Numer. Math. (2006) 104: 1–26
DOI 10.1007/s00211-006-0684-z

Numerische
Mathematik

Mejdi Azaïez · Christine Bernardi ·
Nejmeddine Chorfi

Spectral discretization of the vorticity, velocity and pressure formulation of the Navier–Stokes equations

Received: 6 June 2005 / Revised: 19 January 2006 / Published online: 19 May 2006

Abstract We consider the Navier–Stokes equations in a two- or three-dimensional domain provided with non standard boundary conditions which involve the normal component of the velocity and the tangential components of the vorticity. We write a variational formulation of this problem with three independent unknowns, the vorticity, the velocity and the pressure, and prove the existence of a solution for this problem. Next we propose a discretization by spectral methods which relies on this formulation. In the two-dimensional case, we prove quasi-optimal error estimates for the three unknowns. We conclude with some numerical experiments.
Résumé: Nous considérons les équations de Navier–Stokes dans un domaine bi- ou tri-dimensionnel, munies de conditions aux limites non usuelles portant sur la composante normale de la vitesse et la ou les composantes tangentielles du tourbillon. Nous écrivons une formulation variationnelle de ce problème qui comporte trois inconnues indépendantes: le tourbillon, la vitesse et la pression. Nous prouvons que ce problème admet au moins une solution. Nous proposons une discrétisation par méthodes spectrales construite à partir de cette formulation. Dans le cas bidimensionnel, nous établissons des majorations quasi-optimales de l'erreur pour les trois inconnues. Nous concluons par quelques expériences numériques.

M. Azaïez
Laboratoire TREFLE (UMR C.N.R.S. 8508), Site E.N.S.C.P.B., 16 avenue Pey Berland, 33607 Pessac Cedex, France
E-mail: azaiez@enscpb.fr

C. Bernardi (✉)
Laboratoire Jacques-Louis Lions, C.N.R.S. & Université Pierre et Marie Curie, B.C. 187, 4 place Jussieu, 75252 Paris Cedex 05, France
E-mail: bernardi@ann.jussieu.fr

N. Chorfi
Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 1060 Tunis, Tunisie
E-mail: nejmeddine.chorfi@fst.rnu.tn

1 Introduction

We consider the Navier–Stokes problem in a two- or three-dimensional bounded domain, when provided with boundary conditions on the normal component of the velocity and the vorticity in dimension 2, on the normal component of the velocity and the tangential components of the vorticity in dimension 3. This problem is first studied in the pioneering paper [6], however the formulation that is considered in this work deals with the velocity and the pressure as only unknowns and requires the convexity or some regularity of the domain, both in dimensions 2 and 3. As first proposed in [15] and [21] for the Stokes problem (see also [16] , [1] and [3]), the Navier–Stokes equations with this type of boundary conditions admits an equivalent variational formulation where the unknowns are the vorticity, the velocity and the pressure. Relying on this formulation, we prove that the equations admit a solution with no restriction on the regularity of the domain in dimension 2 and weak limitation in dimension 3. Note however that this existence result is only established for large enough viscosity in dimension 3. We also investigate the uniqueness of the solution.

The numerical analysis of discretizations relying on the vorticity, velocity and pressure formulation has first been performed for finite element methods, see [21] and [2]. In the much simpler case of the Stokes problem, it has been recently extended to the case of spectral methods in [7], where spectral analogues of Nédélec's finite elements [20] are used. Relying on this last work, we propose a discretization of the Navier–Stokes equations in the basic situation where the domain is a square or a cube. More complex geometries can be treated thanks to the arguments in [19], however we prefer to avoid them for simplicity. The numerical analysis of the nonlinear discrete problem makes use of the approach of Brezzi, Rappaz and Raviart [11], the main difficulty being the lack of compactness of the nonlinear term. Nevertheless, we prove the existence of a discrete solution. In the two-dimensional case, by combining the results in [11] and [7], we establish upper bounds on the error concerning the velocity, the vorticity and the pressure. These estimates are fully optimal for the vorticity and the velocity and nearly optimal for the pressure. However extending these results to the three-dimensional case seems much more difficult.

In a final step, the algorithm which is used to solve the nonlinear discrete problem is described. Some two-dimensional numerical experiments turn out to be in good agreement with the numerical analysis and confirm the interest of this formulation.

The extension of this discretization to more complex geometries handled by the spectral element method has been performed for the Stokes problem in [4]. It is under consideration for the Navier–Stokes equations.

An outline of the paper is as follows.

- In Section 2, we write the variational formulation of the problem and prove existence and uniqueness results.
- Section 3 is devoted to the presentation of the spectral discrete problem and to the proof of the error estimates.
- In Section 4, we describe the algorithm that is used to solve the discrete problem and we present some numerical experiments.

2 The velocity, vorticity and pressure formulation

Let Ω be a bounded connected domain in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz–continuous boundary $\partial\Omega$. We assume for simplicity that Ω is simply-connected (we refer to [9, §2.5] for the treatment of more complex geometries in the three-dimensional case). The generic point in Ω is denoted by $\boldsymbol{x} = (x, y)$ or $\boldsymbol{x} = (x, y, z)$ according to the dimension d. We introduce the unit outward normal vector $\boldsymbol{n}$ to Ω on $\partial\Omega$ and we consider the system of Navier–Stokes equations

$$\begin{cases} -\nu\,\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \mathbf{grad}\,P = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega, \\ \gamma_t(\mathbf{curl}\,\boldsymbol{u}) = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

To make precise the sense of the operator γ_t, we recall that

- in dimension $d = 2$, for any vector field $\boldsymbol{v}$ with components v_x and v_y, $\mathbf{curl}\,\boldsymbol{v}$ stands for the scalar function $\partial_x v_y - \partial_y v_x$, so that the operator γ_t is the trace operator on $\partial\Omega$,
- in dimension $d = 3$, for any vector field $\boldsymbol{v}$ with components v_x, v_y and v_z, $\mathbf{curl}\,\boldsymbol{v}$ stands for the vector field with components $\partial_y v_z - \partial_z v_y$, $\partial_z v_x - \partial_x v_z$ and $\partial_x v_y - \partial_y v_x$, and the operator γ_t is the tangential trace operator on $\partial\Omega$, defined by: $\gamma_t(\boldsymbol{w}) = \boldsymbol{w}\times\boldsymbol{n}$.

The unknowns in system (2.1) are the velocity $\boldsymbol{u}$ and the pressure P, while the data $\boldsymbol{f}$ represent a density of body forces. The viscosity ν is a positive constant.

The basic idea in [21] consists in introducing the vorticity $\boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u}$ as a new unknown. Then, it can be noted that the convection term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ can be written as

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \boldsymbol{\omega}\times\boldsymbol{u} + \frac{1}{2}\,\mathbf{grad}\,|\boldsymbol{u}|^2,$$

where

- in dimension $d = 2$, for any scalar function θ and vector field $\boldsymbol{v}$ with components v_x and v_y, $\boldsymbol{\vartheta}\times\boldsymbol{v}$ stands for the vector with components $-\theta v_y$ and θv_x,
- in dimension $d = 3$, for any vector fields $\boldsymbol{\vartheta}$ with components θ_x, θ_y and θ_z and $\boldsymbol{v}$ with components v_x, v_y and v_z, $\boldsymbol{\vartheta}\times\boldsymbol{v}$ stands for the vector with components $\theta_y v_z - \theta_z v_y$, $\theta_z v_x - \theta_x v_z$ and $\theta_x v_y - \theta_y v_x$.

Thus, defining a pseudo-pressure p (usually called the dynamical pressure) by the formula $p = P + \frac{1}{2}\,|\boldsymbol{u}|^2$, we observe that system (2.1) is fully equivalent to

$$\begin{cases} \nu\,\mathbf{curl}\,\boldsymbol{\omega} + \boldsymbol{\omega}\times\boldsymbol{u} + \mathbf{grad}\,p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u} & \text{in } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega, \\ \gamma_t(\boldsymbol{\omega}) = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Note that the operator **curl** in the first line of this system coincides with the previous one in dimension $d=3$ while, in dimension $d=2$, it is applied to scalar functions φ: $\mathbf{curl}\,\varphi$ here denotes the vector field with components $\partial_y\varphi$ and $-\partial_x\varphi$.

We now introduce the variational spaces. We first consider the domain $H(\mathrm{div},\Omega)$ of the divergence operator, namely

$$H(\mathrm{div},\Omega)=\big\{\boldsymbol{v}\in L^2(\Omega)^d;\ \mathrm{div}\,\boldsymbol{v}\in L^2(\Omega)\big\},\tag{2.3}$$

and also its subspace

$$H_0(\mathrm{div},\Omega)=\big\{\boldsymbol{v}\in H(\mathrm{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n}=0 \text{ on } \partial\Omega\big\}.\tag{2.4}$$

Similarly, we define the domain of the **curl** operator

$$H(\mathbf{curl},\Omega)=\big\{\boldsymbol{\varphi}\in L^2(\Omega)^{\frac{d(d-1)}{2}};\ \mathbf{curl}\,\boldsymbol{\varphi}\in L^2(\Omega)^d\big\},\tag{2.5}$$

and its subspace

$$H_0(\mathbf{curl},\Omega)=\big\{\boldsymbol{\varphi}\in H(\mathbf{curl},\Omega);\ \gamma_t(\boldsymbol{\varphi})=\mathbf{0} \text{ on } \partial\Omega\big\}.\tag{2.6}$$

It must be observed that the spaces $H(\mathbf{curl},\Omega)$ and $H_0(\mathbf{curl},\Omega)$ coincide with the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively, in dimension $d=2$, so that their approximation relies on more standard discrete spaces than in dimension $d=3$.

The spaces $H(\mathrm{div},\Omega)$ and $H(\mathbf{curl},\Omega)$ are provided with the graph norm. Moreover, both in dimensions $d=2$ and $d=3$, the normal trace operator $\boldsymbol{v}\mapsto\boldsymbol{v}\cdot\boldsymbol{n}$ and the operator γ_t are continuous on $H(\mathrm{div},\Omega)$ and $H(\mathbf{curl},\Omega)$, respectively, see [17, Chap I, Thms 2.5 & 2.11], so that the spaces $H_0(\mathrm{div},\Omega)$ and $H_0(\mathbf{curl},\Omega)$ are Hilbert spaces for the scalar products associated with these norms. Finally, let $L_0^2(\Omega)$ denote the space of functions in $L^2(\Omega)$ with a null integral on Ω.

We consider the variational problem

Find $(\boldsymbol{\omega},\boldsymbol{u},p)$ *in* $H_0(\mathbf{curl},\Omega)\times H_0(\mathrm{div},\Omega)\times L_0^2(\Omega)$ *such that*

$$\begin{aligned}
&\forall\boldsymbol{v}\in H_0(\mathrm{div},\Omega),\quad a(\boldsymbol{\omega},\boldsymbol{u};\boldsymbol{v})+K(\boldsymbol{\omega},\boldsymbol{u};\boldsymbol{v})+b(\boldsymbol{v},p)=\langle\boldsymbol{f},\boldsymbol{v}\rangle,\\
&\forall q\in L_0^2(\Omega),\quad b(\boldsymbol{u},q)=0,\\
&\forall\boldsymbol{\varphi}\in H_0(\mathbf{curl},\Omega),\quad c(\boldsymbol{\omega},\boldsymbol{u};\boldsymbol{\varphi})=0,
\end{aligned}\tag{2.7}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H_0(\mathrm{div},\Omega)$ and its dual space. The bilinear forms $a(\cdot,\cdot;\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot;\cdot)$ are defined by

$$\begin{aligned}
&a(\boldsymbol{\omega},\boldsymbol{u};\boldsymbol{v})=\nu\int_\Omega(\mathbf{curl}\,\boldsymbol{\omega})(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\,d\boldsymbol{x},\quad b(\boldsymbol{v},q)=-\int_\Omega(\mathrm{div}\,\boldsymbol{v})(\boldsymbol{x})q(\boldsymbol{x})\,d\boldsymbol{x},\\
&c(\boldsymbol{\omega},\boldsymbol{u};\boldsymbol{\varphi})=\int_\Omega\boldsymbol{\omega}(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{x})\,d\boldsymbol{x}-\int_\Omega\boldsymbol{u}(\boldsymbol{x})\cdot(\mathbf{curl}\,\boldsymbol{\varphi})(\boldsymbol{x})\,d\boldsymbol{x}.
\end{aligned}\tag{2.8}$$

In contrast, the form $K(\cdot,\cdot;\cdot)$, defined by

$$K(\boldsymbol{\omega},\boldsymbol{u};\boldsymbol{v})=\int_\Omega(\boldsymbol{\omega}\times\boldsymbol{u})(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\,d\boldsymbol{x},\tag{2.9}$$

is no longer bilinear but trilinear.

Exactly the same arguments as in [7, Prop. 2.1] lead to the next statement.

Proposition 2.1 *Problems* (2.2) *and* (2.7) *are equivalent, in the sense that any triple* $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ *in* $H(\mathbf{curl}, \Omega) \times H(\mathrm{div}, \Omega) \times L^2_0(\Omega)$ *such that* $\boldsymbol{\omega} \times \boldsymbol{u}$ *belongs to* $L^2(\Omega)^d$ *is a solution of problem* (2.2) *if and only if it is a solution of problem* (2.7).

In order to perform the analysis of problem (2.7), we recall some basic results from [7, §2]. First, we note that the forms $a(\cdot, \cdot; \cdot)$, $b(\cdot, \cdot)$ and $c(\cdot, \cdot; \cdot)$ are continuous on $\big(H(\mathbf{curl}, \Omega) \times H(\mathrm{div}, \Omega)\big) \times H(\mathrm{div}, \Omega)$, $H(\mathrm{div}, \Omega) \times L^2_0(\Omega)$ and $\big(H(\mathbf{curl}, \Omega) \times H(\mathrm{div}, \Omega)\big) \times H(\mathbf{curl}, \Omega)$, respectively.

Let V be the kernel

$$V = \big\{\boldsymbol{v} \in H_0(\mathrm{div}, \Omega);\ \forall q \in L^2_0(\Omega),\ b(\boldsymbol{v}, q) = 0\big\}. \tag{2.10}$$

Since the divergence of any function in $H_0(\mathrm{div}, \Omega)$ belongs to $L^2_0(\Omega)$, it is readily checked that V coincides with the space of divergence-free functions in $H_0(\mathrm{div}, \Omega)$. We also introduce the kernel

$$\mathcal{W} = \big\{(\boldsymbol{\vartheta}, \boldsymbol{w}) \in H_0(\mathbf{curl}, \Omega) \times V;\ \forall \boldsymbol{\varphi} \in H_0(\mathbf{curl}, \Omega),\ c(\boldsymbol{\vartheta}, \boldsymbol{w}; \boldsymbol{\varphi}) = 0\big\}. \tag{2.11}$$

As can easily be derived from density results (see [17, Chap. I, §2]), $\mathcal{W}$ coincides with the space of pairs $(\boldsymbol{\vartheta}, \boldsymbol{w})$ in $H_0(\mathbf{curl}, \Omega) \times V$ such that $\boldsymbol{\vartheta}$ is equal to $\mathbf{curl}\, \boldsymbol{w}$ in the distribution sense. Moreover it follows from the continuity properties of the forms $b(\cdot, \cdot)$ and $c(\cdot, \cdot; \cdot)$ that both V and $\mathcal{W}$ are Hilbert spaces.

The following properties are established in [7, Form. (2.14) & (2.17)] thanks to an extension of the arguments in [21]:

(i) There exists a constant $\alpha > 0$ such that the form $a(\cdot, \cdot; \cdot)$ satisfies the positivity and inf-sup conditions

$$\begin{aligned} &\forall \boldsymbol{v} \in V \setminus \{0\}, \quad \sup_{(\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{W}} a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) > 0, \\ &\forall (\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{W}, \quad \sup_{\boldsymbol{v} \in V} \frac{a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v})}{\|\boldsymbol{v}\|_{L^2(\Omega)^d}} \geq \alpha \left(\|\boldsymbol{\omega}\|_{H(\mathbf{curl}, \Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)^d}\right); \end{aligned} \tag{2.12}$$

(ii) There exists a constant $\beta > 0$ such that the form $b(\cdot, \cdot)$ satisfies the inf-sup condition

$$\forall q \in L^2_0(\Omega), \quad \sup_{\boldsymbol{v} \in H_0(\mathrm{div}, \Omega)} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{H(\mathrm{div}, \Omega)}} \geq \beta \, \|q\|_{L^2(\Omega)}. \tag{2.13}$$

We also need the more precise properties

$$\forall (\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{W}, \quad a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{\omega}) \geq \tfrac{\nu}{2} \, \|\boldsymbol{\omega}\|^2_{H(\mathbf{curl}, \Omega)} + \tfrac{\nu}{2c_0^2} \, \|\boldsymbol{u}\|^2_{L^2(\Omega)^d}, \tag{2.14}$$

and

$$\forall (\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{W}, \quad a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{u}) \geq \tfrac{\nu}{2} \, \|\boldsymbol{\omega}\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} + \tfrac{\nu}{2c_0^2} \, \|\boldsymbol{u}\|^2_{L^2(\Omega)^d}, \tag{2.15}$$

where c_0 denotes the smallest constant such that

$$\forall \boldsymbol{v} \in V \cap H(\mathbf{curl}, \Omega), \quad \|\boldsymbol{v}\|_{L^2(\Omega)^d} \leq c_0 \, \|\mathbf{curl}\, \boldsymbol{v}\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}}. \tag{2.16}$$

The existence of such a constant follows from [5, Cor. 3.16] since Ω is simply-connected.

To go further, we now investigate the properties of the form $K(\cdot, \cdot; \cdot)$. We need a further assumption for that in the three-dimensional case.

Assumption 2.2 In dimension $d = 3$, the spaces $H_0(\text{div}, \Omega) \cap H(\textbf{curl}, \Omega)$ and $H(\text{div}, \Omega) \cap H_0(\textbf{curl}, \Omega)$ are compactly imbedded in $H^{3/4}(\Omega)^3$.

Assumption 2.2 holds whenever Ω has a boundary of class $\mathcal{C}^{1,1}$ or is convex, see [5, §2], but seems less restrictive. However, we can build from the ideas in [13] and [14] the following counter-example. Let Ω denote the L–shaped domain

$$\Omega = \big\{(x, y, z);\ (x, y) \in]-1, 1[^2 \setminus [0, 1[^2 \text{ and } z \in]0, 3[\big\},$$

and let Γ be the union of the two faces that share the edge $\gamma = \{0\} \times \{0\} \times]0, 3[$. Let also χ be a smooth function with a compact support in $\Omega \cup \Gamma$ which is equal to 1 in a neighbourhood of a part of γ. Thus, when setting $x = r \cos\theta$ and $y = r \sin\theta$, the gradients of the functions

$$\begin{aligned} & S(x, y, z) = \chi(x, y, z)\, r^{\frac{2}{3}} \cos\left(\frac{2\theta}{3}\right) \\ \text{and} \qquad & S(x, y, z) = \chi(x, y, z)\, r^{\frac{2}{3}} \sin\left(\frac{2\theta}{3}\right), \end{aligned}$$

belong to $H_0(\text{div}, \Omega) \cap H(\textbf{curl}, \Omega)$ and $H(\text{div}, \Omega) \cap H_0(\textbf{curl}, \Omega)$, respectively, but not to $H^{3/4}(\Omega)^3$.

In any case, we need Assumption 2.2 to prove the continuity of the nonlinear term.

Lemma 2.3 *If Assumption* 2.2 *is satisfied,*

(i) the following continuity property holds

$$\begin{aligned} & \forall (\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{W}, \quad \forall \boldsymbol{v} \in L^2(\Omega)^d, \\ & \quad K(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) \le c_* \, \|\boldsymbol{\omega}\|_{H(\textbf{curl},\Omega)} \\ & \qquad \Big(\|\boldsymbol{\omega}\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}} + \|\boldsymbol{u}\|_{H(\text{div},\Omega)}\Big)\, \|\boldsymbol{v}\|_{L^2(\Omega)^d}, \end{aligned} \tag{2.17}$$

for a constant c_* only depending on Ω;

(ii) for any $(\boldsymbol{\vartheta}, \boldsymbol{w})$ in $\mathcal{W}$, the operators: $(\boldsymbol{\omega}, \boldsymbol{u}) \mapsto \boldsymbol{\omega} \times \boldsymbol{w}$ and: $(\boldsymbol{\omega}, \boldsymbol{u}) \mapsto \boldsymbol{\vartheta} \times \boldsymbol{u}$ are compact from $\mathcal{W}$ into $L^2(\Omega)^d$.

Proof Any pair $(\boldsymbol{\omega}, \boldsymbol{u})$ in $\mathcal{W}$ satisfies $\boldsymbol{\omega} = \textbf{curl}\, \boldsymbol{u}$, so that the following imbedding holds

$$\mathcal{W} \subset \big(H(\text{div}, \Omega) \cap H_0(\textbf{curl}, \Omega)\big) \times \big(H_0(\text{div}, \Omega) \cap \widetilde{H}(\textbf{curl}, \Omega)\big), \tag{2.18}$$

with obvious definition for the modified space $\widetilde{H}(\textbf{curl}, \Omega)$ in dimension $d = 2$. Next, in dimension $d = 2$, the space $H_0(\textbf{curl}, \Omega)$ is equal to $H_0^1(\Omega)$, hence is compactly imbedded in $L^p(\Omega)$ for all $p < +\infty$, and the space $H_0(\text{div}, \Omega) \cap \widetilde{H}(\textbf{curl}, \Omega)$ is imbedded in $H^{1/2}(\Omega)^2$, see [12], hence is compactly imbedded in

$L^3(\Omega)^2$ for instance. In dimension $d = 3$, from Assumption 2.2 and the Sobolev imbedding theorem, the space $\mathcal{W}$ is compactly imbedded in $L^4(\Omega)^3 \times L^4(\Omega)^3$. So both assertions of the lemma are a consequence of the Hölder's inequality

$$K(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) \leq c\, \|\boldsymbol{\omega}\|_{L^p(\Omega)^{\frac{d(d-1)}{2}}} \|\boldsymbol{u}\|_{L^q(\Omega)^d} \|\boldsymbol{v}\|_{L^2(\Omega)^d},$$

with $p = 6$ and $q = 3$ in dimension $d = 2$, $p = q = 4$ in dimension $d = 3$.

We skip the proof of the next lemma that relies on simple arguments.

Lemma 2.4 *If Assumption* 2.2 *holds, the form* $K(\cdot, \cdot; \cdot)$ *satisfies the antisymmetry properties*

$$\forall (\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{W}, \quad K(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{u}) = 0, \tag{2.19}$$

and also, in dimension $d = 2$,

$$\forall (\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{W}, \quad K(\boldsymbol{\omega}, \boldsymbol{u}; \mathbf{curl}\, \boldsymbol{\omega}) = 0. \tag{2.20}$$

We observe that, for any solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ of problem (2.7), the pair $(\boldsymbol{\omega}, \boldsymbol{u})$ is a solution of the following reduced problem

Find $(\boldsymbol{\omega}, \boldsymbol{u})$ *in* $\mathcal{W}$ *such that*

$$\forall \boldsymbol{v} \in V, \quad a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) + K(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle. \tag{2.21}$$

The main difficulty consists in proving the existence of a solution for this problem. Moreover, due the previous lemma, the proof is much simpler in dimension $d = 2$. So we begin with this case.

Proposition 2.5 *In dimension* $d = 2$, *for any data* $\boldsymbol{f}$ *in the dual space of* $H_0(\mathrm{div}, \Omega)$, *problem* (2.21) *has a solution* $(\boldsymbol{\omega}, \boldsymbol{u})$ *in* $\mathcal{W}$. *Moreover this solution satisfies*

$$\|\boldsymbol{\omega}\|_{H(\mathbf{curl}, \Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)^2} \leq c\, \nu^{-1}\, \|\boldsymbol{f}\|_{H_0(\mathrm{div}, \Omega)'}, \tag{2.22}$$

where the constant c *only depends on* Ω.

Proof It is performed in several steps.
1) We first define the following mapping Φ from $\mathcal{W}$ onto its dual space by

$$\begin{aligned} &\forall (\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{W}, \ \forall (\boldsymbol{\vartheta}, \boldsymbol{w}) \in \mathcal{W}, \\ &\langle \Phi(\boldsymbol{\omega}, \boldsymbol{u}), (\boldsymbol{\vartheta}, \boldsymbol{w}) \rangle = a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{w} + \mathbf{curl}\, \boldsymbol{\vartheta}) \\ &\qquad\qquad + K(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{w} + \mathbf{curl}\, \boldsymbol{\vartheta}) - \langle \boldsymbol{f}, \boldsymbol{w} + \mathbf{curl}\, \boldsymbol{\vartheta} \rangle. \end{aligned}$$

It follows from part (i) of Lemma 2.3 that the mapping Φ is continuous on $\mathcal{W}$. Moreover, by combining (2.14) and Lemma 2.4, we have the property

$$\begin{aligned} \langle \Phi(\boldsymbol{\omega}, \boldsymbol{u}), (\boldsymbol{\omega}, \boldsymbol{u}) \rangle \geq \frac{\nu}{2} \|\boldsymbol{\omega}\|^2_{H(\mathbf{curl}, \Omega)} + \frac{\nu}{2c_0^2} \|\boldsymbol{u}\|^2_{L^2(\Omega)^d} \\ - \|\boldsymbol{f}\|_{H_0(\mathrm{div}, \Omega)'} \big(\|\boldsymbol{\omega}\|_{H(\mathbf{curl}, \Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)^d} \big). \end{aligned}$$

So, the quantity $\langle \Phi(\boldsymbol{\omega}, \boldsymbol{u}), (\boldsymbol{\omega}, \boldsymbol{u}) \rangle$ is nonnegative on the sphere S_μ with radius

$$\mu = \frac{2\sqrt{2}\, \max\{1, c_0^2\}}{\nu} \|\boldsymbol{f}\|_{H_0(\mathrm{div}, \Omega)'}. \tag{2.23}$$

2) Since $\mathcal{W}$ is included in $L^2(\Omega) \times L^2(\Omega)^2$, it is a separable Hilbert space. So there exists an increasing sequence $(\mathcal{W}_n)_n$ of finite-dimensional subspaces $\mathcal{W}_n$ of $\mathcal{W}$ such that $\cup_n \mathcal{W}_n$ is dense in $\mathcal{W}$. The mapping Φ is continuous from $\mathcal{W}_n$ onto its dual space and satisfies

$$\forall(\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{W}_n \cap S_\mu, \quad \langle \Phi(\boldsymbol{\omega}, \boldsymbol{u}), (\boldsymbol{\omega}, \boldsymbol{u}) \rangle \geq 0.$$

So it follows from Brouwer's fixed point theorem, see [17, Chap. IV, Cor. 1.1], that there exists a $(\boldsymbol{\omega}_n, \boldsymbol{u}_n)$ in $\mathcal{W}_n$ such that

$$\begin{aligned} &\forall(\boldsymbol{\vartheta}_n, \boldsymbol{w}_n) \in \mathcal{W}_n, \quad \langle \Phi(\boldsymbol{\omega}_n, \boldsymbol{u}_n), (\boldsymbol{\vartheta}_n, \boldsymbol{w}_n) \rangle = 0 \\ &\qquad \text{and} \quad \left(\|\boldsymbol{\omega}_n\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}_n\|^2_{L^2(\Omega)^d} \right)^{\frac{1}{2}} \leq \mu. \end{aligned} \tag{2.24}$$

The sequence $(\boldsymbol{\omega}_n, \boldsymbol{u}_n)_n$ is bounded by μ, so that there exists a subsequence still denoted by $(\boldsymbol{\omega}_n, \boldsymbol{u}_n)_n$, which converges to a pair $(\boldsymbol{\omega}, \boldsymbol{u})$ weakly in $\mathcal{W}$. Thanks to the compactness result stated in part (ii) of Lemma 2.3, passing to the limit in equation (2.24) yields for any n

$$\forall(\boldsymbol{\vartheta}_n, \boldsymbol{v}_n) \in \mathcal{W}_n, \quad \langle \Phi(\boldsymbol{\omega}, \boldsymbol{u}), (\boldsymbol{\vartheta}_n, \boldsymbol{w}_n) \rangle = 0. \tag{2.25}$$

Finally, using the density of $\cup_n \mathcal{W}_n$ into $\mathcal{W}$ gives

$$\forall(\boldsymbol{\vartheta}, \boldsymbol{w}) \in \mathcal{W}, \quad \langle \Phi(\boldsymbol{\omega}, \boldsymbol{u}), (\boldsymbol{\vartheta}, \boldsymbol{w}) \rangle = 0.$$

Moreover, the pair $(\boldsymbol{\omega}, \boldsymbol{u})$ has its norm bounded by μ, hence satisfies (2.22).
3) The pair $(\boldsymbol{\omega}, \boldsymbol{u})$ satisfies

$$\begin{aligned} \forall(\boldsymbol{\vartheta}, \boldsymbol{w}) \in \mathcal{W}, \quad a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{w} + \mathbf{curl}\, \boldsymbol{\vartheta}) + K(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{w} + \mathbf{curl}\, \boldsymbol{\vartheta}) \\ = \langle \boldsymbol{f}, \boldsymbol{w} + \mathbf{curl}\, \boldsymbol{\vartheta} \rangle. \end{aligned} \tag{2.26}$$

Let now $\boldsymbol{v}$ be any function in V. We consider the problem
Find $(\boldsymbol{\vartheta}, \boldsymbol{w})$ *in* $\mathcal{W}$ *such that*

$$\forall \boldsymbol{z} \in V, \quad \tilde{a}(\boldsymbol{\vartheta}, \boldsymbol{w}; \boldsymbol{z}) = \int_\Omega \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{z}(\boldsymbol{x})\, d\boldsymbol{x}, \tag{2.27}$$

where the bilinear form $\tilde{a}(\cdot, \cdot)$ is defined by

$$\tilde{a}(\boldsymbol{\vartheta}, \boldsymbol{w}; \boldsymbol{z}) = \int_\Omega \boldsymbol{w}(\boldsymbol{x}) \cdot \boldsymbol{z}(\boldsymbol{x})\, d\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{\vartheta})(\boldsymbol{x}) \cdot \boldsymbol{z}(\boldsymbol{x})\, d\boldsymbol{x}.$$

Exactly the same arguments as for [7, Lemma 2.3] yield that properties (2.12) stil hold with $a(\cdot, \cdot; \cdot)$ replaced by $\tilde{a}(\cdot, \cdot; \cdot)$, so that problem (2.27) has a unique solution. Moreover, since $\boldsymbol{w} + \mathbf{curl}\, \boldsymbol{\vartheta}$ belongs to V, $\boldsymbol{v}$ is equal to $\boldsymbol{w} + \mathbf{curl}\, \boldsymbol{\vartheta}$. Thus, applying equation (2.26) to this pair $(\boldsymbol{\vartheta}, \boldsymbol{w})$ implies that $(\boldsymbol{\omega}, \boldsymbol{u})$ is a solution of problem (2.21).

The existence result in dimension $d = 3$ is only proved for ν large enough with respect to the data $\boldsymbol{f}$. The main reason is that formula (2.20) does not hold in dimension $d = 3$. We begin with a lemma where this difficulty is brought to light.

Lemma 2.6 *In dimension $d = 3$, if Assumption 2.2 holds, there exists a constant $c_\sharp$ only depending on Ω such that, for any data $\boldsymbol{f}$ in the dual space of $H_0(div, \Omega)$ satisfying*

$$c_\sharp\, \nu^{-2}\, \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'} < 1, \tag{2.28}$$

any solution of problem (2.21) *satisfies*

$$\|\boldsymbol{\omega}\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)^3} \le c\, \nu^{-1}\, \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}, \tag{2.29}$$

where the constant c only depends on Ω.

Proof We set

$$m_w(\boldsymbol{\omega}, \boldsymbol{u}) = \big(\|\boldsymbol{\omega}\|^2_{L^2(\Omega)^3} + \|\boldsymbol{u}\|^2_{L^2(\Omega)^3}\big)^{\frac{1}{2}},$$
$$m_s(\boldsymbol{\omega}, \boldsymbol{u}) = \big(\|\boldsymbol{\omega}\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|^2_{L^2(\Omega)^3}\big)^{\frac{1}{2}},$$

where the indices w and s stands for weak and strong, respectively. By taking $\boldsymbol{v}$ equal to $\boldsymbol{u}$ in (2.21) and using (2.15) and (2.19), we have

$$\frac{\nu}{2\, \max\{1, c_0^2\}}\, m_w(\boldsymbol{\omega}, \boldsymbol{u})^2 \le \langle \boldsymbol{f}, \boldsymbol{u}\rangle \le \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'} \|\boldsymbol{u}\|_{L^2(\Omega)^3},$$

whence

$$m_w(\boldsymbol{\omega}, \boldsymbol{u}) \le 2\, \max\{1, c_0^2\}\, \nu^{-1}\, \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}. \tag{2.30}$$

Next, we take $\boldsymbol{v} = \boldsymbol{u} + \mathbf{curl}\, \boldsymbol{\omega}$ in (2.21). Using now (2.14) and (2.19), we obtain

$$\frac{\nu}{2\, \max\{1, c_0^2\}}\, m_s(\boldsymbol{\omega}, \boldsymbol{u})^2 \le \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'} \|\boldsymbol{u}\|_{L^2(\Omega)^3} - K(\boldsymbol{\omega}, \boldsymbol{u}; \mathbf{curl}\, \boldsymbol{\omega}).$$

From (2.17), we derive

$$|K(\boldsymbol{\omega}, \boldsymbol{u}; \mathbf{curl}\, \boldsymbol{\omega})| \le c_*\, m_w(\boldsymbol{\omega}, \boldsymbol{u})\, m_s(\boldsymbol{\omega}, \boldsymbol{u})^2,$$

so that, thanks to (2.30),

$$|K(\boldsymbol{\omega}, \boldsymbol{u}; \mathbf{curl}\, \boldsymbol{\omega})| \le 2c_*\, \max\{1, c_0^2\}\, \nu^{-1}\, \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}\, m_s(\boldsymbol{\omega}, \boldsymbol{u})^2.$$

Combining all this gives

$$\frac{\nu}{2\, \max\{1, c_0^2\}}\, m_s(\boldsymbol{\omega}, \boldsymbol{u})\, \big(1 - 4c_*\, (\max\{1, c_0^2\})^2\, \nu^{-2}\, \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}\big) \le \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}.$$

So, assumption (2.28) with $c_\sharp = 8c_*\, (\max\{1, c_0^2\})^2$ leads to estimate (2.29).

Proposition 2.7 *In dimension $d = 3$, if Assumption 2.2 holds, there exists a constant $c_\flat$ only depending on Ω such that, for any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div}, \Omega)$ satisfying*

$$c_\flat\, \nu^{-2}\, \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'} < 1, \tag{2.31}$$

problem (2.21) *has a solution $(\boldsymbol{\omega}, \boldsymbol{u})$ in $\mathcal{W}$. Moreover this solution satisfies* (2.29).

Proof Setting $(\boldsymbol{\omega}^0, \boldsymbol{u}^0) = (\mathbf{0}, \mathbf{0})$, we iteratively solve the problem, for $n \geq 1$:
Find $(\boldsymbol{\omega}^n, \boldsymbol{u}^n)$ *in* $\mathcal{W}$ *such that*

$$\forall \boldsymbol{v} \in V, \quad a(\boldsymbol{\omega}^n, \boldsymbol{u}^n; \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle - K(\boldsymbol{\omega}^{n-1}, \boldsymbol{u}^{n-1}; \boldsymbol{v}). \tag{2.32}$$

Thanks to (2.12) and also (2.17), this problem admits a unique solution. Now, let μ be defined by

$$\mu = \frac{\nu}{4c_* \sqrt{2} \max\{1, c_0^2\}},$$

where the constant c_* is that in (2.17). We now check by induction on n that, for an appropriate choice of $c_\flat$ in (2.31), the sequence $(\boldsymbol{\omega}^n, \boldsymbol{u}^n)_n$ is bounded by μ in the norm of $\mathcal{W}$, namely that

$$\left(\|\boldsymbol{\omega}^n\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}^n\|^2_{L^2(\Omega)^3} \right)^{\frac{1}{2}} \leq \mu. \tag{2.33}$$

Since this estimate obviously holds for $n = 0$, we now assume that it holds with n replaced by $n - 1$. We now take $\boldsymbol{v}$ equal to $\boldsymbol{u}^n + \mathbf{curl}\, \boldsymbol{\omega}^n$ in problem (2.32) and we easily derive from (2.14) and (2.17) that

$$\frac{\nu}{2 \max\{1, c_0^2\}} \left(\|\boldsymbol{\omega}^n\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}^n\|^2_{L^2(\Omega)^3} \right)^{\frac{1}{2}} \leq \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'} + c_* \sqrt{2}\, \mu^2.$$

Thank to the choice of μ, we have

$$\frac{2c_* \sqrt{2} \max\{1, c_0^2\}}{\nu} \mu^2 = \frac{1}{2} \mu.$$

Similarly, when taking $c_\flat \geq 16 c_* \sqrt{2}\, (\max\{1, c_0^2\})^2$, we obtain

$$\begin{aligned}
\frac{2 \max\{1, c_0^2\}}{\nu} \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'} &\leq \frac{2 \max\{1, c_0^2\}}{c_\flat} \nu \\
&\leq \frac{\nu}{8\, c_* \sqrt{2} \max\{1, c_0^2\}} = \frac{1}{2} \mu.
\end{aligned}$$

So, we have proved the desired estimate. On the other hand, we have for all $n \geq 2$,

$$\begin{aligned}
\forall \boldsymbol{v} \in V, \quad & a(\boldsymbol{\omega}^n - \boldsymbol{\omega}^{n-1}, \boldsymbol{u}^n - \boldsymbol{u}^{n-1}; \boldsymbol{v}) \\
&= K(\boldsymbol{\omega}^{n-2}, \boldsymbol{u}^{n-2}; \boldsymbol{v}) - K(\boldsymbol{\omega}^{n-1}, \boldsymbol{u}^{n-1}; \boldsymbol{v}) \\
&= -K(\boldsymbol{\omega}^{n-1} - \boldsymbol{\omega}^{n-2}, \boldsymbol{u}^{n-2}; \boldsymbol{v}) - K(\boldsymbol{\omega}^{n-1}, \boldsymbol{u}^{n-1} - \boldsymbol{u}^{n-2}; \boldsymbol{v}).
\end{aligned}$$

Using once more (2.14) and (2.17) together with (2.33) thus leads to

$$\begin{aligned}
&\frac{\nu}{2 \max\{1, c_0^2\}} \left(\|\boldsymbol{\omega}^n - \boldsymbol{\omega}^{n-1}\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\|^2_{L^2(\Omega)^3} \right)^{\frac{1}{2}} \\
&\qquad \leq c_* \mu \left(\|\boldsymbol{\omega}^{n-1} - \boldsymbol{\omega}^{n-2}\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}^{n-1} - \boldsymbol{u}^{n-2}\|^2_{L^2(\Omega)^3} \right)^{\frac{1}{2}},
\end{aligned}$$

whence, thanks to the choice of μ,

$$\big(\|\boldsymbol{\omega}^n - \boldsymbol{\omega}^{n-1}\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\|^2_{L^2(\Omega)^3}\big)^{\frac{1}{2}} \leq \frac{1}{2\sqrt{2}}\big(\|\boldsymbol{\omega}^{n-1} - \boldsymbol{\omega}^{n-2}\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}^{n-1} - \boldsymbol{u}^{n-2}\|^2_{L^2(\Omega)^3}\big)^{\frac{1}{2}}.$$

The sequence $(\boldsymbol{\omega}^n, \boldsymbol{u}^n)_n$ is a Cauchy sequence in $\mathcal{W}$, so that it converges to a pair $(\boldsymbol{\omega}, \boldsymbol{u})$. By passing to the limit in (2.32), it is readily checked that $(\boldsymbol{\omega}, \boldsymbol{u})$ is a solution of problem (2.21). Estimate (2.29) is finally derived from Lemma 2.6 since $c_\flat$ is larger than $c_\sharp$.

Theorem 2.8 *In dimension $d = 2$, for any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div}, \Omega)$, problem* (2.7) *has a solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ in $H_0(\mathbf{curl}, \Omega) \times H_0(\mathrm{div}, \Omega) \times L^2_0(\Omega)$. In dimension $d = 3$, if Assumption* 2.2 *holds, for any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div}, \Omega)$ such that* (2.31) *is satisfied, problem* (2.7) *has a solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ in $H_0(\mathbf{curl}, \Omega) \times H_0(\mathrm{div}, \Omega) \times L^2_0(\Omega)$. Moreover this solution satisfies*

$$\|\boldsymbol{\omega}\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|_{H(\mathrm{div},\Omega)} + \nu^{-1}\,\|p\|_{L^2(\Omega)} \leq c\,\nu^{-1}\,\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}\,\big(1 + \nu^{-2}\,\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}\big), \quad (2.34)$$

where the constant c only depends on Ω.

Proof For any data $\boldsymbol{f}$ in $H_0(\mathrm{div}, \Omega)'$, it follows from Proposition 2.5 and 2.7 that there exists a solution $(\boldsymbol{\omega}, \boldsymbol{u})$ of problem (2.21). Moreover, since the norms $\|\cdot\|_{L^2(\Omega)^d}$ and $\|\cdot\|_{H(\mathrm{div},\Omega)}$ coincide on V, this solution satisfies (2.22) or (2.29), whence the first part of (2.34). On the other hand, the pressure p must now satisfy

$$\forall \boldsymbol{v} \in H_0(\mathrm{div}, \Omega), \quad b(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle - a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) - K(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}).$$

Since the right-hand side of the previous line vanishes for all $\boldsymbol{v}$ in V, see (2.21), the existence of a solution p of this equation in $L^2_0(\Omega)$ is a consequence of condition (2.13), see once more [17, Chap. I, Lemma 4.1]. Moreover it satisfies

$$\beta\,\|p\|_{L^2(\Omega)} \leq \|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'} + \nu\,\|\boldsymbol{\omega}\|_{H(\mathbf{curl},\Omega)} + c_*\,\big(\|\boldsymbol{\omega}\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|_{H(\mathrm{div},\Omega)}\big)^2,$$

whence the second part of (2.34).

As usual for the Navier–Stokes equations, the uniqueness of the solution can only be proven for small enough data or large enough viscosity.

Theorem 2.9 *If Assumption 2.2 holds, there exists a constant $c_\natural$ only depending on Ω such that, for any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div}, \Omega)$ satisfying*

$$c_\natural\,\nu^{-2}\,\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'} < 1, \quad (2.35)$$

problem (2.7) *has at most a solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ in $H_0(\mathbf{curl}, \Omega) \times H_0(\mathrm{div}, \Omega) \times L^2_0(\Omega)$.*

Proof Let $(\boldsymbol{\omega}_1, \boldsymbol{u}_1, p_1)$ and $(\boldsymbol{\omega}_2, \boldsymbol{u}_2, p_2)$ be two solutions of problem (2.7). In dimension $d = 2$, it follows from (2.14), (2.19) and (2.20) that both $(\boldsymbol{\omega}_1, \boldsymbol{u}_1)$ and $(\boldsymbol{\omega}_2, \boldsymbol{u}_2)$ satisfy (2.22). Similarly, in dimension $d = 3$, it follows from Assumption 2.2 and Lemma 2.6 that, wehn taking $c_\natural > c_\sharp$, both $(\boldsymbol{\omega}_1, \boldsymbol{u}_1)$ and $(\boldsymbol{\omega}_2, \boldsymbol{u}_2)$ satisfy (2.29), On the other hand, the pair $(\boldsymbol{\omega}, \boldsymbol{u})$, with $\boldsymbol{\omega} = \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2$ and $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ belongs to $\mathcal{W}$ and satisfies

$$\begin{aligned}\forall \boldsymbol{v} \in V, \quad a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) &= K(\boldsymbol{\omega}_2, \boldsymbol{u}_2; \boldsymbol{v}) - K(\boldsymbol{\omega}_1, \boldsymbol{u}_1; \boldsymbol{v}) \\ &= -K(\boldsymbol{\omega}, \boldsymbol{u}_2; \boldsymbol{v}) - K(\boldsymbol{\omega}_1, \boldsymbol{u}; \boldsymbol{v}).\end{aligned}$$

By using once more (2.14), taking $\boldsymbol{v}$ equal to $\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{\omega}$ in the previous line gives

$$\begin{aligned}&\frac{\nu}{2\,\max\{1, c_0^2\}} \left(\|\boldsymbol{\omega}\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|^2_{L^2(\Omega)^d}\right) \\ &\qquad \le |K(\boldsymbol{\omega}, \boldsymbol{u}_2; \boldsymbol{u} + \mathbf{curl}\,\boldsymbol{\omega})| + |K(\boldsymbol{\omega}_1, \boldsymbol{u}; \boldsymbol{u} + \mathbf{curl}\,\boldsymbol{\omega})|.\end{aligned}$$

A simple extension of Lemma 2.4 gives

$$K(\boldsymbol{\omega}_1, \boldsymbol{u}; \boldsymbol{u}) = 0.$$

So applying Lemma 2.3 together with (2.22) or (2.29) yields

$$\begin{aligned}&\frac{\nu}{2\,\max\{1, c_0^2\}} \left(\|\boldsymbol{\omega}\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|^2_{L^2(\Omega)^d}\right) \\ &\qquad \le c\,\nu^{-1}\,\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'} \left(\|\boldsymbol{\omega}\|^2_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}\|^2_{L^2(\Omega)^d}\right).\end{aligned}$$

Thus, if condition (2.35) holds with $c_\natural > 2c\,\max\{1, c_0^2\}$, both $\boldsymbol{\omega}$ and $\boldsymbol{u}$ are zero. Finally, we have

$$\forall \boldsymbol{v} \in H_0(\mathrm{div}, \Omega), \quad b(\boldsymbol{v}, p_1 - p_2) = 0,$$

so that $p_1 - p_2$ is zero thanks to condition (2.13). This yields the uniqueness of the solution.

Clearly condition (2.35) is rather restrictive in dimension $d = 2$, so we try to avoid it in the numerical analysis of the discretization. To conclude, we state some regularity properties of the solution of problem (2.7) which can easily be derived from [5, §2], [13] and [14] together with a bootstrap argument.

Proposition 2.10 *When Ω is convex, the mapping: $\boldsymbol{f} \mapsto (\boldsymbol{\omega}, \boldsymbol{u}, p)$, where $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ is the solution of problem* (2.7) *with data $\boldsymbol{f}$, is continuous from $H^{\max\{0,s-1\}}(\Omega)^d$ into $H^s(\Omega)^{\frac{d(d-1)}{2}} \times H^s(\Omega)^d \times H^s(\Omega)$, for all $s \le 1$.*

A stronger property holds in dimension $d = 2$.

Proposition 2.11 *In dimension $d = 2$, the mapping: $\boldsymbol{f} \mapsto (\boldsymbol{\omega}, \boldsymbol{u}, p)$, where $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ is the solution of problem* (2.7) *with data $\boldsymbol{f}$, is continuous from $H^{\max\{0,s\}}(\Omega)^d$ into $H^{s+1}(\Omega) \times H^s(\Omega)^2 \times H^{s+1}(\Omega)$, for*

(i) all $s \le \frac{1}{2}$ in the general case,
(ii) all $s \le 1$ when Ω is convex,
(iii) all $s < \frac{\pi}{\alpha}$ when Ω is a polygon with largest angle equal to α.

Proof Let ψ denote the stream function associate with $\boldsymbol{u}$, namely the fonction ψ in $H_0^1(\Omega)$ such that $\boldsymbol{u} = \mathbf{curl}\,\psi$. The desired regularity properties follow from the fact that ψ et $\boldsymbol{\omega}$ are a solution of the problems

$$\begin{cases} -\Delta\psi = \boldsymbol{\omega} & \text{in } \Omega, \\ \quad \psi = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\nu\,\Delta\boldsymbol{\omega} = \mathbf{curl}\,(\boldsymbol{f} - \boldsymbol{\omega} \times \boldsymbol{u}) & \text{in } \Omega, \\ \quad \boldsymbol{\omega} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{2.36}$$

combined with the formula

$$\forall \boldsymbol{\omega} \in H^1(\Omega), \quad \forall \boldsymbol{u} \in V, \quad \mathbf{curl}\,(\boldsymbol{\omega} \times \boldsymbol{u}) = \mathbf{grad}\,\boldsymbol{\omega} \cdot \boldsymbol{u}, \tag{2.37}$$

see [18, Chap. 4] for instance.

3 The spectral discrete problem

From now on, we assume that Ω is the square or cube $]-1,1[^d$, $d = 2$ or 3. The discrete spaces are constructed from the finite elements proposed by Nédélec on cubic three-dimensional meshes, see [20, §2]. In order to describe them and for any triple (ℓ, m, n) of nonnegative integers, we introduce

- in dimension $d = 2$, the space $\mathbb{P}_{\ell,m}(\Omega)$ of restrictions to Ω of polynomials with degree $\le \ell$ with respect to x and $\le m$ with respect to y,
- in dimension $d = 3$, the space $\mathbb{P}_{\ell,m,n}(\Omega)$ of restrictions to Ω of polynomials with degree $\le \ell$ with respect to x, $\le m$ with respect to y and $\le n$ with respect to z.

When ℓ and m are equal to n, these spaces are simply denoted by $\mathbb{P}_n(\Omega)$.

Let N be an integer ≥ 2. In all that follows, c denotes a generic constant that may vary from one line to the next but is always independent of N. The discrete spaces that we use are exactly the same as in [7, §3]. The space $\mathbb{D}_N$ which approximates $H_0(\mathrm{div}, \Omega)$ is defined by

$$\mathbb{D}_N = H_0(\mathrm{div}, \Omega) \cap \begin{cases} \mathbb{P}_{N,N-1}(\Omega) \times \mathbb{P}_{N-1,N}(\Omega) & \text{if } d = 2, \\ \mathbb{P}_{N,N-1,N-1}(\Omega) \times \mathbb{P}_{N-1,N,N-1}(\Omega) \times \mathbb{P}_{N-1,N-1,N}(\Omega) & \text{if } d = 3. \end{cases} \tag{3.1}$$

The space $\mathbb{C}_N$ which approximates $H_0(\mathbf{curl}, \Omega)$ is rather different according to the dimension of; it is defined by

$$\mathbb{C}_N = \begin{cases} H_0^1(\Omega) \cap \mathbb{P}_N(\Omega) & \text{if } d = 2, \\ H_0(\mathbf{curl}, \Omega) & \\ \cap\big(\mathbb{P}_{N-1,N,N}(\Omega) \times \mathbb{P}_{N,N-1,N}(\Omega) \times \mathbb{P}_{N,N,N-1}(\Omega)\big) & \text{if } d = 3. \end{cases} \tag{3.2}$$

Finally, for the approximation of $L_0^2(\Omega)$, we consider the space $\mathbb{M}_N$:

$$\mathbb{M}_N = L_0^2(\Omega) \cap \mathbb{P}_{N-1}(\Omega). \tag{3.3}$$

Setting $\xi_0 = -1$ and $\xi_N = 1$, we introduce the $N-1$ nodes ξ_j, $1 \le j \le N-1$, and the $N+1$ weights ρ_j, $0 \le j \le N$, of the Gauss–Lobatto quadrature formula.

Denoting by $\mathbb{P}_n(-1,1)$ the space of restrictions to $[-1,1]$ of polynomials with degree $\leq n$, we recall that the following equality holds

$$\forall \Phi \in \mathbb{P}_{2N-1}(-1,1), \quad \int_{-1}^{1} \Phi(\zeta)\,d\zeta = \sum_{j=0}^{N} \Phi(\xi_j)\,\rho_j. \tag{3.4}$$

We also recall [10, form. (13.20)] the following property, which is useful in what follows

$$\begin{aligned} &\forall \varphi_N \in \mathbb{P}_N(-1,1), \\ &\qquad \|\varphi_N\|^2_{L^2(-1,1)} \leq \sum_{j=0}^{N} \varphi_N^2(\xi_j)\,\rho_j \leq 3\,\|\varphi_N\|^2_{L^2(-1,1)}. \end{aligned} \tag{3.5}$$

Relying on this formula, we introduce the discrete product, defined on continuous functions u and v by

$$(u,v)_N = \begin{cases} \sum_{i=0}^{N}\sum_{j=0}^{N} u(\xi_i,\xi_j)v(\xi_i,\xi_j)\,\rho_i\rho_j & \text{if } d=2, \\ \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} \\ \qquad u(\xi_i,\xi_j,\xi_k)v(\xi_i,\xi_j,\xi_k)\,\rho_i\rho_j\rho_k & \text{if } d=3. \end{cases} \tag{3.6}$$

It follows from (3.5) that it is a scalar product on $\mathbb{P}_N(\Omega)$. Let finally $\mathcal{I}_N$ denote the Lagrange interpolation operator at the nodes (ξ_i,ξ_j), $0 \leq i, j \leq N$, in dimension $d=2$, at the nodes (ξ_i,ξ_j,ξ_k), $0 \leq i, j, k \leq N$, in dimension $d=3$, with values in $\mathbb{P}_N(\Omega)$.

We now assume that $\boldsymbol{f}$ is continuous on $\overline{\Omega}$. The discrete problem is constructed from (2.7) by using the Galerkin method combined with numerical integration. It reads

Find $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ *in* $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v}_N \in \mathbb{D}_N, \quad a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) + K_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) + b_N(\boldsymbol{v}_N, p_N) \\ &\qquad\qquad\qquad\qquad = (\boldsymbol{f}, \boldsymbol{v}_N)_N, \\ &\forall q_N \in \mathbb{M}_N, \quad b_N(\boldsymbol{u}_N, q_N) = 0, \\ &\forall \boldsymbol{\varphi}_N \in \mathbb{C}_N, \quad c_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{\varphi}_N) = 0, \end{aligned} \tag{3.7}$$

where the bilinear forms $a_N(\cdot,\cdot;\cdot)$, $b_N(\cdot,\cdot)$ and $c_N(\cdot,\cdot;\cdot)$ are defined by

$$\begin{aligned} &a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = \nu\,(\mathbf{curl}\,\boldsymbol{\omega}_N, \boldsymbol{v}_N)_N, \quad b_N(\boldsymbol{v}_N, q_N) = -(\operatorname{div} \boldsymbol{v}_N, q_N)_N, \\ &c_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{\varphi}_N) = (\boldsymbol{\omega}_N, \boldsymbol{\varphi}_N)_N - (\boldsymbol{u}_N, \mathbf{curl}\,\boldsymbol{\varphi}_N)_N, \end{aligned} \tag{3.8}$$

while the trilinear form $K_N(\cdot,\cdot;\cdot)$ is now given by

$$K_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = (\boldsymbol{\omega}_N \times \boldsymbol{u}_N, \boldsymbol{v}_N)_N. \tag{3.9}$$

In order to prove that problem (3.7) admits a solution, we introduce the discrete kernels

$$V_N = \big\{\boldsymbol{v}_N \in \mathbb{D}_N;\ \forall q_N \in \mathbb{M}_N,\ b_N(\boldsymbol{v}_N, q_N) = 0\big\}, \tag{3.10}$$

and

$$\mathcal{W}_N = \big\{(\boldsymbol{\vartheta}_N, \boldsymbol{v}_N) \in \mathbb{C}_N \times V_N;\ \forall \boldsymbol{\varphi}_N \in \mathbb{C}_N,\ c_N(\boldsymbol{\vartheta}_N, \boldsymbol{v}_N; \boldsymbol{\varphi}_N) = 0\big\}. \tag{3.11}$$

As noted in [7, Cor. 3.2], the space V_N is contained in V, but the space $\mathcal{W}_N$ is not contained in $\mathcal{W}$ in the general case. We observe that, for any solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ of problem (3.7), the pair $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ is a solution of the reduced problem

Find $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ in $\mathcal{W}_N$ such that

$$\forall \boldsymbol{v}_N \in V_N, \quad a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) + K_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = (\boldsymbol{f}, \boldsymbol{v}_N)_N. \tag{3.12}$$

The existence of a solution to this problem is proved thanks to the same arguments as for Proposition 2.5 for instance.

Proposition 3.1 *For any data $\boldsymbol{f}$ continuous on $\overline{\Omega}$, problem* (3.12) *has a solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ in $\mathcal{W}_N$. Moreover this solution satisfies*

$$\|\boldsymbol{\omega}_N\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}} + \|\boldsymbol{u}_N\|_{L^2(\Omega)^d} \le c\, \nu^{-1}\, \|\mathcal{I}_N \boldsymbol{f}\|_{L^2(\Omega)^d}. \tag{3.13}$$

Proof Here, we introduce the mapping Φ_N defined from $\mathcal{W}_N$ into its dual space by

$$\begin{aligned}
&\forall (\boldsymbol{\omega}_N, \boldsymbol{u}_N) \in \mathcal{W}_N, \quad \forall (\boldsymbol{\vartheta}_N, \boldsymbol{w}_N) \in \mathcal{W}_N,\\
&\qquad \langle \Phi_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N), (\boldsymbol{\vartheta}_N, \boldsymbol{w}_N)\rangle = a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{w}_N)\\
&\qquad\qquad\qquad\qquad + K_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{w}_N) - (\boldsymbol{f}, \boldsymbol{w}_N)_N.
\end{aligned}$$

We provide $\mathcal{W}_N$ with the weak norm

$$\big(\|\boldsymbol{\omega}_N\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} + \|\boldsymbol{u}_N\|^2_{L^2(\Omega)^d}\big)^{\frac{1}{2}}.$$

However, since $\mathcal{W}_N$ is finite-dimensional, it is readily checked that Φ_N is continuous. Next, noting by the same arguments as for Lemma 2.4 that $K_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{u}_N)$ is zero, we have

$$\langle \Phi_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N), (\boldsymbol{\omega}_N, \boldsymbol{u}_N)\rangle = \nu\, (\mathbf{curl}\, \boldsymbol{\omega}_N, \boldsymbol{u}_N)_N - (\mathcal{I}_N \boldsymbol{f}, \boldsymbol{u}_N)_N.$$

Combining (3.5) with the definition of $\mathcal{W}_N$ gives

$$(\mathbf{curl}\, \boldsymbol{\omega}_N, \boldsymbol{u}_N)_N = (\boldsymbol{\omega}_N, \boldsymbol{\omega}_N)_N \ge \|\boldsymbol{\omega}_N\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}}.$$

So using once more (3.5) leads to

$$\langle \Phi_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N), (\boldsymbol{\omega}_N, \boldsymbol{u}_N)\rangle \ge \nu\, \|\boldsymbol{\omega}_N\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} - 3^{\frac{d}{2}}\, \|\mathcal{I}_N \boldsymbol{f}\|_{L^2(\Omega)^d} (\boldsymbol{u}_N, \boldsymbol{u}_N)_N^{\frac{1}{2}}.$$

On the other hand, it follows from [7, Lemmas 3.4 & 3.5] that, for any $\boldsymbol{u}_N$ in V_N, there exists a $\boldsymbol{\psi}_N$ in $\mathbb{C}_N$ such that $\boldsymbol{u}_N = \mathbf{curl}\, \boldsymbol{\psi}_N$ and which moreover satisfies

$$\|\boldsymbol{\psi}_N\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}} \le c\, \|\boldsymbol{u}_N\|_{L^2(\Omega)^d}.$$

Inserting this ψ_N into the definition of $\mathcal{W}_N$ and using once more (3.5) give

$$(\boldsymbol{u}_N, \boldsymbol{u}_N)_N = (\boldsymbol{u}_N, \mathbf{curl}\, \boldsymbol{\psi}_N)_N = (\boldsymbol{\omega}_N, \boldsymbol{\psi}_N)_N \\ \leq 3^d \, \|\boldsymbol{\omega}_N\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}} \|\boldsymbol{\psi}_N\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}},$$

whence

$$(\boldsymbol{u}_N, \boldsymbol{u}_N)_N^{\frac{1}{2}} \leq c \, \|\boldsymbol{\omega}_N\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}}. \tag{3.14}$$

Combining all this yields

$$\langle \Phi_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N), (\boldsymbol{\omega}_N, \boldsymbol{u}_N)\rangle = \nu \, \|\boldsymbol{\omega}_N\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} \\ - c \, \|\mathcal{I}_N \boldsymbol{f}\|_{L^2(\Omega)^d} \|\boldsymbol{\omega}_N\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}}.$$

So, setting

$$\mu_N = \frac{2c \, \max\{1, c\}}{\nu} \, \|\mathcal{I}_N \boldsymbol{f}\|_{L^2(\Omega)^d},$$

and noting from (3.5) and (3.14) that

$$\Big(\|\boldsymbol{\omega}_N\|^2_{L^2(\Omega)^{\frac{d(d-1)}{2}}} + \|\boldsymbol{u}_N\|^2_{L^2(\Omega)^d}\Big)^{\frac{1}{2}} \leq \sqrt{2} \, \max\{1, c\} \, \|\boldsymbol{\omega}_N\|_{L^2(\Omega)^{\frac{d(d-1)}{2}}},$$

we observe that $\langle \Phi_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N), (\boldsymbol{\omega}_N, \boldsymbol{u}_N)\rangle$ is nonnegative on the sphere of $\mathcal{W}_N$ with radius μ_N. So applying Brouwer's fixed point theorem, see [17, Chap. IV, Cor. 1.1], gives the existence result together with estimate (3.13).

The following inf-sup condition is proved in [7, Lemma 3.9]: There exists a positive constant β_* independent of N such that the form $b_N(\cdot, \cdot; \cdot)$ satisfies the inf-sup condition

$$\forall q_N \in \mathbb{M}_N, \quad \sup_{\boldsymbol{v}_N \in \mathbb{D}_N} \frac{b_N(\boldsymbol{v}_N, q_N)}{\|\boldsymbol{v}_N\|_{H(\mathrm{div},\Omega)}} \geq \beta_* \, \|q_N\|_{L^2(\Omega)}. \tag{3.15}$$

So the full existence result follows from Proposition 3.1 and this condition thanks to exactly the same arguments as for Theorem 2.8.

Theorem 3.2 *For any data $\boldsymbol{f}$ continuous on $\overline{\Omega}$, problem* (3.7) *has a solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ in $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$. Moreover, the part $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ of this solution satisfies* (3.13).

Note that the previous existence result still holds when $K_N(\cdot, \cdot; \cdot)$ is replaced by $K(\cdot, \cdot; \cdot)$ in problem (3.7). This means in practice that a more precise quadrature formula, exact on $\mathbb{P}_{3N-1}(\Omega)$, is used to evaluate the integrals that appear in the treatment of the nonlinear term. The corresponding discrete problem reads

Find $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ *in* $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$ *such that*

$$\begin{aligned} \forall \boldsymbol{v}_N \in \mathbb{D}_N, \quad & a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) + K(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) + b_N(\boldsymbol{v}_N, p_N) \\ & \qquad = (\boldsymbol{f}, \boldsymbol{v}_N)_N, \\ \forall q_N \in \mathbb{M}_N, \quad & b_N(\boldsymbol{u}_N, q_N) = 0, \\ \forall \boldsymbol{\varphi}_N \in \mathbb{C}_N, \quad & c_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{\varphi}_N) = 0. \end{aligned} \tag{3.16}$$

Similarly, the reduced problem (3.12) becomes
Find $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ *in* $\mathcal{W}_N$ *such that*

$$\forall \boldsymbol{v}_N \in V_N, \quad a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) + K(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = (\boldsymbol{f}, \boldsymbol{v}_N)_N. \tag{3.17}$$

We now intend to prove an error estimate between the solutions of problems (2.7) and (3.16) only in dimension $d = 2$ for simplicity. Since the proof of this result relies on the theorem due to Brezzi, Rappaz and Raviart [11], we set $\mathcal{X} = H_0(\mathbf{curl}, \Omega) \times V$. We write both problems (2.21) and (3.17) in a different form. Let $\mathcal{S}$ denote the following Stokes operator: For any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div}, \Omega)$, $\mathcal{S}\boldsymbol{f}$ denotes the solution $(\boldsymbol{\omega}, \boldsymbol{u})$ of the reduced problem
Find $(\boldsymbol{\omega}, \boldsymbol{u})$ *in* $\mathcal{W}$ *such that*

$$\forall \boldsymbol{v} \in V, \quad a(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle. \tag{3.18}$$

The fact that $\mathcal{S}$ is well-defined follows from [7, Cor. 2.4]. We also introduce the mapping G defined from $\mathcal{X}$ into the dual space of $H_0(\mathrm{div}, \Omega)$ by

$$\forall (\boldsymbol{\omega}, \boldsymbol{u}) \in \mathcal{X}, \quad \forall \boldsymbol{v} \in H_0(\mathrm{div}, \Omega), \quad \langle G(\boldsymbol{\omega}, \boldsymbol{u}), \boldsymbol{v} \rangle = K(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v} \rangle. \tag{3.19}$$

Then, problem (2.21) can equivalently be written as

$$(\boldsymbol{\omega}, \boldsymbol{u}) + \mathcal{S}G(\boldsymbol{\omega}, \boldsymbol{u}) = 0. \tag{3.20}$$

Similarly, we set $\mathcal{X}_N = \mathbb{C}_N \times V_N$. We thus define the discrete Stokes operator: For any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div}, \Omega)$, $\mathcal{S}_N \boldsymbol{f}$ denotes the solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ of the problem
Find $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ *in* $\mathcal{W}_N$ *such that*

$$\forall \boldsymbol{v}_N \in V_N, \quad a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = \langle \boldsymbol{f}, \boldsymbol{v}_N \rangle. \tag{3.21}$$

It is nearly the same problem as considered in [7, §3], only the discrete product in the right-hand side is replaced by the duality pairing. We also recall from [7, Cor. 3.8, Thm 4.6 & Cor. 4.7] the following results:

(i) The operator $\mathcal{S}_N$ satisfies the stability property

$$\|\mathcal{S}_N \boldsymbol{f}\|_{\mathcal{X}} \le c \sup_{\boldsymbol{v}_N \in V_N} \frac{\langle \boldsymbol{f}, \boldsymbol{v}_N \rangle}{\|\boldsymbol{v}_N\|_{L^2(\Omega)^d}}, \tag{3.22}$$

(ii) The next error estimate for all $\boldsymbol{f}$ such that $\mathcal{S}\boldsymbol{f}$ belongs to $H^{s+1}\Omega) \times H^s(\Omega)^2$, $s \geq 1$,

$$\|(\mathcal{S} - \mathcal{S}_N)\boldsymbol{f}\|_{\mathcal{X}} \leq c\, N^{-s}\, \|\mathcal{S}\boldsymbol{f}\|_{H^{s+1}\Omega)\times H^s(\Omega)^2}. \tag{3.23}$$

Finally we consider the mapping G_N defined from $\mathcal{X}_N$ into the dual space of $\mathbb{D}_N$ by

$$\forall(\boldsymbol{\omega}_N, \boldsymbol{u}_N) \in \mathcal{X}_N, \quad \forall \boldsymbol{v}_N \in \mathbb{D}_N,$$
$$\langle G_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N), \boldsymbol{v}_N\rangle = K(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) - (\boldsymbol{f}, \boldsymbol{v}_N)_N. \tag{3.24}$$

Then, problem (3.17) can equivalently be written as

$$(\boldsymbol{\omega}_N, \boldsymbol{u}_N) + \mathcal{S}_N G_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N) = 0. \tag{3.25}$$

We are led to make the following assumption. Here, D stands for the differential operator.

Assumption 3.3 The pair $(\boldsymbol{\omega}, \boldsymbol{u})$ is a solution of problem (2.21) such that the operator $\mathrm{Id} + \mathcal{S}DG(\boldsymbol{\omega}, \boldsymbol{u})$ is an isomorphism of $\mathcal{X}$.

Note that this assumption can equivalently be written as follows: The operator $\mathrm{Id} + \mathcal{S}DG(\boldsymbol{\omega}, \boldsymbol{u})$ is an isomorphism of $H_0(\mathbf{curl}, \Omega) \times H_0(\mathrm{div}, \Omega)$. It means that, for any data $\boldsymbol{g}$ in the dual space of $H_0(\mathrm{div}, \Omega)$, the linearized problem

Find $(\boldsymbol{\vartheta}, \boldsymbol{w}, r)$ *in* $H_0(\mathbf{curl}, \Omega) \times H_0(\mathrm{div}, \Omega) \times L^2_0(\Omega)$ *such that*

$$\begin{aligned}
&\forall \boldsymbol{v} \in H_0(\mathrm{div}, \Omega), && a(\boldsymbol{\vartheta}, \boldsymbol{w}; \boldsymbol{v}) + K(\boldsymbol{\omega}, \boldsymbol{w}; \boldsymbol{v}) + K(\boldsymbol{\vartheta}, \boldsymbol{u}; \boldsymbol{v}) + b(\boldsymbol{v}, r) \\
& && = \langle \boldsymbol{g}, \boldsymbol{v}\rangle, \\
&\forall q \in L^2_0(\Omega), && b(\boldsymbol{w}, q) = 0, \\
&\forall \boldsymbol{\varphi} \in H_0(\mathbf{curl}, \Omega), && c(\boldsymbol{\vartheta}, \boldsymbol{w}; \boldsymbol{\varphi}) = 0,
\end{aligned} \tag{3.26}$$

has a unique solution with norm bounded by a constant times $\|\boldsymbol{g}\|_{H_0(\mathrm{div},\Omega)'}$. It yields the local uniqueness of the solution $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ but is much less restrictive than the uniqueness condition (2.35). We first prove a basic continuity property.

Lemma 3.4 *In dimension $d = 2$, the following property holds*

$$\forall \boldsymbol{\omega}_N \in \mathbb{C}_N, \quad \forall \boldsymbol{u}_N \in \mathbb{D}_N, \quad \forall \boldsymbol{v}_N \in \mathbb{D}_N,$$
$$|K(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N)| \leq c\, |\log\, N|^{\frac{1}{2}}\, \|\boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} \|\boldsymbol{u}_N\|_{L^2(\Omega)^2} \|\boldsymbol{v}_N\|_{L^2(\Omega)^2}. \tag{3.27}$$

Proof We have

$$|K(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N)| \leq \|\boldsymbol{\omega}_N \times \boldsymbol{u}_N\|_{L^2(\Omega)^d} \|\boldsymbol{v}_N\|_{L^2(\Omega)^2},$$

whence, for all $p > 2$ and $q > 2$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$,

$$|K(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N)| \leq \|\boldsymbol{\omega}_N\|_{L^p(\Omega)} \|\boldsymbol{u}_N\|_{L^q(\Omega)^2} \|\boldsymbol{v}_N\|_{L^2(\Omega)^2}.$$

Finally, we recall from [22] that, for any $p < \infty$, the imbedding from $H(\mathbf{curl}, \Omega) = H^1(\Omega)$ into $L^p(\Omega)$ is continuous, with norm $\leq c\, p^{\frac{1}{2}}$. We also use the following inverse inequality (see [8, Prop. III.3.1]), for $q > 2$,

$$\forall z_N \in \mathbb{P}_N(\Omega), \quad \|z_N\|_{L^q(\Omega)} \leq c\, N^{4(\frac{1}{2}-\frac{1}{q})}\, \|z_N\|_{L^2(\Omega)}.$$

This yields

$$|K(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N)| \leq c\, p^{\frac{1}{2}}\, N^{\frac{4}{p}}\, \|\boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} \|\boldsymbol{u}_N\|_{L^2(\Omega)^2} \|\boldsymbol{v}_N\|_{L^2(\Omega)^2}.$$

Finally, taking $p = \log N$ gives the desired result.

It follows from Proposition 2.11 that, if $\boldsymbol{f}$ is smooth enough, there exists an $s_0 > 1$ such that the solution $(\boldsymbol{\omega}, \boldsymbol{u})$ of problem (2.21) belongs to $H^{s_0+1}(\Omega) \times H^{s_0}(\Omega)^2$. We now prove some lemmas which make use of this property. Let also $\mathcal{L}$ denote the space of linear operators from $\mathcal{X}$ into $\mathcal{X}$.

Lemma 3.5 *In dimension $d = 2$, if Assumption* 3.3 *holds, there exists an integer N_0 such that, for all $N \geq N_0$, the operator $Id + \mathcal{S}_N DG_N(\boldsymbol{\omega}, \boldsymbol{u})$ is an isomorphism of $\mathcal{X}$. Moreover the norm of its inverse operator is bounded independently of N.*

Proof The idea consists in writing the expansion

$$\begin{aligned} \mathrm{Id} + \mathcal{S}_N DG_N(\boldsymbol{\omega}, \boldsymbol{u}) = \mathrm{Id} + \mathcal{S} DG(\boldsymbol{\omega}, \boldsymbol{u}) - \big(\mathcal{S} - \mathcal{S}_N\big) DG(\boldsymbol{\omega}, \boldsymbol{u}) \\ - \mathcal{S}_N \left(DG(\boldsymbol{\omega}, \boldsymbol{u}) - DG_N(\boldsymbol{\omega}, \boldsymbol{u}_N)\right). \end{aligned} \tag{3.28}$$

Moreover it is readily checked from the definitions of G and G_N that the terms $DG(\boldsymbol{\omega}, \boldsymbol{u})$ and $DG_N(\boldsymbol{\omega}, \boldsymbol{u})$ are equal, so that the last term in this expansion disappears. We now check that the second term in the right-hand side tends to zero. We have, for any $(\boldsymbol{\vartheta}, \boldsymbol{w})$ in $\mathcal{X}$,

$$DG(\boldsymbol{\omega}, \boldsymbol{u}) \cdot (\boldsymbol{\vartheta}, \boldsymbol{w}) = \boldsymbol{\omega} \times \boldsymbol{w} + \boldsymbol{\vartheta} \times \boldsymbol{u},$$

whence, owing to (2.36),

$$\mathbf{curl}\left(DG(\boldsymbol{\omega}, \boldsymbol{u}) \cdot (\boldsymbol{\vartheta}, \boldsymbol{w})\right) = \mathbf{grad}\,\boldsymbol{\omega} \cdot \boldsymbol{w} + \mathbf{grad}\,\boldsymbol{\vartheta} \cdot \boldsymbol{u}.$$

Then, the same arguments as used in the proof of Proposition 2.11 yield that $\mathcal{S} DG(\boldsymbol{\omega}, \boldsymbol{u}) \cdot (\boldsymbol{\vartheta}, \boldsymbol{w})$ belongs to $H^2(\Omega) \times H^1(\Omega)^2$ and satisfy

$$\|\mathcal{S} DG(\boldsymbol{\omega}, \boldsymbol{u}) \cdot (\boldsymbol{\vartheta}, \boldsymbol{w})\|_{H^2(\Omega)\times H^1(\Omega)^2} \leq c \left(\|\boldsymbol{\omega}\|_{H^{s_0+1}(\Omega)} + \|\boldsymbol{u}\|_{H^{s_0}(\Omega)^2}\right) \|(\boldsymbol{\vartheta}, \boldsymbol{w})\|_{\mathcal{X}}.$$

Thus, using (3.23) yields

$$\lim_{N \to +\infty} \|\big(\mathcal{S} - \mathcal{S}_N\big) DG(\boldsymbol{\omega}, \boldsymbol{u})\|_{\mathcal{L}} = 0. \tag{3.29}$$

From Assumption 3.3, if γ denotes the norm of the inverse of $\mathrm{Id} + \mathcal{S} DG(\boldsymbol{\omega}, \boldsymbol{u})$, choosing N large enough for the quantity in (3.29) to be smaller than $\frac{1}{2\gamma}$ gives the desired property with the norm of the inverse of $\mathrm{Id} + \mathcal{S}_N DG_N(\boldsymbol{\omega}, \boldsymbol{u})$ smaller than 2γ.

Remark *More technical arguments prove that Lemma 3.5 still holds if the solution $(\boldsymbol{\omega}, \boldsymbol{u})$ of problem* (2.21) *only belongs to $H^{s_0+1}(\Omega) \times H^{s_0}(\Omega)^2$ for some $s_0 > 0$. However we do not need this extension here.*

Lemma 3.6 *In dimension $d = 2$, the following Lipschitz–property holds*

$$\begin{aligned} \forall (\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{u}}) \in \mathcal{X}, \quad & \|\mathcal{S}_N \big(DG_N(\boldsymbol{\omega}, \boldsymbol{u}) - DG_N(\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{u}})\big)\|_{\mathcal{L}} \\ & \leq c\, |\log N|^{\frac{1}{2}} \left(\|\boldsymbol{\omega} - \tilde{\boldsymbol{\omega}}\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^2(\Omega)^d}\right). \end{aligned} \tag{3.30}$$

Proof We have

$$\langle \big(DG_N(\boldsymbol{\omega}, \boldsymbol{u}) - DG_N(\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{u}})\big)\cdot (\boldsymbol{\vartheta}, \boldsymbol{w}), \boldsymbol{v}_N\rangle = K(\boldsymbol{\omega}-\tilde{\boldsymbol{\omega}}, \boldsymbol{w}; \boldsymbol{v}_N)+K(\boldsymbol{\vartheta}, \boldsymbol{u}-\tilde{\boldsymbol{u}}; \boldsymbol{v}_N).$$

So the same arguments as in the proof of Lemma 3.4 (but with the inverse inequality now applied to $\boldsymbol{v}_N$) and (3.22) lead to the desired property.

Lemma 3.7 *In dimension $d = 2$, assume that the data $\boldsymbol{f}$ belong to $H^\sigma(\Omega)^d$, $\sigma > 1$, and that the solution $(\boldsymbol{\omega}, \boldsymbol{u}, \boldsymbol{p})$ of problem* (2.7) *belongs to $H^{s+1}(\Omega) \times H^s(\Omega)^2 \times H^s(\Omega)$, $s > 1$. The following estimate holds*

$$\|(\boldsymbol{\omega}, \boldsymbol{u}) + \mathcal{S}_N G_N(\boldsymbol{\omega}, \boldsymbol{u})\|_{\mathcal{X}} \\ \leq c(\boldsymbol{f})\left(N^{-s}\,\|(\boldsymbol{\omega}, \boldsymbol{u})\|_{H^{s+1}\Omega)\times H^s(\Omega)^2} + N^{-\sigma}\,\|\boldsymbol{f}\|_{H^\sigma(\Omega)^2}\right), \tag{3.31}$$

for a constant $c(\boldsymbol{f})$ only depending on the data $\boldsymbol{f}$.

Proof From equation (3.20), we derive

$$\|(\boldsymbol{\omega}, \boldsymbol{u}) + \mathcal{S}_N G_N(\boldsymbol{\omega}, \boldsymbol{u})\|_{\mathcal{X}} \\ \leq \|\big(\mathcal{S} - \mathcal{S}_N\big)G(\boldsymbol{\omega}, \boldsymbol{u})\|_{\mathcal{X}} + \|\mathcal{S}_N\left(G(\boldsymbol{\omega}, \boldsymbol{u}) - G_N(\boldsymbol{\omega}, \boldsymbol{u})\right)\|_{\mathcal{X}}.$$

The bound for the first term in the right-hand side is now a direct consequence of (3.23). Finally, if Π_{N-1} denotes the orthogonal projection operator from $L^2(\Omega)$ onto $\mathbb{P}_{N-1}(\Omega)$, adding and subtracting the term $\Pi_{N-1}\boldsymbol{f}$ in the last term and using (3.22) lead to

$$\|\mathcal{S}_N\left(G(\boldsymbol{\omega}_N^*, \boldsymbol{u}_N^*) - G_N(\boldsymbol{\omega}_N^*, \boldsymbol{u}_N^*)\right)\|_{\mathcal{X}} \\ \leq c\left(\|\boldsymbol{f} - \Pi_{N-1}\boldsymbol{f}\|_{L^2(\Omega)^2} + \|\boldsymbol{f} - \mathcal{I}_N\boldsymbol{f}\|_{L^2(\Omega)^2}.\right).$$

Thus the standard approximation properties of the operators Π_{N-1} and $\mathcal{I}_N$ [10, Thms 7.1 & 14.2] give the bound for this last term, which concludes the proof.

We are now in a position to prove the error estimate.

Theorem 3.8 *In dimension $d = 2$, assume that the data $\boldsymbol{f}$ belong to $H^\sigma(\Omega)^d$, $\sigma > 1$, and that the solution $(\boldsymbol{\omega}, \boldsymbol{u}, \boldsymbol{p})$ of problem* (2.7) *belongs to $H^{s+1}(\Omega) \times H^s(\Omega)^2 \times H^s(\Omega)$, $s > 1$, and satisfies Assumption* 3.3*. Then, there exists an integer $N_\diamond$ and a constant $c_\diamond$ such that, for all $N \geq N_\diamond$, problem* (3.16) *has a unique solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N, p_N)$ such that*

$$\|\boldsymbol{\omega} - \boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u} - \boldsymbol{u}_N\|_{H(\mathrm{div},\Omega)} \leq c_\diamond\,|\log N|^{-\frac{1}{2}}. \tag{3.32}$$

Moreover this solution satisfies the following error estimate

$$\|\boldsymbol{\omega} - \boldsymbol{\omega}_N\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u} - \boldsymbol{u}_N\|_{H(\mathrm{div},\Omega)} + |\log N|^{-\frac{1}{2}}\,\|p - p_N\|_{L^2(\Omega)} \\ \leq c(\boldsymbol{f})\left(N^{-s}\left(\|\boldsymbol{\omega}\|_{H^{s+1}(\Omega)} + \|\boldsymbol{u}\|_{H^s(\Omega)^2} + \|p\|_{H^s(\Omega)}\right) + N^{-\sigma}\,\|\boldsymbol{f}\|_{H^\sigma(\Omega)^2}\right), \tag{3.33}$$

for a constant $c(\boldsymbol{f})$ only depending on the data $\boldsymbol{f}$.

Proof Combining Lemmas 3.5 to 3.7 with the Brezzi–Rappaz–Raviart theorem [11] (see also [17, Chap. IV, Thm 3.1]) yields that, for N large enough, problem (3.17) has a unique solution $(\boldsymbol{\omega}_N, \boldsymbol{u}_N)$ which satisfies (3.32) and the first part of (3.33). Moreover, thanks to the discrete inf-sup condition (3.15), there exists a unique p_N in $\mathbb{M}_N$ such that

$$\forall \boldsymbol{v}_N \in \mathbb{D}_N, \quad b_N(\boldsymbol{v}_N, p_N) = (\boldsymbol{f}, \boldsymbol{v}_N)_N - a_N(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) - K(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N),$$

whence the existence and local uniqueness result. Moreover, we have for any q_N in $\mathbb{M}_N$

$$\begin{aligned} b_N(\boldsymbol{v}_N, p_N - q_N) = b(\boldsymbol{v}_N, p - q_N) - \langle \boldsymbol{f}, \boldsymbol{v}_N \rangle + (\boldsymbol{f}, \boldsymbol{v}_N)_N \\ + a(\boldsymbol{\omega} - \boldsymbol{\omega}_N, \boldsymbol{u} - \boldsymbol{u}_N; \boldsymbol{v}_N) + (a - a_N)(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N) \\ + K(\boldsymbol{\omega}, \boldsymbol{u}; \boldsymbol{v}_N) - K(\boldsymbol{\omega}_N, \boldsymbol{u}_N; \boldsymbol{v}_N), \end{aligned}$$

so that the estimate for $\|p - p_N\|_{L^2(\Omega)}$ follows from (3.15), a triangle inequality, the same arguments as in the proof of Lemmas 3.6 and 3.7 and the first part of (3.33).

Estimate (3.33) is fully optimal, up to the $|\log N|^{\frac{1}{2}}$ which only occurs for the pressure and is most often neglectable. However proving even a weaker estimate in dimension 3 seems rather difficult, first because the existence of $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ is only proved for ν large enough in Section 2, second because such an estimate would require some regularity properties of this solution which are not likely.

4 The solution algorithm and numerical experiments

In view of the results of the previous analysis, the numerical experiments have only been performed in the two-dimensional case, on the square $\Omega =]-1, 1[^2$. Problem (3.7) is solved via the following iterative algorithm. In what follows, we omit part of the indices N for simplicity.

STEP A. We first solve the Stokes problem
Find $(\boldsymbol{\omega}^0, \boldsymbol{u}^0, p^0)$ *in* $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v}_N \in \mathbb{D}_N, \quad a_N(\boldsymbol{\omega}^0, \boldsymbol{u}^0; \boldsymbol{v}_N) + b_N(\boldsymbol{v}_N, p^0) = (\boldsymbol{f}, \boldsymbol{v}_N)_N, \\ &\forall q_N \in \mathbb{M}_N, \quad b_N(\boldsymbol{u}^0, q_N) = 0, \\ &\forall \boldsymbol{\varphi}_N \in \mathbb{C}_N, \quad c_N(\boldsymbol{\omega}^0, \boldsymbol{u}^0; \boldsymbol{\varphi}_N) = 0. \end{aligned} \tag{4.1}$$

STEP B. Assuming that the triple $(\boldsymbol{\omega}^{n-1}, \boldsymbol{u}^{n-1}, p^{n-1})$ is known, we now solve the problem
Find $(\boldsymbol{\omega}^n, \boldsymbol{u}^n, p^n)$ *in* $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v}_N \in \mathbb{D}_N, \quad a_N(\boldsymbol{\omega}^n, \boldsymbol{u}^n; \boldsymbol{v}_N) + K_N(\boldsymbol{\omega}^{n-1}, \boldsymbol{u}^n; \boldsymbol{v}_N) + K_N(\boldsymbol{\omega}^n, \boldsymbol{u}^{n-1}; \boldsymbol{v}_N) \\ &\qquad + b_N(\boldsymbol{v}_N, p^n) = (\boldsymbol{f}, \boldsymbol{v}_N)_N + K_N(\boldsymbol{\omega}^{n-1}, \boldsymbol{u}^{n-1}; \boldsymbol{v}_N), \\ &\forall q_N \in \mathbb{M}_N, \quad b_N(\boldsymbol{u}^n, q_N) = 0, \\ &\forall \boldsymbol{\varphi}_N \in \mathbb{C}_N, \quad c_N(\boldsymbol{\omega}^n, \boldsymbol{u}^n; \boldsymbol{\varphi}_N) = 0. \end{aligned} \tag{4.2}$$

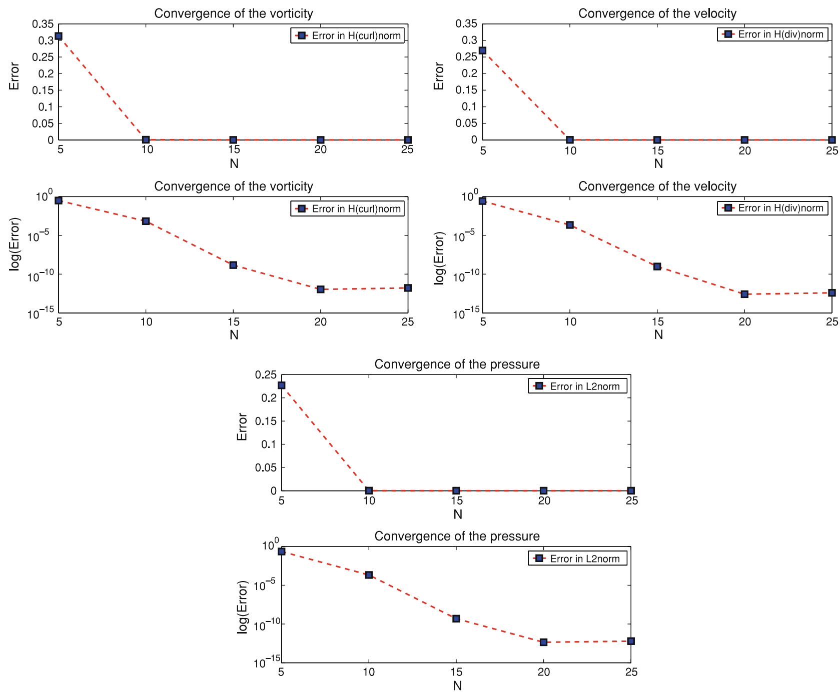

Fig. 1 The errors for the solution issued from (4.5)

Step B is iterated until the following condition holds

$$\|\boldsymbol{\omega}^n - \boldsymbol{\omega}^{n-1}\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\|_{H(\mathrm{div},\Omega)} \leq \eta, \tag{4.3}$$

for a fixed tolerance η.

The convergence of this algorithm is not proved but likely, at least for ν large enough (we recall from [11] that, if the assumptions of Theorem 3.8 hold, Newton's method converges for any initial guess in the neighbourhood of $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ defined in (3.32)). The interest of such an algorithm is that both problems (4.1) and (4.2) are Stokes-like problems. The details concerning the implementation of the discrete Stokes problem (i.e. with $K_N(\cdot, \cdot; \cdot)$ replaced by zero) are given in [7, §6].

The first two experiments are aimed to check the convergence of the method. The viscosity ν and the tolerance η are given by

$$\nu = 5\,.\,10^{-2}, \quad \eta = 10^{-12}. \tag{4.4}$$

We work with two solutions $(\boldsymbol{\omega}, \boldsymbol{u}, p)$ defined by $\boldsymbol{\omega} = -\Delta\psi$, $\boldsymbol{u} = \mathbf{curl}\,\psi$, where ψ and p are

- first, very smooth functions given by

$$\psi(x, y) = \sin(\pi x)\sin(\pi y), \quad p(x, y) = \sin(x + y), \tag{4.5}$$

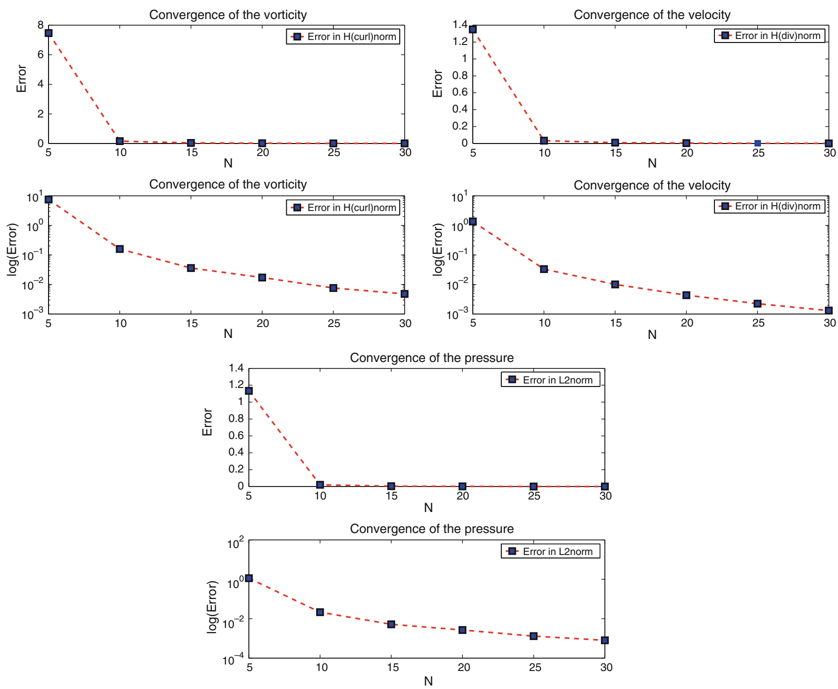

Fig. 2 The errors for the solution issued from (4.6)

- second, functions of limited regularity, defined by

$$\psi(x, y) = (1 - x^2)^3 (1 - y^2)^{\frac{7}{2}},$$
$$p(x, y) = x(1 - x^2)^{\frac{3}{2}} (1 + y^2)^{-\frac{1}{2}}. \tag{4.6}$$

Figure 1 for the solution issued from (4.5) and Figure 2 for the solution issued from (4.6) present the convergence curves of the relative errors on $\boldsymbol{\omega}$, $\boldsymbol{u}$ and p in the $L^2(\Omega)$ or $L^2(\Omega)^2$ norm, both in standard and semi-logarithmic scales, as a function of N, for N varying from 5 to 25 or 30.

It can be noted that the number of iterations in order that condition (4.3) is satisfied does not increase with N. On the other hand, the error is much larger (and the slope of the error as a function of N is lower) for the singular solution issued from (4.6) than for the regular solution issued from (4.5), which is in good coherency with the results of Theorem 3.8. However there is no doubt about the convergence of the discretization.

The last numerical experiments deal with the more realistic case of a Poiseuille like flow, i.e. with the fourth line of problem (2.2) replaced by

$$\boldsymbol{u} \cdot \boldsymbol{n} = g \quad \text{on } \partial\Omega. \tag{4.7}$$

The way of handling this nonhomogeneous boundary condition is the same as proposed in [7, §5] for the Stokes problem.

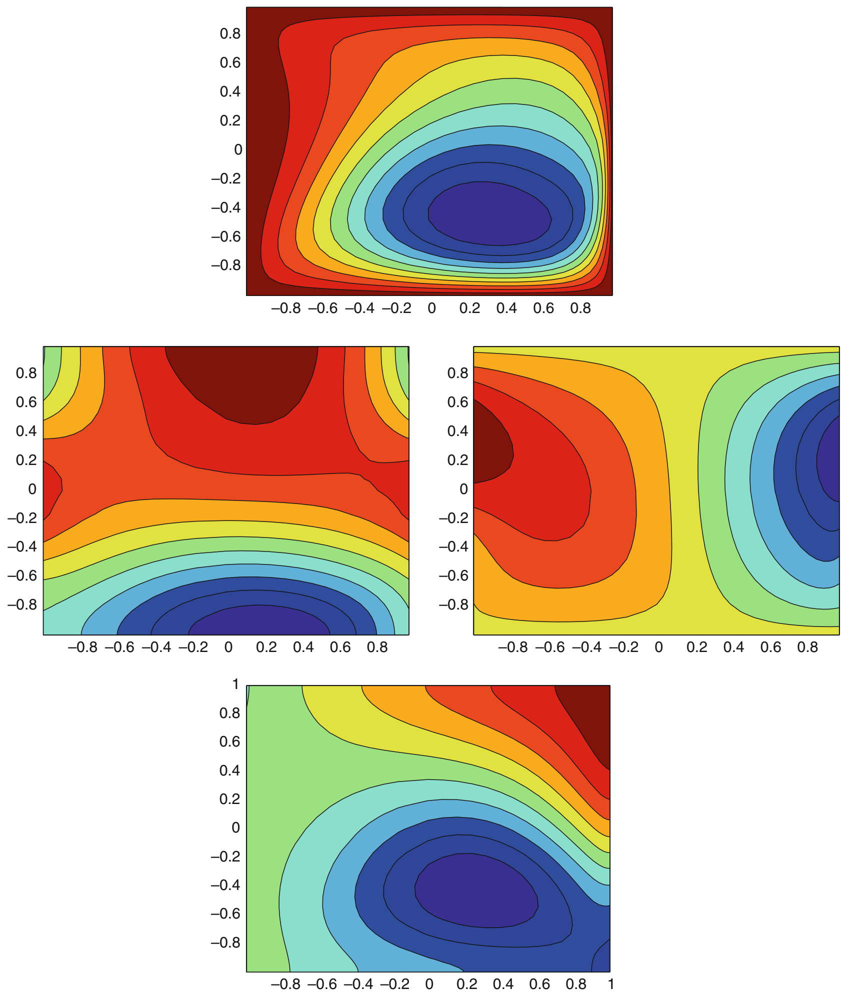

Fig. 3 Isovalues of the vorticity, the two components of the velocity and the pressure

Still for the parameters ν and η given in (4.4), we first work with the data $\boldsymbol{f} = (f_x, f_y)$ and g given by

$$f_x(x, y) = y, \quad f_y(x, y) = 0, \tag{4.8}$$

$$g(\pm 1, y) = \pm(1 - y^2)^3, \quad g(x, \pm 1) = 0. \tag{4.9}$$

Figure 3 presents, from top to bottom, the curves of isovalues of the vorticity, the two components of the velocity and the pressure, as obtained with $N = 38$.

In a second step, we investigate the influence of the viscosity on the flow. We take $\eta = 10^{-12}$ and work with the data g given by

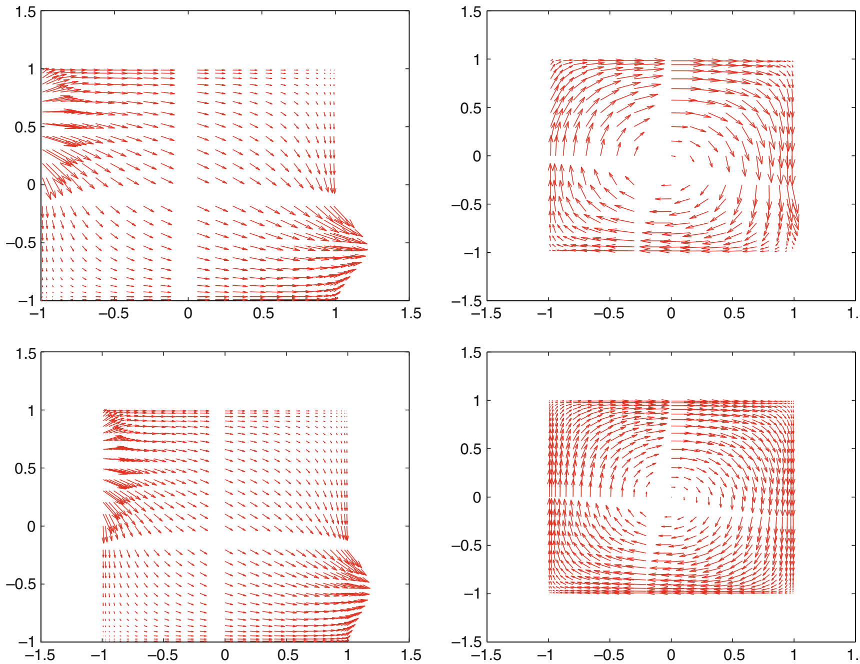

Fig. 4 Curves of the velocity fields for different viscosities

$$
\begin{aligned}
g(-1,y) &= \begin{cases} 0 & \text{if } -1 \le y \le 0, \\ -y(1-y) & \text{if } 0 \le y \le 1, \end{cases} \\
g(1,y) &= \begin{cases} y(1+y) & \text{if } -1 \le y \le 0, \\ 0 & \text{if } 0 \le y \le 1, \end{cases} \qquad g(x,\pm 1) = 0.
\end{aligned} \tag{4.10}
$$

Figure 4 presents, from left to right, the curves of the velocity field, in the case of a zero datum $\boldsymbol{f}$ (in which it is readily checked that the vorticity $\boldsymbol{\omega}$ is zero) and in the case where $\boldsymbol{f}$ is given by (4.8),

(i) in the top part, with $\nu = 10^{-1}$ (obtained with $N = 20$),
(ii) in the bottom part with $\nu = 10^{-2}$ (obtained with $N = 30$).

Note that, as scheduled, the vorticity is zero in the two left curves. The influence of ν is more important in the case where it is not zero (see the equations) and, in this case, the importance of the boundary condition on $\boldsymbol{u}$ diminishes in comparison to that on $\boldsymbol{\omega}$ when ν decreases.

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