Numer. Math. (2006) 103: 413–433
DOI 10.1007/s00211-006-0686-x

Numerische Mathematik

Kerstin Hesse

A lower bound for the worst-case cubature error on spheres of arbitrary dimension

Received: 31 January 2005 / Revised: 14 January 2006 / Published online: 11 April 2006

Abstract This paper is concerned with numerical integration on the unit sphere S^r of dimension $r \geq 2$ in the Euclidean space $\mathbb{R}^{r+1}$. We consider the worst-case cubature error, denoted by $E(Q_m; H^s(S^r))$, of an arbitrary m-point cubature rule Q_m for functions in the unit ball of the Sobolev space $H^s(S^r)$, where $s > \frac{r}{2}$, and show that $E(Q_m; H^s(S^r)) \geq c_{s,r}\, m^{-\frac{s}{r}}$. The positive constant $c_{s,r}$ in the estimate depends only on the sphere dimension $r \geq 2$ and the index s of the Sobolev space $H^s(S^r)$. This result was previously only known for $r = 2$, in which case the estimate is order optimal. The method of proof is constructive: we construct for each Q_m a 'bad' function f_m, that is, a function which vanishes in all nodes of the cubature rule and for which $\| f_m \|_{s,r}^{-1} \, | \int_{S^r} f_m(\mathbf{x})\, d\omega_r(\mathbf{x}) | \geq c_{s,r}\, m^{-\frac{s}{r}}$. Our proof uses a packing of the sphere S^r with spherical caps, as well as an interpolation result between Sobolev spaces of different indices.

Mathematics Subject Classification (1991): Primary 41A55 · Secondary 46B70 · 46E22 · 46E35 · 52C17 · 65D30 · 65D32

1 Introduction

Cubature (or numerical integration) rules on the sphere are needed in applications in geophysics and partial differential equations, where integrals over spherical geometries need to be computed from discrete (measured) data. For example, in

---

Send offprint requests to: Kerstin Hesse

K. Hesse (✉)
School of Mathematics, The University of New South Wales, Sydney NSW 2052, Australia,
Tel: +61 2 9385 7074,
Fax: +61 2 9385 7123,
E-mail: k.hesse@unsw.edu.au

the modelling of the earth's gravitational potential or magnetic field from satellite data on (nearly) spherical orbits, cubature rules on the sphere are an important tool (see for instance [3]). In this paper we consider cubature on general spheres S^r, $r \geq 2$, in a Sobolev space setting and derive a lower bound for the worst-case cubature error in terms of the number of nodes of the cubature rule. Such a lower bound tells us how fast the worst-case cubature error of any sequence $(Q_{m(n)})_{n\in\mathbb{N}_0}$, with $\lim_{n\to\infty} m(n) = \infty$, of cubature rules $Q_{m(n)}$ with $m = m(n)$ nodes could at best decline as $n \to \infty$, and hence gives important information about the 'limitations' of cubature on S^r in a Sobolev space setting.

Let $S^r \subset \mathbb{R}^{r+1}$, where $r \geq 2$, denote the unit sphere in the Euclidean space $\mathbb{R}^{r+1}$. The integral of a continuous function $f : S^r \to \mathbb{R}$, denoted by

$$If := \int_{S^r} f(\mathbf{x})\, d\omega_r(\mathbf{x}),$$

where $d\omega_r(\mathbf{x})$ is the surface element of S^r, can be approximated by an *m-point cubature rule* (or m-point numerical integration rule)

$$Q_m f := \sum_{j=1}^{m} w_j\, f(\mathbf{x}_j),$$

with nodes $\mathbf{x}_1, \ldots, \mathbf{x}_m \in S^r$ and corresponding weights $w_1, \ldots, w_m \in \mathbb{R}$.

The *worst-case (cubature) error* of the cubature rule Q_m in a space H of continuous functions on S^r, with norm $\|\cdot\|_H$, is defined by

$$E(Q_m; H) := \sup_{f\in H,\, \|f\|_H \leq 1} |Q_m f - If|.$$

An important question in relation to the worst-case cubature error in H is how good any m-point cubature rule Q_m in H can *at best* be. Mathematically speaking, we are interested in a lower bound for $\inf\{E(Q_m; H) \mid Q_m\}$ in terms of orders of m, where the infimum is taken over all m-point cubature rules Q_m on S^r. Such a lower bound is sharp or (order) optimal if we can identify sequences $(Q_{m(n)})_{n\in\mathbb{N}_0}$ of $m(n)$-point cubature rules $Q_{m(n)}$, with $\lim_{n\to\infty} m(n) = \infty$, for which the worst-case error $E(Q_{m(n)}; H)$ has an upper bound in terms of orders of $m = m(n)$ that matches the lower bound for $\inf\{E(Q_m; H) \mid Q_m\}$ (in terms of orders of m).

In this work we consider the Sobolev (Hilbert) space $H^s(S^r)$, with norm $\|\cdot\|_{s,r}$, where $r \geq 2$ and $s > \frac{r}{2}$. Intuitively, the space $H^s(S^r)$, with $s > \frac{r}{2}$, is the space of those continuous functions on the sphere S^r whose generalized (distributional) derivatives up to (and including) the order s are square-integrable. We show that the worst-case (cubature) error $E(Q_m; H^s(S^r))$ of an arbitrary m-point cubature rule Q_m in $H^s(S^r)$ has the lower bound

$$E(Q_m; H^s(S^r)) \geq c_{s,r}\, m^{-\frac{s}{r}}, \tag{1}$$

where the positive constant $c_{s,r}$ depends only on s and r, but is independent of the cubature rule Q_m and the number of nodes m. Hence for $s > \frac{r}{2}$

$$\inf\left\{ E(Q_m; H^s(S^r)) \;\middle|\; \begin{array}{l} Q_m \text{ is an } m\text{-point} \\ \text{cubature rule} \end{array} \right\} \geq c_{s,r}\, m^{-\frac{s}{r}}, \quad m \to \infty.$$

(Throughout this paper the positive constants $c_1, c_2, c_3, \ldots$ have fixed values (possibly depending on r), whereas $c_{s,r}$ and c_r are generic positive constants that depend on the sphere dimension r and the Sobolev space index s, and on the sphere dimension r, respectively (as indicated by the indices), and that may take different values at different places.)

The result (1) is an extension of [6, page 794, Theorem 1], where (1) was shown for the special case of the sphere S^2, that is, $r = 2$. For the sphere S^2 we know from [5,7] also a matching upper bound of the same order for infinite sequences of cubature rules with certain properties. Therefore the result (1) is *optimal* with respect to order for the case $r = 2$ and so are the corresponding upper bounds in [5,7]. We conjecture that the estimate (1) is also order optimal for $r > 2$.

The proof of the estimate (1) is constructive: for a fixed arbitrary m-point cubature rule Q_m we construct a 'bad' function f_m, that is, a function which vanishes in all nodes of the cubature rule (hence $Q_m f_m = 0$) and for which the cubature error satisfies

$$\frac{|Q_m f_m - I f_m|}{\|f_m\|_{s,r}} = \frac{\left|\int_{S^r} f_m(\mathbf{x})\, d\omega_r(\mathbf{x})\right|}{\|f_m\|_{s,r}} \geq c_{s,r}\, m^{-\frac{s}{r}}, \tag{2}$$

where the positive constant $c_{s,r}$ depends only on the space index s and the sphere dimension r, but not on f_m, Q_m, and the number of nodes m. In particular, the constant $c_{s,r}$ is independent of the placement of the nodes $\mathbf{x}_1, \ldots, \mathbf{x}_m$ and the corresponding weights $w_1, \ldots, w_m$. The left-hand side of (2) is clearly a lower bound for the worst-case error $E(Q_m; H^s(S^r))$ which implies (1).

The construction of f_m involves a packing of the sphere with (at least) $2m$ spherical caps of radius $\alpha_m = c_1\,(2m)^{-\frac{1}{r}}$, with a suitable constant c_1 independent of m. As the caps of a packing are by definition not allowed to overlap but may only touch at their boundaries and as there are only m nodes of the cubature rule, we can find at least m caps that contain no nodes in the interior. Then we construct f_m such that the support of f_m is contained in the union of m such caps and such that f_m, restricted to any of these caps, is rotationally symmetric with respect to the center of the cap and looks exactly the same for each cap. In fact we use for all cubature rules Q_m (and for all m) the same suitably chosen infinitely often differentiable function on $\mathbb{R}$ with compact support in the construction of f_m. The 'bad' function f_m vanishes at all nodes of the cubature rule Q_m and satisfies $f_m \in H^s(S^r)$ for any $s \geq 0$, and we use the same function f_m in the proof of the estimate (2) for all $s > \frac{r}{2}$. The difficult part in the proof of (2) is then to find an upper bound for $\|f_m\|_{s,r}$ without giving away any orders of m. This is handled by first deriving an upper bound of $\|f_m\|_{s,r}$ for non-negative even integer values of s. After that we use Hölder's inequality to interpolate between these estimates (for the even integer values of s) to obtain a suitable upper bound for $\|f_m\|_{s,r}$ with arbitrary $s > \frac{r}{2}$.

The method of proof is analogous to the one in the special case of S^2 in [6] which was inspired by the ideas with which lower bounds for the worst-case cubature error in certain spaces of continuous functions on the unit cube $[0, 1]^r$ are derived (see [1,8]). While the function with compact support used in the construction of f_m is essentially the same as in [6], the technical details of the estimate of $\|f_m\|_{s,r}$ are quite different from the proof in [6].

2 Notation and formulation of the results

2.1 Background and notation

We denote the Euclidean inner product of $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{r+1}$ by $\mathbf{x} \cdot \mathbf{y}$ and the induced Euclidean norm by $|\mathbf{x}| := \sqrt{\mathbf{x} \cdot \mathbf{x}}$, where $\mathbf{x} \in \mathbb{R}^{r+1}$.

In this paper we use the Pochhammer symbol $(a)_n$, where $n \in \mathbb{N}_0$ and $a \in \mathbb{R}$, defined by

$$(a)_0 := 1, \quad (a)_n := a(a+1) \cdots (a+n-1) \qquad \text{for } n \in \mathbb{N}.$$

The surface area of the unit sphere S^r in $\mathbb{R}^{r+1}$,

$$S^r := \left\{ \mathbf{x} \in \mathbb{R}^{r+1} \mid |\mathbf{x}| = 1 \right\},$$

is given by

$$\omega_r := |S^r| = \frac{2\,\pi^{\frac{r+1}{2}}}{\Gamma\left(\frac{r+1}{2}\right)}.$$

The (Lebesgue) surface measure of S^r is denoted by $d\omega_r(\mathbf{x})$.

Let $L_2(S^r)$ denote the Hilbert space of square-integrable functions on the sphere S^r endowed with the inner product

$$(f, g)_{L_2(S^r)} := \int_{S^r} f(\mathbf{x})\, g(\mathbf{x})\, d\omega_r(\mathbf{x})$$

and the corresponding norm

$$\|f\|_{L_2(S^r)} := \left(\int_{S^r} |f(\mathbf{x})|^2\, d\omega_r(\mathbf{x}) \right)^{\frac{1}{2}}.$$

With $C(S^r)$ we denote the space of all continuous functions on S^r, endowed with the supremum norm $\|f\|_{C(S^r)} := \sup_{\mathbf{x} \in S^r} |f(\mathbf{x})|$. The space $C^\infty(S^r)$ is the space of infinitely often differentiable functions on S^r, and $C^\infty(\mathbb{R})$ is the space of infinitely often differentiable functions on $\mathbb{R}$.

The restriction of a harmonic homogeneous polynomial in $r+1$ real variables of exact degree ℓ to the sphere S^r is called a spherical harmonic of degree ℓ. We denote the space of all spherical harmonics on S^r of degree $\ell \in \mathbb{N}_0$ by $\mathbb{H}_\ell(S^r)$. The dimension of $\mathbb{H}_\ell(S^r)$ is (see [2, Section 11.2, page 237, formula (2)])

$$\dim(\mathbb{H}_\ell(S^r)) = N(r-1, \ell),$$

where

$$N(r-1, 0) := 1, \qquad N(r-1, \ell) := \frac{(2\ell + r - 1)\,(\ell + r - 2)!}{(r-1)!\,\ell!}, \quad \ell \in \mathbb{N}.$$

We observe that

$$N(r-1, \ell) = \frac{(2\ell + r - 1)\,(\ell + 1)_{r-2}}{(r-1)!} \leq c_r \left(\ell + \frac{r-1}{2} \right)^{r-1} \tag{3}$$

with a positive constant c_r that depends only on the sphere dimension r. Any two spherical harmonics of different degree are orthogonal to each other, and the (real-valued) spherical harmonics of degree ℓ satisfy the addition theorem in $\mathbb{H}_\ell(S^r)$ (see [2, Section 11.4, pages 242–243, Theorem 4] and [10, Section 4.1, page 69–71, Lemma 4.5 and Theorem 4.7]): for any $L_2(S^r)$-orthonormal set $\{Y_{\ell k}^{(r)} \mid k = 1, \ldots, N(r-1,\ell)\}$ of real-valued spherical harmonics of degree ℓ

$$\sum_{k=1}^{N(r-1,\ell)} Y_{\ell k}^{(r)}(\mathbf{x})\, Y_{\ell k}^{(r)}(\mathbf{y}) = \frac{N(r-1,\ell)}{\omega_r} \frac{C_\ell^{\frac{r-1}{2}}(\mathbf{x}\cdot\mathbf{y})}{C_\ell^{\frac{r-1}{2}}(1)}, \quad \mathbf{x},\mathbf{y} \in S^r, \tag{4}$$

where $C_\ell^{\frac{r-1}{2}}$ is the Gegenbauer polynomial (or ultraspherical polynomial) of index $\lambda = \frac{r-1}{2}$ and degree ℓ, defined by

$$C_\ell^\lambda(t) := \frac{(2\lambda)_\ell}{\left(\lambda + \frac{1}{2}\right)_\ell} P_\ell^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(t).$$

Here $P_\ell^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}$ is the Jacobi polynomial with indices $\alpha = \beta = \lambda - \frac{1}{2}$ and degree ℓ (see [11, Section 4.7, page 80, formula (4.7.1)]). Note that $C_\ell^\lambda(1) = \frac{(2\lambda)_\ell}{\ell!}$ (see [2, Subsection 11.1.2, page 236, formula (28)] and [11, Section 4.7, page 80, formula (4.7.3)]) and that $\max_{t\in[-1,1]} |C_\ell^\lambda(t)| = C_\ell^\lambda(1)$ (from [11, Section 4.7, page 80, formula (4.7.3) and Section 7.32, page 168, Theorem 7.32.1]).

The space of all spherical polynomials on S^r of degree $\le n$, that is, the restriction to S^r of all polynomials in $r+1$ real variables of degree at most n, is denoted by $\mathbb{P}_n(S^r)$. It can be shown (see [10, Section 4.1, page 77, Theorem 4.11]) that

$$\mathbb{P}_n(S^r) = \bigoplus_{\ell=0}^{n} \mathbb{H}_\ell(S^r).$$

The dimension of $\mathbb{P}_n(S^r)$ is therefore

$$\dim(\mathbb{P}_n(S^r)) = \sum_{\ell=0}^{n} N(r-1,\ell) = N(r,n) = \frac{(2n+r)(n+r-1)!}{r!\,n!}.$$

From now on

$$\left\{ Y_{\ell k}^{(r)} \;\middle|\; k = 1, \ldots, N(r-1,\ell) \right\} \tag{5}$$

always denotes an $L_2(S^r)$-orthonormal basis for $\mathbb{H}_\ell(S^r)$ of real-valued spherical harmonics of degree ℓ. Then the union of the orthonormal sets (5) for all $\ell \in \mathbb{N}_0$ is a complete orthonormal system in $L_2(S^r)$. Thus every function $f \in L_2(S^r)$ can be expanded into a Fourier series (or Laplace series) with respect to this orthonormal system: in the $L_2(S^r)$ sense

$$f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \hat{f}_{\ell k}^{(r)}\, Y_{\ell k}^{(r)}, \tag{6}$$

where the Fourier coefficients $\hat{f}_{\ell k}^{(r)}$ are given by

$$\hat{f}_{\ell k}^{(r)} := \left(f, Y_{\ell k}^{(r)}\right)_{L_2(S^r)} = \int_{S^r} f(\mathbf{x})\, Y_{\ell k}^{(r)}(\mathbf{x})\, d\omega_r(\mathbf{x}).$$

The *Beltrami operator* Δ^* on S^r is the angular part of the Laplacian Δ on $\mathbb{R}^{r+1}$ (see [10, Section 4.2, pages 80–81], and also [2, Subsection 11.1.1, page 234–235] for the Laplace operator). The spherical harmonics on S^r are eigenfunctions of the Beltrami operator Δ^* on S^r (see [10, Section 4.2, page 81, Theorem 4.13]): for any spherical harmonic $Y_\ell^{(r)} \in \mathbb{H}_\ell(S^r)$

$$\Delta^* Y_\ell^{(r)} = -\ell(\ell + r - 1)\, Y_\ell^{(r)}. \tag{7}$$

The Beltrami operator Δ^* is a self-adjoint operator, that is, for sufficiently often differentiable functions f and g on S^r,

$$\int_{S^r} \left(\Delta^* f(\mathbf{x})\right)\, g(\mathbf{x})\, d\omega_r(\mathbf{x}) = \int_{S^r} f(\mathbf{x})\, \left(\Delta^* g(\mathbf{x})\right)\, d\omega_r(\mathbf{x}).$$

In this work we will eventually need local coordinates which are a slight modification of the usual spherical coordinates:

$$\begin{aligned}
x_1 &= \sqrt{1-t^2}\, \sin\theta_{r-2} \cdots \sin\theta_2\, \sin\theta_1\, \sin\phi \\
x_2 &= \sqrt{1-t^2}\, \sin\theta_{r-2} \cdots \sin\theta_2\, \sin\theta_1\, \cos\phi \\
x_3 &= \sqrt{1-t^2}\, \sin\theta_{r-2} \cdots \sin\theta_2\, \cos\theta_1 \\
&\vdots \\
x_{r-1} &= \sqrt{1-t^2}\, \sin\theta_{r-2}\, \cos\theta_{r-3}, \\
x_r &= \sqrt{1-t^2}\, \cos\theta_{r-2}, \\
x_{r+1} &= t,
\end{aligned} \tag{8}$$

where $\theta_1, \ldots, \theta_{r-2} \in [0, \pi]$, $t \in [-1, 1]$, and $\phi \in [0, 2\pi)$. With the substitution $t = \cos\theta_{r-1}$, where $\theta_{r-1} \in [0, \pi]$, we obtain the usual spherical coordinates (see [2, Subsection 11.1.1, page 233, formulas (7) and (8)]). In the local coordinates (8) the surface element $d\omega_r$ has the representation (see [2, Subsection 11.1.1, page 233, formula (10)])

$$\begin{aligned}
&d\omega_r(\mathbf{x}) \\
&= \sin\theta_1\, (\sin\theta_2)^2 \cdots (\sin\theta_{r-2})^{r-2}\, (1-t^2)^{\frac{r-2}{2}}\, d\phi\, d\theta_1\, d\theta_2 \cdots d\theta_{r-2}\, dt.
\end{aligned}$$

For an integrable function $f : [-1, 1] \to \mathbb{R}$ this implies that for any fixed $\mathbf{y} \in S^r$

$$\int_{S^r} f(\mathbf{z}\cdot\mathbf{y})\, d\omega_r(\mathbf{z}) = \omega_{r-1} \int_{-1}^{1} f(t)\, (1-t^2)^{\frac{r-2}{2}}\, dt, \tag{9}$$

where we have used the local coordinates (8) with $x_{r+1} = t := \mathbf{z}\cdot\mathbf{y}$, that is, $\mathbf{y}$ is considered as the north pole.

The Beltrami operator Δ^* on S^r has in the local coordinates (8) the following representation (see [2, Subsection 11.1.1, page 235, formula (15)] for the

Laplacian in the usual polar coordinates): for a function g on S^r with $g(\mathbf{x}) = G(\theta_1, \ldots, \theta_{r-2}, t, \phi)$

$$\begin{aligned}\Delta^* g = &\left[(-r)\, t\, \frac{\partial G}{\partial t} + (1-t^2)\, \frac{\partial^2 G}{\partial t^2}\right] \\ &+ (1-t^2)^{-1}\, (\sin\theta_{r-2})^{2-r}\, \frac{\partial}{\partial \theta_{r-2}} \left[(\sin\theta_{r-2})^{r-2} \frac{\partial G}{\partial \theta_{r-2}}\right] \\ &+ (1-t^2)^{-1}\, (\sin\theta_{r-2})^{-2}\, (\sin\theta_{r-3})^{3-r}\, \frac{\partial}{\partial \theta_{r-3}} \left[(\sin\theta_{r-3})^{r-3} \frac{\partial G}{\partial \theta_{r-3}}\right] \\ &+ \ldots + (1-t^2)^{-1} (\sin\theta_{r-2} \cdots \sin\theta_2)^{-2}\, (\sin\theta_1)^{-1}\, \frac{\partial}{\partial \theta_1} \left[(\sin\theta_1)^1 \frac{\partial G}{\partial \theta_1}\right] \\ &+ (1-t^2)^{-1}\, (\sin\theta_{r-2} \cdots \sin\theta_1)^{-2}\, \frac{\partial^2 G}{\partial \phi^2}. \end{aligned} \tag{10}$$

We observe that the first term on the right-hand side of (10) can also be written as

$$(-r)\, t\, \frac{\partial G}{\partial t} + (1-t^2)\, \frac{\partial^2 G}{\partial t^2} = (1-t^2)^{-\frac{r-2}{2}}\, \frac{\partial}{\partial t} \left[(1-t^2)^{\frac{r}{2}}\, \frac{\partial G}{\partial t}\right].$$

However, the representation on the left-hand side is more useful to us. This first term is the only term in the Beltrami operator (10) in the local coordinates (8) that contains derivatives with respect to the coordinate t.

Now we can introduce the Sobolev spaces $H^s(S^r)$ and derive some of their properties (see [10, Section 6.1, pages 181–182, Definition 6.2 and Theorem 6.3] for S^r and [4, Sections 5.1 and 5.2, pages 81–92] for S^2).

The *Sobolev space* $H^s(S^r)$, for $s \geq 0$, is the completion of $\bigoplus_{\ell=0}^{\infty} \mathbb{H}_\ell(S^r)$ with respect to the norm

$$\|f\|_{s,r} := \left(\sum_{\ell=0}^{\infty} \left(\ell + \frac{r-1}{2}\right)^{2s} \sum_{k=1}^{N(r-1,\ell)} |\hat{f}_{\ell k}^{(r)}|^2 \right)^{\frac{1}{2}}. \tag{11}$$

The space $H^s(S^r)$ is a Hilbert space with the inner product

$$(f, g)_{s,r} := \sum_{\ell=0}^{\infty} \left(\ell + \frac{r-1}{2}\right)^{2s} \sum_{k=1}^{N(r-1,\ell)} \hat{f}_{\ell k}^{(r)}\, \hat{g}_{\ell k}^{(r)}, \qquad f, g \in H^s(S^r). \tag{12}$$

This is relatively easily seen because (12) is exactly the inner product which induces the norm (11). (A detailed discussion of the spaces $H^s(S^r)$ is given in [4, Sections 5.1 and 5.2, pages 81–92] for the case $r = 2$, and the proofs in [4] transfer directly to the case of $r > 2$ with the appropriate modifications.) It is often useful to think of functions in $H^s(S^r)$ in terms of their Fourier series expansion (6). Note that $H^0(S^r) = L_2(S^r)$, and that the spaces $H^s(S^r)$ are nested, that is, if $t \geq s$ then $H^t(S^r) \subset H^s(S^r)$.

For $s > \frac{r}{2}$ the space $H^s(S^r)$ is embedded into the space $C(S^r)$. This can be seen as follows: for $f \in H^s(S^r)$ and any $M \in \mathbb{N}_0$, the Cauchy-Schwarz inequality and the addition theorem (4) imply

$$\begin{aligned}
&\left| \sum_{\ell=M}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \hat{f}_{\ell k}^{(r)} \, Y_{\ell k}^{(r)}(\mathbf{x}) \right| \\
&\quad \leq \sum_{\ell=M}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \left[\left(\ell + \frac{r-1}{2} \right)^{s} |\hat{f}_{\ell k}^{(r)}| \right] \left[\left(\ell + \frac{r-1}{2} \right)^{-s} |Y_{\ell k}^{(r)}(\mathbf{x})| \right] \\
&\quad \leq \left(\sum_{\ell=M}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \left(\ell + \frac{r-1}{2} \right)^{2s} |\hat{f}_{\ell k}^{(r)}|^2 \right)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{\ell=M}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \left(\ell + \frac{r-1}{2} \right)^{-2s} |Y_{\ell k}^{(r)}(\mathbf{x})|^2 \right)^{\frac{1}{2}} \\
&\quad \leq \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \left(\ell + \frac{r-1}{2} \right)^{2s} |\hat{f}_{\ell k}^{(r)}|^2 \right)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{\ell=M}^{\infty} \left(\ell + \frac{r-1}{2} \right)^{-2s} \frac{N(r-1,\ell)}{\omega_r} \right)^{\frac{1}{2}} \\
&\quad = C_{s,r,M} \, \|f\|_{s,r},
\end{aligned} \tag{13}$$

with the positive constant

$$C_{s,r,M} := \frac{1}{\sqrt{\omega_r}} \left(\sum_{\ell=M}^{\infty} \frac{N(r-1,\ell)}{\left(\ell + \frac{r-1}{2}\right)^{2s}} \right)^{\frac{1}{2}}.$$

As we have $N(r-1,\ell) \leq c_r \, (\ell + \frac{r-1}{2})^{r-1}$ from (3), the sum in the definition of $C_{s,r,M}$ converges if $r - 1 - 2s < -1$, that is, if $s > \frac{r}{2}$. Also for $s > \frac{r}{2}$ we have $\lim_{M\to\infty} C_{s,r,M} = 0$. Thus, (13) implies that for $s > \frac{r}{2}$ the Fourier series of $f \in H^s(S^r)$ converges uniformly toward a continuous function. This uniform limit coincides almost everywhere with the $L_2(S^r)$ limit of the Fourier series (because they have the same Fourier coefficients), and we can choose f to be the uniform limit. Hence, $H^s(S^r) \subset C(S^r)$ for all $s > \frac{r}{2}$, and from (13) with $M = 0$

$$\|f\|_{C(S^r)} = \sup_{\mathbf{x}\in S^r} |f(\mathbf{x})| \leq C_{s,r,0} \, \|f\|_{s,r}, \qquad f \in H^s(S^r), \tag{14}$$

that is, $H^s(S^r)$ with $s > \frac{r}{2}$ is embedded into $C(S^r)$. In particular, (14) also implies that point evaluation in $H^s(S^r)$, with $s > \frac{r}{2}$, is bounded, and consequently that $H^s(S^r)$, with $s > \frac{r}{2}$, is a reproducing kernel Hilbert space.

The following alternative representation of the norm $\|f\|_{s,r}$ plays an important role in the proof of our result.

For $f \in H^s(S^r)$, where $s \geq 0$, we can formally define the self-adjoint operator $((\frac{r-1}{2})^2 - \Delta^*)^{\frac{s}{2}}$ by

$$\left(\left(\frac{r-1}{2}\right)^2 - \Delta^*\right)^{\frac{s}{2}} f := \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \left(\ell + \frac{r-1}{2}\right)^s \hat{f}_{\ell k}^{(r)} Y_{\ell k}^{(r)}. \qquad (15)$$

This definition is motivated by the fact that for any spherical harmonic $Y_\ell^{(r)}$ of degree ℓ (from (7))

$$\left(\left(\frac{r-1}{2}\right)^2 - \Delta^*\right) Y_\ell^{(r)} = \left(\ell + \frac{r-1}{2}\right)^2 Y_\ell^{(r)}.$$

Note that for $f \in H^s(S^r)$, the function $((\frac{r-1}{2})^2 - \Delta^*)^{\frac{s}{2}} f$ is in $L_2(S^r)$, because the series on the right-hand side of (15) converges in the $L_2(S^r)$ sense. With (15) we can represent the norm of $f \in H^s(S^r)$ as (see [4, Section 5.1, page 83, formula (5.1.9)] for the case of S^2)

$$\begin{aligned} \|f\|_{s,r} &= \left\| \left(\left(\frac{r-1}{2}\right)^2 - \Delta^*\right)^{\frac{s}{2}} f \right\|_{L_2(S^r)} \\ &= \left(\int_{S^r} \left| \left(\left(\frac{r-1}{2}\right)^2 - \Delta^*\right)^{\frac{s}{2}} f(\mathbf{x}) \right|^2 d\omega_r(\mathbf{x}) \right)^{\frac{1}{2}}. \end{aligned} \qquad (16)$$

For even s and $f \in C^\infty(S^r)$ (or sufficiently smooth) the left-hand side in (15) can also be considered as a differential operator in the classical sense, and we find that we are allowed to interchange in the Fourier series of f the order of differentiation and summation and obtain from this also the right-hand side. This shows that the definition (15) for even s is consistent with the classical definition of the differential operator $\left((\frac{r-1}{2})^2 - \Delta^*\right)^{\frac{s}{2}}$, and that for $f \in C^\infty(S^r)$ and even s, $\left((\frac{r-1}{2})^2 - \Delta^*\right)^{\frac{s}{2}} f$ in (15) is a differentiation in the classical sense.

For more detailed background information about the unit sphere S^r in $\mathbb{R}^{r+1}$, spherical harmonics, and the Beltrami operator see for example [2, Chapter XI], [9], and [10, Chapters 4 and 6] for S^r, $r \geq 2$, and [4] for S^2. More information about orthogonal polynomials can for example be found in [11] (see also [10, Chapter 2] for Gegenbauer polynomials). We are not aware of a reference where the Sobolev spaces $H^s(S^r)$ for general $r \geq 2$ (defined the same way as here) are discussed in great detail. Reimer [10, Section 6.1, pages 181–182] briefly discusses the spaces $H^s(S^r)$ for $r \geq 2$ and s which is an even integer, and Freeden, Gervens and Schreiner [4, Sections 5.1 and 5.2, pages 81–92] discuss the spaces $H^s(S^r)$ in detail for $r = 2$. However, all of the proofs of the properties of $H^s(S^r)$, $r \geq 2$, are completely analogous to the case $r = 2$. Since we do not know of a detailed reference for the case of $H^s(S^r)$ with $r > 2$, some of the proofs have been included in this section.

2.2 The results

Now we can formulate the main result in the following theorem.

Theorem 1 *Let $r \geq 2$. For each $s > \frac{r}{2}$ there exists a positive constant $c_{s,r}$ such that the worst-case cubature error $E(Q_m; H^s(S^r))$ in $H^s(S^r)$ of an arbitrary m-point cubature rule Q_m for S^r has the lower bound*

$$E(Q_m; H^s(S^r)) \geq c_{s,r}\, m^{-\frac{s}{r}}. \tag{17}$$

The constant $c_{s,r}$ depends only on s and r, but is independent of the rule Q_m and the number of nodes $m \in \mathbb{N}$ of the rule. For the case $r = 2$ the order $m^{-\frac{s}{2}}$ in (17) cannot be improved.

We stress that the positive constant $c_{s,r}$ in (17) depends only on the sphere dimension r and the Sobolev space index s. In particular, $c_{s,r}$ is independent of the placement and the number of the nodes and of the corresponding weights.

Since the lower bound in (17) is true (with a *universal* positive constant $c_{s,r}$) for *every* cubature rule Q_m on S^r which has m nodes, we can also formulate the statement of the theorem as

$$\inf \left\{ E(Q_m; H^s(S^r)) \;\middle|\; \begin{array}{l} Q_m \text{ is an } m\text{-point} \\ \text{cubature rule} \end{array} \right\} \geq c_{s,r}\, m^{-\frac{s}{r}}, \quad m \to \infty.$$

For the special case of the sphere S^2 Theorem 1 was proved in [6, page 794, Theorem 1]. In this case results from [5,7] (see also [6, page 794, Theorem 2]) imply the order optimality of (17). We conjecture that (17) is also optimal with respect to orders of m for $r \geq 3$.

3 Preparations

Before we actually give the proof of Theorem 1, we state three lemmas that will be needed for the proof. The first (Lemma 1 below) provides information about the packing of S^r with M_m spherical caps of a spherical angle α_m, where $\alpha_m = c_1\,(2m)^{-\frac{1}{r}}$ and $2m \leq M_m \leq c_2\, 2m$ with suitable positive constants $c_1 > 0$ and $c_2 \geq 1$ independent of m. The second lemma (Lemma 2 below) gives an interpolation result between the norms $\|\cdot\|_{s,r}$ of Sobolev spaces with different indices, which can easily be verified with the help of Hölder's inequality. We include the short proof for completeness. The last lemma (Lemma 3 below) gives an expansion of integer powers of a differential operator, which is essentially a constant minus the t-derivative component of Δ^* in the local coordinates (8). The proof of this last lemma follows by induction.

Definition 1 *Let $r \geq 2$. The (closed) spherical cap $S(\mathbf{y}; \alpha) \subset S^r$ with center $\mathbf{y} \in S^r$ and angular radius $\alpha \in [0, \pi]$ is defined by*

$$S(\mathbf{y}; \alpha) = \left\{ \mathbf{x} \in S^r \;\middle|\; \mathbf{x} \cdot \mathbf{y} \geq \cos\alpha \right\}.$$

We note that, from (9), for any $\mathbf{y} \in S^r$ the surface area of $S(\mathbf{y}; \alpha)$ is given by

$$\begin{aligned} |S(\mathbf{y}; \alpha)| &= \int_{S(\mathbf{y};\alpha)} d\omega_r(\mathbf{x}) \\ &= \omega_{r-1} \int_{\cos\alpha}^{1} (1 - t^2)^{\frac{r-2}{2}}\, dt \\ &= \omega_{r-1} \int_0^{\alpha} (\sin\theta)^{r-1}\, d\theta. \end{aligned} \tag{18}$$

The elementary estimates $\sin\theta \le \theta$ for all $\theta \in [0, \pi]$, and $\sin\theta \ge \frac{2}{\pi}\theta$ for all $\theta \in [0, \frac{\pi}{2}]$, and (18) imply the following upper and lower bound for the surface area of a spherical cap $S(\mathbf{y}; \alpha)$ with $\alpha \in [0, \frac{\pi}{2}]$.

$$\left(\frac{2}{\pi}\right)^{r-1} \frac{\omega_{r-1}}{r}\, \alpha^r \le |S(\mathbf{y}; \alpha)| \le \frac{\omega_{r-1}}{r}\, \alpha^r. \tag{19}$$

Now we can formulate the packing result.

Lemma 1 *Let $r \ge 2$. For any $m \in \mathbb{N}$, there exist M_m points $\mathbf{y}_1, \ldots, \mathbf{y}_{M_m}$ on S^r and an angle α_m, with*

$$\begin{aligned} \alpha_m &= c_1\, (2m)^{-\frac{1}{r}}, \\ 2m &\le M_m \le c_2\, 2m, \end{aligned}$$

such that the caps $S(\mathbf{y}_i; \alpha_m)$, $i = 1 \ldots, M_m$, form a packing of S^r (that is, $S(\mathbf{y}_i; \alpha_m)$ and $S(\mathbf{y}_j; \alpha_m)$ with $i \ne j$ touch at most at their boundaries). The positive constants $c_1 > 0$ and $c_2 \ge 1$ depend only on r, but not on m.

Remark 1 This lemma is essentially well-known, but an explicit proof is not easy to find. Our proof follows from a lower bound due to Wyner [12, Subsection 5.1, pages 1089–1091] and an elementary volume argument. Since Wyner's lower bound follows from a rather short and elegant argument and also for making this paper more self-contained, we will briefly explain Wyner's argument in the proof below instead of just quoting his lower bound.

Proof If the caps $S(\mathbf{y}_i; \alpha_m)$, $i = 1, \ldots, M_m$, form a packing their combined surface area is bounded from above by the the surface area ω_r of the sphere S^r. Thus

$$M_m\, |S(\mathbf{y}_1; \alpha_m)| = \sum_{i=1}^{M_m} |S(\mathbf{y}_i; \alpha_m)| \le |S^r| = \omega_r,$$

and with (19) and $\alpha_m = c_1\, (2m)^{-\frac{1}{r}}$

$$M_m \le \frac{\omega_r}{|S(\mathbf{y}_1; \alpha_m)|} \le \frac{\omega_r}{\omega_{r-1} \left(\frac{2}{\pi}\right)^{r-1} \frac{1}{r}\, \alpha_m^r} = r\, \frac{\omega_r}{\omega_{r-1}} \left(\frac{\pi}{2}\right)^{r-1} c_1^{-r}\, 2m.$$

Thus

$$M_m \leq c_2\, 2m \quad \text{with } c_2 := r\, \frac{\omega_r}{\omega_{r-1}} \left(\frac{\pi}{2}\right)^{r-1} c_1^{-r}. \tag{20}$$

Now we show Wyner's argument [12, Subsection 5.1, pages 1089–1091] that leads to a lower bound for M_m.

Since the spherical caps $S(\mathbf{y}_i; \alpha_m)$, $i = 1, \ldots, M_m$, form a packing of S^r their centers $\mathbf{y}_1, \ldots, \mathbf{y}_{M_m}$ satisfy $\arccos(\mathbf{y}_i \cdot \mathbf{y}_j) \geq 2\alpha_m$ for $i \neq j$. Now let us consider a *maximal* set $\{\mathbf{y}_1, \ldots, \mathbf{y}_{M_m}\}$ of points on S^r such that $\arccos(\mathbf{y}_i \cdot \mathbf{y}_j) \geq 2\alpha_m$ for $i \neq j$. (Maximal means here that, if we add any other point $\mathbf{y}$ to the set $\{\mathbf{y}_1, \ldots, \mathbf{y}_{M_m}\}$, then the property that the angle between two distinct point is $\geq 2\alpha_m$ would be violated.) Then the spherical caps $S(\mathbf{y}_i; 2\alpha_m)$, $i = 1, \ldots, M_m$, form a covering of S^r.

This can be seen in the following way: If they were not a covering there would exist $\mathbf{z} \in S^r$ which is in none of the spherical caps $S(\mathbf{y}_i; 2\alpha_m)$, $i = 1, \ldots, M_m$, that is, with $\arccos(\mathbf{z} \cdot \mathbf{y}_i) > 2\alpha_m$ for all $i = 1, \ldots, M_m$. But then we could add this point to $\{\mathbf{y}_1, \ldots, \mathbf{y}_{M_m}\}$, and the enlarged set $\{\mathbf{y}_1, \ldots, \mathbf{y}_{M_m}, \mathbf{z}\}$ would have the property that the angle between two distinct points is $\geq 2\alpha_m$, in contradiction to the maximality of the set $\{\mathbf{y}_1, \ldots, \mathbf{y}_{M_m}\}$.

Because $S(\mathbf{y}_i; 2\alpha_m)$, $i = 1, \ldots, M_m$, form a covering of S^r we have

$$\omega_r = |S^r| \leq \sum_{i=1}^{M_m} |S(\mathbf{y}_i; 2\alpha_m)| = M_m\, |S(\mathbf{y}_1; 2\alpha_m)|,$$

which is [12, Subsection 5.1, page 1091, formula above formula (98)]. With the help of (19) we obtain an upper bound for $|S(\mathbf{y}_1; 2\alpha_m)|$ leading to

$$M_m \geq \frac{\omega_r}{|S(\mathbf{y}_1; 2\alpha_m)|} \geq r\, \frac{\omega_r}{\omega_{r-1}} (2\alpha_m)^{-r} = 2^{-r} r\, \frac{\omega_r}{\omega_{r-1}}\, c_1^{-r}\, 2m,$$

where we have used $\alpha_m = c_1\, (2m)^{-\frac{1}{r}}$. Thus we finally get

$$M_m \geq c_3\, 2m \quad \text{with } c_3 := 2^{-r} r\, \frac{\omega_r}{\omega_{r-1}}\, c_1^{-r}. \tag{21}$$

Combination of (20) and (21) yields

$$c_3\, 2m \leq M_m \leq c_2\, 2m \quad \text{for all } m \in \mathbb{N},$$

with constants c_2 and c_3 that depend only on c_1 and r, but not on m. The constant c_1 is now chosen such that $c_3 = 1$. □

Remark 2 As we consider a packing with $2m \geq 2$ caps in Lemma 1 the angle α_m can be at most $\frac{\pi}{2}$ (which is achieved for a packing with 2 caps with opposite centers).

Lemma 2 *Consider $r \geq 2$. Let $s \geq 0$ and choose $n \in \mathbb{N}_0$ such that $2n \leq s \leq 2(n+1)$. Then for $f \in H^{2(n+1)}(S^r)$,*

$$\|f\|_{s,r} \leq \|f\|_{2n,r}^{\frac{2n+2-s}{2}}\, \|f\|_{2(n+1),r}^{\frac{s-2n}{2}}. \tag{22}$$

The proof of this lemma is analogous to the proof given in [6, pages 801–802] for the special case of S^2 with the appropriate changes. We include the proof for completeness.

Proof First we note that $f \in H^{2(n+1)}(S^r)$ implies that $f \in H^t(S^r)$ for all $t \leq 2(n+1)$, in particular, $f \in H^{2n}(S^r)$ and $f \in H^s(S^r)$. For $s = 2n$ and $s = 2(n+1)$ we obtain equality in (22) and nothing needs to be shown. For $2n < s < 2(n+1)$ we use Hölder's inequality: for $f \in H^{2(n+1)}(S^r)$ and $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}
\|f\|_{s,r}^2 &= \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \left(\ell + \frac{r-1}{2}\right)^{2s} |\hat{f}_{\ell k}^{(r)}|^2 \\
&= \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \left[\left(\ell + \frac{r-1}{2}\right)^{2s-\lambda} |\hat{f}_{\ell k}^{(r)}|^{2-\eta}\right]\left[\left(\ell + \frac{r-1}{2}\right)^{\lambda} |\hat{f}_{\ell k}^{(r)}|^{\eta}\right] \\
&\leq \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \left(\ell + \frac{r-1}{2}\right)^{(2s-\lambda)p} |\hat{f}_{\ell k}^{(r)}|^{(2-\eta)p}\right)^{\frac{1}{p}} \\
&\quad \times \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{N(r-1,\ell)} \left(\ell + \frac{r-1}{2}\right)^{\lambda q} |\hat{f}_{\ell k}^{(r)}|^{\eta q}\right)^{\frac{1}{q}}.
\end{aligned} \tag{23}$$

Now we choose in (23)

$$p := \frac{2}{2n+2-s}, \quad q := \frac{2}{s-2n}, \quad \lambda := 2(n+1)(s-2n), \quad \eta := s-2n.$$

Substitution of the constants p, q, λ, and η into (23) yields (22). □

The following lemma gives an expansion of integer powers of the constant and t-derivative component of the differential operator $(\frac{r-1}{2})^2 - \Delta^*$.

Lemma 3 *Let $r \in \mathbb{N}$ with $r \geq 2$. For $k \in \mathbb{N}$,*

$$\begin{aligned}
&\left(\left(\frac{r-1}{2}\right)^2 + r\,t\,\frac{d}{dt} - (1-t^2)\,\frac{d^2}{dt^2}\right)^k \\
&\quad = \sum_{j=0}^{k} q_{j,k}(t) \left(\frac{d}{dt}\right)^j + \sum_{j=1}^{k} p_{j,k}(t)\,(1-t^2)^j \left(\frac{d}{dt}\right)^{k+j},
\end{aligned} \tag{24}$$

where $q_{j,k}$ is a real polynomial of degree j and $p_{j,k}$ is a real polynomial of degree $k-j$.

The special case of (24) for $r = 2$ was not used in [6], and in fact the part of the proof of Theorem 1, where Lemma 3 comes into play, is rather different from the proof in [6].

Proof The proof of this lemma is straight-forward by induction on k. Since it is rather lengthy we do not include it here. □

4 Proof of the lower bound of the worst-case cubature error

After these preparations we can prove Theorem 1.

Proof The idea behind the proof is to construct for an arbitrarily given m-point cubature rule Q_m,

$$Q_m f := \sum_{j=1}^{m} w_j\, f(\mathbf{x}_j),$$

a 'bad' function $f_m \in H^s(S^r)$ for which

$$\left| Q_m\left(\frac{f_m}{\|f_m\|_{s,r}}\right) - I\left(\frac{f_m}{\|f_m\|_{s,r}}\right)\right| \geq c_{s,r}\, m^{-\frac{s}{r}}, \tag{25}$$

where the constant $c_{s,r}$ depends only on s and r, but not on the function f_m, the cubature rule Q_m (that is, the placement of the nodes $\mathbf{x}_i$ and the corresponding weights w_i, $i = 1, \ldots, m$), and the number of nodes m. Since the worst-case cubature error $E(Q_m; H^s(S^r))$ is the supremum over the cubature errors of all functions in the unit ball of $H^s(S^r)$, the left-hand side in (25) is clearly a lower bound for $E(Q_m; H^s(S^r))$. Thus (25) implies (17) in Theorem 1.

Lemma 1 guarantees that there exist M_m points $\mathbf{y}_1, \ldots, \mathbf{y}_{M_m}$ on S^r, where $2m \leq M_m \leq c_2\, 2m$, and a spherical angle

$$\alpha_m := c_1\, (2m)^{-\frac{1}{r}}$$

such that the spherical caps $S(\mathbf{y}_i; \alpha_m)$, $i = 1, \ldots, M_m$, touch at most at their boundaries. The positive constants c_1 and c_2 do not depend on m.

As there are $M_m \geq 2m$ spherical caps and only m nodes $\mathbf{x}_i$, we know that at most m of the spherical caps $S(\mathbf{y}_i; \alpha_m)$, $i = 1, \ldots, M_m$, contain a node of the cubature rule in the interior. Consequently, there exist at least m spherical caps that contain no node of the cubature rule in their interior. After relabelling we may assume that $S(\mathbf{y}_i; \alpha_m)$, $i = 1, \ldots, m$, contain no node of the cubature rule in their interior. We know that $\alpha_m \leq \frac{\pi}{2}$ as we consider a packing with $2m \geq 2$ caps (see Remark 2).

Now we will construct the 'bad' function $f_m \in H^s(S^r)$ such that the support of f_m is a subset of $\bigcup_{i=1}^{m} S(\mathbf{y}_i; \alpha_m)$. Restricted to each cap $S(\mathbf{y}_i; \alpha_m)$, the function f_m will look the same, and $f_m|S(\mathbf{y}_i; \alpha_m)$ will be a rotationally symmetric function with respect to the center $\mathbf{y}_i$ of the cap. Therefore $f_m|S(\mathbf{y}_i; \alpha_m)$ can in fact be considered as a one-dimensional function.

We start the construction with a function $\Phi \in C^\infty(\mathbb{R})$ with the following properties:

(i) $\Phi(t) \geq 0$ for all $t \in \mathbb{R}$,
(ii) $\max_{t\in\mathbb{R}} \Phi(t) = \Phi(0) = 1$, and
(iii) Φ has the compact support $\mathrm{supp}(\Phi) = [-1, 1]$.

For example we could choose $\Phi \in C^\infty(\mathbb{R})$ to be

$$\Phi(t) := \begin{cases} e^{1-\frac{1}{1-t^2}} & \text{if } t \in (-1, 1), \\ 0 & \text{if } |t| \geq 1. \end{cases}$$

The concrete definition of Φ with the above properties is unimportant, but it is essential that we use the *same* function Φ in the construction of f_m for all cubature rules Q_m.

In a first step we scale the argument of the function Φ in such a way that the support of the function with the scaled argument is $[\cos\alpha_m, \cos\frac{\alpha_m}{2}]$. The linear map that maps $[\cos\alpha_m, \cos\frac{\alpha_m}{2}]$ onto $[-1, 1]$ is given by

$$g_m(t) := \frac{2t - \left(\cos\frac{\alpha_m}{2} + \cos\alpha_m\right)}{\cos\frac{\alpha_m}{2} - \cos\alpha_m} = \frac{2t - \left(\cos\frac{\alpha_m}{2} + \cos\alpha_m\right)}{2\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}}, \tag{26}$$

where we have used the trigonometric identity

$$\cos\alpha - \cos\beta = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}.$$

The scaled version of Φ is then given by

$$\Phi_m(t) := \Phi(g_m(t)) = \Phi\left(\frac{2t - \left(\cos\frac{\alpha_m}{2} + \cos\alpha_m\right)}{2\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}}\right), \quad t \in \mathbb{R}.$$

The function Φ_m is in $C^\infty(\mathbb{R})$, and

(i) $\Phi_m(t) \geq 0$ for all $t \in \mathbb{R}$,
(ii) $\max_{t\in\mathbb{R}} \Phi_m(t) = \Phi_m\left(\frac{1}{2}(\cos\frac{\alpha_m}{2} + \cos\alpha_m)\right) = \Phi(0) = 1$, and
(iii) Φ_m has the support $\operatorname{supp}(\Phi_m) = [\cos\alpha_m, \cos\frac{\alpha_m}{2}]$.

Now we lift Φ_m onto the caps $S(\mathbf{y}_i; \alpha_m)$, $i = 1, \ldots, m$. We define our 'bad' function $f_m \in C^\infty(S^r)$ by

$$f_m(\mathbf{x}) := \sum_{i=1}^{m} \Phi_m(\mathbf{x}\cdot\mathbf{y}_i), \qquad \mathbf{x} \in S^r. \tag{27}$$

The properties of Φ_m imply the following properties of f_m:

(i) $f_m(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in S^r$,
(ii) $\max_{\mathbf{x}\in S^r} f_m(\mathbf{x}) = \max_{\mathbf{x}\in S(\mathbf{y}_i,\alpha_m)} \Phi_m(\mathbf{x}\cdot\mathbf{y}_i) = 1$ for $i = 1, 2, \ldots, m$,
(iii) f_m has the support $\operatorname{supp}(f_m) = \overline{\bigcup_{i=1}^m \left(S(\mathbf{y}_i; \alpha_m) \setminus S(\mathbf{y}_i; \frac{\alpha_m}{2})\right)}$, and
(iv) $f_m(\mathbf{x})|_{\mathbf{x}\in S(\mathbf{y}_i,\alpha_m)} = \Phi_m(\mathbf{x}\cdot\mathbf{y}_i)$ for $i = 1, 2, \ldots, m$.

Furthermore, f_m is in $H^s(S^r)$ for any $s \geq 0$. We will prove this later by verifying the estimate

$$\|f_m\|_{s,r} \leq c_{s,r}\, m^{\frac{s}{r}}, \tag{28}$$

with a constant $c_{s,r}$ that depends only on $s \geq 0$ and r, but not on Q_m, f_m, and the number of points m. The proof of (28) is rather delicate. Instead of interrupting the flow of the proof to derive (28) now, we will defer the proof of (28) for the moment and continue with our argument.

Due to the construction, f_m vanishes in all nodes of the cubature rule, that is, $Q_m f_m = 0$. Hence

$$\left|Q_m\left(\frac{f_m}{\|f_m\|_{s,r}}\right) - I\left(\frac{f_m}{\|f_m\|_{s,r}}\right)\right| = \frac{I f_m}{\|f_m\|_{s,r}}. \tag{29}$$

Now we compute $I f_m$.

$$
\begin{aligned}
I f_m &= \int_{S^r} f_m(\mathbf{x})\, d\omega_r(\mathbf{x}) \\
&= \sum_{i=1}^{m} \int_{S(\mathbf{y}_i;\alpha_m)} \Phi_m(\mathbf{x}\cdot\mathbf{y}_i)\, d\omega_r(\mathbf{x}) \\
&= m\,\omega_{r-1} \int_{\cos\alpha_m}^{\cos\frac{\alpha_m}{2}} \Phi_m(t)\,(1-t^2)^{\frac{r-2}{2}}\, dt \\
&= m\,\omega_{r-1} \int_{\cos\alpha_m}^{\cos\frac{\alpha_m}{2}} \Phi(g_m(t))\,(1-t^2)^{\frac{r-2}{2}}\, dt,
\end{aligned} \tag{30}
$$

where we have used (9) for each integral in the second line. Since $\alpha_m \le \frac{\pi}{2}$ (see Remark 2), $0 \le \cos\alpha_m < \cos\frac{\alpha_m}{2} < 1$, and for $t \in [\cos\alpha_m, \cos\frac{\alpha_m}{2}]$ we obtain that

$$
\left(\sin\frac{\alpha_m}{2}\right)^2 \le 1 - t^2 \le (\sin\alpha_m)^2. \tag{31}
$$

Thus from (30) and (31)

$$
I f_m \ge m\,\omega_{r-1} \left(\sin\frac{\alpha_m}{2}\right)^{r-2} \int_{\cos\alpha_m}^{\cos\frac{\alpha_m}{2}} \Phi(g_m(t))\, dt.
$$

The substitution $u := g_m(t)$, with the linear function g_m defined in (26), yields

$$
I f_m \ge m\,\omega_{r-1} \left(\sin\frac{\alpha_m}{2}\right)^{r-2} \sin\frac{3\alpha_m}{4}\, \sin\frac{\alpha_m}{4} \int_{-1}^{1} \Phi(u)\, du. \tag{32}
$$

The elementary estimate $\sin\theta \ge \frac{2}{\pi}\theta$, $\theta \in [0, \frac{\pi}{2}]$, and $\alpha_m = c_1\,(2m)^{-\frac{1}{r}}$ imply now that

$$
\begin{aligned}
\left(\sin\frac{\alpha_m}{2}\right)^{r-2} \sin\frac{3\alpha_m}{4}\, \sin\frac{\alpha_m}{4} &\ge \left(\frac{\alpha_m}{\pi}\right)^{r-2} \frac{3\alpha_m}{2\pi}\, \frac{\alpha_m}{2\pi} \\
&= \frac{3}{4\pi^r}\,\alpha_m^r \\
&= \frac{3c_1^r}{8\pi^r}\, m^{-1}.
\end{aligned}
$$

Thus from (32),

$$
I f_m \ge \omega_{r-1} \frac{3c_1^r}{8\pi^r} \int_{-1}^{1} \Phi(u)\, du = c_r, \tag{33}
$$

with a constant c_r that depends only on r.

From (28) and (33) we obtain

$$\frac{I f_m}{\| f_m \|_{s,r}} \geq c_{s,r}\, m^{-\frac{s}{r}}$$

which (from (29)) implies (25) and hence (17) in Theorem 1.

It remains to prove (28). We do this by showing (28) first for nonnegative even integer s, and then use Lemma 2 to interpolate between the even integer cases.

For $s = 0$ and from the definition (27) of f_m

$$\begin{aligned}
\| f_m \|_{0,r}^2 &= \int_{S^r} |f_m(\mathbf{x})|^2 \, d\omega_r(\mathbf{x}) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{S^r} \Phi_m(\mathbf{x} \cdot \mathbf{y}_i)\, \Phi_m(\mathbf{x} \cdot \mathbf{y}_j)\, d\omega_r(\mathbf{x}) \\
&= \sum_{i=1}^{m} \int_{S(\mathbf{y}_i;\alpha_m)} |\Phi_m(\mathbf{x} \cdot \mathbf{y}_i)|^2 \, d\omega_r(\mathbf{x}) \\
&= m\, \omega_{r-1} \int_{\cos\alpha_m}^{\cos\frac{\alpha_m}{2}} |\Phi(g_m(t))|^2 \, (1-t^2)^{\frac{r-2}{2}} \, dt, \qquad (34)
\end{aligned}$$

where we have used in the second last step the fact that for $i \neq j$ the supports $\mathrm{supp}(\Phi_m(\mathbf{x} \cdot \mathbf{y}_i))$ and $\mathrm{supp}(\Phi_m(\mathbf{x} \cdot \mathbf{y}_j))$ at most touch at the boundaries and that $\Phi_m(\mathbf{x} \cdot \mathbf{y}_i)$ and $\Phi_m(\mathbf{x} \cdot \mathbf{y}_j)$ vanish at these points. In the last step we have used (9). Applying (31) in (34) yields with the substitution $u = g_m(t)$ (see (26)) and with $\sin\theta \leq \theta$, $\theta \in [0, \pi]$, and $\alpha_m = c_1\, (2m)^{-\frac{1}{r}}$

$$\begin{aligned}
\| f_m \|_{0,r}^2 &\leq m\, \omega_{r-1}\, (\sin\alpha_m)^{r-2} \int_{\cos\alpha_m}^{\cos\frac{\alpha_m}{2}} |\Phi(g_m(t))|^2 \, dt \\
&= m\, \omega_{r-1}\, (\sin\alpha_m)^{r-2}\, \sin\frac{3\alpha_m}{4}\, \sin\frac{\alpha_m}{4} \int_{-1}^{1} |\Phi(u)|^2 \, du \\
&\leq m\, \omega_{r-1}\, \alpha_m^{r-2}\, \frac{3\alpha_m}{4}\, \frac{\alpha_m}{4} \int_{-1}^{1} |\Phi(u)|^2 \, du \\
&= \frac{3}{16}\, \omega_{r-1}\, m\, \alpha_m^r \int_{-1}^{1} |\Phi(u)|^2 \, du \\
&= \frac{3}{32}\, \omega_{r-1}\, c_1^r \int_{-1}^{1} |\Phi(u)|^2 \, du.
\end{aligned}$$

The obtained bound for $\| f_m \|_{0,r}^2$ is independent of m. Thus, for $s = 0$,

$$\| f_m \|_{0,r} \leq c_{0,r} = c_{0,r}\, m^{\frac{0}{r}}. \qquad (35)$$

Next we show (28) for s being a positive even integer. For a positive even integer s, (16) and (27) yield

$$
\begin{aligned}
\| f_m \|_{s,r}^2 &= \int_{S^r} \left| \left(\left(\frac{r-1}{2} \right)^2 - \Delta^* \right)^{\frac{s}{2}} f_m(\mathbf{x}) \right|^2 d\omega_r(\mathbf{x}) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{S^r} \left(\left(\left(\frac{r-1}{2} \right)^2 - \Delta^* \right)^{\frac{s}{2}} \Phi_m(\mathbf{x} \cdot \mathbf{y}_i) \right) \\
&\qquad \times \left(\left(\left(\frac{r-1}{2} \right)^2 - \Delta^* \right)^{\frac{s}{2}} \Phi_m(\mathbf{x} \cdot \mathbf{y}_j) \right) d\omega_r(\mathbf{x}). \quad (36)
\end{aligned}
$$

Because s is an even integer and $\Phi_m(\mathbf{x} \cdot \mathbf{y}_i) \in C^\infty(S^r)$, the function $\left((\frac{r-1}{2})^2 - \Delta^*\right)^{\frac{s}{2}} \Phi_m(\mathbf{x} \cdot \mathbf{y}_i)$ is a classical derivative of $\Phi_m(\mathbf{x} \cdot \mathbf{y}_i)$ and its support is therefore contained in that of $\Phi_m(\mathbf{x} \cdot \mathbf{y}_i)$. Hence for $i \neq j$, the supports of $\left((\frac{r-1}{2})^2 - \Delta^*\right)^{\frac{s}{2}} \Phi_m(\mathbf{x} \cdot \mathbf{y}_i)$ and $\left((\frac{r-1}{2})^2 - \Delta^*\right)^{\frac{s}{2}} \Phi_m(\mathbf{x} \cdot \mathbf{y}_j)$ have at most boundary points in common, where both functions vanish. Thus, from (36), (9), (10), and (31)

$$
\begin{aligned}
\| f_m \|_{s,r}^2 &= \sum_{i=1}^{m} \int_{S(\mathbf{y}_i;\alpha_m)} \left| \left(\left(\frac{r-1}{2} \right)^2 - \Delta^* \right)^{\frac{s}{2}} \Phi_m(\mathbf{x} \cdot \mathbf{y}_i) \right|^2 d\omega_r(\mathbf{x}) \\
&= m\, \omega_{r-1} \int_{\cos \alpha_m}^{\cos \frac{\alpha_m}{2}} \left| \left(\left(\frac{r-1}{2} \right)^2 + r\, t \frac{d}{dt} - (1-t^2) \frac{d^2}{dt^2} \right)^{\frac{s}{2}} \Phi(g_m(t)) \right|^2 \\
&\qquad \times (1-t^2)^{\frac{r-2}{2}}\, dt \\
&\leq m\, \omega_{r-1} (\sin \alpha_m)^{r-2} \\
&\qquad \times \int_{\cos \alpha_m}^{\cos \frac{\alpha_m}{2}} \left| \left(\left(\frac{r-1}{2} \right)^2 + r\, t \frac{d}{dt} - (1-t^2) \frac{d^2}{dt^2} \right)^{\frac{s}{2}} \Phi(g_m(t)) \right|^2 dt. \quad (37)
\end{aligned}
$$

Note that in the second step in (37) we only need to take the component of Δ^* which contains derivatives with respect to t into account (see (10)) because $\Phi(g_m(t))$ does not depend on the other coordinates.

Now we use (24) in Lemma 3 to estimate the integrand in the last line of (37). We first note that from the chain rule

$$
\frac{d}{dt} \Phi(g_m(t)) = \Phi'(g_m(t))\, g_m'(t) = \frac{\Phi'(g_m(t))}{\sin \frac{3\alpha_m}{4} \sin \frac{\alpha_m}{4}},
$$

and differentiating j times gives

$$
\left(\frac{d}{dt} \right)^j \Phi(g_m(t)) = \frac{\Phi^{(j)}(g_m(t))}{\left(\sin \frac{3\alpha_m}{4} \sin \frac{\alpha_m}{4} \right)^j}.
$$

This and (24) imply that

$$\begin{aligned}
&\left(\left(\frac{r-1}{2}\right)^2 + r\,t\,\frac{d}{dt} - (1-t^2)\,\frac{d^2}{dt^2}\right)^{\frac{s}{2}} \Phi(g_m(t)) \\
&= \sum_{j=0}^{\frac{s}{2}} q_{j,\frac{s}{2}}(t) \left(\frac{d}{dt}\right)^j \Phi(g_m(t)) \\
&\quad + \sum_{j=1}^{\frac{s}{2}} p_{j,\frac{s}{2}}(t)\,(1-t^2)^j \left(\frac{d}{dt}\right)^{\frac{s}{2}+j} \Phi(g_m(t)) \\
&= \sum_{j=0}^{\frac{s}{2}} q_{j,\frac{s}{2}}(t)\, \frac{\Phi^{(j)}(g_m(t))}{\left(\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}\right)^j} \\
&\quad + \sum_{j=1}^{\frac{s}{2}} p_{j,\frac{s}{2}}(t)\, \frac{(1-t^2)^j}{\left(\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}\right)^j}\, \frac{\Phi^{(\frac{s}{2}+j)}(g_m(t))}{\left(\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}\right)^{\frac{s}{2}}},
\end{aligned}$$

with real polynomials $q_{j,\frac{s}{2}}$ of degree j, where $j = 0, 1, \ldots, \frac{s}{2}$, and real polynomials $p_{j,\frac{s}{2}}$ of degree $\frac{s}{2} - j$, where $j = 1, \ldots, \frac{s}{2}$. Thus

$$\begin{aligned}
&\left|\left(\left(\frac{r-1}{2}\right)^2 + r\,t\,\frac{d}{dt} - (1-t^2)\,\frac{d^2}{dt^2}\right)^{\frac{s}{2}} \Phi(g_m(t))\right| \\
&\le \left(\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}\right)^{-\frac{s}{2}} \\
&\quad \times \left(\sum_{j=0}^{\frac{s}{2}} |q_{j,\frac{s}{2}}(t)| \left(\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}\right)^{\frac{s}{2}-j} |\Phi^{(j)}(g_m(t))|\right. \\
&\quad \left. + \sum_{j=1}^{\frac{s}{2}} |p_{j,\frac{s}{2}}(t)|\, \frac{(1-t^2)^j}{\left(\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}\right)^j} |\Phi^{(\frac{s}{2}+j)}(g_m(t))|\right). \qquad (38)
\end{aligned}$$

The functions $|q_{j,\frac{s}{2}}(t)|$, $j = 0, 1, \ldots, \frac{s}{2}$, and $|p_{j,\frac{s}{2}}(t)|$, $j = 1, \ldots, \frac{s}{2}$, are bounded uniformly on $[-1, 1]$ with bounds independent of Q_m and m (but dependent on s and r). The function Φ and its derivatives on $[-1, 1]$ up to (and including) the order s are all bounded by a constant that depends only on s. Furthermore for $j = 0, 1, \ldots, \frac{s}{2}$

$$\left(\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}\right)^{\frac{s}{2}-j} \le 1,$$

and for $j = 1, \ldots, \frac{s}{2}$ and $t \in [\cos\alpha_m, \cos\frac{\alpha_m}{2}]$, from (31) and $\sin\theta \le \theta, \theta \in [0, \pi]$, as well as $\sin\theta \ge \frac{2}{\pi}\theta, \theta \in [0, \frac{\pi}{2}]$,

$$\frac{(1-t^2)^j}{\left(\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}\right)^j} \le \left(\frac{(\sin\alpha_m)^2}{\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}}\right)^j \le \left(\frac{\alpha_m^2}{\frac{3\alpha_m}{2\pi}\frac{\alpha_m}{2\pi}}\right)^j = \left(\frac{4\pi^2}{3}\right)^j \le \left(\frac{4\pi^2}{3}\right)^{\frac{s}{2}}.$$

Thus (38) implies for $t \in [\cos\alpha_m, \cos\frac{\alpha_m}{2}]$

$$\left|\left(\left(\frac{r-1}{2}\right)^2 + r\,t\,\frac{d}{dt} - (1-t^2)\,\frac{d^2}{dt^2}\right)^{\frac{s}{2}} \Phi(g_m(t))\right| \le c_{s,r}\left(\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}\right)^{-\frac{s}{2}}. \tag{39}$$

Application of (39) and $\sin\theta \le \theta, \theta \in [0, \pi]$, as well as $\sin\theta \ge \frac{2}{\pi}\theta, \theta \in [0, \frac{\pi}{2}]$, in (37) yield

$$\begin{aligned}
\|f_m\|_{s,r}^2 &\le \omega_{r-1}\,c_{s,r}^2\,m\,\frac{(\sin\alpha_m)^{r-2}}{\left(\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}\right)^s}\int_{\cos\alpha_m}^{\cos\frac{\alpha_m}{2}} dt \\
&= \omega_{r-1}\,c_{s,r}^2\,m\,\frac{(\sin\alpha_m)^{r-2}}{\left(\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}\right)^s}\left(\cos\frac{\alpha_m}{2} - \cos\alpha_m\right) \\
&= 2\,\omega_{r-1}\,c_{s,r}^2\,m\,(\sin\alpha_m)^{r-2}\left(\sin\frac{3\alpha_m}{4}\sin\frac{\alpha_m}{4}\right)^{1-s} \\
&\le 2\,\omega_{r-1}\,c_{s,r}^2\,m\,\alpha_m^{r-2}\left(\frac{3\alpha_m}{2\pi}\,\frac{\alpha_m}{2\pi}\right)^{1-s} \\
&= 2\left(\frac{3}{4\pi^2}\right)^{1-s}\omega_{r-1}\,c_{s,r}^2\,m\,\alpha_m^{r-2s} \\
&= 2^{\frac{2s}{r}}\left(\frac{3}{4\pi^2}\right)^{1-s}\omega_{r-1}\,c_{s,r}^2\,c_1^{r-2s}\,m^{\frac{2s}{r}}.
\end{aligned}$$

Thus for positive even s,

$$\|f_m\|_{s,r} \le c_{s,r}\,m^{\frac{s}{r}}. \tag{40}$$

It remains to show that (28) also holds for $s > 0$ which is not an even integer. For $s > 0$ which is not an even integer there exists $n \in \mathbb{N}_0$ such that $2n < s < 2(n+1)$.

From (22) in Lemma 2 and from (35) and (40) we obtain

$$\begin{aligned}
\|f_m\|_{s,r} &\le \|f_m\|_{2n,r}^{\frac{2n+2-s}{2}} \, \|f_m\|_{2(n+1),r}^{\frac{s-2n}{2}} \\
&\le \left(c_{2n,r}\, m^{\frac{2n}{r}}\right)^{\frac{2n+2-s}{2}} \left(c_{2(n+1),r}\, m^{\frac{2(n+1)}{r}}\right)^{\frac{s-2n}{2}} \\
&= c_{2n,r}^{\frac{2n+2-s}{2}} \, c_{2(n+1),r}^{\frac{s-2n}{2}} \, m^{\frac{s}{r}} \\
&= c_{s,r}\, m^{\frac{s}{r}}.
\end{aligned} \tag{41}$$

Thus, (35), (40), and (41) imply (28) for all $s \ge 0$. □

Acknowledgements The support of the Australian Research Council is gratefully acknowledged.

References

1. Bakhvalov, N. S.: On approximate computation of integrals. Vestnik MGV, Ser. Math. Mech. Astron. Phys. Chem. **4**, 3–18 (1959), in Russian
2. Erdélyi, A. (ed.), Magnus, W., Oberhettinger, F., Tricomi, F. G. (research associates): Higher Transcendental Functions, Volume II. California Institute of Technology, Bateman Manuscript Project, McGraw-Hill Book Company, Inc., New York, Toronto, London, 1953
3. Freeden, W.: Multicsale Modelling of Spaceborne Geodata. B. G. Teubner, Stuttgart, Leipzig, 1999
4. Freeden, W., Gervens, T., Schreiner, M.: Constructive Approximation on the Sphere with Applications to Geomathematics. Oxford University Press, Oxford, 1998
5. Hesse, K., Sloan, I. H.: Worst-case errors in a Sobolev space setting for cubature over the sphere S^2. Bull. Austral. Math. Soc. **71**, 81–95 (2005)
6. Hesse, K., Sloan, I. H.: Optimal lower bounds for cubature error on the sphere S^2. J. Complexity **21**, 790–803 (2005)
7. Hesse, K., Sloan, I. H.: Cubature over the sphere S^2 in Sobolev spaces of arbitrary order. J. Approx. Theory, to appear
8. Novak, E.: Deterministic and Stochastic Error Bounds in Numerical Analysis. In Lecture Notes in Mathematics 1349, Springer–Verlag, Berlin, Heidelberg, 1988
9. Reimer, M.: Constructive Theory of Multivariate Functions. BI Wissenschaftsverlag, Mannheim, Wien, Zürich, 1990
10. Reimer, M.: Multivariate Polynimial Approximation. Birkhäuser Verlag, Basel, Bosten, Berlin, 2003
11. Szegö, G.: Orthogonal Polynomials. American Mathematical Society Colloquium Publications **23**, American Mathematical Society, Providence, 1975, 4th edn.
12. Wyner, A. D.: Capabilities of bounded discrepancy decoding. The Bell System Technical Journal **44**, 1061–1122 (1965), July–August