$\sum_{E\in\mathcal{E}(T)} h_E^{-1}\|v - Jv\|_{L^2(E)}^2 \lesssim \|Dv\|_{L^2(\omega_T)}^2. \tag{2.7}$$

Remark 2.2 The conforming functions are given as those with vanishing jumps, i.e., $v_h \in V_h^c$ implies $[v_h]_E = 0$ for all $E \in \mathcal{E}$.

The aforementioned standard assumptions are typical in finite element simulations. The innovative condition on the nonstandard finite element space V_h^{nc} and the conforming counterpart V_h^c of (H1) and (H2) is the following.

(H3) There exists some operator $\Pi : V_h^c \to V_h^{nc}$ such that, for all $v_h \in V_h^c$ and all $T \in \mathcal{T}$, there holds

$$\|\nabla(\Pi v_h)\|_{L^2(T)} \lesssim \|\nabla v_h\|_{L^2(\omega_T)} \quad \text{and} \quad \int_T v_h\,dx = \int_T \Pi v_h\,dx. \tag{2.8}$$

Moreover, for some given discrete approximation $p_h \in L^2(\Omega;\mathbb{R}^{m\times n})$ and the $\mathcal{T}$-piecewise gradient $D_\mathcal{T}$, there holds

$$\int_\Omega p_h : D_\mathcal{T} v_h\,dx = \int_\Omega p_h : D_\mathcal{T}(\Pi v_h)\,dx. \tag{2.9}$$

A direct consequence of (2.8) is

$$h_T^{-1}\|v_h - \Pi v_h\|_{L^2(T)} \lesssim \|Dv_h\|_{L^2(\omega_T)} \quad \text{for all } T \in \mathcal{T}. \tag{2.10}$$

Given $g \in L^2(\Omega)^m$ and p_h as above, the residual $\mathcal{R}es_V \in V^*$ is, for $v \in V + V_h^{nc} \subset L^2(\Omega, \mathbb{R}^{m\times n})$ defined by

$$\mathcal{R}es_V(v) := \int_\Omega g\cdot v\,dx - \int_\Omega p_h : D_\mathcal{T} v\,dx. \tag{2.11}$$

The residual is supposed to stem from a nonstandard finite element scheme with V_h^{nc} and hence

$$\mathcal{R}es_V(v_h) = 0 \quad \text{for all } v_h \in V_h^{nc}. \tag{2.12}$$

With the abbreviation $g_T := |T|^{-1}\int_T g(x)dx \in \mathbb{R}^m$, the data oscillation reads

$$\mathrm{osc}(g) := \left(\sum_{T\in\mathcal{T}} h_T^2\|g - g_T\|_{L^2(T)}^2\right)^{1/2}. \tag{2.13}$$

Under the assumptions of (H1)–(H3), the residual-based error estimator

$$\eta := \left(\sum_{T\in\mathcal{T}} h_T^2 \|g + \operatorname{div} p_h\|_{L^2(T)}^2\right)^{1/2} + \left(\sum_{E\in\mathcal{E}} h_E \|[p_h]_E \cdot \nu_E\|_{L^2(E)}^2\right)^{1/2} \tag{2.14}$$

is reliable in the following sense.

Theorem 2.1 *There holds* $\|\mathcal{R}es_V\|_{V^*} \lesssim \eta + \operatorname{osc}(g)$.

Proof Given any $v \in V$ with $\Pi J v \in V_h^{nc}$, (2.12) leads to

$$\mathcal{R}es_V(v) = \int_\Omega g \cdot (v - \Pi J v)\, dx - \int_\Omega p_h : D_\mathcal{T}(v - \Pi J v)\, dx.$$

An elementwise integration by parts and a careful re-arrangement of boundary pieces leads to

$$\begin{aligned}\int_\Omega p_h : D(v - Jv)\, dx = &-\int_\Omega (\operatorname{div}_\mathcal{T} p_h) \cdot (v - Jv)\, dx \\ &+ \sum_{E\in\mathcal{E}} \int_E [p_h] \cdot \nu_E (v - Jv)\, ds.\end{aligned}$$

The combination of the two identities with (2.9), i.e., $\int_\Omega p_h : D_\mathcal{T}(Jv - \Pi J v)\, dx = 0$, where v_h is replaced by $Jv \in V_h^c$, reads

$$\begin{aligned}\mathcal{R}es_V(v) &= \int_\Omega (g + \operatorname{div}_\mathcal{T} p_h) \cdot (v - Jv)\, dx + \int_\Omega g \cdot (Jv - \Pi J v)\, dx \\ &\quad - \sum_{E\in\mathcal{E}} \int_E [p_h] \cdot \nu_E (v - Jv)\, ds \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

The first integral I_1 on the right-hand side is controlled with (2.2) and (2.3), Hölder and Cauchy inequalities. This leads to

$$I_1 \lesssim \left(\sum_{T\in\mathcal{T}} h_T^2 \|g + \operatorname{div} p_h\|_{L^2(T)}^2\right)^{1/2} \|Dv\|_{L^2(\Omega)}.$$

The second term I_2 requires (2.8), (2.10) and (2.13). This yields

$$\begin{aligned}
I_2 &= \sum_{T\in\mathcal{T}} \int_T g\cdot (Jv - \Pi Jv)\,dx \\
&= \sum_{T\in\mathcal{T}} \int_T (g-g_T)\cdot (Jv - \Pi Jv)\,dx \\
&\leq \sum_{T\in\mathcal{T}} h_T \|g-g_T\|_{L^2(T)} h_T^{-1} \|Jv - \Pi Jv\|_{L^2(T)} \\
&\lesssim \operatorname{osc}(g) \left(\sum_{T\in\mathcal{T}} \|Dv_h\|^2_{L^2(\omega_T)} \right)^{1/2} \\
&\lesssim \operatorname{osc}(g) \|Dv\|_{L^2(\Omega)}.
\end{aligned}$$

Standard arguments with (2.1)—(2.3) and (2.7) control the last term

$$\begin{aligned}
I_3 &\leq \sum_{E\in\mathcal{E}} h_E^{1/2} \|[p_h]_E \cdot \nu_E\|_{L^2(E)} h_E^{-1/2} \|v - Jv\|_{L^2(E)} \\
&\lesssim \left(\sum_{E\in\mathcal{E}} h_E \|[p_h]_E \cdot \nu_E\|^2_{L^2(E)} \right)^{1/2} \|Dv\|_{L^2(\Omega)}.
\end{aligned}$$

Altogether, there follows the assertion

$$\mathcal{R}es_V(v) = I_1 + I_2 + I_3 \lesssim (\eta + \operatorname{osc}(g)) \|Dv\|_{L^2(\Omega)}.$$

□

3 Reliability control of the consistency residual

This section establishes a general control of the consistency residual (1.11). Given $u_h \in V_h^{nc}$ with $D_{\mathcal{T}} u_h \in L^2(\Omega; \mathbb{R}^{m\times n})$ and the conforming finite element space V_h^c from (H1)–(H3), let $(\psi_z : z \in \mathcal{K})$ denote a Lipschitz continuous partition of unity,

$$\sum_{z\in\mathcal{K}} \psi_z = 1 \quad \text{in } \Omega. \tag{3.1}$$

Moreover, for any $z \in \mathcal{K}$, suppose that, ψ_z vanishes outside an open and connected set $\Omega_z \subseteq \Omega$

$$\operatorname{supp}\psi_z \subseteq \overline{\Omega}_z \quad \text{and} \quad \max_{x\in\overline{\Omega}} \operatorname{card}\{z \in \mathcal{K} : x \in \Omega_z\} \lesssim 1. \tag{3.2}$$

Given $z \in \mathcal{K}$, let $\mathcal{E}(z) := \{E \in \mathcal{E} : \psi_z|_E \not\equiv 0\}$ denote the set of edges, where ψ_z is nonvanishing. For any edge E let $\mathcal{K}(E)$ denote the set of all $z \in \mathcal{K}$ with

$E \in \mathcal{E}(z)$. The tangential component of a vector $v \in \mathbb{R}^n$ is defined as

$$\gamma_{\tau_E}(v) := \begin{cases} v \cdot \tau_E & \text{if } n = 2, \\ v \times \nu_E & \text{if } n = 3. \end{cases} \tag{3.3}$$

The general estimator

$$\mu := \left(\sum_{E \in \mathcal{E}} \sum_{z \in \mathcal{K}(E)} h_E \| \gamma_{\tau_E}([D_\mathcal{T}(\psi_z\, u_h)]) \|^2_{L^2(E)} \right)^{1/2} \tag{3.4}$$

is reliable in the following sense.

Theorem 3.1 *For $n = 2, 3$, there holds* $\min_{\tilde{u}_h \in V} \| D_\mathcal{T} u_h - D\tilde{u}_h \|_{L^2(\Omega)} \lesssim \mu$.

Remark 3.1 In the examples below, $0 \le \psi_z \le 1$ is a finite sum of hat functions and continuous such that $\gamma_{\tau_E}([D_\mathcal{T}(\psi_z u_h)]) = \gamma_{\tau_E}(D\psi_z)[u_h] + \psi_z \gamma_{\tau_E}([D_\mathcal{T} u_h])$. Moreover, the polynomial $[u_h]$ has some zero on E and allows an estimate

$$\| [u_h] \|_{L^2(E)} \lesssim h_E \| \gamma_{\tau_E}([D_\mathcal{T} u_h]) \|_{L^2(E)}. \tag{3.5}$$

With $\| D\psi_z \|_{L^\infty} \approx h_E^{-1}$, one deduces

$$\mu \lesssim \left(\sum_{E \in \mathcal{E}} h_E \| \gamma_{\tau_E}([D_\mathcal{T} u_h]) \|^2_{L^2(E)} \right)^{1/2}. \tag{3.6}$$

This estimator is the frequently found version of the consistency error control [23,25,28,29].

Remark 3.2 Theorem 3.1 generalizes [25]. To control the nonconformity, it was assumed therein that

$$\int_E [v_h]\, ds = 0 \quad \text{for } E \in \mathcal{E} \quad \text{and} \quad \int_E v_h\, ds = 0 \quad \text{for } E \quad \text{on } \partial\Omega \quad \text{for all } v_h \in V_h^{nc}. \tag{3.7}$$

The condition (3.7) is removed in Theorem 3.1 of the present paper.

Proof of Theorem 3.1. Given $z \in \mathcal{K}$ let a_z and b_z denote the functions of the Helmholtz decomposition of $D_\mathcal{T}(\psi_z u_h)$, i.e.,

$$D_\mathcal{T}(\psi_z u_h) = Da_z + \operatorname{curl} b_z \in L,$$

Here $a_z \in H_0^1(\Omega_z)$, $b_z \in H^1(\Omega_z)^k$ with $\int_{\Omega_z} b_z(x)\,dx = 0$, and $k = 1$ for $n = 2$ while $k = 3$ for $n = 3$. Since $\int_{\Omega_z} \operatorname{curl} b_z : Da\,dx = 0$ for any $a \in H_0^1(\Omega_z)$,

$$\|\operatorname{curl} b_z\|^2_{L^2(\Omega_z)} = \min_{a \in H_0^1(\Omega_z)} \|Da - D_{\mathcal{T}}(\psi_z u_h)\|^2_{L^2(\Omega_z)} = \int_{\Omega_z} (\operatorname{curl} b_z) : D_{\mathcal{T}}(\psi_z u_h)\,dx.$$

An elementwise integration by parts followed by $\operatorname{curl}_{\mathcal{T}} D_{\mathcal{T}} \equiv 0$ yields

$$\int_{\Omega_z} (\operatorname{curl} b_z) : D_{\mathcal{T}}(\psi_z u_h)\,dx = \pm \int_{\bigcup \mathcal{E}(z)} b_z \cdot \gamma_{\tau_E}([D_{\mathcal{T}}(\psi_z u_h)])\,ds \le \|\gamma_{\tau_E}([D_{\mathcal{T}}(\psi_z u_h)])\|_{L^2(\bigcup \mathcal{E}(z))} \|b_z\|_{L^2(\bigcup \mathcal{E}(z))},$$

where $\mathcal{E}(z) := \{E \in \mathcal{E} : \psi_z|_E \not\equiv 0\}$. The well-known trace theorem on each element domain K, namely

$$\|b_z\|_{L^2(\partial K)} \le h_K^{-1/2} \|b_z\|_{L^2(K)} + h_K^{1/2} \|Db_z\|_{L^2(K)},$$

leads to the estimate

$$\|b_z\|_{L^2(\bigcup \mathcal{E}(z))} \lesssim h_z^{-1/2} \|b_z\|_{L^2(\Omega_z)} + h_z^{1/2} \|Db_z\|_{L^2(\Omega_z)}.$$

A Poincaré inequality gives

$$\|b_z\|_{L^2(\Omega_z)} \lesssim h_z \|Db_z\|_{L^2(\Omega_z)} \lesssim h_z \|\operatorname{curl} b_z\|_{L^2(\Omega_z)}.$$

The latter inequality results from the stability of the Helmholtz decomposition [23,35] with an h_z-independent constant; it reads $\|Db_z\|_{L^2(\Omega_z)} = \|\operatorname{curl} b_z\|_{L^2(\Omega_z)}$ in $2D$. The combination of the proceeding three inequalities leads to

$$\|b_z\|_{L^2(\bigcup \mathcal{E}(z))} \lesssim h_z^{1/2} \|\operatorname{curl} b_z\|_{L^2(\Omega_z)}.$$

Since $h_{\mathcal{T}}|_{\Omega_z} \approx h_z := \operatorname{diam}(\Omega_z)$, $z \in \mathcal{K}$, the aforementioned arguments imply

$$\|Da_z - D_{\mathcal{T}}(\psi_z u_h)\|_{L^2(\Omega_z)} \lesssim \|h_{\mathcal{E}}^{1/2} \gamma_{\tau_E}([D_{\mathcal{T}}(\psi_z u_h)])\|_{L^2(\bigcup \mathcal{E}(z))}.$$

Since $\sum_{z\in\mathcal{K}} \psi_z \equiv 1$ and $\tilde{u}_h := \sum_{z\in\mathcal{K}} a_z \in H_0^1(\Omega)$, this estimate plus the finite overlap of all Ω_z and $\mathcal{E}(z)$ prove the assertion. In fact,

$$\begin{aligned}\|D_{\mathcal{T}}u_h - D\tilde{u}_h\|_L^2 &= \|\sum_{z\in\mathcal{K}}(D_{\mathcal{T}}(\psi_z u_h) - Da_z)\|_L^2 \\ &\lesssim \sum_{z\in\mathcal{K}} \|Da_z - D_{\mathcal{T}}(\psi_z u_h)\|_{L^2(\Omega_z)}^2 \\ &\lesssim \sum_{z\in\mathcal{K}} \|h_{\mathcal{E}}^{1/2}\gamma_{\tau_E}([D_{\mathcal{T}}(\psi_z u_h)])\|_{L^2(\bigcup \mathcal{E}(z))}^2 \\ &\approx \sum_{E\in\mathcal{E}}\sum_{z\in\mathcal{K}(E)} h_E\|\gamma_{\tau_E}([D_{\mathcal{T}}(\psi_z u_h)])\|_{L^2(E)}^2.\end{aligned}$$

□

4 Application to Laplace equation

This section is devoted to the Poisson problem and its residual-based a posteriori finite element error control. Subsection 4.1 introduces the model problem and Subsection 4.2 some required notations. Subsection 4.3 presents a list of examples. Subsections 4.4 and 4.5 present the applications of the theory to the mortar and dG finite element methods. Subsection 4.6 concerns the extension of the present theory to the high-order NCFEM.

4.1 Model problem

The Lebesgue and Sobolev spaces $L^2(\Omega)$ and $H^1(\Omega)$ are defined as usual and

$$L := L^2(\Omega)^n \quad \text{and} \quad V := H_0^1(\Omega) := \{w \in H^1(\Omega) : w = 0 \text{ on } \partial\Omega\}. \tag{4.1}$$

The gradient operator ∇ maps V into L. Given $g \in L^2(\Omega)$ let $u \in V$ denote the solution to the *Poisson Problem*

$$\Delta u + g = 0 \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega. \tag{4.2}$$

Then, the flux $p := \nabla u \in L$ and $u \in V$ satisfy

$$\begin{aligned}(A(p,u))(q,v) &:= a(p,q) + b(p,v) + b(q,u) \\ &\stackrel{!}{=} -\int_\Omega gv\,dx \quad \text{for all } (q,v) \in X := L \times V.\end{aligned} \tag{4.3}$$

Throughout this section, (1.1)–(1.7) hold for

$$a(p,q) := \int_\Omega p \cdot q \, dx \quad \text{and} \quad b(p,v) := -\int_\Omega p \cdot \nabla v \, dx. \tag{4.4}$$

The operator $A : X \to X^*$ is bounded, linear, and bijective [20].

4.2 Nonconforming finite element methods and unified a posteriori error estimators

Let $P_k(T)$ and $Q_k(T)$ denote the space of algebraic polynomials of total and partial degree $\leq k$, respectively, and set $\mathcal{P}_k(T) = P_k(T)$ and $\mathcal{P}_k(T) = Q_k(T)$ for a triangle (or tetrahedron) and parallelogram (or parallelepiped), respectively. Define

$$\begin{aligned} &\mathcal{P}_k(\mathcal{T}) := \{v \in L^2(\Omega) : \forall T \in \mathcal{T}, \, v|_T \in \mathcal{P}_k(T)\} \quad \text{for } k = 0, 1; \\ &\mathcal{S}^1(\mathcal{T}) := \mathcal{P}_1(\mathcal{T}) \cap C(\overline{\Omega}) \quad \text{and} \quad V_h^c := \mathcal{S}_0^1(\mathcal{T}) := \mathcal{S}^1(\mathcal{T}) \cap V. \end{aligned} \tag{4.5}$$

Let $\mathcal{N}$ denote the set of nodes (i.e., vertices of elements in $\mathcal{T}$). $h_\mathcal{T}$ and $h_\mathcal{E}$ denote $\mathcal{T}$- and $\mathcal{E}$-piecewise constant functions on Ω and $\cup\mathcal{E} = \cup_{E\in\mathcal{E}} E$ defined by $h_\mathcal{T}|_T := h_T := \mathrm{diam}(T)$ and $h_\mathcal{E}|_E := h_E := \mathrm{diam}(E)$ for $T \in \mathcal{T}$ and $E \in \mathcal{E}$. For a given quadrilateral or parallelepiped element $T \in \mathcal{T}$, $\mathcal{F}_T : \hat{T} = [-1,1]^n \to T$ denotes the canonical bilinear (for $n = 2$) or trilinear (for $n = 3$) transformation.

Let V_h^{nc} denote some nonconforming finite element space specified in Table 1. For the moment solely suppose that $\nabla_\mathcal{T} v_h \in L$ for any $v_h \in V_h^{nc}$, where $\nabla_\mathcal{T}$ denote the $\mathcal{T}$-piecewise action of the gradient operator. The finite element solution $u_h \in V_h^{nc}$ is the unique solution to

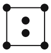
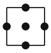
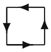
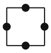
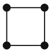
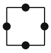
$$\int_\Omega \nabla_\mathcal{T} u_h \cdot \nabla_\mathcal{T} v_h \, dx = \int_\Omega g v_h \, dx \quad \text{for all } v_h \in V_h^{nc}. \tag{4.6}$$

The aim is to estimate the flux error $p - p_h$ for the discrete flux $p_h := \nabla_\mathcal{T} u_h \in L = L^2(\Omega)^n$.

For any $\tilde{u}_h \in V$ there holds (1.7) for $\mathcal{R}es_L \in L^*$ and $\mathcal{R}es_V \in V^*$ defined, for all $q \in L$ and $v \in V$, by

$$\begin{aligned} \mathcal{R}es_L(q) &:= \int_\Omega q \cdot (\nabla\tilde{u}_h - p_h) \, dx \quad \text{and} \\ \mathcal{R}es_V(v) &:= -\int_\Omega g v \, dx + \int_\Omega p_h \cdot \nabla v \, dx. \end{aligned} \tag{4.7}$$

Table 1 Nonconforming elements for the Laplace equation (4.2) with (H1)–(H3) and the error estimate (4.8)

Picture	Name	Reference	Space
	Crouzeix and Raviart	[27]	V_h^{CR}
	Wilson	[59,54]	V_h^{Wil}
	Han	[36]	V_h^{Han}
	NR (midpoint)	[50]	$V_h^{RT,P}$
	NR (average)	[50]	$V_h^{RT,A}$
	CNR	[41]	V_h^{CRT}
	DSSY	[31]	V_h^{DSSY}

4.3 Examples

This subsection presents a list of $2D$ and $3D$ nonconforming finite element spaces V_h^{nc} of Table 1 with (H1)–(H3), so that

$$\|p - p_h\|_{L^2(\Omega)} \lesssim \eta + \mu + \operatorname{osc}(g) \tag{4.8}$$

with η from (2.14), μ from (3.4), and $\operatorname{osc}(g)$ from (2.13). This list below is not comprehensive. In fact, we conjecture that all known NCFEMs could be analyzed in the present framework. Only the triangular Crouzeix–Raviart element has already been analyzed in [20]. The present unifying theory leads to new error control (4.8) for all nonconforming finite elements of Subsubsects. 4.3.2–4.3.6.

4.3.1 The triangular Crouzeix–Raviart element

Based on the regular triangulation $\mathcal{T}$ into simplices, the set of midpoints $\mathcal{M}$ of edges (or faces), the non-conforming *Crouzeix–Raviart* finite element space reads (in 2D and 3D)

$$V_h^{CR} := \{v \in \mathcal{P}_1(\mathcal{T}) : v \text{ continuous at } \mathcal{M} \cap \Omega \text{ and } v = 0 \text{ at } \mathcal{M} \cap \partial\Omega\}. \tag{4.9}$$

Since $V_h^C \subset V_h^{CR}$, then there holds (H1)–(H3) with $\Pi = \mathrm{id}$; cf. Sect. 4 of [25] for proofs. Similar arguments verify (H1)–(H3) in 3D as well; we therefore omit the details.

4.3.2 The quadrilateral Wilson element

Let B denote one of the nonconforming quadratic bubble function spaces on the reference element $\hat{T} = [-1,1]^n$, i.e.,

$$B := \begin{cases} \text{span}\{1-(\xi^2+\eta^2)/2\} \quad \text{or} \quad \text{span}\{1-\xi^2, 1-\eta^2\} & \text{for } n=2, \\ \text{span}\{1-\xi^2, 1-\eta^2, 1-\zeta^2\} & \text{for } n=3. \end{cases}$$

The nonconforming quadrilateral *Wilson* finite element space V_h^{Wil} [54,59] reads

$$\begin{aligned} V_h^{Wil} &= S^h \oplus B^h \text{ with the factors} \\ S^h &:= \{v \in H_0^1(\Omega) : \forall T \in \mathcal{T}, \quad \hat{v} = v \circ \mathcal{F}_T \in Q_1(\hat{T})\}, \\ B^h &:= \{v \in L^2(\Omega) : \forall T \in \mathcal{T}, \quad \hat{v} = v \circ \mathcal{F}_T \in B\}. \end{aligned} \tag{4.10}$$

This element is excluded from the analysis of [20,25] since (3.7) is violated. However, there holds (H1)–(H3) with $\Pi = \text{id}$, the proof is immediate since $V_h^c \subset V_h^{Wil}$.

4.3.3 The parallelogram nonconforming Han element

Consider the functional

$$\mathcal{F}_E(v) = |E|^{-1} \int_E v\, ds \quad \text{for all } E \in \mathcal{E}(T) \quad \text{and} \quad T \in \mathcal{T}. \tag{4.11}$$

The parametric formulation of rectangular and parallelogram elements of Han [36] is introduced by

$$\mathcal{Q}_{\mathcal{H}}^{nc} := \text{span}\left\{1,\, \xi,\, \eta,\, \xi^2 - \frac{5}{3}\xi^4,\, \eta^2 - \frac{5}{3}\eta^4\right\}. \tag{4.12}$$

The nonconforming *Han* finite element space then reads (with $[\cdot] := \cdot$ along $\partial\Omega$)

$$V_h^{Han} := \left\{v \in L^2(\Omega) : \forall T \in \mathcal{T}, v|_T \circ \mathcal{F}_T \in \mathcal{Q}_{\mathcal{H}}^{nc} \text{ and } \forall E \in \mathcal{E}, \mathcal{F}_E([v]) = 0\right\}. \tag{4.13}$$

Then there holds (H1)–(H3) with the associated interpolation operator Π for V_h^{Han} [36], the proof follows from $\Pi V_h^c = V_h^{CRT} \subset V_h^{Han}$ [41] with V_h^{CRT} from Subsubsect. 4.3.5 below. Further details for the properties of Π can be found in Sect. 4 of [25], Remark 2.5 and Lemma 3.1 of [41].

4.3.4 The parallelogram nonconforming rotated Q_1 elements

Rannacher and Turek [50] introduce two types of parallelogram nonconforming elements, called the *NR* elements. The first element *RTA* uses the average of the function over the edge (or face) as the local degree of freedom, and the second one *RTP* uses the value at the midside point (or midpoint) of the edge (or face) instead. Define

$$\mathcal{Q}_{\mathcal{R}}^{nc} := \begin{cases} \text{span}\{1,\, \xi,\, \eta,\, \xi^2 - \eta^2\} & \text{for } n = 2, \\ \text{span}\{1,\, \xi,\, \eta,\, \zeta,\, \xi^2 - \eta^2,\, \xi^2 - \zeta^2\} & \text{for } n = 3 \end{cases} \tag{4.14}$$

then nonconforming space $V_h^{RT,A}$ is defined in (4.13) with $\mathcal{Q}_{\mathcal{H}}^{nc}$ replaced by $\mathcal{Q}_{\mathcal{R}}^{nc}$, and $V_h^{RT,P}$ is defined in (4.9) with $\mathcal{P}_1(\mathcal{T})$ replaced by $\mathcal{Q}_{\mathcal{R}}^{nc}$.

For $2D$, following a similar argument for the *Han* element, one proves that the average version element satisfies (H1)–(H3) with the canonical interpolation operator Π for $V_h^{RT,A}$; [25] contains further details.

The midside point version element is not included in [25] since the condition (3.7) is violated by this element. However, there holds equally (H1)–(H3) for it with the canonical interpolation operator Π of $V_h^{RT,P}$. In fact, we have

$$\Pi V_h^c = V_h^{CRT} \subset V_h^{RT,P}, \tag{4.15}$$

and V_h^{CRT} contains the linear part of V_h^c, and only the nonlinear part is excluded [41]. With this fact, (H3) follows from straight forward investigations.

For 3D, define the local interpolation operator $\Pi_T : H^1(T) \to \mathcal{Q}_{\mathcal{R}}^{nc} \circ \mathcal{F}_T^{-1}$ by

$$\mathcal{F}_E(\Pi_T v) = \mathcal{F}_E(v) \quad \text{for } E \in \mathcal{E}(T) \quad \text{for all } v \in H^1(T). \tag{4.16}$$

Since $\mathcal{F}_{\hat{E}}(v) = 0$ for $v = \xi\eta, \xi\zeta, \eta\zeta, \xi\eta\zeta$ with $\hat{E} \in \mathcal{E}(\hat{T})$, we conclude for any $v = a_0 + a_1\xi + a_2\eta + a_3\zeta + a_4\xi\eta + a_4\xi\zeta + a_6\eta\zeta + a_7\xi\eta\zeta$ that

$$\Pi_T v = a_0 + a_1\xi + a_2\eta + a_3\zeta, \tag{4.17}$$

with some interpolation constants $a_0, \ldots, a_7$. The global interpolation operator Π is defined by $\Pi|_T = \Pi_T$ for any $T \in \mathcal{T}$. Then (H1)–(H3) eventually follows from (4.17).

Remark 4.1 The analysis does not cover the non-parametric variant of this element except on parallelogram meshes.

Remark 4.2 Notice the remarks in [25] on earlier references [2,42] and corrections in progress on [2].

4.3.5 The parallelogram constrained nonconforming rotated Q_1 elements

The constrained rotated nonconforming finite element (referred to as *CNR* element) introduced in [41] is obtained by enforcing a constraint on the *NR* element on each element for $2D$. The space of the *CNR* element reads

$$V_h^{CRT} := \left\{ v \in V_h^{RT,A} : \forall T \in \mathcal{T}, \int_{E_1} v\,ds + \int_{E_3} v\,ds = \int_{E_2} v\,ds + \int_{E_4} v\,ds \right.$$

$$\left. \text{with } \{E_1, \cdots, E_4\} = \mathcal{E}(T) \text{ numbered counterclockwise} \right\}. \quad (4.18)$$

For rectangular and parallelogram meshes, the element is equivalent to the P_1-quadrilateral element of [33,49]. Then there holds (H1)–(H3) with the interpolation operator Π of V_h^{CRT}. The proof follows from the argument for the *NR* element with the midside point version. We refer to Sect. 4 of [25] for more details. The goal-oriented error control of this element is given in [34].

4.3.6 The parallelogram DSSY elements

The *DSSY* element is obtained by introducing on the reference element [31] with $\theta_1(t) = t^2 - \frac{5}{3}t^4$ and $\theta_2(t) = t^2 - \frac{25}{6}t^4 + \frac{7}{2}t^6$ and

$$\mathcal{Q}_{\mathcal{D}}^{nc} := \begin{cases} \text{span}\{1,\ \xi,\ \eta,\ \theta_\ell(\xi) - \theta_\ell(\eta)\} \quad \text{for } \ell = 1,2 & \text{for } n = 2, \\ \text{span}\{1,\ \xi,\ \eta,\ \zeta,\ (\xi^2 - \frac{5}{3}\xi^4) - (\eta^2 - \frac{5}{3}\eta^4),\ (\xi^2 - \frac{5}{3}\xi^4) - (\zeta^2 - \frac{5}{3}\zeta^4)\} & \text{for } n = 3. \end{cases} \quad (4.19)$$

The nonconforming finite element spaces V_h^{DSSY} are defined as in (4.13) with $\mathcal{Q}_{\mathcal{H}}^{nc}$ replaced by $\mathcal{Q}_{\mathcal{D}}^{nc}$. There holds (H1)–(H3) with the interpolation operator Π of V_h^{DSSY}, cf. the proof in Sect. 4 of [25] for 2D. Arguments similar to those of Subsection 4.3.4 verify (H1)–(H3) for 3D.

Remark 4.3 The parallelogram nonconforming element of [47] can also be analyzed by this unifying theory.

4.4 Comments on mortar finite element methods

Another class of nonconforming FEM is known as mortar FEM [10,11] where the continuity of u_h over the common side of two subdomains K^- and K^+ in some locally quasi-uniform regular decomposition $\mathcal{T}_H$ of Ω into triangles is enforced by Lagrange multipliers. The a posteriori error estimates with the saturation assumptions are presented in [12,60]. A more general one is analyzed in [9]. For the ease of the discussion, suppose that $n = 2$ and that the partition $\mathcal{T}_h$ is obtained from $\mathcal{T}_H$ by refining some of the triangles in $\mathcal{T}_H$ by some finite number $\leq k$ of successive red-refinements (i.e., cutting a triangle into 4 congruent subtriangles by connecting its edges' midpoints) so that the ratio of the diameters of two neighbouring triangles with adjoined edges is bounded by 2^{-k}. Notice that (2.5) holds for all edges E of T while the equivalence with ω_T depends on k.

Let V_h^{nc} be the mortar finite element space with respect to $\mathcal{T}_h$ as in [9]. With $V_h^c := V \cap P_1(\mathcal{T}_H)$ one can prove (H1) by along the lines of [21]. Since $V_h^c \subseteq V_h^{nc}$, (H3) holds for $\Pi = \mathrm{id}$. Then, Theorem 2.1 reads

$$\|\mathrm{Res}\|_{V^*}^2 \lesssim \sum_{T\in\mathcal{T}_h} H_T^2 \|g + \operatorname{div} p_h\|_{L^2(T)}^2 + \sum_{E\in\mathcal{E}} H_E \|[p_h]\cdot \nu_E\|_{L^2(E)}^2$$

for $H_T := \max\{\mathrm{diam}(K) : T \subseteq K \in \mathcal{T}_H\}$ and $H_E := \max\{\mathrm{diam}(K) : E \subset \partial K, K \in \mathcal{T}_H\}$. Moreover, Theorem 3.1 yields (with $\mathcal{T} = \mathcal{T}_h$ etc.)

$$\min_{\tilde{u}_h \in V} \|\mathrm{Res}_L\|_{L^*}^2 \lesssim \sum_{E\in\mathcal{E}} \sum_{z\in\mathcal{K}(E)} H_E \|\gamma_{\tau_E}([D_{\mathcal{T}}((\psi_z u_h)])\|_{L^2(E)}^2.$$

Therein, ψ_z is the partition of unity with respect to $\mathcal{T}_H$ and $\|H_E D\psi_z\|_{L^\infty} \approx 1$.

This reliability error estimate is essential Theorem 3.4 in [9]. In fact, since (in 2D)

$$\frac{\partial}{\partial s}[\psi_z u_h] = \Big(\frac{\partial}{\partial s}\psi_z\Big)[u_h] + \psi_z[\partial u_h/\partial s],$$

there holds (with an inverse estimate $\|\partial[u_h]/\partial s\|_{L^2(E)} \lesssim H_E^{-1}\|[u_h]\|_{L^2(E)}$) that

$$\begin{aligned} H_E \Big\|\frac{\partial}{\partial s}[\psi_z u_h]\Big\|_{L^2(E)}^2 &\lesssim H_E H_T^{-2}\|[u_h]\|_{L^2(E)}^2 + H_E \cdot \|[\partial u_h/\partial s]\|_{L^2(E)}^2 \\ &\lesssim (H_E H_T^{-2} + H_E^{-1})\|[u_h]\|_{L^2(E)}^2 \lesssim H_E^{-1}\|[u_h]\|_{L^2(E)}^2 \,. \end{aligned}$$

Altogether, the upper bounds for (1.7) with $p_h := D_{\mathcal{T}} u_h$ and $p = \nabla u$ reads

$$\begin{aligned} \|p - p_h\|_{L^2(\Omega)} \lesssim &\sum_{T\in\mathcal{T}_h} H_T^2 \|g + \operatorname{div} p_h\|_{L^2(T)}^2 + \sum_{E\in\mathcal{E}} H_E \|[p_h]\cdot \nu_E\|_{L^2(E)}^2 \\ &+ \sum_{E\in\mathcal{E}\cup\mathcal{E}_{\partial\Omega}} H_E^{-1}\|[u_h]\|_{L^2(E)}^2 \,. \end{aligned}$$

Therein, $\mathcal{E}_{\partial\Omega}$ denote the set of edges on the boundary $\partial\Omega$. Notice $[u_h] = 0$ on edges interior to $T \in \mathcal{T}_H$.

In comparison to [9, Theorem 3.4], the factor 2^{-k} therein is hidden herein the mesh-sizes H_T, H_E.

4.5 Comments on discontinuous Galerkin methods

The feature for the discontinous Galerkin(abbreviated dG hereafter) methods [3–6,30,43,57] lies in that the trial and test spaces consist of piecewise discontinuous polynomials. A posteriori error estimates for dG type methods are considered in [7,18,39,44,51,52] for second order elliptic problems, in [37] for the Stokes problem, and in [38,58] for plane elasticity. This subsection comments on the extension of the unifying theory to dG FEM. For any $v_h \in \mathcal{P}_k(\mathcal{T})$, the average across $E = T_1 \cap T_2$ reads

$$\langle v\rangle_E(x) := 1/2((v_h|_{T_1})(x) + (v_h|_{T_2})(x)) \quad \text{for } x \in E.$$

With some appropriately chosen constant γ, the modified bilinear form is defined as

$$\begin{aligned} a_h^{\gamma}(u_h, v_h) := & \sum_{T\in\mathcal{T}} \int_T \nabla u_h \cdot \nabla v_h \, dx + \gamma \sum_{E\in\mathcal{E}} h_E^{-1} \int_E [u_h]_E [v_h]_E \, ds \\ & - \sum_{E\in\mathcal{E}} \int_E (\langle \nabla_h u_h\rangle_E \cdot \nu_E \, [v_h]_E + \langle \nabla_h v_h\rangle_E \cdot \nu_E \, [u_h]_E) \, ds \\ & - \sum_{E\subset\partial\Omega} \int_E (\nabla_h u_h \cdot \nu_E v_h + \nabla_h v_h \cdot \nu_E \, u_h) \, ds + \gamma \sum_{E\subset\partial\Omega} h_E^{-1} \int_E u_h v_h \, ds \end{aligned}$$

for any $u_h, v_h \in \mathcal{P}_k(\mathcal{T}) + H_0^1(\Omega)$. This is the symmetric dG method from [5,6,43,44]. The discontinuous Galerkin solution $u_h \in \mathcal{P}_k(\mathcal{T})$ is characterized by

$$a_h^{\gamma}(u_h, v_h) = (g, v_h)_{L^2(\Omega)} \quad \text{for any } v_h \in \mathcal{P}_k(\mathcal{T}). \tag{4.20}$$

From $V_h^c \subset \mathcal{P}_k(\mathcal{T})$, there holds (H3) with $\Pi = \text{id}$. Theorem 3.1 yields

$$\min_{\tilde{u}_h \in V} \|\text{Res}_L\|_{L^*}^2 \lesssim \sum_{E\in\mathcal{E}} \sum_{z\in\mathcal{K}(E)} h_E \|\gamma_{\tau_E}([D_{\mathcal{T}}((\psi_z u_h)])\|_{L^2(E)}^2.$$

To bound $\|\mathcal{R}es_V\|_{V^*}$, let $v \in V$ and deduce

$$\begin{aligned} \mathcal{R}es_V(v) &= -\int_\Omega gv \, dx + \int_\Omega \nabla_{\mathcal{T}} u_v \cdot \nabla v \, dx \\ &= -\int_\Omega gv \, dx + \int_\Omega \nabla_{\mathcal{T}} u_v \cdot \nabla v \, dx \end{aligned}$$

$$- \sum_{E\in\mathcal{E}} \int_E \langle \nabla_h u_h \rangle_E \cdot \nu_E [v]_E \, ds + \gamma \sum_{E\in\mathcal{E}} h_E^{-1} \int_E [u_h]_E [v]_E \, dx$$
$$- \sum_{E\subset\partial\Omega} \int_E \langle \nabla_h u_h \rangle_E \cdot \nu_E [v]_E \, ds + \gamma \sum_{E\subset\partial\Omega} h_E^{-1} \int_E u_h v \, ds.$$

It follows from $Jv \in V \cap \mathcal{P}_k(\mathcal{T})$ that

$$\|\mathcal{R}es_V\|_{V^*} \lesssim \eta + \operatorname{osc}(g) + \left(\sum_{E\in\mathcal{E}} h_E^{-1} \|[u_h]_E\|_{L^2(E)}^2 + \sum_{E\subset\partial\Omega} h_E^{-1} \|u_h\|_{L^2(E)}^2 \right)^{1/2}.$$

Remark 4.4 A combination of the above estimates for $\|\mathcal{R}es_V\|_{V^*}$ and $\min_{\tilde{u}_h\in V} \|\mathcal{R}es_L\|_{L^*}$ with (1.7) recovers the estimate

$$\|p - p_h\|_{L^2(\Omega)} \lesssim \|\mathcal{R}es_V\|_{V^*} + \min_{\tilde{u}_h\in V} \|\mathcal{R}es_L\|_{L^*},$$

which appeared in Theorem 3.1 of [7] and Theorem 3.1 from [44] without the assumption $u \in H^2(\Omega)$. Where $p = \nabla u$ and $p_h = \nabla_{\mathcal{T}} u_h$.

Remark 4.5 For brevity, we only consider the a posteriori error estimate of the symmetric dG methods for the Poisson equation, the analysis with corresponding modifications can equally apply to the Stokes problem in Sect. 5, and the elasticity in Sect. 6. In particular, this yields the a posteriori error control from Theorem 4.1 of [58] for the plane elasticity, and Theorem 3.1 of [37] for the Stokes problem. Moreover, the unifying theory can be generalized to other dG methods reviewed in [4].

4.6 Comments on high-order nonconforming schemes

In this paper, we focus on the first-order nonconforming finite element method. The present unifying theory can be extended to high-order NCFEMs with the corresponding modifications in (H1)–(H3). In fact, Theorem 3.1 holds equally for all nonstandard finite element methods. We only need to modify the the conforming space V_h^c in (H1) and (H3) and its associated Clemént interpolation operator. For instance, (H3) reads

$$\int_T \Pi v_h q dx = \int_T v_h q dx \quad \text{for any } q \in P_{k-1}(T) \text{ and for any } v_h \in V_h^c.$$

5 Applications to the Stokes problem

5.1 The Stokes problem

The unsymmetric formulation of the Stokes problem reads: given $g \in L^2(\Omega)^n$ seek $(u,p) \in H_0^1(\Omega)^n \times L_0^2(\Omega)$, such that for all $(v,q) \in H_0^1(\Omega)^n \times L_0^2(\Omega)$,

$$\mu \int_\Omega Du : Dv\, dx - \int_\Omega p \operatorname{div} v\, dx - \int_\Omega q \operatorname{div} u\, dx = \int_\Omega g \cdot v\, dx. \tag{5.1}$$

Here, $L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q\, dx = 0\} \equiv L^2(\Omega)/\mathbb{R}$ fixes a global additive constant in the pressure p (note that p is *not* the flux from the previous section). The unique existence of solution to (5.1) is well known. Set

$$a(\sigma,\tau) := \int_\Omega \frac{1}{\mu} \operatorname{dev}\sigma : \operatorname{dev}\tau \; dx$$
$$\text{for all } \sigma,\tau \in L := \left\{ \tau \in L^2(\Omega, \mathbb{R}^{n\times n}), \int_\Omega \operatorname{tr}\tau\, dx = 0 \right\}. \tag{5.2}$$

The deviatoric-part operator dev is defined as

$$\operatorname{dev} F = F - (\operatorname{tr}(F)/n)\ \mathrm{id} \quad \text{for any } F \in \mathbb{R}^{n\times n}. \tag{5.3}$$

with $\operatorname{tr}(F) = F_{11} + \cdots + F_{nn}$. It is known that the operator $A : X = L \times V \to X^*$, defined for $(\sigma,u) \in X$ by

$$(A(\sigma,u))(\tau,v) := a(\sigma,\tau) - (\sigma, Dv)_{L^2(\Omega)} - (\tau, Du)_{L^2(\Omega)} \tag{5.4}$$

is a linear, bounded and bijective, cf. e.g., [20].

5.2 Nonconforming finite element methods and unified a posteriori error estimators

Given some nonconforming finite element space V_h^{nc} for $V := H_0^1(\Omega)^n$ and $\mathcal{Q}_h \subset L_0^2(\Omega)$, the finite element solution $(u_h, p_h) \in V_h^{nc} \times \mathcal{Q}_h$ to (5.1) satisfies, for all $(v_h, q_h) \in V_h^{nc} \times \mathcal{Q}_h$,

$$\mu \int_\Omega D_\mathcal{T} u_h : D_\mathcal{T} v_h\, dx - \int_\Omega p_h \operatorname{div}_\mathcal{T} v_h\, dx - \int_\Omega \operatorname{div}_\mathcal{T} u_h\, q_h\; dx = \int_\Omega g\, v_h\, dx. \tag{5.5}$$

Given the unique discrete solution $u_h \in H^1(\mathcal{T})^n$ and $p_h \in L^2_0(\Omega)$, set

$$\sigma_h := \mu D_{\mathcal{T}} u_h - p_h \,\mathrm{id} \in L \tag{5.6}$$

and define the *linear functional* $\mathcal{R}es_V : V := H^1_0(\Omega)^n \to \mathbb{R}$ by

$$\mathcal{R}es_V(v) = \int_\Omega (g \cdot v - \sigma_h : Dv)\, dx \quad \text{for } v \in V := H^1_0(\Omega)^n. \tag{5.7}$$

The theory of Sect. 3 shows that the norm of the residual $\mathcal{R}es_L$ reads

$$\|\mathcal{R}es_L\|_{L^*} \approx \|D(\tilde{u}_h) - \frac{1}{\mu}\,\mathrm{dev}\, D_{\mathcal{T}}(u_h)\|_{L^2(\Omega)}. \tag{5.8}$$

Given any $\tilde{u}_h \in V$ with $\sigma := \mu Du - p\,\mathrm{id}$, the unifying theory in the form of (1.7) and (5.4) prove

$$\begin{aligned}\|\sigma - \sigma_h\|_L + \|u - \tilde{u}_h\|_V \lesssim \|D(\tilde{u}_h) - D_{\mathcal{T}}(u_h)\|_{L^2(\Omega)} + \|\,\mathrm{div}_{\mathcal{T}}\, u_h\|_{L^2(\Omega)}\\ + \|\mathcal{R}es_V(v)\|_{V^*}.\end{aligned} \tag{5.9}$$

5.3 Examples

This subsection lists some examples of nonconforming finite element schemes with (H1)–(H3) from the literature displayed in Table 2. Then, it follows from (5.9), the definitions of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_h$ with a straightforward investigation, Theorems 2.1 and 3.1, that

$$\begin{aligned}&\|Du - D_{\mathcal{T}} u_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)}\\ &\quad\lesssim \min_{\tilde{u}_h \in V} \|D(\tilde{u}_h) - D_{\mathcal{T}}(u_h)\|_{L^2(\Omega)} + \|\,\mathrm{div}_{\mathcal{T}}\, u_h\|_{L^2(\Omega)} + \|\mathcal{R}es_V(v)\|_{V^*}\\ &\quad\lesssim \eta + \mu + \|\,\mathrm{div}_{\mathcal{T}}\, u_h\|_{L^2(\Omega)} + \mathrm{osc}(g).\end{aligned} \tag{5.10}$$

This recovers the result from [26,28] for the Crouzeix–Raviart element, and is new for four parallelogram elements of Subsubsect. 5.3.2.

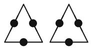
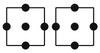
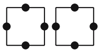
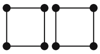
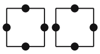
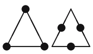
5.3.1 The Crouzeix–Raviart element

This is a triangular element with the velocity space

$$V_h^{nc} := V_h^{CR} \times V_h^{CR}$$

for the space V_h^{CR} from Subsection 4.3.1, and the piecewise constant pressure space $\mathcal{Q}_h \subset L^2_0(\Omega)$. Since $V_h^c \times V_h^c \subset V_h^{CR} \times V_h^{CR}$, there holds (H1)–(H3) with $\Pi = \mathrm{id}$.

Table 2 Nonconforming elements for the Stokes problem (5.1) with (H1)–(H3) and the error estimate (5.10)

Picture	Name	Reference	Space
	Crouzeix and Raviart	[27]	$V_h^{CR} \times V_h^{CR}$
	Han	[36]	$V_h^{Han} \times V_h^{Han}$
	NR	[50]	$V_h^{RT,A} \times V_h^{RT,A}$
	Hu et al.	[40]	$V_h^{CRT} \times V_h^{CRT}$
	CJY	[19]	$V_h^{DSSY} \times V_h^{DSSY}$
	Kouhia and Stenberg	[45]	$V_h^{c} \times V_h^{CR}$

5.3.2 Four parallelogram elements

There are four parallelogram elements in the literature including the parallelogram Han element, the parallelogram nonconforming rotated (*NR*) element of Rannacher and Turek [50], the parallelogram *CJY* element [19], and the parallelogram constrained nonconforming rotated element of Hu et al. [40]. These elements employ the piecewise constant pressure space $Q_h \subset L_0^2(\Omega)$. The velocity spaces for these methods are chosen from the following list.

$$V_h^{nc} := V_h^{Han} \times V_h^{Han}, V_h^{RT,A} \times V_h^{RT,A}, \\ V_h^{CRT} \times V_h^{CRT}, V_h^{DSSY} \times V_h^{DSSY}.$$

Herein V_h^{Han}, $V_h^{RT,A}$, V_h^{CRT} and V_h^{DSSY} denote the nonconforming finite element spaces from the respective Subsubsects. 4.3.3–4.3.6. Then there holds (H1)–(H3) with the canonical interpolation operators Π for these nonconforming finite element spaces. The proof follows with the results of Sect. 4; further details are omitted.

Remark 5.1 The parallelogram nonconforming finite elements from [26] can also be analyzed in the present framework to recover the a posteriori error estimation on for the isotropic mesh therein.

5.4 The Kouhia–Stenberg element

The Stokes problem in its form (5.1) is equivalent to the symmetric form with $\varepsilon(u) := \text{sym}(Du) := 1/2(Du + Du^T)$ replacing Du in (5.1). The velocity space

[45] reads

$$V_h^{nc} := V_h^c \times V_h^{CR}.$$

Since $V_h^c \times V_h^c \subset V_h^c \times V_h^{CR}$, there holds (H1)–(H3) with $\Pi = \mathrm{id}$, cf. [20].

6 Linear elasticity

This section is devoted to the Navier–Lamé equation and its locking-free nonconforming finite element approximation. The presented unifying theory leads to a posteriori error estimates which are robust with respect to the Lamé parameter $\lambda \to \infty$. Subsection 6.1 displays the model problem and Subsection 6.2 NCFEMs and their unifying error control. Subsection 6.3 presents some examples. Subsection 6.4 discusses the unsymmetric formulation for linear elasticity and the examples for this case are given in Subsection 6.5

6.1 Model problem

Adopt the notation of the previous sections and the following linear stress-strain relation, for λ, $\mu > 0$,

$$\begin{aligned}&\mathbb{C}F := \lambda\ \mathrm{tr}(F)\ \mathrm{id} + 2\mu F \quad \text{and}\\ &\mathbb{C}^{-1}F := \frac{1}{2\mu}F - \frac{\lambda}{2\mu(n\lambda + 2\mu)}\,\mathrm{tr}(F)\ \mathrm{id}, \quad \text{for } F \in \mathbb{R}^{n\times n}.\end{aligned} \tag{6.1}$$

The weak form of the linear elasticity problem reads: Given $g \in L^2(\Omega)^n$ find $u \in V := H_0^1(\Omega)^n$ with

$$\int_\Omega \varepsilon(v) : \sigma\, dx = \int_\Omega g \cdot v\, dx \quad \text{and} \quad \sigma = \mathbb{C}\varepsilon(u) \quad \text{for all } v \in V. \tag{6.2}$$

Define the operator $A : X = L \times V \to X^*$ for any $(\sigma, u) \in X$ by

$$(A(\sigma, u))(\tau, v) := (\mathbb{C}^{-1}\sigma, \tau)_{L^2(\Omega)} - (\sigma, \varepsilon(v))_{L^2(\Omega)} - (\tau, \varepsilon(u))_{L^2(\Omega)}. \tag{6.3}$$

Here, $L := \{\sigma \in L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{sym}}), \int_\Omega \mathrm{tr}\,\sigma\, dx = 0\}$. The operator A is linear, bounded, and bijective with λ-independent operator norms of A and A^{-1} [14,24].

6.2 Nonconforming finite element methods and unified a posteriori error estimators

With the nonconforming finite element approximation $u_h \in V_h^{nc}$ to u and the discrete Green strain $\varepsilon_{\mathcal{T}}(v) := (D_{\mathcal{T}} v + D_{\mathcal{T}} v^T))/2 \in L^2(\Omega; \mathbb{R}^{n\times n}_{\text{sym}})$, set

$$\sigma_h = 2\mu \varepsilon_{\mathcal{T}}(u_h) + \lambda \Pi_2 \operatorname{div}_{\mathcal{T}} u_h \operatorname{id}. \tag{6.4}$$

Throughout this section, $\Pi_2 : L^2(\Omega) \to L^2(\Omega)$ denotes some reduction operators in the context of the locking phenomena, and the discrete stress σ_h is supposed to satisfy

$$\int_\Omega \sigma_h : \varepsilon_{\mathcal{T}}(v_h)\, dx = \int_\Omega g \cdot v_h \, dx \quad \text{for all } v_h \in V_h^{nc}. \tag{6.5}$$

We define the continuous and discrete pressures as

$$p = \lambda \operatorname{div} u \quad \text{and} \quad p_h = \lambda \Pi_2 \operatorname{div}_{\mathcal{T}} u_h. \tag{6.6}$$

Theorem 6.1 *For any $\tilde{u}_h \in V$ there holds*

$$\begin{aligned} &\|\varepsilon(u) - \varepsilon_{\mathcal{T}}(u_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} + \|\varepsilon(u - \tilde{u}_h)\|_{L^2(\Omega)} \\ &\quad \approx \|\varepsilon_{\mathcal{T}}(u_h) - \varepsilon(\tilde{u}_h)\|_{L^2(\Omega)} + \|\mathcal{R}es_V\|_{V^*} + \|\operatorname{div}_{\mathcal{T}} u_h - \Pi_2 \operatorname{div}_{\mathcal{T}} u_h\|_{L^2(\Omega)}. \end{aligned} \tag{6.7}$$

Proof The unifying theory with (1.7) and (6.3) reads in the present notations

$$\|\sigma - \sigma_h\|_L + \|\varepsilon(u - \tilde{u}_h)\|_{L^2(\Omega)} \approx \|\mathbb{C}^{-1}\sigma_h - \varepsilon(\tilde{u}_h)\|_{L^2(\Omega)} + \|\mathcal{R}es_V\|_{V^*}. \tag{6.8}$$

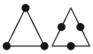
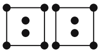
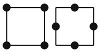
Then the assertion follows from the definitions of σ_h, $\mathbb{C}^{-1}$, p, and p_h. □

6.3 Examples

This subsection analyzes finite element methods depicted in Table 3 for the planar elasticity problem. These schemes satisfy (H1)–(H3). Then, the estimate (6.7) with Theorems 2.1 and 3.1 leads to

$$\begin{aligned} &\|\varepsilon(u) - \varepsilon_{\mathcal{T}}(u_h)\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \\ &\quad \lesssim \min_{\tilde{u}_h \in V} \|\varepsilon_{\mathcal{T}}(u_h) - \varepsilon(\tilde{u}_h)\|_{L^2(\Omega)} + \|\mathcal{R}es_V\|_{V^*} + \|\operatorname{div}_{\mathcal{T}} u_h - \Pi_2 \operatorname{div}_{\mathcal{T}} u_h\|_{L^2(\Omega)} \\ &\quad \lesssim \mu + \eta + \|\operatorname{div}_{\mathcal{T}} u_h - \Pi_2 \operatorname{div}_{\mathcal{T}} u_h\|_{L^2(\Omega)} + \operatorname{osc}(g). \end{aligned} \tag{6.9}$$

Table 3 Nonconforming elements for the linear elasticity problem (6.2) with (H1)–(H3) and the error estimate (6.9)

Picture	Name	Reference	Space
	Kouhia and Stenberg	[45]	$V_h^c \times V_h^{CR}$
	Zhang	[61]	$V_h^{Wil} \times V_h^{Wil}$
	Ming	[48]	$V_h^c \times V_h^{RT,A}$

The error control for the Kouhia–Stenberg element has already been analyzed in [20]. The a posteriori error estimator (6.9) for the Falk elements, the Zhang element, and the Ming element is new.

6.3.1 The Falk elements

Two nonconforming triangular finite element methods are proposed in [32] for the linear elasticity equation for $k = 2, 3$ with $\Pi_2 = \text{id}$ and

$$V_h^{nc} := \{v \in L^2(\Omega)^2 : \forall T \in \mathcal{T}_h, v|_T \in P_k(T)^2 \text{ and } v \text{ is continuous (resp. vanishes) at the k Gauss points on each interior (resp. boundary) edge}\}. \quad (6.10)$$

Since $V_h^c \times V_h^c \subset V_h^{nc}$ there holds (H1)–(H3) with $\Pi = \text{id}$.

6.3.2 The Kouhia–Stenberg element

This triangular element for the symmetric formulation (6.1) and $\Pi_2 = \text{id}$ [45] is defined by the nonconforming finite element space

$$V_h^{nc} := V_h^c \times V_h^{CR}. \quad (6.11)$$

Since $V_h^c \times V_h^c \subset V_h^c \times V_h^{CR}$ there holds (H1)–(H3) with $\Pi = \text{id}$, cf. also [20].

6.3.3 The Zhang element

This element is proposed in [61] based on the nonconforming quadrilateral Wilson element [54,59] with $\Pi_2 = \text{id}$. In this element,

$$V_h^{nc} := V_h^{Wil} \times V_h^{Wil}. \quad (6.12)$$

Since $V_h^c \times V_h^c \subset V_h^{Wil} \times V_h^{Wil}$ there holds (H1)–(H3) with $\Pi = \text{id}$.

6.3.4 The Ming element

In Ming's dissertation [48], a parallelogram nonconforming element is proposed based on the nonconforming rotated Q_1 space from [50] for planar elasticity. The nonconforming finite element space reads

$$V_h^{nc} := V_h^c \times V_h^{RT,A} \tag{6.13}$$

where $\Pi_2 = \Pi_0 : L^2(\Omega) \to Q_0$ denotes the piecewise constant projection operator with Q_0 the piecewise constant space. Following the arguments in Subsubsection 4.3.4 and [25], one proves (H1)–(H3) for the associated interpolation operator Π.

6.4 The unsymmetric formulation

For the pure Dirichlet boundary condition under consideration, one can use the equivalent unsymmetric formulation and then define the following formal stress-strain relation, for $F \in \mathbb{R}^{n\times n}$,

$$\mathbb{C}F := (\lambda+\mu)\,\mathrm{tr}(F)\,\mathrm{id} + \mu F \quad \text{and} \quad \mathbb{C}^{-1}F := \frac{1}{\mu}F - \frac{\lambda+\mu}{\mu(n\lambda+(n+1)\mu)}\,\mathrm{tr}(F)\,\mathrm{id}\,. \tag{6.14}$$

Given some nonconforming finite element space V_h^{nc}, the finite element solution $u_h \in V_h^{nc}$ satisfies

$$\int_\Omega \sigma_h : D_\mathcal{T} v_h\,dx = \int_\Omega g\cdot v_h\,dx \quad \text{for all } v_h \in V_h^{nc}. \tag{6.15}$$

Given the unique discrete solution $u_h \in V_h^{nc}$, set

$$\sigma_h = \mu D_\mathcal{T} u_h + (\lambda+\mu)\Pi_2\,\mathrm{div}_\mathcal{T}\, u_h\,\mathrm{id}, \tag{6.16}$$

The continuous and discrete pressures read

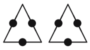
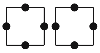
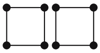
$$p = (\lambda+\mu)\,\mathrm{div}\, u \quad \text{and} \quad p_h = (\lambda+\mu)\Pi_2\,\mathrm{div}_\mathcal{T}\, u_h. \tag{6.17}$$

Define the operator $A : X = L \times V := \{\tau \in L^2(\Omega, \mathbb{R}^{n\times n}), \int_\Omega \mathrm{tr}\,\tau\,dx = 0\} \times H_0^1(\Omega)^n \to X^*$ for any $(\sigma, u) \in X$ as

$$(A(\sigma,u))(\tau,v) := (\mathbb{C}^{-1}\sigma,\tau)_{L^2(\Omega)} - (\sigma, Dv)_{L^2(\Omega)} - (\tau, Du)_{L^2(\Omega)}.$$

The arguments for the symmetric case in [14] show that the operator A is linear, bounded, and bijective with λ-independent operator norms of A and A^{-1}. Following the argument for the symmetric case, one proves

Table 4 Nonconforming elements for the linear elasticity problem in unsymmetric formulation with (H1)–(H3) and the error estimate (6.19)

Picture	Name	Reference	Space
	Brenner and Sung	[16]	$V_h^{CR} \times V_h^{CR}$
	Lee et al.	[46]	$V_h^{RT,A} \times V_h^{RT,A}$
	Hu-Man-Shi	[40]	$V_h^{CRT} \times V_h^{CRT}$

Theorem 6.2 *For any $\tilde{u}_h \in V$ there holds that*

$$\begin{aligned}\|Du - D_{\mathcal{T}} u_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} + \|D(u - \tilde{u}_h)\|_{L^2(\Omega)} \\ \approx \|D_{\mathcal{T}} u_h - D\tilde{u}_h\|_{L^2(\Omega)} + \|\mathcal{R}es_V\|_{V^*} + \|\operatorname{div}_{\mathcal{T}} u_h - \Pi_2 \operatorname{div}_{\mathcal{T}} u_h\|_{L^2(\Omega)}.\end{aligned} \tag{6.18}$$

6.5 Examples

Three nonconforming finite elements are listed below as examples with the unsymmetric formulation and are summarized in Table 4. There holds that

$$\begin{aligned}\|Du - D_{\mathcal{T}} u_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \\ \lesssim \mu + \eta + \|\operatorname{div}_{\mathcal{T}} u_h - \Pi_2 \operatorname{div}_{\mathcal{T}} u_h\|_{L^2(\Omega)} + \operatorname{osc}(g).\end{aligned} \tag{6.19}$$

This a posteriori error estimator is brand new for these elements.

6.5.1 The Brenner–Sung element

This triangular element is proposed in [16] with $\Pi_2 = \mathrm{id}$, and

$$V_h^{nc} := V_h^{CR} \times V_h^{CR}. \tag{6.20}$$

Since $V_h^c \times V_h^c \subset V_h^{CR} \times V_h^{CR}$ there holds (H1)–(H3) with $\Pi = \mathrm{id}$.

6.5.2 The Lee–Lee–Sheen element

In this parallelogram element [46], both components of the displacement are approximated by the nonconforming rotated Q_1 space from [50], namely

$$V_h^{nc} := V_h^{RT,A} \times V_h^{RT,A}. \tag{6.21}$$

The reduction integration operator is the same as in the Ming elements. (H1)–(H3) is satisfied by this element with the canonical interpolation operator Π for V_h^{nc}. It follows the arguments for the nonconforming rotated Q_1 element in Subsubsection 4.3.4.

6.5.3 *The Hu–Man–Shi element*

This parallelogram element is designed in [40] without reduction integration. The nonconforming finite element space is the constrained nonconforming rotated Q_1 from [41]. There also holds (H1)–(H3) with the canonical interpolation operator Π. The proof can be found in Subsubsection 5.3.2.

Remark 6.1 Our conditions and therefore analysis in this paper can be extended to other nonstandard finite element methods for the elasticity, for instance, the Wang-Qi element from [56] and the enhanced strain finite element from [14,53].

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