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Efficient Filon-type methods for $\int_a^b f(x)\, e^{i\omega g(x)}\, dx$

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Abstract Based on the transformation $y = g(x)$, some new efficient Filon-type methods for integration of highly oscillatory function $\int_a^b f(x)\, e^{i\omega g(x)}\, dx$ with an irregular oscillator are presented. One is a moment-free Filon-type method for the case that $g(x)$ has no stationary points in $[a, b]$. The others are based on the Filon-type method or the asymptotic method together with Filon-type method for the case that $g(x)$ has stationary points. The effectiveness and accuracy are tested by numerical examples.

AMS Subject Classifications 65D32 · 65D30

1 Introduction

Highly oscillatory integral $\int_a^b f(x)\, e^{i\omega g(x)}\, dx$ occurs in a wide range of practical problems and applications ranging from nonlinear optics to fluid dynamics, plasma transport, computerized tomography, celestial mechanics, computation of Schrödinger spectra, Bose–Einstein condensates (cf. [10]). Highly oscillatory integrals are allegedly difficult to calculate by the standard classic integration formulas when the frequency is significantly larger than the number of quadrature points. Many methods have been developed since Filon (cf. [2–11,13–20,23]).

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The Filon quadrature of the form $\int_a^b f(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$, is achieved by approximating $f(x)$ by a polynomial $p(x)$ with degree v at interpolation nodes $c_0, c_1, c_2, \ldots, c_v$ and calculating $\int_a^b p(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$ instead of $\int_a^b f(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$. Recently Iserles and Nørsett [9,10] extended the approach of Iserles [7,8], defined an asymptotic method and a Filon-type method for $\int_a^b f(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$ and analyzed the convergent behavior in a range of frequency regimes and showed that the accuracy increases when oscillation becomes faster.

The foundation of the asymptotic method [10] lies in the observation that for $g'(x) \neq 0$ for all $x \in [a, b]$

$$\begin{aligned} I[f] &= \int_a^b f(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x \\ &= \frac{1}{\mathrm{i}\omega}\int_a^b \frac{f(x)}{g'(x)}\,\mathrm{d}\mathrm{e}^{\mathrm{i}\omega g(x)} \\ &= \frac{1}{\mathrm{i}\omega}\left[\frac{f(x)}{g'(x)}\,\mathrm{e}^{\mathrm{i}\omega g(x)}\right]_a^b - \frac{1}{\mathrm{i}\omega}\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g'(x)}\right]\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x \\ &=: Q_1^A[f] - \frac{1}{\mathrm{i}\omega}\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g'(x)}\right]\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x. \end{aligned}$$

Continue this process of approximating by integrating by parts and define σ_k as

$$\sigma_1[f](x) = \frac{f(x)}{g'(x)}, \quad \sigma_{k+1}[f](x) = \frac{\sigma_k[f]'(x)}{g'(x)}, \quad k \geq 1 \ ([17]).$$

Then we get the s-step asymptotic quadrature [10]

$$Q_s^A[f] = -\sum_{k=1}^{s} \frac{1}{(-\mathrm{i}\omega)^k}\left\{\sigma_k[f](b)\,\mathrm{e}^{\mathrm{i}\omega g(b)} - \sigma_k[f](a)\,\mathrm{e}^{\mathrm{i}\omega g(a)}\right\}. \tag{1.1}$$

The error $I[f] - Q_s^A[f] \sim O(\omega^{-s-1})$ [10]. The shortcoming with using an asymptotic expansion as an approximation is that the quadrature in general diverges for fixed ω as $s \to \infty$. In other words, for fixed ω the accuracy of approximating an integral the asymptotic method is limited.

To work around this weakness, Iserles and Nørsett [9,10] derived a Filon-type method which extends the work of Filon: let s be some positive integer and let $\{m_k\}_0^v$ be a set of multiplicities associated with the node points $a = c_0 < c_1 < \cdots < c_v = b$ such that $m_0, m_v \geq s$. Suppose that $v(x) = \sum_{k=0}^{n} a_k x^k$, where

$n = \sum_{k=0}^{v} m_k - 1$, is the solution of the system of equations

$$v(c_k) = f(c_k), \quad v'(c_k) = f'(c_k), \ldots, v^{(m_k-1)}(c_k) = f^{(m_k-1)}(c_k) \tag{1.2}$$

for every integer $0 \le k \le v$. Then

$$I[f] - Q_s^F[f] \sim O(\omega^{-s-1}),$$

where $Q_s^F[f] \equiv I[v(x)] = \sum_{k=0}^{n} a_k I[x^k]$ and $I[x^k] = \int_a^b x^k\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$, $k = 0, 1, \ldots, n$ (cf. [10]).

In many situations the accuracy of the Filon-type method is significantly higher than that of the asymptotic method, even though it is of the same order. An entirely different approach without computing the moments $I[x^k]$, the Levin collocation method, was proposed by Levin [13]. Based on the Levin collocation method, Olver [17] developed a Levin-type method that collocates function values and derivatives of f at the interpolation nodes. The Filon-type method and Levin-type method with the approach of the standard basis of polynomials are identical for the case that the oscillator $g(x)$ is a linear function. Otherwise, the two quadratures are not identical [18,25]. Filon-type methods are often more accurate, affordable and much simpler to construct than the corresponding Levin-type methods [12,18]. Unfortunately, the Filon-type method requires that the moments are easily computable, which is not necessarily the case.

Once g' vanishes at one of more points in $[a,b]$, the Levin-type method fails for computing $\int_a^b f(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$. For treatment of stationary points, without loss of generality, assume $\xi \in [a,b]$ is the unique stationary point of g. Iserles and Nørsett [10] generalized the asymptotic method $Q_s^A[f]$ and Filon-type method $Q_s^F[f]$ and extended to the case for any integer $r \ge 1$

$$g'(\xi) = g''(\xi) = \cdots = g^{(r)}(\xi) = 0, \quad g^{(r+1)}(\xi) \neq 0, \quad g'(x) \neq 0 \text{ for } x \neq \xi,\ x \in [a,b]$$

with error bounds

$$Q_s^A[f] - I[f] \sim O(\omega^{-s-1/(r+1)}), \quad Q_s^F[f] - I[f] \sim O(\omega^{-s-1/(r+1)}),$$

based on the first few generalized moments

$$\mu_m(\omega,\xi) = \int_a^b (x-\xi)^m\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x, \quad m = 0, 1, \ldots, r-1$$

can be computed explicitly. In the important case $g(x) = (x-\xi)^p$, where $\xi \in [a,b]$ and $p \ge 1$ is an integer, the generalized moments are easily calculated. In general, however, the moments are often unknown.

In this paper, by the transformation $y = g(x)$, we present new efficient Filon-type quadratures for highly oscillatory integral $I[f] = \int_a^b f(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$.

- For the case that $g'(x) \neq 0$ for all $x \in [a,b]$: let $p(y) = \sum_{k=0}^{n} a_k y^k$ with $n = \sum_{k=0}^{v} m_k - 1$ satisfy that for $k = 0, 1, \ldots, v$,

$$p(g(c_k)) = \sigma_1[f](c_k),\ \ p'(g(c_k)) = \sigma_2[f](c_k), \ldots, p^{(m_k-1)}(g(c_k)) = \sigma_{m_k}[f](c_k), \tag{1.3}$$

and define

$$Q^F_{v,m}[f] = \int_{g(a)}^{g(b)} p(y)\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y. \tag{1.4}$$

From (1.3) we know that the interpolation polynomial $p(y)$ at nodes $g(c_0)$, $g(c_1), \ldots, g(c_v)$ is related to the values of $\sigma_j[f](c_k)$ for $j = 1, 2, \ldots, m_k$ and $k = 0, 1, \ldots, v$. From (1.4) the quadrature can be computed explicitly. The advantage of this approach is that there is no need to compute moments $I[x^k]$ and the inverse of $g(x)$. The error bound for the new quadrature is

$$Q^F_{v,m}[f] - I[f] \sim O(\omega^{-s-1}). \tag{1.5}$$

- For the case that $g'(\xi) = g''(\xi) = \cdots = g^{(r)}(\xi) = 0$, $g^{(r+1)}(\xi) \neq 0$ and $g'(x) \neq 0$ for $x \neq \xi$, $x \in [a,b]$, without loss of generality, assume $g^{(r+1)}(\xi) > 0$, otherwise we consider $I[f] = \int_a^b f(x)\,\mathrm{e}^{-\mathrm{i}\omega(-g(x))}\,\mathrm{d}x$ for $-g(x)$ instead of $g(x)$. Then $g(\xi) = \min\limits_{x\in[a,b]} g(x)$ for r being an odd integer: let $t^{r+1} = g(x) - g(\xi)$. Then t^{r+1} is well-defined in each subinterval $[a,\xi]$ and $[\xi, b]$ for any positive integer r, and $I[f]$ can be represented by

$$\begin{aligned} I[f] &= \mathrm{e}^{\mathrm{i}\omega g(\xi)} \left(\int_a^{\xi} + \int_{\xi}^{b} \right) f(x)\,\mathrm{e}^{\mathrm{i}\omega(g(x)-g(\xi))}\,\mathrm{d}x \\ &= \mathrm{e}^{\mathrm{i}\omega g(\xi)} \left[\int_0^{(g(b)-g(\xi))^{\frac{1}{r+1}}} \widetilde{f}_2(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t - \int_0^{(g(a)-g(\xi))^{\frac{1}{r+1}}} \widetilde{f}_1(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t \right]. \end{aligned}$$

$\int_0^{(g(a)-g(\xi))^{\frac{1}{r+1}}} \widetilde{f}_1(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t$ and $\int_0^{(g(b)-g(\xi))^{\frac{1}{r+1}}} \widetilde{f}_2(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t$ can be approximated by the Filon-type method respectively with forms

$$\int_0^{(g(a)-g(\xi))^{\frac{1}{r+1}}} p_1(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t, \qquad \int_0^{(g(b)-g(\xi))^{\frac{1}{r+1}}} p_2(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t$$

with an error bound $O(\omega^{-s-1/(r+1)})$, where $p_k(t)$ is the interpolation polynomial corresponding to $\widetilde{f}_k(t)$ $(k=1,2)$ satisfying a linear system like (1.2) with $m_0, m_v \geq s(r+1)$.

- For the case that $g'(\xi)=g''(\xi)=\cdots=g^{(r)}(\xi)=0$, $g^{(r+1)}(\xi)>0$, $g'(x)\neq 0$ for $x\neq\xi$, $x\in[a,b]$ and the expressions $x_j(y)=g^{-1}(y-g(\xi))$ $(j=1,2)$ are available for $x\in[a,\xi]$ and $x\in[\xi,b]$ respectively: the generalized moment $\mu_m(\omega,\xi)$ can be written by $y=g(x)-g(\xi)$ as

$$\mu_m(\omega,\xi)=\mathrm{e}^{\mathrm{i}\omega g(\xi)}\left(\int_a^{\xi}+\int_{\xi}^{b}\right)(x-\xi)^m\,\mathrm{e}^{\mathrm{i}\omega(g(x)-g(\xi))}\,\mathrm{d}x$$

$$=\mathrm{e}^{\mathrm{i}\omega g(\xi)}\left[\int_0^{g(b)-g(\xi)} h_2(y)y^{-\frac{r}{r+1}}\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y-\int_0^{g(a)-g(\xi)} h_1(y)y^{-\frac{r}{r+1}}\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y\right].$$

$\int_0^{g(a)-g(\xi)} h_1(y)y^{-\frac{r}{r+1}}\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y$ and $\int_0^{g(b)-g(\xi)} h_2(y)y^{-\frac{r}{r+1}}\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y$ can be approximated by the Filon-type method with forms

$$\int_0^{g(a)-g(\xi)} p_1(y)y^{-\frac{r}{r+1}}\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y,\qquad \int_0^{g(b)-g(\xi)} p_2(y)y^{-\frac{r}{r+1}}\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y$$

with an error bound $O(\omega^{-s-1/(r+1)})$ respectively, where $p_k(y)$ is the interpolation polynomial corresponding to $h_k(y)(k=1,2)$ satisfying a linear system like (1.2) with $m_0, m_v \geq s$. Together with the asymptotic method [10], it presents a method with an error bound $O(\omega^{-s-1/(r+1)})$.

The above methods for $g(x)$ involving stationary points are based on that the generalized moments

$$v_m(y)=\int_0^y t^m\,\mathrm{e}^{\mathrm{i}\omega t^p}\,\mathrm{d}t=\frac{1}{p(-\mathrm{i}\omega)^{(m+1)/p}}\left[\Gamma\left(\frac{m+1}{p}\right)-\Gamma\left(\frac{m+1}{p},-\mathrm{i}\omega y^p\right)\right],$$

can be computed explicitly by the incomplete Gamma function $\Gamma(z,\alpha)$ (Abramowitz and Stegun [1], p. 260, Iserles and Nørsett [10]).

In this paper, we assume f and g are smooth functions and without loss of generality, assume $\omega>0$.

2 Numerical analysis for the moment-free method (1.4) in case that $g'(x) \neq 0$ for all $x \in [a,b]$

Let $I[f]$ denote the following integral

$$I[f] = \int_a^b f(x)\, \mathrm{e}^{\mathrm{i}\omega g(x)}\, \mathrm{d}x, \tag{2.1}$$

where f and g are suitably smooth functions and $g'(x) \neq 0$, $\forall x \in [a,b]$.

Lemma 2.1 *Let s be some positive integer and let $\{m_k\}_0^v$ be a set of multiplicities associated with the node points $a = c_0 < c_1 < \cdots < c_v = b$ such that $m_0, m_v \geq s$. Suppose that $\phi(x) \in C^\infty[a,b]$ satisfies*

$$\phi(c_k) = \phi'(c_k) = \cdots = \phi^{(m_k-1)}(c_k) = 0, \quad k = 0, 1, \ldots, v.$$

Then $\phi(x)$ can be represented in the form

$$\phi(x) = \frac{\phi^{(n+1)}(\xi)}{(n+1)!} \prod_{k=0}^{v} (x - c_k)^{m_k}, \tag{2.2}$$

$$\phi'(x) = \frac{\phi^{(n+1)}(\zeta)}{n!} \prod_{k=0}^{v} (x - c_k)^{m_k - 1} \prod_{j=0}^{v-1} (x - d_j), \tag{2.3}$$

and

$$\phi^{(s)}(x) = \frac{\phi^{(n+1)}(\varsigma)}{(n-s+1)!} \prod_{k \in J} (x - c_k)^{m_k - s} \prod_{m=0}^{s|J|-s+\sum_{j \notin J} m_k} (x - x_m) \quad x \in [a,b], \tag{2.4}$$

for some $\xi, \zeta, \varsigma \in [a,b]$ depending on x and $d_j, x_m \in [a,b]$ for $j = 0, 1, \ldots, v-1$ and $m = 0, 1, \ldots, s|J| - s + \sum_{j \notin J} m_k$, where $J = \{i | m_i > s\}$, $|J|$ denotes the cardinal number of J and $n = \sum_{k=0}^{v} m_k - 1$.

Proof Since (2.2) is trivially satisfied if x coincides with one of the interpolation points $c_0, c_1, \ldots, c_v$, we need be concerned only with the case where x does not coincide with one of the interpolation points. We define $w(t) = \Pi_{k=0}^{v} (t - c_k)^{m_k}$ and, keeping x fixed, consider

$$\varphi(t) = \frac{\phi(x)}{w(x)} w(t).$$

By the assumption on ϕ, $\varphi(t)$ is also in $C^\infty[a,b]$ and $\eta(t)$ defined by $\eta(t) = \varphi(t) - \phi(t)$ satisfies

$$\eta(c_k) = \eta'(c_k) = \cdots = \eta^{(m_k-1)}(c_k) = 0, \quad k = 0, 1, \ldots, v,\ \eta(x) = 0.$$

Then $\eta(t)$ has $n+2$ zeros in $[a,b]$ if we consider the multiplicities. By Rolle's theorem ([22], p. 288) inductively, it is not difficult to show that there is a $\xi \in [a,b]$ such that

$$0 = \eta^{(n+1)}(\xi) = \frac{\phi(x)}{w(x)}(n+1)! - \phi^{(n+1)}(\xi).$$

From this we obtain (2.2).

Since $\phi(c_0) = \phi(c_1) = \cdots = \phi(c_v) = 0$, by Rolle's theorem in each subinterval $[c_j, c_{j+1}]$, there is a $d_j \in (c_j, c_{j+1})$ such that $\phi'(d_j) = 0$, $j = 0, 1, \ldots, v-1$. Thus $\phi'(x)$ has n zeros in $[a,b]$ if we consider the multiplicities. Similar to the proof of (2.2), for $\phi'(x)$ we have

$$\begin{aligned}\phi'(x) &= \frac{(\phi')^{(n)}(\zeta)}{n!}\prod_{k=0}^{v}(x-c_k)^{m_k-1}\prod_{m=0}^{v-1}(x-d_m)\\ &= \frac{\phi^{(n+1)}(\zeta)}{n!}\prod_{k=0}^{v}(x-c_k)^{m_k-1}\prod_{m=0}^{v-1}(x-d_m),\end{aligned}$$

for some $\zeta \in [a,b]$ depending on x. By induction, it is not difficult to prove that $\phi^{(s)}(x)$ has $n-s+1$ zeros in $[a,b]$ if we consider the multiplicities. Similarly, we have

$$\phi^{(s)}(x) = \frac{\phi^{(n+1)}(\varsigma)}{(n-s+1)!}\prod_{k\in J}(x-c_k)^{m_k-s}\prod_{m=0}^{s|J|-s+\sum_{j\notin J}m_k}(x-x_m) \quad x \in [a,b],$$

for some $\varsigma \in [a,b]$ depending on x and $x_m \in [a,b]$, $m = 0, 1, \ldots, s|J| - s + \sum_{j\notin J} m_k$. □

Lemma 2.2 (van der Corput, [21]) *Suppose $g(x)$ is real-valued and smooth in (a,b), and that $|g^{(k)}(x)| \geq 1$ for all $x \in (a,b)$ for a fixed value of k. Then*

$$\left|\int_a^b \mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x\right| \leq c(k)\omega^{-1/k}$$

holds when

(i) *$k \geq 2$, or* (ii) *$k = 1$ and $g'(x)$ is monotonic.*

The bound $c(k)$ is independent of g and ω, $c(k) = 5 \cdot 2^{k-1} - 2$.

Lemma 2.3 ([21], p. 334) *Under the assumptions on $g(x)$ in Lemma 2.2, we can conclude that*

$$\left|\int_a^b e^{i\omega g(x)}\varphi(x)\,dx\right| \le c(k)\omega^{-1/k}\left[|\varphi(b)| + \int_a^b |\varphi'(x)|dx\right].$$

Theorem 2.1 *Let s be some positive integer and let $\{m_k\}_0^v$ be a set of multiplicities associated with the node points $a = c_0 < c_1 < \cdots < c_v = b$ such that $m_0, m_v \ge s$. Suppose that $p(y) = \sum_{k=0}^{n} a_k y^k$ with $n = \sum_{k=0}^{v} m_k - 1$ satisfies (1.3) for $k = 0, 1, \ldots, v$. Define $Q_{v,m}^F[f]$ by (1.4). If $g'(x)$ is finitely N piecewise monotonic in $[a, b]$. Then*

$$E[f] = |I[f] - Q_{v,m}^F[f]| \le \min\left\{\frac{C_1(b-a)^{n+1}}{(n+1)!\omega\delta}, \frac{C_2(b-a)^{n+1-s}}{(n+1-s)!\omega^{s+1}\delta}\right\}, \tag{2.5}$$

where $C_1 = 3(n+2)N\|[p(g(x))g'(x) - f(x)]^{(n+1)}\|_\infty$, $C_2 = 3(n-s+2)N\|\sigma_s[p(g)g' - f]^{(n-s+1)}(x)\|_\infty$ and $\delta = \min_{x\in[a,b]}|g'(x)|$.

Proof First we show that for $k = 1, \ldots, m_j - 1$ and $j = 0, 1, \ldots, v$

$$p(g(x))g'(x)|_{x=c_j} = f(x)|_{x=c_j}, \quad (p(g(x))g'(x))^{(k)}|_{x=c_j} = f^{(k)}(x)|_{x=c_j}. \tag{2.6}$$

Since $\sigma_1[f](x) = \frac{f(x)}{g'(x)}$, then by the definition of $\sigma_k[f](x)$, for $y = g(x)$ we have

$$\frac{d\left(\frac{f(x)}{g'(x)}\right)}{dy} = \frac{d\sigma_1[f](x)}{dx}\cdot\frac{dx}{dy} = \frac{\sigma_1[f]'(x)}{g'(x)} = \sigma_2[f](x).$$

Assume $\frac{d^k\left(\frac{f(x)}{g'(x)}\right)}{dy^k} = \sigma_{k+1}[f](x)$, then

$$\frac{d^{k+1}\left(\frac{f(x)}{g'(x)}\right)}{dy^{k+1}} = \frac{d\sigma_{k+1}[f](x)}{dx}\cdot\frac{dx}{dy} = \frac{\sigma_{k+1}[f]'(x)}{g'(x)} = \sigma_{k+2}[f](x).$$

Therefore by induction, $\frac{d^k\left(\frac{f(x)}{g'(x)}\right)}{dy^k} = \sigma_{k+1}[f](x)$ for $k = 1, 2, \ldots, m_k - 1$. By the assumption (1.3), we get

$$\left[\frac{f(x)}{g'(x)} - p(y)\right]_{x=c_j}^{y=g(c_j)} = 0, \quad \frac{d^k\left[\frac{f(x)}{g'(x)} - p(y)\right]}{dy^k}\Big|_{x=c_j}^{y=g(c_j)} = 0, \quad k = 1, 2, \ldots, m_k - 1. \tag{2.7}$$

Then $(p(g(x))g'(x))|_{x=c_j} = f(x)|_{x=c_j}$ and

$$\frac{\mathrm{d}\left(\frac{f(x)-p(g(x))g'(x)}{g'(x)}\right)}{\mathrm{d}x}|_{x=c_j} = \left[\frac{\mathrm{d}\left(\frac{f(x)}{g'(x)} - p(y)\right)}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{x=c_j}^{y=g(c_j)} = 0. \qquad (2.8)$$

Since $\frac{\mathrm{d}^k\left(\frac{f(x)-p(g(x))g'(x)}{g'(x)}\right)}{\mathrm{d}x^k} = \frac{\mathrm{d}^k\left(\frac{f(x)}{g'(x)}-p(y)\right)}{\mathrm{d}x^k}$, from the observation of (2.8), $\frac{\mathrm{d}^k\left(\frac{f(x)}{g'(x)}-p(y)\right)}{\mathrm{d}x^k}$ can be written as a linear combination of $\frac{\mathrm{d}^j\left[\frac{f(x)}{g'(x)}-p(y)\right]}{\mathrm{d}y^j}$ for $j = 1, 2, \ldots, k$. From (2.7), we get

$$\frac{\mathrm{d}^k\left(\frac{f(x)-p(g(x))g'(x)}{g'(x)}\right)}{\mathrm{d}x^k}|_{x=c_j} = 0.$$

Since $(f(x) - p(g(x))g'(x))|_{x=c_j} = 0$ and for $k = 1, 2\ldots, m_j - 1$

$$0 = \frac{\mathrm{d}^k\left(\frac{f(x)-p(g(x))g'(x)}{g'(x)}\right)}{\mathrm{d}x^k}|_{x=c_j} = \left[\frac{\mathrm{d}^k(f(x)-p(g(x))g'(x))}{\mathrm{d}x^k}\frac{1}{g'(x)} - \sum_{\ell=0}^{k-1}\binom{k}{\ell}\frac{\mathrm{d}^\ell(f(x)-p(g(x))g'(x))}{\mathrm{d}x}\cdot\frac{\mathrm{d}^{k-\ell}\left(\frac{1}{g'(x)}\right)}{\mathrm{d}x^{k-\ell}}\right]_{x=c_j},$$

we have $(p(g(x))g'(x))'|_{x=c_j} = f'(x)|_{x=c_j}$ for $j = 0, 1, \ldots, v$ and by induction we have

$$(p(g(x))g'(x))''|_{x=c_j} = f''(x)|_{x=c_j}, \ldots,\ (p(g(x))g'(x))^{(m_j-1)}|_{x=c_j} = f^{(m_j-1)}(x)|_{x=c_j}.$$

To analyze the error bound (2.5), we first assume that $g'(x)$ is monotonic in $[a, b]$. Since $Q^F_{v,m}[f] = \int_{g(a)}^{g(b)} p(y)\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y = \int_a^b p(g(x))g'(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$ and

$$E[f] = |I[f] - Q^F_{v,m}[f]| = \left|\int_a^b \left(f(x) - p(g(x))g'(x)\right)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x\right|.$$

Then by Lemma 2.3 for $k = 1$ and $c(1) = 3$, and noting that $\frac{|g'(x)|}{\delta} \geq 1$ for all $x \in [a, b]$, we have

$$E[f] \leq \frac{c(1)}{\omega\delta}\left(|\left(f(b) - p(g(b))g'(b)\right)| + \int_a^b |\left(f(x) - p(g(x))g'(x)\right)'|\mathrm{d}x\right). \qquad (2.9)$$

By Lemma 2.1 and (2.6),

$$|f(b) - p(g(b))g'(b)| \leq \frac{\|[p(g(x))g'(x) - f(x)]^{(n+1)}\|_\infty (b-a)^{n+1}}{(n+1)!}$$

and

$$|\left(f(x) - p(g(x))g'(x)\right)'| \leq \frac{\|[p(g(x))g'(x) - f(x)]^{(n+1)}\|_\infty (b-a)^{n}}{n!}.$$

These together with (2.9) complete the first part of (2.5) with $N = 1$.

Notice that $\left(f(x) - p(g(x))g'(x)\right)|_{x=a} = \left(f(x) - p(g(x))g'(x)\right)|_{x=b} = 0$ and

$$\left(f(x) - p(g(x))g'(x)\right)^{(k)}|_{x=a} = \left(f(x) - p(g(x))g'(x)\right)^{(k)}|_{x=b} = 0, \quad k = 1, 2, \ldots, s-1.$$

Integrating by parts, we have

$$\begin{aligned} E[f] &= \frac{1}{\omega}\left|\int_a^b \sigma_1[p(g)g' - f]'(x)\, \mathrm{e}^{\mathrm{i}\omega g(x)}\, \mathrm{d}x\right| \\ &= \cdots = \frac{1}{\omega^s}\left|\int_a^b \sigma_s[p(g)g' - f]'(x)\, \mathrm{e}^{\mathrm{i}\omega g(x)}\, \mathrm{d}x\right|. \end{aligned} \tag{2.10}$$

By the assumption on $\{m_k\}_{k=0}^{v}$, it is not difficult to show that $\sigma_s[f - p(g)g'](x)$ has $n - s + 1$ zeros if we consider multiplicities. By Lemma 2.1 $\sigma_s[f - p(g)g'](x)$ can be written as

$$\sigma_s[f - p(g)g'](x) = \frac{\sigma_s^{(n-s+1)}(\varsigma)}{(n-s+1)!}\prod_{k\in J}(x - c_k)^{m_k - s}\prod_{m=0}^{s|J|-s+\sum_{j\notin J} m_k}(x - x_m), \quad x \in [a, b],$$

For some $\varsigma \in [a, b]$ depending on x and $x_m \in [a, b]$, $m = 0, 1, \ldots, s|J| - s + \sum_{j\notin J} m_k$. Similarly, by Lemma 2.1 and Lemma 2.3 for $k = 1$ and $c(1) = 3$, we have

$$E[f] \leq \frac{3(n - s + 2)\|\sigma_s[p(g)g' - f]^{(n-s+1)}(x)\|_\infty (b-a)^{n-s+1}}{(n-s+1)!\,\omega^{s+1}\delta}.$$

(2.5) is satisfied in case that $g'(x)$ is monotonic in $[a, b]$.

For the general case, without loss of generality, we assume $g'(x)$ is monotonic in $[a, a_1]$ and $[a_1, b]$ with $a \le a_1 \le b$. Note that for any positive integer k, $(a_1 - a)^k + (b - a_1)^k \le (b - a)^k$ and

$$E[f] \le \left| \int_a^{a_1} \big(f(x) - p(g(x))g'(x)\big)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x \right| + \left| \int_{a_1}^{b} \big(f(x) - p(g(x))g'(x)\big)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x \right|.$$

Together with the above analysis, (2.5) is satisfied for $N = 2$. We complete the proof. □

Remark 1 From the proof of Theorem 2.1, quadrature (1.4) is a Filon-type method based on the interpolation function $p(g(x))g'(x)$ which satisfies

$$(p(g(x))g'(x))|_{x=c_k} = f(x)|_{x=c_k}, \ldots, \; (p(g(x))g'(x))^{(m_k-1)}|_{x=c_k} = f^{(m_k-1)}(x)|_{x=c_k}$$

for $k = 0, 1, \ldots, v$, where $p(y) = \sum_{k=0}^{n} a_k y^k$.

Let $\ell_k(y) \in \mathbf{P_v}$ be the kth cardinal polynomial of Lagrangian interpolation,

$$\ell_k(g(c_j)) = \begin{cases} 1, & j = k, \\ 0, & j \ne k, \end{cases} \quad k, j = 0, 1, 2, \ldots, v.$$

Set

$$p(y) = \sum_{k=0}^{v} \frac{f(g(c_k))}{g'(c_k)} \ell_k(y). \tag{2.11}$$

A quadrature of Filon-type method (1.4) for $\{m_k\}_{k=0}^{v}$ being all ones is given by

$$Q_v^F[f] = \int_{g(a)}^{g(b)} p(y)\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y. \tag{2.12}$$

Especially, for $g(x) = x$ we have

Corollary 2.1 *Let $a = c_0 < c_1 < \cdots < c_v = b$ be interpolation nodes in $[a, b]$, $g(x) = x$, $p(x) = \sum_{k=0}^{v} f(c_k)\ell_k(x)$ and $Q_v^F[f] = \int_a^b p(x)\,\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x$. Then*

$$E[f] = |I[f] - Q_v^F[f]| \le \frac{\beta(b-a)^v}{v!\,\omega^2}, \tag{2.13}$$

where $\beta = 3(v+1)\|f^{(v+1)}(x)\|_\infty$.

However, the condition that "$g'(x)$ is finitely N piecewise monotonic in $[a,b]$" in Theorem 2.1 is strict and in some cases, N may be very large. In the following, we give another error analysis for the Filon-type quadrature (1.4) without the condition that "$g'(x)$ is finitely piecewise monotonic in $[a,b]$" in Theorem 2.1.

Theorem 2.2 *Under the assumptions on* s, $\{m_k\}_0^v$, $p(y)=\sum_{k=0}^n a_k y^k$ *and* $Q^F_{v,m}[f]$ *in Theorem 2.1, we can conclude that*

$$E[f]=|I[f]-Q^F_{v,m}[f]|\le \frac{3(n-s+2)\|\Psi^{(n+1)}(y)\|_\infty |g(b)-g(a)|^{n-s+1}}{(n-s+1)!\omega^{s+1}}, \tag{2.14}$$

where $\Psi(y)=\frac{f(g^{-1}(y))}{g'(g^{-1}(y))}=\frac{f(x)}{g'(x)}$.

Proof Since

$$\begin{aligned} E[f] &= |I[f]-Q^F_{v,m}[f]| \\ &= \left|\int_a^b \left(f(x)-p(g(x))g'(x)\right) \mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x\right| \\ &= \left|\int_{g(a)}^{g(b)} \left(\frac{f(x)}{g'(x)}-p(y)\right) \mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y\right| \\ &= \left|\int_{g(a)}^{g(b)} (\Psi(y)-p(y))\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y\right|. \end{aligned} \tag{2.15}$$

From

$$p(g(c_k))=\sigma_1[f](c_k), p'(g(c_k))=\sigma_2[f](c_k),\ldots,p^{(m_k-1)}(g(c_k))=\sigma_{m_k}[f](c_k),$$

for $k=0,1,\ldots,v$ and the first part of the proof of Theorem 2.1, we have $p(y)|_{y=g(c_k)}=\Psi(y)|_{y=g(c_k)}$ and $p^{(j)}(y)|_{y=g(c_k)}=\Psi^{(j)}(y)|_{y=g(c_k)}$ for $j=1,2,\ldots,m_k-1$ and $k=0,1,\ldots,v$. By Lemma 2.1, similar to the proof of Theorem 2.1 and noticing that $p^{(n+1)}(y)\equiv 0$, we derive that for (2.14)

$$E[f]\le \frac{3(n-s+2)\|\Psi^{(n+1)}(y)\|_\infty |g(b)-g(a)|^{n-s+1}}{(n-s+1)!\omega^{s+1}}. \qquad \square$$

Corollary 2.2 *Let $a = c_0 < c_1 < \cdots < c_v = b$ be interpolation nodes in $[a, b]$, let $p(y) = \sum_{k=0}^{v} \frac{f(g(c_k))}{g'(c_k)} \ell_k(y)$ and $Q_v^F[f]$ be defined by (1.4) for $\{m_k\}_{k=0}^{v}$ being all ones. Then*

$$E[f] = |I[f] - Q_v^F[f]| \le \frac{3(v+1)(b-a)^v \|\Psi^{(v+1)}(y)\|_\infty \|g'\|_\infty^v}{v!\omega^2}, \tag{2.16}$$

where $\Psi(y) = \frac{f(g^{-1}(y))}{g'(g^{-1}(y))} = \frac{f(x)}{g'(x)}$.

The Filon-type quadrature with the linear interpolation or Hermite interpolation approximating is simple and commonly used. The Filon-type quadrature with the linear interpolation approximating at $c_0 = a$ and $c_1 = b$ is as follows: let $y_0 = \frac{f(a)}{g'(a)}$, $y_1 = \frac{f(b)}{g'(b)}$, $d_1 = \frac{y_1 - y_0}{g(b)-g(a)}$ and $d_2 = \frac{y_0 g(b) - y_1 g(a)}{g(b)-g(a)}$, then

$$\begin{aligned} Q_{1,L}^F[f] = {} & \frac{(-\mathrm{i}\omega g(b)+1)\,\mathrm{e}^{\mathrm{i}\omega g(b)} + \mathrm{e}^{\mathrm{i}\omega g(a)}(\mathrm{i}\omega g(a) - 1)}{\omega^2} d_1 \\ & + \frac{-\mathrm{i}\mathrm{e}^{\mathrm{i}\omega g(b)} + i\mathrm{e}^{\mathrm{i}\omega g(a)}}{\omega} d_2. \end{aligned} \tag{2.17}$$

The Filon-type quadrature with the Hermite interpolation approximating at $c_0 = a$ and $c_1 = b$ is as follows: let

$$\begin{aligned} & y_0 = \frac{f(a)}{g'(a)}, \quad y_1 = \frac{f(b)}{g'(b)}, \quad y_2 = \frac{f'(a)g'(a) - f(a)g''(a)}{(g'(a))^3}, \\ & y_3 = \frac{f'(b)g'(b) - f(b)g''(b)}{(g'(b))^3}, \end{aligned}$$

and

$$\begin{aligned} d_1 &= \frac{y_2}{(g(b)-g(a))^2} + \frac{2y_0}{(g(b)-g(a))^3} - \frac{2y_1}{(g(b)-g(a))^3} + \frac{y_3}{(g(b)-g(a))^2}, \\ d_2 &= \frac{6y_1 g(a)}{(g(b)-g(a))^3} + \frac{3y_1}{(g(b)-g(a))^2} - \frac{y_3}{g(b)-g(a)} - \frac{6y_0 g(a)}{g(b)-g(a))^3} \\ &\quad - \frac{3y_3 g(a)}{(g(b)-g(a))^2} - \frac{3y_0}{(g(b)-g(a))^2} - \frac{2y_2}{g(b)-g(a)} - \frac{3y_2 g(a)}{(g(b)-g(a))^2} \\ d_3 &= y2 + \frac{6y_0 g(a)}{(g(b)-g(a))^3} + \frac{3y_3 g(a)^2}{(g(b)-g(a))^2} + \frac{6y_0 g(a)}{(g(b)-g(a))^2} + \frac{4y_2 g(a)}{g(b)-g(a)} \\ &\quad + \frac{2y_3 g(a)}{g(b)-g(a)} + \frac{3y_2 g(a)^2}{(g(b)-g(a))^2} - \frac{6y_1 g(a)^2}{(g(b)-g(a))^3} - \frac{6y_1 g(a)}{(g(b)-g(a))^2} \end{aligned}$$

$$d_4 = y_0 + \frac{3y_1 g(a)^2}{(g(b)-g(a))^2} - \frac{3y_0 g(a)^2}{(g(b)-g(a))^2} - \frac{2y_0 g(a)^3}{(g(b)-g(a))^3} + \frac{2y_1 g(a)^3}{(g(b)-g(a))^3}$$
$$- y_2 g(a) - \frac{2y_2 g(a)^2}{g(b)-g(a)} - \frac{y_3 g(a)^2}{g(b)-g(a)} - \frac{y_3 g(a)^3}{(g(b)-g(a))^2} - \frac{y_2 g(a)^3}{(g(b)-g(a))^2}.$$

Then

$$\begin{aligned} Q^F_{1,H}[f] = & \frac{(-\mathrm{i}\omega^3 g(b)^3 + 3\omega^2 g(b)^2 + 6\mathrm{i}\omega g(b) - 6)\,\mathrm{e}^{\mathrm{i}\omega g(b)}}{\omega^4} d_1 \\ & + \frac{(\mathrm{i}\omega^3 g(a)^3 - 3\omega^2 g(a)^2 - 6\mathrm{i}\omega g(a) + 6)\,\mathrm{e}^{\mathrm{i}\omega g(a)}}{\omega^4} d_1 \\ & + \frac{(-\mathrm{i}\omega^2 g(b)^2 + 2\omega g(b) + 2\mathrm{i})\,\mathrm{e}^{\mathrm{i}\omega g(b)} + (\mathrm{i}\omega^2 g(a)^2 - 2\omega g(a) - 2\mathrm{i})\,\mathrm{e}^{\mathrm{i}\omega g(a)}}{\omega^3} d_2 \\ & + \frac{(-\mathrm{i}\omega g(b) + 1)\,\mathrm{e}^{\mathrm{i}\omega g(b)} + (\mathrm{i}\omega g(a) - 1)\,\mathrm{e}^{\mathrm{i}\omega g(a)}}{\omega^2} d_3 + \frac{-\mathrm{i}\mathrm{e}^{\mathrm{i}\omega g(b)} + \mathrm{i}\mathrm{e}^{\mathrm{i}\omega g(a)}}{\omega} d_4. \end{aligned} \tag{2.18}$$

Example 2.1 Let us consider the Filon-type quadrature (1.4) with the linear interpolation or Hermite interpolation at endpoints respectively to approximate $\int_0^1 \cos(x)\,\mathrm{e}^{\mathrm{i}\omega(x^2+x)}\,\mathrm{d}x$ and compare with the corresponding asymptotic method (see Figs. 1, 2) (Tables 1, 2).

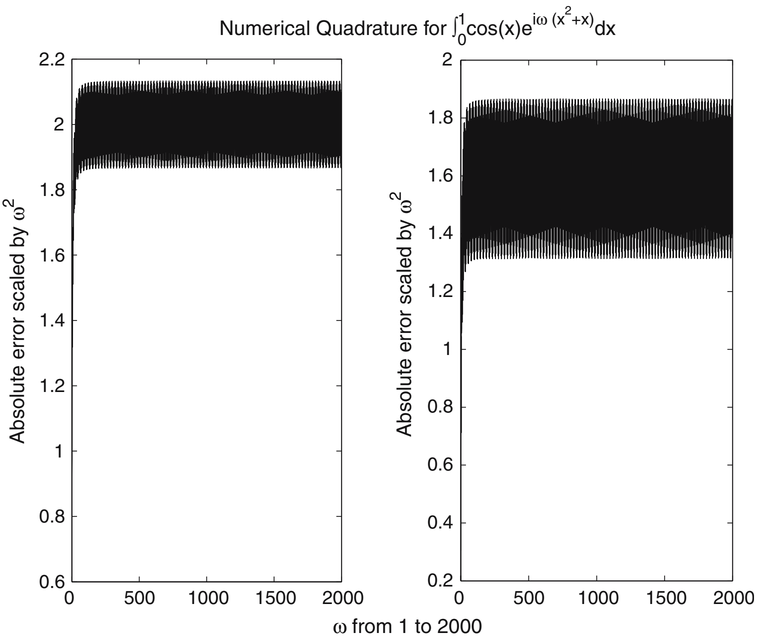

Fig. 1 The absolute error scaled by ω^2 for quadrature (1.4) with the linear interpolation at the endpoints, compared with the asymptotic method $Q_1^A[\cos(x)]$ (on the *left*)

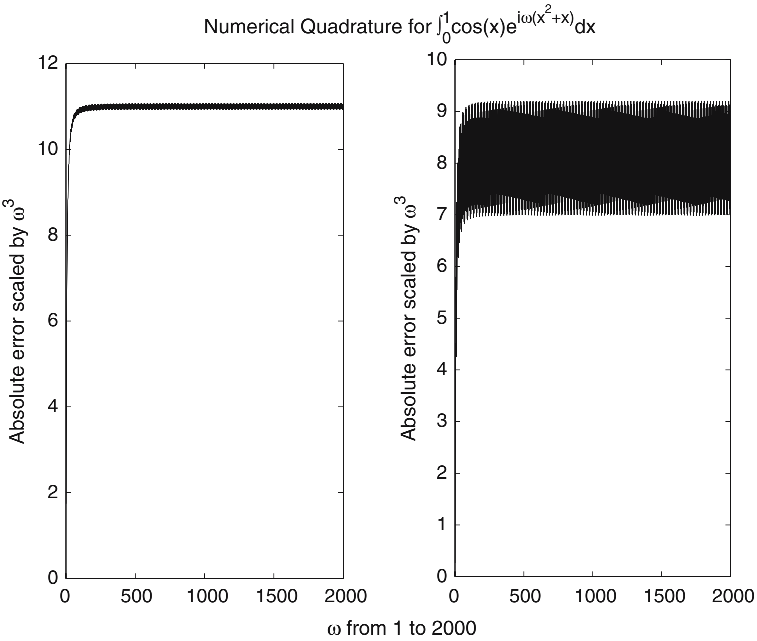

Fig. 2 The absolute error scaled by ω^3 for quadrature (1.4) with the Hermite interpolation at the endpoints, compared with the asymptotic method $Q_2^A[\cos(x)]$ (on the *left*)

Table 1 Error of Example 2.1 by the asymptotic method $Q_1^A[f]$ and the Filon-type quadrature (1.4) with the linear interpolation

ω	1	10	10^2	10^3	10^4
$E_1^A[f]$	0.7084	0.0166	1.9250e−4	2.0533e−6	1.8931e−8
Relative error	0.9842	0.1832	0.0209	0.0019	2.2013e−4
$E_{\mathrm{New}}[f]$	0.3050	0.0130	1.7501e−4	1.5085e−6	1.8219e−8
Relative error	0.4238	0.1429	0.0190	0.0014	2.1185e−4

3 Treatment of stationary points

Once g' vanishes at one of more points in $[a,b]$, without loss of generality, assume $\xi \in [a,b]$ is the unique stationary point of g and

$$g'(\xi) = g''(\xi) = \cdots = g^{(r)}(\xi) = 0, \quad g^{(r+1)}(\xi) > 0, \quad g'(x) \neq 0 \quad \text{for } x \neq \xi.$$

By Darboux Theorem for derivatives [24], $g(x)$ is strictly monotonic in $[a,\xi]$ and $[\xi,b]$ respectively and $g(\xi) = \min_{x\in[a,b]} g(x)$ for r being an odd integer.

Table 2 Error of Example 2.1 by the asymptotic method $Q_2^A[f]$ and the Filon-type quadrature (1.4) with the Hermite interpolation

ω	1	10	10^2	10^3	10^4
$E_2^A[f]$	1.7455	0.0078	1.0868e−5	1.1025e−8	1.0944e−11
Relative error	2.4252	0.0861	0.0012	1.0214e−5	1.2726e−7
$E_{\mathrm{New}}[f]$	0.2631	0.0063	7.3966e−6	8.5784e−9	7.2267e−12
Relative error	0.3655	0.0695	8.0160e−4	7.9470e−6	8.4035e−8

In the following, we will consider two cases for treatment of the stationary point based on that

$$v_m(y) = \int_0^y t^m \,\mathrm{e}^{\mathrm{i}\omega t^p}\,\mathrm{d}t = \frac{1}{p(-\mathrm{i}\omega)^{(m+1)/p}}\left[\Gamma\left(\frac{m+1}{p}\right) - \Gamma\left(\frac{m+1}{p}, -\mathrm{i}\omega y^p\right)\right]$$

can be computed explicitly by the incomplete Gamma function $\Gamma(z,\alpha)$ ([1,10]).

For the general case that $g'(\xi) = 0$: let $t^{r+1} = g(x) - g(\xi)$, then t^{r+1} is well-defined in each subinterval $[a,\xi]$ and $[\xi,b]$, and $I[f]$ can be represented by

$$\begin{aligned}
I[f] &= \mathrm{e}^{\mathrm{i}\omega g(\xi)}\left(\int_a^\xi + \int_\xi^b\right) f(x)\,\mathrm{e}^{\mathrm{i}\omega(g(x)-g(\xi))}\,\mathrm{d}x \\
&= \mathrm{e}^{\mathrm{i}\omega g(\xi)}\left[\int_{(g(a)-g(\xi))^{\frac{1}{r+1}}}^{0} f(x_1(t))x_1'(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t + \int_0^{(g(b)-g(\xi))^{\frac{1}{r+1}}} f(x_2(t))x_2'(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t\right] \\
&= \mathrm{e}^{\mathrm{i}\omega g(\xi)}\left[\int_{(g(a)-g(\xi))^{\frac{1}{r+1}}}^{0} \widetilde{f_1}(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t + \int_0^{(g(b)-g(\xi))^{\frac{1}{r+1}}} \widetilde{f_2}(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t\right] \\
&= \mathrm{e}^{\mathrm{i}\omega g(\xi)}\left[\int_0^{(g(b)-g(\xi))^{\frac{1}{r+1}}} \widetilde{f_2}(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t - \int_0^{(g(a)-g(\xi))^{\frac{1}{r+1}}} \widetilde{f_1}(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t\right] \\
&=: I_2 - I_1,
\end{aligned} \tag{3.1}$$

where $\widetilde{f_1}(t)$ and $\widetilde{f_2}(t)$ are smooth functions ([21], pp. 336–337).

For simplicity, assume $\xi = a$, $g(a) = 0$ and $g'(x) \geq 0$ in $[a, b]$, then

$$I[f] = \int_0^{\sqrt[r+1]{g(b)}} \widetilde{f}(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t$$

can be approximated by the Filon-type method with the form $\int_0^{\sqrt[r+1]{g(b)}} p(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t$ which can be calculated by the incomplete Gamma function, where $p(t)$ is the interpolation polynomial with degree $n = \sum_{k=0}^{v} m_k - 1$ and $m_0, m_v \geq s(r+1)$ such that for $0 = \sqrt[r+1]{g(a)} = d_0 < d_1 < \cdots < d_v = \sqrt[r+1]{g(b)}$

$$p(d_k) = \widetilde{f}(d_k), \quad p'(d_k) = \widetilde{f}'(d_k), \ldots, p^{(m_k-1)}(d_k) = \widetilde{f}^{(m_k-1)}(d_k). \tag{3.2}$$

Since $\widetilde{f}(t) - p(t) \sim O(t^{s(r+1)})$. From van der Corput Lemma, similar to the proof of Theorem 2.1, we obtain

$$Q_s^F[\widetilde{f}(t)] - I[f] \sim O(\omega^{-s-1/(r+1)}), \tag{3.3}$$

where $Q_s^F[\widetilde{f}(t)] = \int_0^{\sqrt[r+1]{g(b)}} p(t)\,\mathrm{e}^{\mathrm{i}\omega t^{r+1}}\,\mathrm{d}t$.

Example 3.1 Let $I[f] = \int_{-1}^{1} \mathrm{e}^x \mathrm{e}^{\mathrm{i}\omega\cos(x)}\,\mathrm{d}x$. $I[f]$ can be transferred by $t^2 = 1 - \cos(x)$ into

$$\begin{aligned} I[f] &= \mathrm{e}^{\mathrm{i}\omega}\left(\int_{-1}^{0} \mathrm{e}^x \mathrm{e}^{-\mathrm{i}\omega(1-\cos(x))}\,\mathrm{d}x + \int_0^1 \mathrm{e}^x\,\mathrm{e}^{-\mathrm{i}\omega(1-\cos(x))}\,\mathrm{d}x\right) \\ &= 2\,\mathrm{e}^{\mathrm{i}\omega}\left(\int_0^{\sqrt{1-\cos(1)}} \frac{\mathrm{e}^{-\arccos(1-t^2)}\,\mathrm{e}^{-\mathrm{i}\omega t^2}}{\sqrt{2-t^2}}\,\mathrm{d}t + \int_0^{\sqrt{1-\cos(1)}} \frac{\mathrm{e}^{\arccos(1-t^2)}\,\mathrm{e}^{-\mathrm{i}\omega t^2}}{\sqrt{2-t^2}}\,\mathrm{d}t\right) \\ &= 2\,\mathrm{e}^{\mathrm{i}\omega}\left(\int_0^{\sqrt{1-\cos(1)}} \frac{\mathrm{e}^{-\arccos(1-t^2)} + \mathrm{e}^{\arccos(1-t^2)}}{\sqrt{2-t^2}}\mathrm{e}^{-\mathrm{i}\omega t^2}\,\mathrm{d}t\right). \end{aligned}$$

$\widetilde{f}(t) = \frac{\mathrm{e}^{-\arccos(1-t^2)} + \mathrm{e}^{\arccos(1-t^2)}}{\sqrt{2-t^2}}$ is smooth in $[0, \sqrt{1-\cos(1)}]$, which can be approximated by polynomial $p(t)$ satisfying (3.2). $I[f]$ can be approximated by the

Filon-type method

$$Q_s^F\left[\frac{e^{-\arccos(1-t^2)}+e^{\arccos(1-t^2)}}{\sqrt{2-t^2}}\right]$$

with an error bound $O(\omega^{-s-1/2})$ (see Fig. 3).

Remark 2 If the inverse of $g(x)$ for $t^{r+1}=g(x)-g(\xi)$ is unavailable piecewise, the derivatives of $\widetilde{f}^{(\ell)}(t)$ can be computed by

$$\frac{d^\ell\left[\frac{(r+1)f(x)t^r}{g'(x)}\right]}{dt^\ell},$$

where x is considered as a function of t and $x'(t)=\frac{(r+1)t^r}{g'(x)}$.

Another approach to approximating $I[f]$, which is of the same order as (3.3) with the lower degree of the interpolation polynomials, is base on the asymptotic method $Q_s^A[f]$ [10] and that the inverse of $y=g(x)$ is available piecewise.

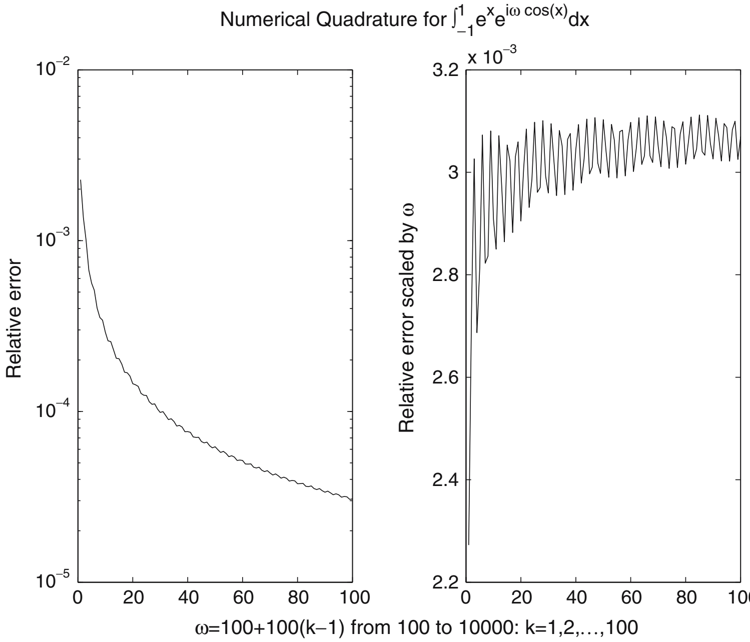

Fig. 3 The relative error and the relative error scaled by ω for $\int_{-1}^1 e^x e^{i\omega\cos(x)}dx$ approximated by $Q_2^F\left[\frac{e^{-\arccos(1-t^2)}+e^{\arccos(1-t^2)}}{\sqrt{2-t^2}}\right]$ at nodes 0 and $\sqrt{1-\cos(1)}$

The asymptotic method (1.1) can be extended to the case for any integer $r \geq 1$ [10]

$$g'(\xi) = g''(\xi) = \cdots = g^{(r)}(\xi) = 0, \quad g^{(r+1)}(\xi) \neq 0, \quad g'(x) \neq 0 \quad \text{for } x \neq \xi$$

as follows

$$\begin{aligned} Q_s^A[f] = & \sum_{j=0}^{r-1} \frac{1}{j!} \mu_j(\omega,\xi) \sum_{m=0}^{s-j-1} \frac{1}{(-\mathrm{i}\omega)^m} \rho_m^{(j)}[f](\xi) \\ & - \sum_{m=1}^{s-1} \frac{1}{(-\mathrm{i}\omega)^m} \left(\frac{\mathrm{e}^{\mathrm{i}\omega g(b)}}{g'(b)} \{\rho_{m-1}[f](b) - \rho_{m-1}[f](\xi)\} \right. \\ & \left. - \frac{\mathrm{e}^{\mathrm{i}\omega g(a)}}{g'(a)} \{\rho_{m-1}[f](a) - \rho_{m-1}[f](\xi)\} \right) \end{aligned} \tag{3.4}$$

with an error bound

$$Q_s^A[f] - I[f] \sim O(\omega^{-s-1/(r+1)}), \tag{3.5}$$

where $\mu_m(\omega,\xi) = \int_a^b (x-\xi)^m\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$, $m = 0, 1, \ldots, r-1$ and

$$\rho_0[f](x) = f(x), \quad \rho_{k+1}[f](x) = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\rho_k[f](x) - \sum_{j=0}^{r-1} \frac{1}{j!} \rho_k[f]^{(j)}(\xi)(x-\xi)^j}{g'(x)}, \ k \geq 0. \tag{3.6}$$

For the case that $g'(\xi) = 0$ and the expressions $x_j(y) = g^{-1}(y - g(\xi))(j = 1, 2)$ are available for $x \in [a, \xi]$ and $[\xi, b]$, respectively: the generalized moment $\mu_m(\omega,\xi)$ can be written by $y = g(x) - g(\xi)$ as

$$\begin{aligned} & \mu_m(\omega,\xi) \\ & = \mathrm{e}^{\mathrm{i}\omega g(\xi)} \left(\int_a^\xi + \int_\xi^b \right) (x-\xi)^m\,\mathrm{e}^{\mathrm{i}\omega(g(x)-g(\xi))}\,\mathrm{d}x \\ & = \mathrm{e}^{\mathrm{i}\omega g(\xi)} \left[\int_{g(a)-g(\xi)}^{0} (x_1(y)-\xi)^m x_1'(y)\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y + \int_0^{g(b)-g(\xi)} (x_2(y)-\xi)^m x_2'(y)\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y \right] \\ & = \mathrm{e}^{\mathrm{i}\omega g(\xi)} \left[\int_0^{g(b)-g(\xi)} (x_2(y)-\xi)^m x_2'(y)\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y - \int_0^{g(a)-g(\xi)} (x_1(y)-\xi)^m x_1'(y)\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y \right] \\ & =: I_2 - I_1. \end{aligned}$$

Notice that by Taylor's expansion,

$$\begin{aligned} y = g(x) - g(\xi) &= \frac{g^{(r+1)}(\xi)(x-\xi)^{r+1}}{(r+1)!} + o((x-\xi)^{r+1}), \\ y' = g'(x) &= \frac{g^{(r+1)}(\xi)(x-\xi)^{r}}{r!} + o((x-\xi)^{r}). \end{aligned} \tag{3.7}$$

Then

$$\lim_{y\to 0} x'(y) y^{\frac{r}{r+1}} = \lim_{x\to\xi} \frac{y^{\frac{r}{r+1}}}{g'(x)} = \frac{1}{(r+1)\sqrt[r+1]{\frac{g^{(r+1)}(\xi)}{(r+1)!}}}$$

and

$$\begin{aligned} I_j &= e^{i\omega g(\xi)} \int_0^{g(x_j)-g(\xi)} (x_j(y)-\xi)^m \big[x_j'(y) y^{\frac{r}{r+1}}\big] y^{-\frac{r}{r+1}} e^{i\omega y}\, dy \\ &=: e^{i\omega g(\xi)} \int_0^{g(x_j)-g(\xi)} h_j(y) y^{-\frac{r}{r+1}} e^{i\omega y}\, dy \quad (j=1,2,\ x_1=a,\ x_2=b) \end{aligned} \tag{3.8}$$

can be approximated by the Filon-type method with the form $\int_0^{g(x_j)-g(\xi)} p_j(y)$ $y^{-\frac{r}{r+1}} e^{i\omega y}\, dy$, which can be calculated by the incomplete Gamma function, where $p_j(y)$ is the interpolation polynomial with degree $n = \sum_{k=0}^{v} m_k - 1$ and $m_0, m_v \geq s$ satisfying that for $0 = d_0 < d_1 < \cdots < d_v = g(x_j) - g(\xi)$

$$p_j(d_k) = h_j(d_k), \quad p_j'(d_k) = h_j'(d_k), \ldots, p_j^{(m_k-1)}(d_k) = h_j^{(m_k-1)}(d_k). \tag{3.9}$$

Since $\frac{h_j(y)-p_j(y)}{y^{\frac{r}{r+1}}} = O\big(y^{s+\frac{1}{r+1}}\big)$. From van der Corput Lemma, the proof of Theorem 2.1 and the observation

$$\left| \int_0^{g(x_j)-g(\xi)} y^{-\frac{r}{r+1}} e^{i\omega y}\, dy \right| = \left| \int_0^{\sqrt[r+1]{g(x_j)-g(\xi)}} (r+1)\, e^{i\omega t^{r+1}}\, dt \right| \sim O\left(\omega^{-1/(r+1)}\right), \tag{3.10}$$

similarly we obtain

$$\begin{aligned} &Q_s^F[h_j(y)] - I_j \sim O(\omega^{-s-1/(r+1)}), \\ &Q_s^F[h_2(y)] - Q_s^F[h_1(y)] - \mu_m(\omega,\xi) \sim O(\omega^{-s-1/(r+1)}), \end{aligned}$$

where $Q_s^F[h_j(y)] = \int_0^{g(x_j)-g(\xi)} p_j(y) y^{-\frac{r}{r+1}}\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y$. Hence together with the asymptotic method $Q_s^A[f]$, it presents a quadrature with an error bound $O(\omega^{-s-1/(r+1)})$.

For example, for $I[f] = \int_{-1}^{1} \mathrm{e}^x\,\mathrm{e}^{\mathrm{i}\omega\cos(x)}\,\mathrm{d}x$, the generalized moment can be transferred by $y = 1-\cos(x)$ into

$$\begin{aligned}\mu_0(\omega,0) &= \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega\cos(x)}\,\mathrm{d}x = 2\,\mathrm{e}^{\mathrm{i}\omega}\int_0^1 \mathrm{e}^{-\mathrm{i}\omega(1-\cos(x))}\,\mathrm{d}x\\ &= 2\,\mathrm{e}^{\mathrm{i}\omega}\int_0^{1-\cos(1)} \frac{\mathrm{e}^{-\mathrm{i}\omega y}}{\sqrt{1-(1-y)^2}}\,\mathrm{d}y = 2\,\mathrm{e}^{\mathrm{i}\omega}\int_0^{1-\cos(1)} \frac{1}{\sqrt{2-y}}\frac{\mathrm{e}^{-\mathrm{i}\omega y}}{y^{\frac{1}{2}}}\,\mathrm{d}y.\end{aligned}$$

$h(y) = \frac{1}{\sqrt{2-y}}$ is smooth in $[0, 1-\cos(1)]$, which can be approximated by a polynomial $p(y)$ satisfying (3.9). Thus $\mu_0(\omega,0)$ can be approximated by Filon-type method $Q_s^F[\frac{1}{\sqrt{2-y}}] = 2\,\mathrm{e}^{\mathrm{i}\omega}\cdot\int_0^{1-\cos(1)} p(y) y^{-\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}\omega y}\,\mathrm{d}y$ and $I[f]$ can be approximated by the asymptotic method $Q_s^A[\cos(x)]$ defined by (3.4) with an error bound $O(\omega^{-s-1/(r+1)})$ (see Fig. 4).

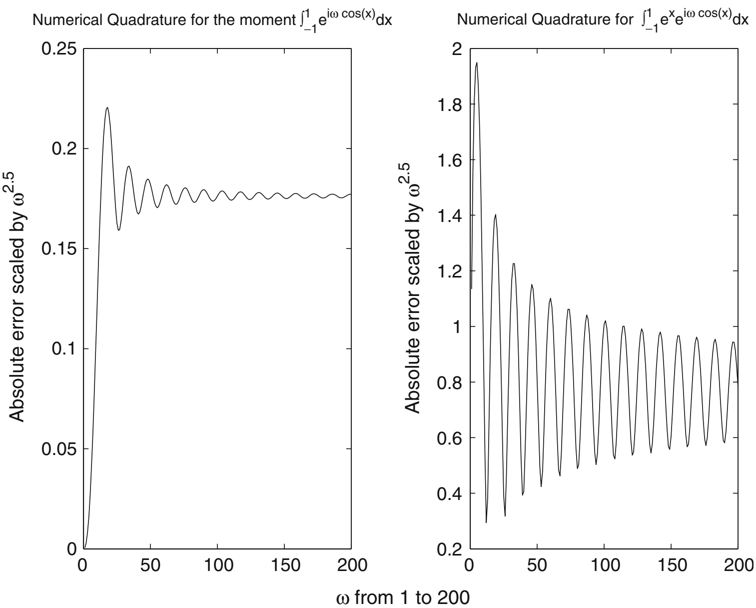

Fig. 4 The absolute error scaled by $\omega^{2.5}$ for the generalized moment $\int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega\cos(x)}\,\mathrm{d}x$ approximated by $Q_2^F\left[\frac{1}{\sqrt{2-y}}\right]$ at nodes 0 and $1-\cos(1)$, and $\int_{-1}^{1} \mathrm{e}^x\,\mathrm{e}^{\mathrm{i}\omega\cos(x)}\,\mathrm{d}x$ approximated by $Q_2^A[\mathrm{e}^x]$

Example 3.2 Let us consider an example with the oscillator $g(x) = \alpha x + \beta\sqrt{1-x^2}$, $a = -1$ and $b = 1$, where $\alpha^2 + \beta^2 = 1$, which occurs in papers of Huybrechs and Vandewalle [6] and of Iserles and Nørsett [11] on multivariate quadrature. The moments of such g are unknown but it is easy to derive the inverse function. Without loss of generality, we assume $\alpha > 0$ and $\beta > 0$, and obtain

$$\mu_0(\omega,\alpha) = \int_{-1}^{1} e^{i\omega\left(\alpha x+\beta\sqrt{1-x^2}\right)}\,dx = e^{i\omega}\int_{-1}^{1} e^{-i\omega\left(1-\alpha x-\beta\sqrt{1-x^2}\right)}\,dx$$

$$:= e^{i\omega}\int_{-1}^{1} e^{-i\omega g_1(x)}\,dx,$$

where $g_1(x) = 1 - \alpha x - \beta\sqrt{1-x^2}$, $g_1(\alpha) = 0$, $y' = g_1'(\alpha) = 0$, $g_1''(\alpha) > 0$ and $g_1'(x) \neq 0$ for $x \neq \alpha$. The generalized moment $\mu_0(\omega,\alpha)$ can be rewritten as

$$\mu_0(\omega,\alpha) = e^{i\omega}\int_{-1}^{\alpha} e^{-i\omega g_1(x)}\,dx + e^{i\omega}\int_{\alpha}^{1} e^{-i\omega g_1(x)}\,dx := I_1 + I_2.$$

Set $x_1(y) = g_1^{-1}(y) = \alpha(1-y) - \beta\sqrt{1-(1-y)^2}$ for $y \in [0, 1+\alpha]$, then I_1 can be represented by

$$I_1 = e^{i\omega}\int_{1+\alpha}^{0} x_1'(y)\,e^{-i\omega y}\,dy = \alpha\,e^{i\omega}\int_{0}^{1+\alpha} e^{-i\omega y}\,dy + \beta\,e^{i\omega}\int_{0}^{1+\alpha} \frac{1-y}{\sqrt{2-y}}\frac{e^{-i\omega y}}{\sqrt{y}}\,dy.$$

Set $x_2(y) = g_1^{-1}(y) = \alpha(1-y) + \beta\sqrt{1-(1-y)^2}$ for $y \in [0, 1-\alpha]$, then I_2 can be represented by

$$I_2 = e^{i\omega}\int_{0}^{1-\alpha} x_2'(y)\,e^{-i\omega y}\,dy = -\alpha\,e^{i\omega}\int_{0}^{1-\alpha} e^{-i\omega y}\,dy + \beta\,e^{i\omega}\int_{0}^{1-\alpha} \frac{1-y}{\sqrt{2-y}}\frac{e^{-i\omega y}}{\sqrt{y}}\,dy.$$

Here $h(y) = \frac{1-y}{\sqrt{2-y}}$ is smooth in $[0, 1 \pm \alpha]$ and can be approximated by polynomial $p_k(y)$ $(k = 1, 2)$ in each subinterval respectively, which satisfies (3.9) in each subinterval. Note that the first term in I_1 and I_2 respectively can be computed explicitly. Then the generalized moment $\mu_0(\omega, 0)$ can be approximated by Filon-type method $Q_s^F\left[\frac{1-y}{\sqrt{2-y}}\right]$, where $Q_s^F\left[\frac{1-y}{\sqrt{2-y}}\right] = Q_{s,1}^F\left[\frac{1-y}{\sqrt{2-y}}\right] + Q_{s,2}^F\left[\frac{1-y}{\sqrt{2-y}}\right]$ and

$$Q_{s,1}^F\left[\frac{1-y}{\sqrt{2-y}}\right] = \alpha\,\mathrm{e}^{\mathrm{i}\omega}\int_0^{1+\alpha}\mathrm{e}^{-\mathrm{i}\omega y}\,\mathrm{d}y + \beta\,\mathrm{e}^{\mathrm{i}\omega}\int_0^{1+\alpha} p_1(y)\frac{\mathrm{e}^{-\mathrm{i}\omega y}}{\sqrt{y}}\,\mathrm{d}y,$$

$$Q_{s,2}^F\left[\frac{1-y}{\sqrt{2-y}}\right] = -\alpha\,\mathrm{e}^{\mathrm{i}\omega}\int_0^{1-\alpha}\mathrm{e}^{-\mathrm{i}\omega y}\,\mathrm{d}y + \beta\,\mathrm{e}^{\mathrm{i}\omega}\int_0^{1-\alpha} p_2(y)\frac{\mathrm{e}^{-\mathrm{i}\omega y}}{\sqrt{y}}\,\mathrm{d}y$$

(see Fig. 5).

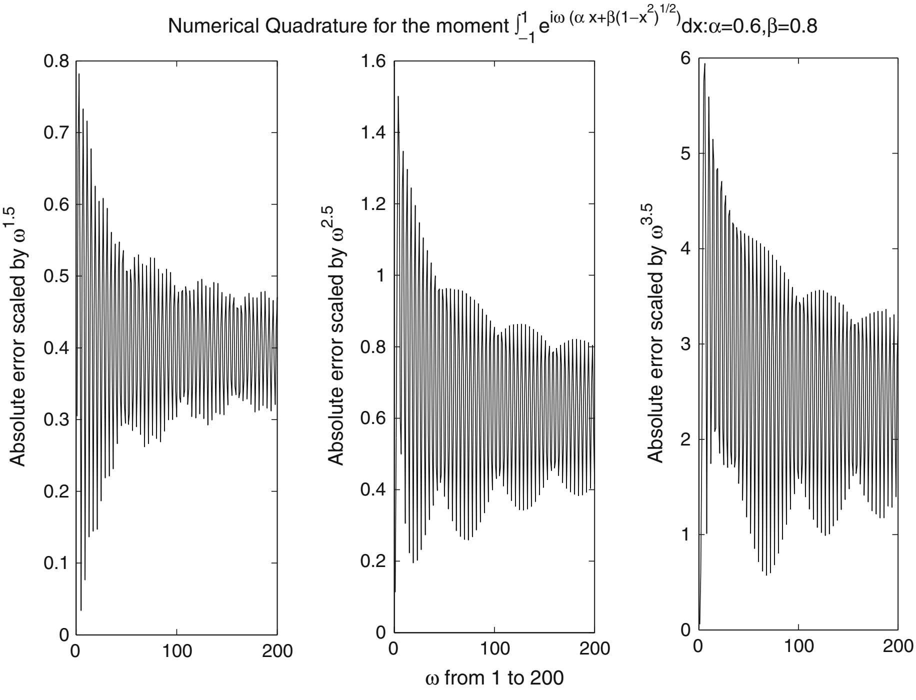

Fig. 5 Scaled error for the moment $\mu_0(\omega, \alpha) = \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega(\alpha x + \beta\sqrt{1-x^2})}\,\mathrm{d}x$ approximated by $Q_1^F\left[\frac{1-y}{\sqrt{2-y}}\right]$ (on the *left*) $Q_2^F\left[\frac{1-y}{\sqrt{2-y}}\right]$ and $Q_3^F\left[\frac{1-y}{\sqrt{2-y}}\right]$ (on the *right*), respectively

For the case that $g'(\xi) = 0$ and the expression $x(y) = g^{-1}(y - g(\xi))$ is not available piecewise, similar method can also be used to compute $I[f]$. In this case, $I[f] = \int_a^b f(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x$ can be rewritten as

$$I[f] = \left(\int_a^{\xi-h} + \int_{\xi-h}^{\xi} + \int_{\xi}^{\xi+h} + \int_{\xi+h}^{b} \right) f(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x := I_1 + I_2 + I_3 + I_4$$

for some fixed small h. I_1 and I_4 can be efficiently computed by the moment-free method (1.4). And I_2 and I_3 can be transferred by $y = g(x) - g(\xi)$ into

$$I_2 = \mathrm{e}^{\mathrm{i}\omega g(\xi)} \int_{g(\xi-h)-g(\xi)}^{0} \frac{f(x)(g(x)-g(\xi))^{\frac{r}{r+1}}}{g'(x)} y^{-\frac{r}{r+1}}\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y \tag{3.11}$$

$$I_3 = \mathrm{e}^{\mathrm{i}\omega g(\xi)} \int_{0}^{g(\xi+h)-g(\xi)} \frac{f(x)(g(x)-g(\xi))^{\frac{r}{r+1}}}{g'(x)} y^{-\frac{r}{r+1}}\,\mathrm{e}^{\mathrm{i}\omega y}\,\mathrm{d}y. \tag{3.12}$$

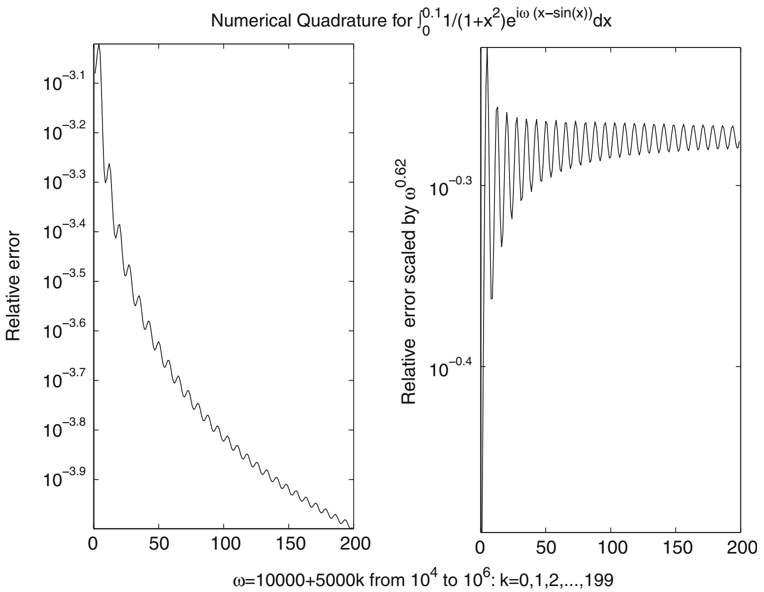

Fig. 6 Relative error analysis for $I[f] = \int_0^{0.1} \frac{1}{1+x^2}\,\mathrm{e}^{\mathrm{i}\omega(x-\sin(x))}\,\mathrm{d}x$ approximated by $Q_1^F\left[\frac{(x-\sin(x))^{\frac{2}{3}}}{(1+x^2)(1-\cos(x))}\right]$ with the linear interpolation at $y = 0$ and $y = 0.1 - \sin(0.1)$

Since from (3.7),

$$\lim_{x\to\xi} \frac{f(x)(g(x)-g(\xi))^{\frac{r}{r+1}}}{g'(x)} = \frac{f(\xi)}{(r+1)\sqrt[r+1]{\frac{g^{(r+1)}(\xi)}{(r+1)!}}}.$$

We can select linear interpolations at the endpoints to approximate $\frac{f(x)(g(x)-g(\xi))^{\frac{r}{r+1}}}{g'(x)}$ in each subinterval, respectively. Then I_2 and I_3 can be approximated by the Filon-type method. Here, x is considered as a function of t.

Example 3.3 Let $I[f] = \int_0^{0.1} \frac{1}{1+x^2}\, e^{i\omega(x-\sin(x))}\, dx$. Define $y = x - \sin(x)$, then $I[f]$ can be rewritten as

$$I[f] = \int_0^{0.1-\sin(0.1)} \frac{(x-\sin(x))^{\frac{2}{3}}}{(1+x^2)(1-\cos(x))} y^{-\frac{2}{3}}\, e^{i\omega y}\, dy = \int_0^{0.1-\sin(0.1)} h(x) y^{-\frac{2}{3}}\, e^{i\omega y}\, dy.$$

Let $p(y) = a_0 + a_1 y$ such that $h(0) = p(0)$ and $h(0.1) = p(0.1 - \sin(0.1))$ and consider the Filon-type quadrature $\int_0^{0.1-\sin(0.1)} p(y) y^{-\frac{2}{3}}\, e^{i\omega y}\, dy$ to approximate $I[f]$ (see Fig. 6).

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References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. National Bureau of Standards, Washington (1964)
2. Davis, P.I., Rabinowitz, P.: Methods of Numerical Integral Integration, 2nd edn. Academic, New York (1984)
3. Evans, G.A.: Practical Numerical Integration. Wiley, New York (1993)
4. Evans, G.A.: An alternative methods for irregular oscillatory integrals. Int. J. Comput. Math. **53**, 185–193 (1994)
5. Filon, L.N.G.: On a quadrature formula for trigonometric integrals. Proc. R. Soc. Edinb. **49**, 38–47 (1928)
6. Huybrechs, D., Vandewalle, S.: The construction of cubature rules for multivariate highly oscillatory integrals. Report TW442 Katholieke University Leuven (2005)
7. Iserles, A.: On the numerical quadrature of highly-oscillating integrals I: Fourier transforms. IMA J. Numer. Anal. **24**, 365–391 (2004)
8. Iserles, A.: On the numerical quadrature of highly-oscillating integrals II: Irregular oscillators. IMA J. Numer. Anal. **25**, 25–44 (2005)
9. Iserles, A., Nørsett, S.P.: On quadrature methods for highly oscillatory integrals and their implementation. BIT **44**, 755–772 (2004)
10. Iserles, A., Nørsett, S.P.: Efficient quadrature of highly-oscillatory integrals using derivatives. Proc. R. Soc. A **461**, 1383–1399 (2005)

11. Iserles, A., Nørsett, S.P.: Quadrature methods for multivariate highly oscillatory integrals using derivatives. Math. Comput. **75**, 1233–1259 (2006)
12. Iserles, A., Nørsett, S.P., Olver, S.: Highly oscillatory quadrature: the story so far. In: Proceedings of ENumath, Santiago de Compostela (2006). Springer, Berlin, pp. 97–118 (2006)
13. Levin, D.: Procedures for computing one-and-two dimensional integrals of functions with rapid irregular oscillations. Math. Comput. **38**, 531–538 (1982)
14. Levin, D.: Fast integration of rapidly oscillatory functions. J. Comput. Appl. Math. **67**, 95–101 (1996)
15. Longman, I.M.: A method for numerical evaluation of finite integrals of oscillatory functions. Math. Comput. **14**, 53–59 (1960)
16. Luke, Y.K.: On the computation of oscillatory integrals. Proc. Camb. Philos. Soc. **50**, 269–277 (1954)
17. Olver, S.: Moment-free numerical integration of highly oscillatory functions. IMA. J. Numer. Anal. **26**, 213–227 (2006)
18. Olver, S.: On the quadrature of multivariate highly oscillatory integrals over nonpolytope domains. Reports no. NA2005/07, DAMTP, University of Cambridge, Numer. Math. (to appear)
19. Piessens, R.: Gaussian quadrature formulas for the integration of oscillatory functions. ZAMM **50**, 698–700 (1970)
20. Piessens, R., Poleunis, F.: A numerical method for the integration of oscillatory functions. BIT **11**, 317–327 (1971)
21. Stein, E.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton (1993)
22. Stewart, J.: Calculus. Brooks/Cole Publishing Company, Belmont (1993)
23. van der Vooren, A.I., Linde, H.J.: Numerical calculation of integrals with strongly oscillating integrand. Math. Comput. **20**, 232–245 (1966)
24. Whittaker, E.T., Waston, G.N.: A Course of Modern Analysis. Cambridge University Press, Cambridge (1993)
25. Xiang, S.: On the Filon and Levin methods for highly oscillatory integral $\int_a^b f(x)\,e^{i\omega g(x)}\,dx$. J. Comput. Appl. Math. (to appear)