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A posteriori error estimators, gradient recovery by averaging, and superconvergence

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Abstract For the linear finite element solution to a linear elliptic model problem, we derive an error estimator based upon appropriate gradient recovery by local averaging. In contrast to popular variants like the ZZ estimator, our estimator contains some additional terms that ensure reliability also on coarse meshes. Moreover, the enhanced estimator is proved to be (locally) efficient and asymptotically exact whenever the recovered gradient is superconvergent. We formulate an adaptive algorithm that is directed by this estimator and illustrate its aforementioned properties, as well as their importance, in numerical tests.

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1 Motivation and introduction

In order to motivate what follows, we start with a remark concerning the standard residual a posteriori error estimator for anisotropic elliptic problems. Let u_S be the finite element approximation of u satisfying appropriate homogeneous boundary conditions and

$$-\operatorname{div}(A_\epsilon \nabla u) = f \text{ in } \Omega \subset \mathbb{R}^2 \tag{1.1}$$

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where $A_\epsilon, \epsilon \in (0, 1]$, is a diagonal matrix with diagonal $(\epsilon, 1/\epsilon)$. We are interested in the so-called energy norm error $\|\nabla(u_S - u)\|_{A_\epsilon} := \|A_\epsilon^{1/2}\nabla(u_S - u)\|_{L_2(\Omega)}$. The standard approach for the explicit residual estimator

$$\eta := \Big[\sum_{T\in\mathcal{T}} \eta_T^2\Big]^{1/2} \quad \text{with} \quad \eta_T^2 = h_T \int_{\partial T} |\,[\![A_\epsilon \nabla u_S \cdot \nu]\!]\,|^2 + h_T^2 \int_T |f|^2 \quad (1.2)$$

(cf. e.g. [2,3,26] and define the jump appropriately) yields global upper and local lower bounds such that in particular (suppose f to be piecewise polynomial)

$$\|\nabla(u_S - u)\|_{A_\epsilon} \le \frac{C_1}{\sqrt{\epsilon}}\,\eta \le \frac{C_2}{\epsilon}\,\|\nabla(u_S - u)\|_{A_\epsilon}. \quad (1.3)$$

Both bounds are essentially sharp; see §6.4. The ratio between left and right hand side, which essentially equals the condition number $1/\epsilon$ of $A_\epsilon^{1/2}$, increases with decreasing ϵ. This may be expressed by saying that the estimator η is not robust with respect to $\epsilon \in (0, 1)$. Consequently, stopping the adaptive algorithm when $C_1\eta/\sqrt{\epsilon} \le$ tol (with suitably chosen C_1) is reliable but maybe quite inefficient for small ϵ. The first and second row of Table 1 illustrate the resulting relationship between CPU time/number of degree of freedoms (DOFs) and tolerance for the corresponding worst case of $\epsilon = 0.1$. The symbol '$*$' indicates that the corresponding values could not be obtained because of "out of memory"; for more details see §6.4. The third row is obtained by stopping (and adapting) with the help of the estimator $\mathcal{E}$ that is proposed and analyzed in this article. Notice that there is a relevant improvement for smaller tolerances. This improvement gets particularly important when the approximate solution of anisotropic equations like (1.1) constitutes an inner iteration of an infinite-dimensional solver as e.g. in Deuflhard/Weiser [12] or generalizations of Bänsch/Morin/Nochetto [5].

We derive the estimator $\mathcal{E}$ for the model problem and the finite element discretization that are described in §2. Although the model problem is more general, we shall stick with (1.1) for the rest of this introduction.

Inspired by the superconvergence results in Xu/Zhang [28] and the observed asymptotic exactness in Carstensen/Bartels [9] for (1.1) with $\epsilon = 1$, we choose as main building block ζ of $\mathcal{E}$ the estimator in Zienkiewicz/Zhu [29] or, more generally, the energy norm of $Gu_S - \nabla u_S$ where Gu_S is the continuous outcome of a gradient recovery procedure operating on the typically discontinuous ∇u_S. The required properties of this smoothing procedure are discussed in §3.1. In particular, we introduce the notion of local nondeteriorating smoothing procedures and characterize them with the help of local consistency and stability conditions. The latter ones are related to those in Ainsworth/Craig [1] or [2, Chapter 4].

Table 1 CPU times (left columns, in seconds) and #DOFs (right columns) that are necessary to guarantee that the energy norm error is below given tolerances using the standard residual estimator $\eta/\sqrt{\epsilon}$ and the estimator $\mathcal{E}$ of this article

tol	0.5		0.05		0.005		0.001	
$\sqrt{10}\,\eta$	0.19	447	11.3	23 691	1 973	1 658 081	$*$	
$\mathcal{E}$	0.05	145	0.8	1 741	79	97 267	1 663	1 524 851

Since the main building block ζ stems from a post-processing procedure, it cannot be reliable (see, e.g., §6.1) and thus should not be used in a 'black box algorithm'. To remedy without loosing the nice properties of ζ, we propose in §3.2 to derive an a posteriori estimator for (the relevant part of) the error of the recovered gradient Gu_S. To this end, we use a corresponding residual estimator ρ and an additional term γ that accounts for the fact that Gu_S in general does not satisfy the Galerkin orthogonality and that is derived by adapting techniques from Chen/Wu [11].

We add these terms to the estimator and prove global upper and local lower bounds in §4 that imply in particular the following counterpart of (1.3):

$$\begin{aligned}\|\nabla(u_S - u)\|_{A_\epsilon} \le \mathcal{E} = \zeta + \frac{C_1}{\sqrt{\epsilon}}(\rho + \gamma) &\le \|\nabla(u_S - u)\|_{A_\epsilon} + \frac{\tilde{C}_2}{\epsilon}\,\|Gu_S - \nabla u\|_{A_\epsilon} \\ &\le \frac{C_2}{\epsilon^2}\,\|\nabla(u_S - u)\|_{A_\epsilon}\,.\end{aligned}$$

The first two inequalities illustrate the main difference between the a posteriori analysis in §4 and the preceding ones in Rodriguez [20], Carstensen/Bartels [9], and Carstensen [8]: the part of the error missed by ζ is treated in such a way that the estimator consists of a 'constant-free' principal part and a not robust but (hopefully) superconvergent 'security part' that also involves some interpolation constant. The last inequality and its local counterparts, which ensures efficiency regardless of the concrete quality of Gu_S, hinge on the fact that the smoothing procedure is locally nondeteriorating.

In §5 we formulate an adaptive algorithm that takes advantage of the structure of $\mathcal{E}$ in stopping and marking. We apply this algorithm to four test problems in §6 and obtain, in particular, Table 1.

The disadvantages in (1.3) may be also remedied by a posteriori error estimates on suitable anisotropic elements; see Bernardi/Verfürth [6, §3]. For more general problems, such an approach requires mesh generation/refining techniques which are more sophisticated than the bisection algorithm of §5. Nevertheless, it may be worthwhile combining such techniques with the approach given here.

Constant-free upper bounds have been constructed with the help of infinite-dimensional local problems. In order to obtain compatible local boundary data, one can use equilibrated fluxes, see [2,3] and the references therein, or employ the built-in cancellation of the Galerkin solution with the help of suitable weights; see Carstensen/Funken [10] and Morin/Nochetto/Siebert [19]. Upper bounds with constant-free 'leading' term have been obtained by Strouboulis/Babuška/Gangaraj [24] basing upon finite-dimensional local problems with equlibrated fluxes and by Luce/Wohlmuth [17] with the help of a nonconforming local gradient recovery. The complementing (local) lower bounds in these approaches involve interpolation constants or require additional computations. If the anisotropic equation (1.1) is covered, the estimators (or their leading term) are not proved to be robust with respect to $\epsilon \in (0, 1)$.

2 Model problem and discretization

We start by introducing the linear elliptic model problem and its discretization with the help of finite elements.

Throughout this article, we denote the norm of a space X by $\|\cdot\|_X$. For weighted L_2-norms like $\|\cdot\|_X = (\int_D w\,|\cdot|^2)^{1/2}$, we write also $\|\cdot\|_{w;D}$ where w stands for the weight and D for the domain of integration. The weight $w \equiv 1$ and the domain $D = \Omega$ will be often omitted; in particular, $\|\cdot\|$ is an abbreviation for $\|\cdot\|_{L_2(\Omega)}$.

2.1 Model problem and assumptions on the data

Let Ω be a bounded polygonal Lipschitz (in the sense of [7, (1.4.4)]) domain of the plane $\mathbb{R}^2$. The boundary $\partial\Omega$ is divided into two disjoint parts: Γ_D for Dirichlet boundary conditions and Γ_N for Neumann boundary conditions. The outward normal vector of $\partial\Omega$ is denoted by n. For convenience, we exclude the pure Neumann problem by supposing that the Dirichlet part Γ_D is closed and has strictly positive length: $\mathcal{H}^1(\Gamma_D) > 0$. The corresponding boundary conditions are given with the help of the functions v and g satisfying

$$v \in H^1(\Gamma_D) \quad \text{and} \quad g \in L_2(\Gamma_N). \tag{2.1}$$

The load term fulfills

$$f \in L_2(\Omega). \tag{2.2}$$

Finally, let the variable coefficients of the linear elliptic operator be given by a matrix-valued mapping $A : \overline{\Omega} \to \mathbb{R}^{2\times 2}$. We suppose that

$$\forall x \in \overline{\Omega} \quad A(x) \text{ is symmetric} \tag{2.3a}$$

and that there are given two continuous functions $\lambda, \Lambda : \overline{\Omega} \to \mathbb{R}$ such that

$$\forall x \in \overline{\Omega},\ \xi \in \mathbb{R}^2 \setminus \{0\} \quad 0 < \lambda(x)|\xi|^2 \le A(x)\xi\cdot\xi \le \Lambda(x)|\xi|^2. \tag{2.3b}$$

We define $\lambda_\Omega := \inf_\Omega \lambda > 0$ and $\Lambda_\Omega := \sup_\Omega \Lambda < \infty$. Moreover, we assume that

$$A \text{ is continuous and piecewise affine.} \tag{2.3c}$$

Let u be the typically unknown weak solution of

$$-\operatorname{div}\big(A\nabla u\big) = f \text{ in } \Omega, \quad u = v \text{ on } \Gamma_D, \quad A\nabla u\cdot n = g \text{ on } \Gamma_N.$$

In other words:

$$u \in X^v \quad \text{and} \quad \forall \varphi \in X^0 \quad \int_\Omega A\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi + \int_{\Gamma_N} g\varphi, \tag{2.4}$$

where $X^v := \{w \in H^1(\Omega) \mid w = v\ \mathcal{H}^1\text{-a.e. on } \Gamma_D\}$. In view of assumptions (2.3a) and (2.3b), problem (2.4) is a linear, symmetric, uniformily elliptic boundary value problem. Thus, the other assumptions on Γ_D, v, g, and f, ensure existence and uniqueness of u; see e.g. [16, Theorem 7.3.5].

Both conditions in (2.3c) affect the results on the efficiency of the estimator in §3.2. The first condition, the continuity of A, will be crucial for the lower bound in Proposition 4.2, while the second one avoids only the appearance of additional higher order terms in the lower bounds of Lemma 4.1. The first part of (2.1) allows in particular to use Lagrange interpolation for the approximation of the Dirichlet boundary values; for an approach with $v \in H^{1/2}(\Gamma_D)$ only, we refer to Sacchi/Veeser [21]. The second part of (2.1) and (2.2) exclude the more involved cases of pure functionals.

2.2 Discretization, approximate solution, and error notion

We shall use linear finite elements in order to approximate the function u in (2.4). Suppose that $\mathcal{T}_0$ is a conforming (admissibile) triangulation of Ω subordinated to the subdivision of $\partial\Omega$ and to the mapping A in the following sense: for every triangle $T \in \mathcal{T}_0$ and any edge E of T, there holds

$$A_{|T} \text{ is affine,} \tag{2.5a}$$

$$\mathcal{H}^1(E \cap \Gamma_D) > 0 \implies \mathcal{H}^1(E \cap \Gamma_D) = \mathcal{H}^1(E \cap \partial\Omega). \tag{2.5b}$$

In what follows, we shall refer to $\mathcal{T}_0$ as the macro triangulation. The following two quantities of $\mathcal{T}_0$ will be important:

$$\alpha_{\min} := \text{ smallest angle occuring in } \mathcal{T}_0 \quad \text{and} \quad \mu := \frac{\max_{T \in \mathcal{T}_0} h_T}{\min_{T \in \mathcal{T}_0} h_T}, \tag{2.6}$$

where $h_T := \operatorname{diam} T$ denotes the diameter of a triangle T.

Let $\mathcal{T}$ be any (not necessarily quasi-uniform) refinement of $\mathcal{T}_0$ that was obtained with the help of the newest-vertex bisection; see e.g. [22]. Hereafter, we suppose that, together with $\mathcal{T}_0$ itself, we are given an appropriate fixed set of refinement edges. The triangles and edges of $\mathcal{T}$ also satisfy (2.5) and their minimum angle is bounded away from 0 in terms of $\alpha_{\min}$. The set of the nodes (or vertices) of $\mathcal{T}$ is denoted by $\mathcal{N}$. Let S be the space of continuous piecewise affine finite elements over $\mathcal{T}$:

$$S := \left\{ w \in C(\bar{\Omega}) \mid \forall T \in \mathcal{T} \; w_{|T} \in P_1(T) \right\},$$

where $P_k(T)$, $k \in \mathbb{N}$, stands for the polynomials over T with degree $\leq k$.

In view of (2.5), the triangulation $\mathcal{T}$ induces a subdivision of Γ_D with nodes $\mathcal{N}_D := \mathcal{N} \cap \Gamma_D$. Let I denote the corresponding piecewise affine Lagrange interpolation operator associated with these nodes. Thanks to (2.1), we can approximate the Dirichlet boundary values by Iv in a computationally convenient way.

The finite element approximation $\bar{u}_S$ of u in (2.4) is then characterized by

$$\bar{u}_S \in S^v \quad \text{and} \quad \forall \chi \in S^0 \quad \int_\Omega A\nabla\bar{u}_S \cdot \nabla\chi = \int_\Omega f\chi + \int_{\Gamma_N} g\chi, \tag{2.7}$$

where $S^v := \{w \in S \mid \forall z \in \mathcal{N}_D \; w(z) = v(z)\}$. Similarly to u, the finite element approximation $\bar{u}_S$ exists and is unique.

In practice, one often does not solve the linear system resulting from (2.7) exactly. We therefore suppose that $u_S \in S$ is an approximation of $\bar{u}_S$ and will provide an a posteriori error analysis for the *approximate* finite element solution u_S. Notice that, if v is not piecewise affine and thus (2.7) a nonconforming method, $u_S - u \in X^0$ does not hold.

The properties (2.3a) and (2.3b) of A imply that the so-called *energy norm*

$$\|\nabla w\|_A := \Big(\int_\Omega A\nabla w \cdot \nabla w \Big)^{1/2} \quad \text{for } w \in H^1(\Omega) \tag{2.8}$$

is a norm on X^0, but only a seminorm on $H^1(\Omega)$. However, [21, (2.7)] and the natural assumption $u_S = Iv$ on Γ_D imply that

$$\|\nabla(u_S - u)\|_A \geq \sqrt{\lambda_\Omega}\Big(C_1 \, \|u_S - u\|_{H^1(\Omega)} - C_2 \, \|Iv - v\|_{\Gamma_D} \Big),$$

where C_1 and C_2 depend on the minimum angle $\alpha_{\min}$ of $\mathcal{T}_0$ and Ω. We will see in Remark 4.1 that $\|Iv - v\|_{\Gamma_D}$ is formally of higher order and so, if $u_S = Iv$ on Γ_D holds, $\|\nabla(u_S - u)\|_A$ is also in the nonconforming case an error notion that is essentially 'equivalent' to $\|u_S - u\|_{H^1(\Omega)}$.

3 Gradient smoothing and error estimator

The purpose of this section is to introduce the computable quantities that will be used to bound the energy norm error $\|\nabla(u_S - u)\|_A$. The definition of these quantities involves a procedure that removes the discontinuities in the gradient of the approximate solution u_S and will be discussed beforehand.

3.1 Assumptions on gradient smoothing

Let us start with a motivation for the use of gradient recovery by smoothing in a posteriori error estimation. Since the approximate solution u_S from §2.2 is an element of the space S, its gradient ∇u_S is constant on each triangle of $\mathcal{T}$ and thus, typically, has jumps across interelement edges. Due to (2.3c) and (2.2), the exact gradient ∇u does not have such jumps. Therefore, one may hope that a suitable continuous (or smoothed) version Gu_S of ∇u_S approximates ∇u better than the discontinuous ∇u_S. This can be actually achieved in many cases, even for singular solutions; see e.g. §6.3.

Having this in mind, one may build up an error estimator with the help of the computable distance $\|\nabla u_S - Gu_S\|_A$ between untreated and smoothed gradient. The properties of the smoothing procedure for obtaining Gu_S will then affect the properties of the corresponding estimator.

We now introduce the assumptions on the smoothing procedure that will be used in the a posteriori analysis of §4. In view of the preceding motivation, the smoothing procedure has to be of the form

$$\sigma : \nabla S \to S \times S, \ \ \nabla w \mapsto \sigma[\nabla w], \tag{3.1a}$$

where $\nabla S := \{\nabla w \mid w \in S\}$ is a subspace of the discontinuous piecewise constant mappings into $\mathbb{R}^2$ over the triangulation $\mathcal{T}$. Of course, one wants σ to be simple and therefore it seems to be reasonable to suppose that

$$\sigma \text{ is linear.} \tag{3.1b}$$

Remarkably, property (3.1a) is the only assumption on the smoothing procedure σ that is needed in deriving an a posteriori upper bound, i.e. in ensuring the (asymptotic) reliability of the corresponding error estimator. This fact was already observed by Carstensen/Bartels [9].

However, the estimator may overestimate the error, that is, may not be efficient. In order to exclude this, one needs to show a complementing a posteriori global lower bound. It is desirable to do this by passing *local* lower bounds, because the latter ones play a crucial role for the convergence of adaptive finite elements in the works Dörfler [13], Morin/Nochetto/Siebert [18,19], and Veeser [25].

In view of the triangle inequality, the distance between untreated and smoothed gradient bounds the energy norm error up to a constant (locally) from below iff the same holds for the error of the smoothed gradient. If the latter is true, we shall say that the smoothing procedure is *(locally) nondeteriorating*. The meaning of "local" may be specified in various ways; here, we shall use triangles along with their "smallest balls" $\mathcal{B}(T) := \cup\{T' \in \mathcal{T} : T' \cap T \neq \emptyset\}$, $T \in \mathcal{T}$. In the following theorem we determine the corresponding class of locally nondeteriorating smoothing procedures for Laplace's equation.

Theorem 3.1 (**Locally nondeteriorating smoothers**) *Let σ be a smoothing procedure satisfying* (3.1). *Then the following two statements are equivalent:*

(i) If $f = 0$ and $A = \mathrm{Id}$ is the identity matrix, then σ is locally nondeteriorating in the following sense: for any triangle $T \in \mathcal{T}$,

$$\|\sigma[\nabla u_S] - \nabla u\|_T \leq C\, \|\nabla(u_S - u)\|_{\mathcal{B}(T)}\,,$$

with C depending only on the minimum angle α_{min} of the macro triangulation $\mathcal{T}_0$.

(ii) σ is locally consistent and stable in the following sense: for any $w \in S$ and any triangle $T \in \mathcal{T}$,

$$\nabla w \textit{ is constant in } \mathcal{B}(T) \implies \sigma[\nabla w] = \nabla w \textit{ on } T, \tag{3.2a}$$

$$\|\sigma[\nabla w]\|_T \leq C\, \|\nabla w\|_{\mathcal{B}(T)} \tag{3.2b}$$

with C depending only on α_{min}.

Proof Here we prove only the implication (i) $\Longrightarrow$ (ii); the opposite one follows from Proposition 4.2 below. Let us first prove (3.2b). Given $w \in S$, take homogeneous boundary data in the continuous problem (2.4) such that $u = 0$ and choose $u_S = w$ as approximate solution. Then (i) implies (3.2b). In view of the linearity of σ, this entails that

$$\sigma[\nabla w]_{|T} \text{ depends only on } \nabla w_{|\mathcal{B}(T)}. \tag{3.3}$$

In fact, if $\tilde{w} \in S$ is such that $\nabla\tilde{w} = \nabla w$ in $\mathcal{B}(T)$ but $\sigma[\nabla\tilde{w}] \neq \sigma[\nabla w]$, then

$$\begin{aligned} |\alpha|\|\sigma[\nabla\tilde{w}] - \sigma[\nabla w]\|_T - \|\sigma[\nabla w]\|_T &\leq \|\sigma[\nabla(w + \alpha(\tilde{w} - w))]\|_T \\ &\leq C\|\nabla w + \alpha\nabla(\tilde{w} - w)\|_{\mathcal{B}(T)} \\ &= C\|\nabla w\|_{\mathcal{B}(T)} \end{aligned} \tag{3.4}$$

produces a contradiction for $\alpha \nearrow \infty$.

To prove the consistency (3.2a), suppose that $\nabla w = c$ on $\mathcal{B}(T)$. Arrange the data of (2.4) such that $u(x) = c \cdot x$ for all $x \in \Omega$ and choose $u_S = u$. Consequently, (3.3) and (i) lead to $\sigma[\nabla w] = \sigma[\nabla u_S] = \nabla u = c = \nabla w$ on T. □

The assumptions (3.1) and (3.2) on the gradient smoothing are closely related to the general framework of gradient recovery in Ainsworth/Craig [1] or [2, Chapter 4]. The main difference is the weaker invariance (or consistency) condition (3.2a): [1,2] require invariance on T also if ∇w is affine on $\mathcal{B}(T)$. This stronger version was used to derive superconvergence of the smoothed gradient under certain circumstances.

The local nature of the additional assumptions (3.2) imply that the smoothed gradient can be computed locally. Consequently, global procedures as in Bank/Xu [4] do not directly enter in the given framework.

If A is not the identity, statement (ii) still ensures that the smoother is locally nondeteriorating, but with a constant that involves the condition number of A. This dependence gets less sensitive, if we suppose in addition:

$$\sigma \text{ can be extended to } D \times D,\ \ D := \{w : \Omega \to \mathbb{R} \mid \forall\, T \in \mathcal{T}\ w_{|T} \in P_0(T)\}, \tag{3.5a}$$

such that, for each triangle $T \in \mathcal{T}$,

$$A \text{ is constant in } \mathcal{B}(T) \implies A\sigma[\nabla w] = \sigma[A\nabla w] \text{ on } T. \tag{3.5b}$$

This relies on the following observation: given (3.5b) and an A that is constant in $\mathcal{B}(T)$, (3.2b) implies its energy norm counterpart

$$\|\sigma[\nabla w]\|_{A;T} \le C\, \|\nabla w\|_{A;\mathcal{B}(T)} \tag{3.6}$$

without intervention of the condition number of A.

We conclude this subsection by discussing the smoothing procedure that is associated to the ZZ estimator and that will be used for our numerical tests in §6. Let $(\phi_z)_{z\in\mathcal{N}}$ be the canonical nodal basis of S and denote the star around a node $z \in \mathcal{N}$ by $\omega_z = \operatorname{supp}\phi_z = \cup\{T \in \mathcal{T} : T \ni z\}$. Given $w \in S$, we then define $\sigma[\nabla w] \in S \times S$ by the following local averaging:

$$\forall\, z \in \mathcal{N}\quad \sigma[\nabla w](z) = \mathcal{L}^2(\omega_z)^{-1} \int_{\omega_z} \nabla w \in \mathbb{R}^2. \tag{3.7}$$

In other words: $\sigma[\nabla w]$ is the projection on $S \times S$ of ∇w with the help of the 'lumped scalar product'; see also e.g. [26, §1.5].

The conditions (3.1a), (3.1b), (3.2a), and (3.5) are readily verified. In order to show (3.2b), let $w \in S$, $T \in \mathcal{T}$, and estimate

$$\begin{aligned}\|\sigma[\nabla w]\|_T &\le \sum_{z\in\mathcal{N}\cap T} \left|\sigma[\nabla w](z)\right|\, \|\phi_z\|_T \\ &\le \sum_{z\in\mathcal{N}\cap T} \|\phi_z\|_T\, \mathcal{L}^2(\omega_z)^{-1/2}\, \|\nabla w\|_{\omega_z} \le C\, \|\nabla w\|_{\mathcal{B}(T)}\end{aligned}$$

by using the Cauchy-Schwarz inequality as well as the inequalities $\|\phi_z\|_T \le C h_T$ and $\mathcal{L}^2(\omega_z) \ge C h_T^2$, where $h_T := \operatorname{diam} T$ stands for the diameter of T.

Remark 3.1 (Consistency and stability on stars) Suppose that the smoothing procedure satisfies (3.1). Analyzing the verification of (3.2) for (3.7), we see that

$$\nabla w \text{ is constant in } \omega_z \implies \sigma[\nabla w](z) = \nabla w(z) \tag{3.8a}$$
$$\left|\sigma[\nabla w](z)\right| \leq C\mathcal{L}^2(\omega_z)^{-1/2} \, \|\nabla w\|_{\omega_z} \tag{3.8b}$$

for all nodes $z \in \mathcal{N}$ implies (3.2). The opposite implication is also true but we shall omitt the proof here; cf. also [2, Lemma 4.5].

3.2 Estimator for the energy norm error

We now build up an error estimator for the energy norm error $\|\nabla(u_S - u)\|_A$ that is based on the difference $\nabla u_S - Gu_S$, $Gu_S := \sigma[\nabla u_S]$, between untreated and smoothed discrete gradient. Doing so, we will also provide guidelines for the a posteriori error analysis in §4.

Let us temporarily suppose that there is no error in the Dirichlet boundary values: $u_S - u \in X^0$. To estimate $\|\nabla(u_S - u)\|_A$ or, equivalently, $\int_\Omega A\nabla(u_S - u)\cdot\nabla\varphi$ for any $\varphi \in X^0$ with $\|\nabla\varphi\|_A \leq 1$, the motivation from §3.1 suggests to write

$$\int_\Omega A\nabla(u_S - u)\cdot\nabla\varphi = \int_\Omega A(\nabla u_S - Gu_S)\cdot\nabla\varphi + \int_\Omega A(Gu_S - \nabla u)\cdot\nabla\varphi. \tag{3.9}$$

The first integral can be easily, and in a ‘constant-free’ manner, estimated by the computable quantity

$$\zeta := \|\nabla u_S - Gu_S\|_A = \left[\int_\Omega (\nabla u_S - Gu_S)\cdot A(\nabla u_S - Gu_S)\right]^{1/2}, \tag{3.10}$$

which will be the principal part of our error estimator. If the smoothing procedure (3.7) is used, then ζ is the well-known estimator of Zienkiewicz/Zhu [29]. To split ζ into local contributions, we use the partition of unity $\sum_{z\in\mathcal{N}}\phi_z = 1$ provided by the canonical basis functions of S: for all $z \in \mathcal{N}$, we set

$$\zeta_z^2 := \|\nabla u_S - Gu_S\|^2_{A\phi_z} = \int_{\omega_z} (\nabla u_S - Gu_S)\cdot A(\nabla u_S - Gu_S)\phi_z. \tag{3.11}$$

The quantity ζ alone, as any post-processing technique, cannot be a reliable estimator. In fact, consider $A = \mathrm{Id}$, $\Gamma_D = \partial\Omega$, $v = 0$, and a load term $f \neq 0$ that is $L_2(\Omega)$-orthogonal to S. Then $u \neq 0$ but $u_S := \bar{u}_S = 0$, whence $\|\nabla(u_S - u)\|_A > 0$. However, due to (3.2a), we have $Gu_S = \sigma[\nabla u_S] = 0$ and thus $\zeta = 0$. For a concrete example and related underestimation, see [2, §4.7] or §6.1 and §6.2.

In order to obtain a reliable estimator, we have to add terms to ζ. We therefore turn to the second integral in (3.9), which may be superconvergent according to the aforementioned motivation. Consequently, if we estimate this integral in a sharp way, we get additional terms for the estimator that may be superconvergent.

Using equation (2.4) for the exact solution u, the second integral in (3.9) can be viewed as the residual of the smoothed gradient Gu_S:

$$\int_\Omega A(Gu_S - \nabla u)\cdot\nabla\varphi = \langle R, \varphi\rangle$$
$$:= \int_\Omega AGu_S\cdot\nabla\varphi - \int_\Omega f\varphi - \int_{\Gamma_N} g\varphi. \quad (3.12)$$

Writing

$$\langle R, \varphi\rangle = \langle R, \varphi - \chi\rangle + \langle R, \chi\rangle \quad (3.13)$$

with a suitable $\chi \in S^0$ allows us to control the first term on the right hand side with standard techniques. Recalling $Gu_S \in S \times S$ and (2.3c), we observe that AGu_S does not jump across interior (or interelement) edges. The corresponding estimates will thus involve only the element residual r that is defined by $r := f + \mathrm{div}(AGu_S)$, the Neumann residual $AGu_S\cdot n - g$, and the eigenvalues of A. More precisely, $\langle R, \varphi - \chi\rangle$ will be bounded from above in terms of

$$\overline{\rho} := \Big[\sum_{z\in\mathcal{N}} \lambda_z^{-1}\rho_z^2\Big]^{1/2}, \quad (3.14)$$

where

$$\lambda_z := \inf_{\omega_z}\lambda,$$

$$\rho_z^2 := h_z^2\int_{\omega_z}|r - \bar{r}_z|^2\phi_z + h_z\int_{\partial\omega_z\cap\Gamma_N}|AGu_S\cdot n - g|^2\phi_z,$$

$$h_z := \mathrm{diam}\,\omega_z, \qquad \bar{r}_z := \begin{cases}\left(\int_{\omega_z}\phi_z\right)^{-1}\int_{\omega_z} r\phi_z & \text{if } z \in \mathcal{N}\setminus\mathcal{N}_D,\\ 0 & \text{if } z\in\mathcal{N}_D.\end{cases}$$

The corresponding lower bounds will involve also

$$\Lambda_z := \sup_{\omega_z}\Lambda.$$

For the second term of the right hand side in (3.13), we shall use a multilevel decomposition of S. The bisections generating $\mathcal{T}$ from $\mathcal{T}_0$ can be recorded by a forest of binary trees $\mathcal{F}$, where each triangle corresponds to a node, the triangles of $\mathcal{T}_0$ are roots, and those of $\mathcal{T}$ are leafs; see e.g. [22] and Figure 1 for an example. Let $\mathcal{F}_\ell$ be the maximal subforest of $\mathcal{F}$ with depth equal or smaller than $\ell \geq 0$ such that its leafs constitute a conforming triangulation, which will be called $\mathcal{T}_\ell$. We denote by $\mathcal{N}_\ell$ the vertices (or nodes) in $\mathcal{T}_\ell$ and by S_ℓ the continuous linear finite elements over $\mathcal{T}_l$. Clearly, there holds

$$S_\ell \subset S_{\ell+1}, \quad S = \bigcup_{\ell\geq 0} S_\ell \quad\text{and}\quad \mathcal{N}_\ell\subset\mathcal{N}_{\ell+1}, \quad \mathcal{N} = \bigcup_{\ell\geq 0}\mathcal{N}_\ell. \quad (3.15)$$

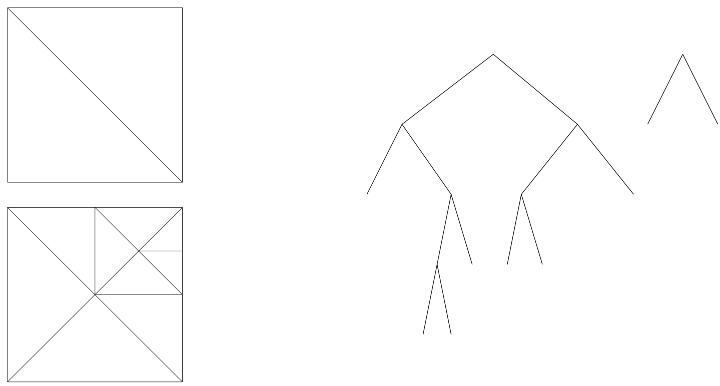

Fig. 1 A macro triangulation (top left), a refinement (bottom left), and its corresponding forest of binary trees (right), which has maximal depth 4

The indicators of the ℓ^{th} level are given by

$$\gamma_{\ell z} := \begin{cases} |\langle R, \phi_{\ell z}\rangle|, & z \in \mathcal{N}_\ell \setminus \Gamma_{\mathrm{D}}, \\ 0, & z \in \mathcal{N}_\ell \cap \Gamma_{\mathrm{D}}, \end{cases} \tag{3.16}$$

where $(\phi_{\ell z})_{z\in\mathcal{N}_\ell}$ are the canonical basis function in S_ℓ satisfying $\phi_{\ell z}(z) = 1$ and $\phi_{\ell z}(y) = 0$ for all $y \in \mathcal{N}_\ell \setminus \{z\}$. Moreover, we define

$$\tilde{\mathcal{N}}_\ell := \big(\mathcal{N}_\ell \setminus \mathcal{N}_{\ell-1}\big) \cup \{z \in \mathcal{N}_{\ell-1} \mid \phi_{\ell z} \neq \phi_{\ell-1,z}\}$$

for $\ell \geq 1$ and $\tilde{\mathcal{N}}_0 := \mathcal{N}_0$. To any node $z \in \tilde{\mathcal{N}}_\ell$ with $\ell \geq 1$, there corresponds a hat function $\phi_{\ell z}$ that is not contained in $S_{\ell-1}$. Consequently the corresponding indicators $\gamma_{\ell z}$, $z \in \tilde{\mathcal{N}}_\ell$ provide the information on the residual R that cannot be seen on the previous level $\ell - 1$. This will allow to bound the term $\langle R, \chi\rangle$ from above in terms of

$$\overline{\gamma} := \Big[\lambda_\Omega^{-1} \sum_{\ell\geq 0} \sum_{z\in\tilde{\mathcal{N}}_\ell} \gamma_{\ell z}^2\Big]^{1/2}, \tag{3.17}$$

which measures how much $Gu_S - \nabla u$ misses to mimic the Galerkin orthogonality of $\nabla(\bar{u}_S - u)$.

To finish this subsection, let us turn back to general Dirichlet boundary values. If we suppose that $u_S = Iv$ on Γ_{D}, the corresponding error will be bounded by

$$\overline{\delta} := \Big[\sum_{z\in\mathcal{N}_{\mathrm{D}}} \overline{\delta}_z^2\Big]^{1/2} \quad \text{with} \quad \overline{\delta}_z^2 = \Lambda_z h_z \int_{\Gamma_{\mathrm{D}}} |\partial_\tau (v - Iv)|^2 \phi_z, \tag{3.18}$$

where ∂_τ denotes the tangential derivative on Γ_{D}.

4 Error control

In this section we derive the theoretical main results of this article: upper and lower bounds for the energy norm error in terms of the computable quantities that have been introduced in §3.

4.1 Upper bound

Let us first establish a global upper bound for the error in the energy norm (2.8) between the exact solution u and the approximate finite element solution u_S of §2.

Theorem 4.1 (**Global upper bound**) *Suppose that the smoothing procedure satisfies* (3.1a) *and that* $u_S = Iv$ *on* Γ_D. *Then the energy norm error of the approximate finite element solution* u_S *is globally bounded in the following way:*

$$\|\nabla(u_S - u)\|_A \leq \zeta + C(\overline{\rho} + \overline{\gamma} + \overline{\delta}),$$

where ζ, $\overline{\rho}$, $\overline{\gamma}$, $\overline{\delta}$ *are given by* (3.10), (3.14), (3.17), (3.18), *and the constant* C *depends only on* α_{min} *and* μ *in* (2.6) *of the macro triangulation* $\mathcal{T}_0$.

Proof **1.** We start by decomposing the error appropriately. Let $\tilde{u}$ be the solution of (2.4), where the Dirichlet boundary values v are replaced by their approximation $v_S = Iv$. Since $\int_\Omega A\nabla(\tilde{u} - u) \cdot \nabla\varphi = 0$ for all $\varphi \in X^0$ and $u_S - \tilde{u} \in X^0$, the energy norm error then splits into two parts:

$$\|\nabla(u_S - u)\|_A^2 = \|\nabla(u_S - \tilde{u})\|_A^2 + \|\nabla(\tilde{u} - u)\|_A^2 , \tag{4.1}$$

where the second term $\|\nabla(\tilde{u} - u)\|_A$ measures the error due to the approximation of the Dirichlet boundary values, while the first term $\|\nabla(u_S - \tilde{u})\|_A$ can be estimated with the help of (3.9), (3.12), and (3.13). Thus, given any $\varphi \in X^0$ with $\|\nabla\varphi\|_A \leq 1$ and any $\chi \in S^0$, we write

$$\int_\Omega A\nabla(u_S - \tilde{u}) \cdot \nabla\varphi = \int_\Omega A(\nabla u_S - Gu_S) \cdot \nabla\varphi + \langle R, \varphi - \chi\rangle + \langle R, \chi\rangle \tag{4.2}$$

As already observed in §3.2, the first term is easily estimated by ζ in a 'constant-free' manner. Estimates for the other two terms and $\|\nabla(\tilde{u} - u)\|_A$ will be provided in the following steps of proof.

2. In order to prepare for the proper estimate of $\langle R, \varphi - \chi\rangle$, we derive an appropriate representation formula, thereby choosing a particular $\chi \in S^0$. To this end, we follow the lines of [15, Lemma 4.1]. Integration by parts on each triangle $T \in \mathcal{T}$ and the use of $\sum_{z\in\mathcal{N}} \phi_z = 1$ as well as $\chi = \sum_{z\in\mathcal{N}} \chi(z)\phi_z$ gives

$$\langle R, \varphi - \chi\rangle = \sum_{z\in\mathcal{N}} \Big(\int_{\partial\omega_z\cap\Gamma_N} [AGu_S \cdot n - g][\varphi - \chi(z)]\phi_z - \int_{\omega_z} r[\varphi - \chi(z)]\phi_z \Big).$$

The choice

$$\chi(z) = \begin{cases} \left(\int_{\omega_z} \phi_z\right)^{-1} \int_{\omega_z} \varphi\phi_z & \text{if } z \in \mathcal{N} \setminus \mathcal{N}_D, \\ 0 & \text{if } z \in \mathcal{N}_D, \end{cases} \tag{4.3}$$

guarantees $\chi \in S^0$ as well as $\int_{\omega_z} [\varphi - \chi(z)]\phi_z = 0$ for all $z \in \mathcal{N} \setminus \mathcal{N}_D$ and thus

$$\langle R, \varphi - \chi \rangle = \sum_{z \in \mathcal{N}} \Big(\int_{\partial\omega_z \cap \Gamma_N} [AGu_S \cdot n - g][\varphi - \chi(z)]\phi_z - \int_{\omega_z} [r - \bar{r}_z][\varphi - \chi(z)]\phi_z \Big). \tag{4.4}$$

3. We proceed by estimating the right hand side of (4.4) and start by recalling two auxiliary inequalities. If E is an edge of a triangle $T \in \mathcal{T}$ containing the node $z \in \mathcal{N}$, then there holds the following trace theorem (use scaling arguments and, e.g., [16, Theorems 6.2.40 and 6.2.25] on the reference triangle): for all $\psi \in H^1(T)$,

$$\|\psi\|_E \le C\Big(h_z^{-1/2} \|\psi\|_T + h_z^{1/2} \|\nabla\psi\|_T\Big). \tag{4.5}$$

Moreover, we have the following Poincaré inequality; see e.g. [21, Lemma 3.5]: Suppose that $\int_{\omega_z} \psi\phi_z = 0$ if $z \in \mathcal{N} \setminus \mathcal{N}_D$ or $\psi = 0$ on $\partial\omega_z \cap \Gamma_D$ if $z \in \mathcal{N}_D$. Then

$$\|\psi\|_{\omega_z} \le C h_z \|\nabla\psi\|_{\omega_z}. \tag{4.6}$$

Furthermore, thanks to (2.3b), there holds

$$\|\nabla\psi\|_{\omega_z} \le \lambda_z^{-1/2} \|\nabla\psi\|_{A;\omega_z}. \tag{4.7}$$

Combined with (4.3), the two inequalities (4.6) and (4.7) imply

$$\|\varphi - \chi(z)\|_{\omega_z} \le C h_z \lambda_z^{-1/2} \|\nabla\varphi\|_{A;\omega_z}.$$

Thus, by means of a 'weighted' Cauchy-Schwarz inequality and $\phi_z \le 1$, we derive

$$\Big|\int_{\omega_z} [r - \bar{r}_z][\varphi - \chi(z)]\phi_z\Big| \le \|r - \bar{r}_z\|_{\phi_z} \|\varphi - \chi(z)\|_{\phi_z} \le C h_z \lambda_z^{-1/2} \|\nabla\varphi\|_{A;\omega_z}$$

and, using also the scaled trace theorem (4.5),

$$\begin{aligned}
&\Big|\int_{\partial\omega_z \cap \Gamma_N} [AGu_S \cdot n - g][\varphi - \chi(z)]\phi_z\Big| \\
&\le \|AGu_S \cdot n - g\|_{\phi_z;\partial\omega_z\cap\Gamma_N} \|\varphi - \chi(z)\|_{\phi_z;\partial\omega_z\cap\Gamma_N} \\
&\le C \|AGu_S \cdot n - g\|_{\phi_z;\Gamma_N} \big[h_z^{-1/2} \|\varphi - \chi(z)\|_{\omega_z} + h_z^{1/2} \|\nabla\varphi\|_{\omega_z}\big] \\
&\le C h_z^{1/2} \lambda_z^{-1/2} \|AGu_S \cdot n - g\|_{\phi_z;\Gamma_N} \|\nabla\varphi\|_{A;\omega_z}.
\end{aligned}$$

In view of the last two inequalities, each term in the sum of (4.4) is bounded by $C\lambda_z^{-1/2}\rho_z \|\nabla\varphi\|_{A;\omega_z}$. We thus arrive at

$$\langle R, \varphi - \chi \rangle \le C_1 \overline{\rho} \tag{4.8}$$

by summing up, applying Cauchy-Schwarz, observing that each triangle is contained in three stars, and recalling $\|\nabla\varphi\|_A \le 1$.

4. In order to bound $\langle R, \chi\rangle$, we first show the following estimate, the proof of which is inspired by the one of Chen/Wu [11, Lemma 2.4]:

$$\langle R, \chi\rangle \le C\Big[\sum_{\ell\ge 0}\sum_{z\in\tilde{\mathcal{N}}_\ell}\gamma_{\ell z}^2\Big]^{1/2}\|\nabla\chi\|. \tag{4.9}$$

To this end, we introduce an interpolation operator as in Scott/Zhang [23] for each level ℓ of the decomposition (3.15): given $w \in H^1(\Omega)$, set $\Pi_{-1}w := 0$ and, for $\ell \ge 0$,

$$\Pi_\ell w := \sum_{z\in\mathcal{N}_\ell}\Pi_\ell w(z)\phi_{\ell z} \in S_\ell \quad\text{with}\quad \Pi_\ell w(z) = \int_{\sigma_{\ell z}}\psi_{\ell z} w,$$

where $\sigma_{\ell z}$ is a certain edge containing z and $\psi_{\ell z}$ the affine function such that

$$w_{|\sigma_{\ell z}} \in P_1(\sigma_{\ell z}) \implies \int_{\sigma_{\ell z}}\psi_{\ell z} w = w(z). \tag{4.10}$$

The choice of $\sigma_{\ell z}$ is subject to the following conditions:

- if $z \in \Gamma_D$, then $\sigma_{\ell z} \subset \Gamma_D$ is a Dirichlet edge;
- if $z \in \mathcal{N}_\ell \setminus \tilde{\mathcal{N}}_\ell$ (and so $\ell \ge 1$), then $\sigma_{\ell z} = \sigma_{\ell-1,z}$;
- if $z \in \tilde{\mathcal{N}}_\ell$ then, $\sigma_{\ell z}$ is an edge of level ℓ, i.e. an edge of $\mathcal{T}_\ell$ containing z and a 'new' node from $\mathcal{N}_\ell \setminus \mathcal{N}_{\ell-1}$, with the convention $\mathcal{N}_{-1} := \emptyset$.

Thus, we have the following properties for $\chi \in S^0$:

$$\Pi_L\chi(z) = \chi(z) \text{ for } L = L(z) = \max\{\ell \ge 0 \mid z \in \tilde{\mathcal{N}}_\ell\},$$
$$z \in \mathcal{N}_\ell \cap \Gamma_D \implies \Pi_\ell\chi(z) = 0,$$
$$z \in \mathcal{N}_\ell \setminus \tilde{\mathcal{N}}_\ell \implies \Pi_\ell\chi(z) = \Pi_{\ell-1}\chi(z).$$

In view of these properties, we have $\chi = \sum_{\ell\ge 0}(\Pi_\ell - \Pi_{\ell-1})\chi$ and can deduce

$$\begin{aligned}\langle R, \chi\rangle &= \sum_{\ell\ge 0}\langle R, (\Pi_\ell - \Pi_{\ell-1})\chi\rangle\\ &= \sum_{\ell\ge 0}\sum_{z\in\tilde{\mathcal{N}}_\ell}(\Pi_\ell - \Pi_{\ell-1})\chi(z)\langle R, \phi_{\ell z}\rangle\\ &\le \Big[\sum_{\ell\ge 0}\sum_{z\in\tilde{\mathcal{N}}_\ell}\gamma_{\ell z}^2\Big]^{1/2}\Big[\sum_{\ell\ge 0}\sum_{z\in\tilde{\mathcal{N}}_\ell}|(\Pi_\ell - \Pi_{\ell-1})\chi(z)|^2\Big]^{1/2}.\end{aligned} \tag{4.11}$$

To estimate the second factor, it is convenient to embed the multilevel decomposition (3.15) into a quasi-uniform one. Denote by $\tilde{\mathcal{T}}_i$ the finest conforming triangulation obtained by bisecting every triangle of $\mathcal{T}_0$ up to level i, by $\tilde{S}_i$ the space of continuous piecewise affine finite elements over $\tilde{\mathcal{T}}_i$, and by $\tilde{P}_i$ the orthogonal $H_0^1(\Omega)$-projection onto $\tilde{S}_i \cap X^0$, with the convention $\tilde{P}_{-1} := 0$. We then may write

$$\chi = \textstyle\sum_{i\ge 0}\chi_i \quad\text{with}\quad \chi_i := (\tilde{P}_i - \tilde{P}_{i-1})\chi.$$

We fix a $z \in \tilde{\mathcal{N}}_\ell$ and claim that

$$(\Pi_\ell - \Pi_{\ell-1})\chi_i(z) = 0 \quad \text{whenever} \quad i \leq \ell - 2. \tag{4.12}$$

Consider first the case $z \in \tilde{\mathcal{N}}_\ell \cap \mathcal{N}_{\ell-1}$. Then there holds $z \in \tilde{\mathcal{N}}_{\ell-1} \cup \tilde{\mathcal{N}}_{\ell-2}$. In fact, we can suppose $\ell \geq 3$ without loss of generality and so it is sufficient to analyze the possible refinements in Figure 2 of a triangle of $\mathcal{T}_{\ell-3}$ containing z and a new node from $\mathcal{N}_\ell \setminus \mathcal{N}_{\ell-1}$. The edges $\sigma_{\ell-1,z}$ and $\sigma_{\ell z}$ are therefore at least of level $l-2$. Consequently, χ_i is affine on these edges and we obtain (4.12) with the help of the property (4.10) of $\psi_{\ell z}$:

$$\Pi_{\ell-1}\chi_i(z) = \chi_i(z) = \Pi_\ell \chi_i(z).$$

Consider now the other case $z \in \tilde{\mathcal{N}}_\ell \setminus \mathcal{N}_{\ell-1}$ and let z_1, z_2 be the two adjacent nodes in $\tilde{\mathcal{N}}_\ell \cap \mathcal{N}_{\ell-1}$ such that $z = \frac{1}{2}(z_1 + z_2)$. Then $\sigma_{\ell z}$ and the edge given by z_1 and z_2 are at least of level ℓ and $\ell - 1$, respectively. Consequently, χ_i is affine on both edges and $\Pi_{\ell-1}\chi_i$ on the one given by z_1 and z_2. Combing this with (4.10) and the first case, we get

$$\begin{aligned} \Pi_\ell \chi_i(z) &= \chi_i(z) \\ &= \tfrac{1}{2}\big[\chi_i(z_1) + \chi_i(z_2)\big] = \tfrac{1}{2}\big[\Pi_{\ell-1}\chi_i(z_1) + \Pi_{\ell-1}\chi_i(z_2)\big] = \Pi_{\ell-1}\chi_i(z) \end{aligned}$$

and (4.12) is proved.

On the other hand, for $i > \ell - 2$, we claim

$$|(\Pi_\ell - \Pi_{\ell-1})\chi_i(z)| \leq C h_\ell^{-\alpha} |\chi_i|_{H^{1-\alpha}(\mathcal{B}(z;\ell-2))}, \tag{4.13}$$

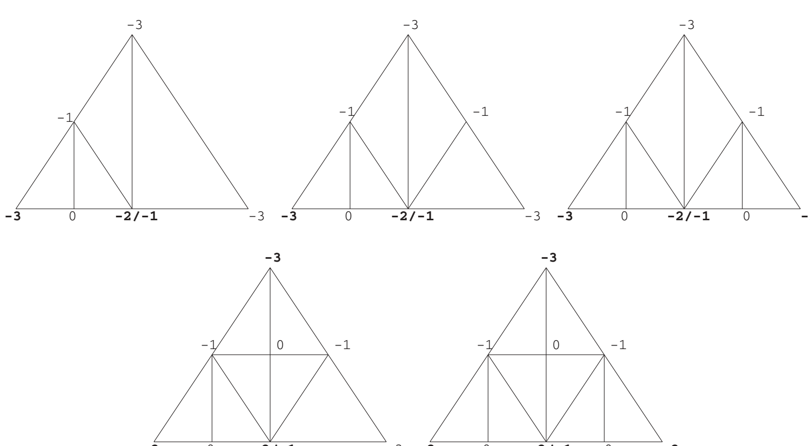

Fig. 2 Possibile refinements (apart from symmetric variants of the first, second, and fourth one) of a triangle in $\mathcal{T}_{\ell-3}$ containing a node from $\mathcal{N}_\ell \setminus \mathcal{N}_{\ell-1}$. Other possibilities are outruled by the conforming newest-vertex bisection. The numbers associated to the nodes indicate their level relative to ℓ. If a node is contained in $\tilde{\mathcal{N}}_\ell \cap \mathcal{N}_{\ell-1}$, then its associated number is in boldface.

where $\alpha \in (0, \frac{1}{2})$ will be chosen in a moment, $|\cdot|_{H^{1-\alpha}(\mathcal{B}(z;\ell-2))}$ is the seminorm of $H^{1-\alpha}\big(\mathcal{B}(z;\ell-2)\big)$, h_ℓ denotes the typical edge length of level ℓ, and $\mathcal{B}(z;\ell-2)$ is the union of all stars in $\mathcal{T}_{\ell-2}$ containing z. To prove (4.15), we start by writing

$$\begin{aligned}|(\Pi_\ell - \Pi_{\ell-1})\chi_i(z)| &= |(\Pi_\ell - \Pi_{\ell-1})(\chi_i - c)(z)| \\ &\le |\Pi_\ell(\chi_i - c)(z)| + |\Pi_{\ell-1}(\chi_i - c)(z)|. \end{aligned} \tag{4.14}$$

with $c \in \mathbb{R}$ arbitrary. For the first term, we employ $\|\psi_{\ell z}\|_{L_\infty(\sigma_{\ell z})} \le Ch_\ell^{-1}$ (see [23, Lemma 3.1]), a scaled trace inequality similar to (4.5), as well as the Poincaré inequality [14, Lemma 3.4] and derive

$$\begin{aligned}|\Pi_\ell(\chi_i - c)(z)| &\le \Big|\int_{\sigma_{\ell z}} \psi_{\ell z}(\chi_i - c)\Big| \\ &\le Ch_\ell^{-1}\|\chi_i - c\|_{L_1(\sigma_{\ell z})} \\ &\le Ch_\ell^{-1/2}\|\chi_i - c\|_{\sigma_{\ell z}} \\ &\le Ch_\ell^{-1/2}\Big[h_\ell^{-1/2}\|\chi_i - c\|_T + h_\ell^{1/2-\alpha}|\chi_i|_{H^{1-\alpha}(T)}\Big] \\ &\le Ch_\ell^{-\alpha}|\chi_i|_{H^{1-\alpha}(T)},\end{aligned}$$

where $T \in \mathcal{T}_\ell$ is chosen such that it contains $\sigma_{\ell z}$ and $c = \fint_T \chi_i$. If $z \in \tilde{\mathcal{N}}_\ell \cap \mathcal{N}_{\ell-1}$, then one similarly obtains the same estimate for the second term with T from $\mathcal{T}_{\ell-1} \cup \mathcal{T}_{\ell-2}$. The remaining estimate of the second term for $z \in \tilde{\mathcal{N}}_\ell \setminus \mathcal{N}_{\ell-1}$ follows from $\Pi_{\ell-1}\chi_i(z) = \frac{1}{2}\big[\Pi_{\ell-1}\chi_i(z_1) + \Pi_{\ell-1}\chi_i(z_2)\big]$, where $z_1, z_2 \in \tilde{\mathcal{N}}_\ell \cap \mathcal{N}_{\ell-1}$ are defined as before. Inserting these estimates into (4.14) then yields (4.13).

We square (4.13), sum it over $z \in \tilde{\mathcal{N}}_\ell$ and, after noting $\tilde{P}_{i-1}\chi_i = 0$, apply [27, Theorem 5.1] with an admissibile α to arrive at

$$\begin{aligned}\sum_{z\in\tilde{\mathcal{N}}_\ell} |(\Pi_\ell - \Pi_{\ell-1})\chi_i(z)|^2 &\le Ch_\ell^{-2\alpha}|\chi_i|^2_{H^{1-\alpha}(\Omega)} \\ &\le C\left(\frac{h_i}{h_\ell}\right)^{2\alpha}\|\nabla\chi_i\|^2.\end{aligned} \tag{4.15}$$

In view of (4.12), (4.15), and $h_\ell = q^\ell h_0$ with $q = 1/\sqrt{2} < 1$, we obtain

$$\begin{aligned}&\sum_{\ell\ge 0}\sum_{z\in\tilde{\mathcal{N}}_\ell} |(\Pi_\ell - \Pi_{\ell-1})\chi(z)|^2 \\ &\quad= \sum_{\ell\ge 0}\sum_{i,j\ge\ell-1}\sum_{z\in\tilde{\mathcal{N}}_\ell}(\Pi_\ell - \Pi_{\ell-1})\chi_i(z)(\Pi_\ell - \Pi_{\ell-1})\chi_j(z) \\ &\quad\le \sum_{\ell\ge 0}\sum_{i,j\ge\ell-1}\Big[\sum_{z\in\tilde{\mathcal{N}}_\ell}|(\Pi_\ell - \Pi_{\ell-1})\chi_i(z)|^2\Big]^{1/2}\Big[\sum_{z\in\tilde{\mathcal{N}}_\ell}|(\Pi_\ell - \Pi_{\ell-1})\chi_j(z)|^2\Big]^{1/2} \\ &\quad\le C\sum_{\ell\ge 0}\sum_{i,j\ge\ell-1} q^{\alpha[(i+j)-2\ell]}\|\nabla\chi_i\|\|\nabla\chi_j\|.\end{aligned}$$

To estimate further, we proceed similarly to the proof of [27, Theorem 5.8] by reordering, exploiting $\sum_{\ell=0}^{\min\{i,j\}+1} q^{-2\alpha\ell} \le Cq^{-2\alpha\min\{i,j\}}$ and $\sum_{i\ge0} \|\nabla\chi_i\|^2 = \|\nabla\chi\|^2$ in the following way:

$$\begin{aligned}
\sum_{\ell\ge0}\sum_{z\in\tilde{\mathcal{N}}_\ell} |(\Pi_\ell - \Pi_{\ell-1})\chi(z)|^2 &\le C \sum_{i,j\ge0} \sum_{\ell=0}^{\min\{i,j\}+1} q^{\alpha[(i+j)-2\ell]} \|\nabla\chi_i\|\,\|\nabla\chi_j\| \\
&\le C \sum_{i,j\ge0} q^{\alpha|i-j|} \|\nabla\chi_i\|\,\|\nabla\chi_j\| \\
&\le C \sum_{m\ge0} q^{\alpha m} \sum_{i\ge0} \|\nabla\chi_i\|\,\|\nabla\chi_{i+m}\| \\
&\le C \sum_{m\ge0} q^{\alpha m} \Big[\sum_{i\ge0} \|\nabla\chi_i\|^2\Big]^{1/2} \Big[\sum_{i\ge m} \|\nabla\chi_i\|^2\Big]^{1/2} \\
&\le \frac{C}{1-q^\alpha} \|\nabla\chi\|^2
\end{aligned}$$

This and (4.11) verify (4.9).

5. To finish the bound of $\langle R, \chi\rangle$, we have to prove an appropriate stability bound for $\|\nabla\chi\|$. Let $T \in \mathcal{T}$ be arbitrary. Since, for any constant c, (4.3) implies

$$\begin{aligned}
\|\nabla\chi\|_T = \|\nabla(\chi - c)\|_T &\le \sum_{z\in\mathcal{N}\cap T} |\chi(z) - c|\, \|\nabla\phi_z\|_T \\
&\le C \sum_{z\in\mathcal{N}\cap T} \mathcal{L}^2(\omega_z)^{-1/2}\, \|\varphi - c\|_{\omega_z} \\
&\le C h_T^{-1}\, \|\varphi - c\|_{\mathcal{B}(T)},
\end{aligned}$$

we obtain the local stability estimate

$$\|\nabla\chi\|_T \le C\, \|\nabla\varphi\|_{\mathcal{B}(T)}$$

with the help of the Bramble-Hilbert Lemma [23, (4.2)]. Its global counterpart is derived by squaring and then summing over $T \in \mathcal{T}$. Taking also (2.3b) into account, we have

$$\|\nabla\chi\| \le C\, \|\nabla\varphi\| \le C\lambda_\Omega^{-1/2}\, \|\nabla\varphi\|_A$$

which, together with (4.9) and $\|\nabla\varphi\|_A \le 1$, leads to

$$\langle R, \chi\rangle \le C_2\overline{\gamma}. \tag{4.16}$$

6. Next, we estimate the error $\|\nabla(\tilde{u} - u)\|_A$ due to the approximation of the Dirichlet boundary values. Since $\int_\Omega A\nabla(\tilde{u} - u)\cdot\nabla\varphi = 0$ for all $\varphi \in X^0$, there holds $\|\nabla(\tilde{u} - u)\|_A \le \|\nabla w\|_A$ for any extension $w \in X^{Iv-v}$ of the approximation error $Iv - v$ in the Dirichlet boundary values. We choose w as in [19, Lemma 3.4] (replace $\partial\Omega$ there by Γ_{D}). Then w satisfies

$$\int_T |\nabla w|^2 \le C h_T \int_{\Gamma_{\mathrm{D}}\cap\partial T} |\partial_\tau(Iv - v)|^2$$

for all $T \in \mathcal{T}_D := \{T \in \mathcal{T} \mid \mathcal{H}^1(\Gamma_\mathrm{D} \cap \partial T) > 0\}$ and the support of w is contained in the union $\bigcup_{T\in\mathcal{T}_D} T$ of these triangles; see the proof of [19, Lemma 3.4]. Consequently,

$$\|\nabla(\tilde{u}-u)\|_A^2 \le \|\nabla w\|_A^2 = \sum_{T\in\mathcal{T}_D} \|\nabla w\|_{A;T}^2 \le C \sum_{T\in\mathcal{T}_D} \Lambda_T h_T \, \|\partial_\tau(Iv - v)\|_{\Gamma_\mathrm{D}\cap\partial T}^2$$

with $\Lambda_T := \sup_T \Lambda$. Since $\Lambda_T h_T \le C\Lambda_z h_z$ whenever $T \subset \omega_z$, this implies

$$\|\nabla(\tilde{u}-u)\|_A \le C_3\overline{\delta}. \tag{4.17}$$

7. Combining (4.1), (4.2), (4.8), (4.16), and (4.17) finally yields

$$\|\nabla(u_S - u)\|_A \le \left[\left(\zeta + C_1\overline{\rho} + C_2\overline{\gamma}\right)^2 + C_3^2\overline{\delta}^2\right]^{1/2}, \tag{4.18}$$

which implies the claimed upper bound. □

Remark 4.1 (*Error due to interpolated Dirichlet boundary values*) We saw in §2.2, that, if $u_S = Iv$ on Γ_D, the energy norm error $\|\nabla(u_S - u)\|_A$ is 'equivalent' to $\|u_S - u\|_{H^1(\Omega)}$ up to the perturbation term $\sqrt{\lambda_\Omega}\,\|Iv - v\|_{\Gamma_\mathrm{D}}$. The latter term is dominated by $\overline{\delta}$ and thus will be automatically taken into account by the error estimator. In fact, $Iv(z) - v(z) = 0$ for all $z \in \mathcal{N}_\mathrm{D}$ implies

$$\begin{aligned} \lambda_\Omega \,\|Iv - v\|_{\Gamma_\mathrm{D}}^2 &\le C\lambda_\Omega \sum_{\substack{T\in\mathcal{T}\\ \mathcal{H}^1(\Gamma_\mathrm{D}\cap\partial T)>0}} h_T^2 \,\|\partial_\tau(Iv - v)\|_{\Gamma_\mathrm{D}\cap\partial T}^2 \\ &\le C\frac{\lambda_\Omega}{\Lambda_\Omega} \sum_{z\in\mathcal{N}_\mathrm{D}} h_z \overline{\delta}_z^2, \end{aligned}$$

where the constant C depends on the minimum angle $\alpha_{\min}$ of the macro triangulation $\mathcal{T}_0$. Moreover, since $\overline{\delta}$ formally has at least the order of $\|\nabla(u_S - u)\|_A$, the perturbation $\sqrt{\lambda_\Omega}\,\|Iv - v\|_{\Gamma_\mathrm{D}}$ is formally of higher order.

4.2 Local lower bounds

In order to assess the local and global sharpness of the upper bound in Theorem 4.1, we derive complementing lower bounds. In this subsection we establish *local* lower bounds, while in the following §4.3 *global* ones. In the light of the motivation of §3.1, we proceed in two steps. First, we provide lower bounds that involve also the error of the possibly superconvergent smoothed gradient. Second, we show that this error is bounded by the energy norm error in any event. The lower bound for the energy norm error is then an immediate consequence.

In step 3 of the proof of Theorem 4.1, we estimated the local dual norm of the residual by stronger and thus overestimating integral norms. In order to recover partially from this overestimation by means of an inverse inequality also for non-discrete integrands, we let $\bar{f}$ and $\bar{g}$ be possibly discontinuous approximations of f and g, respectively, such that $\bar{f}$ is affine on every triangle and $\bar{g}$ quadratic on every

Neumann edge. The lower bounds will then involve also the oscillation measures defined by

$$\operatorname{osc}_{\mathcal{T}}(f;\omega)^2 = \sum_{\substack{T\in\mathcal{T}\\ T\subset\omega}} \frac{h_T^2}{\Lambda_T} \left\| f - \bar{f} \right\|_T^2, \quad \operatorname{osc}_{\mathcal{T}}(g;\omega)^2 = \sum_{\substack{E\in\mathcal{E}\\ E\subset\partial\omega\cap\Gamma_{\mathrm{N}}}} \frac{h_E}{\Lambda_E} \|g - \bar{g}\|_E^2 ,$$

where ω is a subdomain of Ω, $\Lambda_E := \Lambda_T$ with T being the triangle that contains E, and $h_E := \operatorname{diam} E$ is the diameter of the edge E. Notice that these oscillation terms formally have at least the order of the error of the possibly superconvergent smoothed gradient.

Also the Dirichlet indicators $\overline{\delta}_z$, $z \in \mathcal{N} \cap \Gamma_{\mathrm{D}}$, are made up by means of a too strong and thus overestimating norm. However, these indicators already measure an oscillation of the Dirichlet boundary values and we therefore do not derive corresponding lower bounds.

Proposition 4.1 **(Local lower bounds with smoothed gradient)** *The indicators ζ_z, ρ_z, $z \in \mathcal{N}$, and $\gamma_{\ell z}$, $\ell \geq 0$, $z \in \mathcal{N}_\ell$, are bounded in the following way:*

$$\begin{aligned} \zeta_z &\leq \|\nabla(u_S - u)\|_{A\phi_z} + \|Gu_S - \nabla u\|_{A\phi_z} , \\ \Lambda_z^{-1/2}\rho_z &\leq C_1\big[\|Gu_S - \nabla u\|_{A;\omega_z} + \operatorname{osc}_{\mathcal{T}}(f;\omega_z) + \operatorname{osc}_{\mathcal{T}}(g;\omega_z)\big], \\ \Lambda_{\ell z}^{-1/2}\gamma_{\ell z} &\leq C_2 \|Gu_S - \nabla u\|_{A;\omega_{\ell z}} , \end{aligned}$$

where the constants C_1 and C_2 depend only on α_{min}, $\omega_{\ell z} = \operatorname{supp}\phi_{\ell z}$ indicates a star on the level ℓ, and $\Lambda_{\ell z} := \sup_{\omega_{\ell z}} \Lambda$.

Proof **1.** The first inequality is an immediate consequences of the triangle inequality. The third one readily follows from (3.12) and $\|\nabla\phi_{\ell z}\|_A \leq C\Lambda_{\ell z}^{1/2}$. To prove the second one, we use the constructive argument of Verfürth, see e.g. [26], to derive

$$h_T \|r\|_T \leq C\big[\Lambda_T^{1/2} \|Gu_S - \nabla u\|_{A;T} + h_T \left\|\bar{f} - f\right\|_T \big]$$

for any triangle T and

$$\begin{aligned} &h_E^{1/2} \|AGu_S \cdot n - g\|_E \\ &\quad\leq C_3\big[\Lambda_E^{1/2} \|Gu_S - \nabla u\|_{A;T} + h_E^{1/2} \|\bar{g}_E - g\|_E + h_E \left\|\bar{f} - f\right\|_T \big], \end{aligned}$$

where E is a Neumann edge and T the triangle containing E. For convenience of the reader and completeness, we prove these estimates in the next two steps. Since $\|r - \bar{r}_z\|_{\phi_z} \leq \|r\|_{\omega_z}$, $h_E \leq h_T$ and $h_z \leq Ch_K$ whenever the edge or triangle K contains z, the proof is then complete.

2. We introduce $\bar{r}_T := \operatorname{div}(AGu_S)_{|T} + \bar{f}_{|T} \in P_1(T)$ and let $\psi_T := \prod_{z\in\mathcal{N}\cap T} \phi_z$ be the bubble function of T. Since $\varphi_T := \bar{r}_T\psi_T \in X^0$, an integration by parts on T and (3.12) imply

$$-\int_T r\varphi_T = \langle R, \varphi_T\rangle = \int_T A(Gu_S - \nabla u)\cdot\nabla\varphi_T .$$

Since

$$\|\nabla\varphi_T\|_{A;T} \le \Lambda_T^{1/2}\,\|\nabla\varphi_T\|_T \tag{4.19}$$

thanks to (2.3b) and $\|\nabla\varphi_T\|_T \le Ch_T^{-1}\,\|\bar{r}_T\|_T$ thanks to inverse inequalities, we obtain

$$\begin{aligned}\left|\int_T r\bar{r}_T\psi_T\right| &\le \|Gu_S - \nabla u\|_{A;T}\,\|\nabla\varphi_T\|_{A;T}\\ &\le Ch_T^{-1}\Lambda_T^{1/2}\,\|Gu_S - \nabla u\|_{A;T}\,\|\bar{r}_T\|_T.\end{aligned}$$

and hence

$$\int_T |\bar{r}_T|^2\psi_T \le \big(Ch_T^{-1}\Lambda_T^{1/2}\,\|Gu_S - \nabla u\|_{A;T} + \|\bar{f} - f\|_T\big)\|\bar{r}_T\|_T.$$

We estimate the left hand side from below by $C\|\bar{r}\|_T^2$ with the help of [2, Theorem 2.2] and arrive at the claimed bound for the element residual:

$$\begin{aligned}h_T\,\|r\|_T &\le h_T\,\|\bar{r}_T\|_T + h_T\,\big\|\bar{f} - f\big\|_T\\ &\le C\big(\Lambda_T^{1/2}\,\|Gu_S - \nabla u\|_{A;T} + h_T\|\bar{f} - f\|_T\big).\end{aligned}$$

3. The bound for the Neumann residual can be proved in quite similar way. In fact, we write

$$AGu_S\cdot n - g = \big[AGu_S\cdot n - \bar{g}\big] + \big[\bar{g} - g\big]$$

use the bubble function $\psi_E = \prod_{z\in\mathcal{N}\cap E}\phi_z$ of the edge E, and [2, Theorem 2.4]. □

Let us compare the upper and local lower bounds in Theorem 4.1 and Proposition 4.1. The indicators controlling the error of the smoothed gradient differ by multiplicative factors that depend on the local condition number of the matrix A, e.g. $\sqrt{\Lambda_z/\lambda_z}$. This disturbing gap, which originates from the use of worst case a priori estimates like (4.7) and (4.19), is however avoided for the principal indicators ζ_z, $z\in\mathcal{N}$.

As announced, we now show that the local error of the smoothed gradient Gu_S is essentially bounded by the local energy norm error. Since (3.6), which we will use for this purpose, requires a locally constant coefficient matrix A, this bound will involve oscillations of A. To measure them, we define

$$\operatorname{osc}_{\mathcal{T}}(A;\omega) := \max_{T\subset\omega}\frac{\sup_{\mathcal{B}(T)}|\bar{A}_T - A|}{\inf_{\mathcal{B}(T)}\lambda}, \tag{4.20}$$

where ω is a subdomain of Ω, $|\cdot|$ stands also for the matrix norm induced by the Euclidean norm $|\cdot|$, and $\bar{A}_T := \mathcal{L}^2(\mathcal{B}(T))^{-1}\int_{\mathcal{B}(T)} A$ is the mean value of A in the patch $\mathcal{B}(T)$ around T.

Proposition 4.2 **(Asymptotic nondeterioration of smoothed gradient)** *Suppose the smoothing procedure σ has the properties* (3.1), (3.2), *and* (3.5). *Then σ is asymptotically nondeteriorating in that, for any $T \in \mathcal{T}$, the smoothed gradient $Gu_S = \sigma[\nabla u_S]$ satisfies*

$$\|Gu_S - \nabla u\|_{A;T} \le C\kappa_{\mathcal{B}(T)}\Big[\, \|\nabla(u_S - u)\|_{A;\mathcal{B}(T)} + \operatorname{osc}_{\mathcal{T}}\big(f; \mathcal{B}(T)\big)\Big],$$

where C depends only on α_{min} and

$$\kappa_{\mathcal{B}(T)} = \frac{\sup_{\mathcal{B}(T)} \Lambda^{1/2}}{\inf_{\mathcal{B}(T)} \lambda^{1/2}}\Big[1 + O\big(\operatorname{osc}_{\mathcal{T}}(A; T)\big)\Big] \tag{4.21}$$

is, up to oscillation of A, the local condition number of $A^{1/2}$.

Note that this proposition implies the missing implication in the proof of Theorem 3.1.

Proof Thanks to (3.2a) and (3.1b), we may start with

$$\|Gu_S - \nabla u\|_{A;T} \le \|\nabla(u - p)\|_{A;T} + \|\sigma[\nabla(u_S - p)]\|_{A;T}$$

where $p \in P_1$ is any polynomial of degree 0 or 1. Invoking (3.6), we derive

$$\begin{aligned}\|\sigma[\nabla(u_S - p)]\|_{A;T} &\le \Big\{1 + \operatorname{osc}_{\mathcal{T}}(A; T)\Big\}^{1/2} \|\sigma[\nabla(u_S - p)]\|_{\bar{A}_T;T}\\ &\le C\Big\{1 + \operatorname{osc}_{\mathcal{T}}(A; T)\Big\}^{1/2} \|\nabla(u_S - p)\|_{\bar{A}_T;\mathcal{B}(T)}\end{aligned}$$

for the second term on the right hand side and thus arrive at

$$\begin{aligned}\|Gu_S - \nabla u\|_{A;T} \le \|\nabla(u - p)\|_{A;T} &+ C\Big\{1 + \operatorname{osc}_{\mathcal{T}}(A; T)\Big\}^{1/2}\\ &\times \|\nabla(u_S - p)\|_{\bar{A}_T;\mathcal{B}(T)}\,.\end{aligned} \tag{4.22}$$

To choose the polynomial p, let us first observe the following: if $T_1, T_2 \subset \mathcal{B}(T)$ are two different triangles sharing an edge $E \in \mathcal{E}$ with normal ν and $p_i := u_{S|T_i} \in P_1$, $i = 1, 2$, then we have the identity $\nabla(p_1 - p_2) = \big[\nabla(p_1 - p_2)\cdot\nu\big]\nu$ and therefore the inequality

$$\begin{aligned}\nabla(p_1 - p_2)\cdot\bar{A}_T\nabla(p_1 - p_2) &= |\nabla(p_1 - p_2)\cdot\nu|^2\,\nu\cdot\bar{A}_T\nu\\ &\le \lambda_{\mathcal{B}(T)}^{-1}|\nabla(p_1 - p_2)\cdot\nu|^2\big[\nu\cdot\bar{A}_T\nu\big]^2\\ &\le \lambda_{\mathcal{B}(T)}^{-1}\big[\bar{A}_T\nabla(p_1 - p_2)\cdot\nu\big]^2,\end{aligned}$$

with $\lambda_{\mathcal{B}(T)} := \inf_{\mathcal{B}(T)}\lambda$. This, together with the first part of (2.3c), yields

$$\begin{aligned}|\bar{A}_T^{1/2}\nabla(p_1 - p_2)| &\le Ch_E^{-1/2}\,\|\nabla(p_1 - p_2)\|_{\bar{A}_T;E}\\ &\le Ch_E^{-1/2}\lambda_{\mathcal{B}(T)}^{-1/2}\,\big\|\bar{A}_T\nabla(p_1 - p_2)\cdot\nu\big\|_E\\ &\le Ch_E^{-1/2}\lambda_{\mathcal{B}(T)}^{-1/2}\Big\{1 + \operatorname{osc}_{\mathcal{T}}(A; T)\Big\}\,\|J\|_E\,,\end{aligned} \tag{4.23}$$

where J denotes the jump in the normal component of $A\nabla u_S$ across E. Moreover, there holds the local lower bound

$$\Lambda_E^{-1/2} h_E^{1/2} \, \|J\|_E \le C\big[\, \|\nabla(u_S - u)\|_{A;\omega_E} + \operatorname{osc}_{\mathcal{T}}(f;\omega_E)\big],$$

where $\Lambda_E := \sup_{\omega_E} \Lambda$ and $\omega_E = T_1 \cup T_2$; see e.g. [18, Lemma 3.4]. Inserting this inequality into (4.23), we obtain, for any pair $T_1, T_2 \subset \mathcal{B}(T)$ sharing one side E,

$$|\bar{A}_T^{1/2}\nabla(p_1 - p_2)| \le C h_T^{-1} \frac{\Lambda_{\mathcal{B}(T)}^{1/2}}{\lambda_{\mathcal{B}(T)}^{1/2}} \Big\{ 1 + \operatorname{osc}_{\mathcal{T}}(A;T) \Big\}$$
$$\times \Big\{ \|\nabla(u_S - u)\|_{A;\omega_E} + \operatorname{osc}_{\mathcal{T}}(f;\omega_E) \Big\} \qquad (4.24)$$

with $\Lambda_{\mathcal{B}(T)} := \sup_{\mathcal{B}(T)} \Lambda$.

To finish the proof, we take $p := u_{S|T} \in P_1$ in (4.22). We then have to estimate the second term on the right hand side in (4.22) appropriately. Let $T' \in \mathcal{T}$ be an arbitrary triangle in $\mathcal{B}(T)$. Then there exists a path of triangles in $\mathcal{B}(T)$ from T to T'. More precisely, there exists a finite sequence $(T_i)_{i=0}^n$ such that

- $T_0 = T$, $T_n = T'$,
- each triangle T_i is in $\mathcal{B}(T)$, there holds $T_i \neq T_j$ whenever $i \neq j$, and
- each pair (T_{i-1}, T_i) has one common side.

We now set $p_i := u_{S|T_i}$, $i = 0, \dots, n$, and devise

$$\begin{aligned}
\|\nabla(u_S - p)\|_{\bar{A}_T;T'} &\le \mathcal{L}^2(T')^{1/2} |\bar{A}_T^{1/2}\nabla(p_n - p_0)| \\
&\le C h_T \sum_{i=1}^n |\bar{A}_T^{1/2}\nabla(p_i - p_{i-1})| \\
&\le C \frac{\Lambda_{\mathcal{B}(T)}^{1/2}}{\lambda_{\mathcal{B}(T)}^{1/2}} \Big\{ 1 + \operatorname{osc}_{\mathcal{T}}(A;T) \Big\} \\
&\quad \times \Big\{ \|\nabla(u_S - u)\|_{A;\mathcal{B}(T)} + \operatorname{osc}_{\mathcal{T}}(f;\mathcal{B}(T)) \Big\}
\end{aligned}$$

by noting that all triangles in $\mathcal{B}(T)$ have about the same diameter and employing (4.24). We 'sum' over $T' \subset \mathcal{B}(T)$ and observe that the number of triangles in $\mathcal{B}(T)$ is bounded in terms of $\alpha_{\min}$. Inserting the resulting inequality in (4.22) then finally yields the claimed estimate. □

Remark 4.2 (Continuity of coefficient matrix) The continuity of the coefficient matrix A plays a crucial role in the proof of Proposition 4.2. In fact, arranging the data of (2.4) and (2.7) such that

$$A(x_1, x_2) = \begin{cases} \mathrm{Id} \\ 2\,\mathrm{Id} \end{cases} \quad \text{and} \quad u(x_1, x_2) = \bar{u}_S(x_1, x_2) = \begin{cases} 1 + x_1 & \text{if } x_1 \le 0, \\ 1 - \frac{1}{2}x_1 & \text{if } x_1 \ge 0 \end{cases}$$

are exact and approximate solutions reveals that a smoothing procedure satisfying (3.1a) cannot be locally nondeteriorating if A has jumps.

The combination of Propositions 4.1 and 4.2 implies local lower bounds for the energy norm error. For their formulations, it is convenient to use the indicators

$$\underline{\rho}_z := \Lambda_z^{-1/2} \rho_z \qquad \underline{\gamma}_z := \Lambda_z^{-1/2} \gamma_z \tag{4.25}$$

for all $z \in \mathcal{N}$, where $\gamma_z := \gamma_{\ell z}$ with ℓ so large such that $\mathcal{N}_\ell = \mathcal{N}$.

Theorem 4.2 (Local lower bounds) *The indicators ζ_z, $\underline{\rho}_z$, $\underline{\gamma}_z$, are bounded by the local energy norm error. More precisely, for any $z \in \mathcal{N}$,*

$$\zeta_z + \underline{\rho}_z + \underline{\gamma}_z \le C\kappa_{\mathcal{B}(\omega_z)} \Big[\|\nabla(u_S - u)\|_{A;\mathcal{B}(\omega_z)} + \operatorname{osc}_{\mathcal{T}}\big(f; \mathcal{B}(\omega_z)\big) + \operatorname{osc}_{\mathcal{T}}(g; \omega_z) \Big],$$

with C depending only on α_{min} and $\kappa_{\mathcal{B}(\omega_z)}$ satisfies (4.21) *where T is replaced by ω_z and $\mathcal{B}(\omega_z) := \bigcup\{T \in \mathcal{T} : T \cap \omega_z \neq \emptyset\}$.*

4.3 Global lower bounds

In this subsection we derive *global* lower bounds and thus analyze the global sharpness of the upper bound in Theorem 4.1. Most of these global bounds readily follow from the *local* ones in the preceding subsection 4.2.

Let us start with the counterpart of Proposition 4.1.

Proposition 4.3 (Global lower bounds with smoothed gradient) *The contributions ζ, $\overline{\rho}$, and $\overline{\gamma}$ of the estimator are bounded in the following way:*

$$\zeta \le \|\nabla(u_S - u)\|_A + \|Gu_S - \nabla u\|_A\,,$$
$$\overline{\rho} \le C_1 \max_{z\in\mathcal{N}} \sqrt{\Lambda_z/\lambda_z} \Big[\|Gu_S - \nabla u\|_A + \operatorname{osc}_{\mathcal{T}}(f;\Omega) + \operatorname{osc}_{\mathcal{T}}(g;\Omega) \Big],$$
$$\overline{\gamma} \le C_2 \sqrt{\Lambda_\Omega/\lambda_\Omega}\, \|Gu_S - \nabla u\|_A\,,$$

where both constants C_1 and C_2 depend only on α_{min} and μ in (2.6).

Proof The first inequality is an immediate consequence of the triangle inequality. The second one follows from squaring, multiplying with λ_z, and summing up its local counterparts in Proposition 4.1.

The third inequality is more involved. Defining $\chi := \sum_{\ell\ge 0} \sum_{z\in\tilde{\mathcal{N}}_\ell\setminus\Gamma_D} \langle R, \phi_{\ell z}\rangle \phi_{\ell z}$, we obtain

$$\gamma^2 := \sum_{\ell\ge 0} \sum_{z\in\tilde{\mathcal{N}}_\ell} \gamma_{\ell z}^2 = \langle R, \chi\rangle \le \|Gu_S - \nabla u\|_A\, \Lambda_\Omega^{1/2}\, \|\nabla\chi\|$$

and so we can conclude by showing $\|\nabla\chi\| \le \gamma$. To this end, we adapt some lines from the proof of Chen/Wu [11, Theorem 2.2] and start with

$$\|\nabla\chi\|^2 \le 2 \sum_{\ell\ge 0} \sum_{z\in\tilde{\mathcal{N}}_\ell} \sum_{m\ge\ell} \sum_{y\in\tilde{\mathcal{N}}_m} \gamma_{\ell z}\gamma_{my} \left| \int_\Omega \nabla\phi_{\ell z} \cdot \nabla\phi_{my} \right|. \tag{4.26}$$

Let us denote by $\mathcal{E}_{\ell z}$ the union of edges of $\mathcal{T}_\ell$ that are contained in the support of $\phi_{\ell z}$. Given $z \in \tilde{\mathcal{N}}_\ell$ and $y \in \tilde{\mathcal{N}}_m$ such that $m \geq \ell$, we observe

$$\int_\Omega \nabla\phi_{\ell z} \cdot \nabla\phi_{my} = 0 \quad \text{whenever} \quad y \notin \mathcal{E}_{\ell z} \tag{4.27a}$$

and, otherwise,

$$\begin{aligned}\left|\int_\Omega \nabla\phi_{\ell z} \cdot \nabla\phi_{my}\right| &\leq \|\nabla\phi_{\ell z}\|_{L_\infty(\Omega)} \left\|\nabla\phi_{my}\right\|_{L_1(\Omega)} \\ &\leq C\frac{h_m}{h_\ell} = Cq^{m-\ell} \end{aligned}\tag{4.27b}$$

in view of

$$\|\nabla\phi_{\ell z}\|_{L_\infty(T)} \leq Ch_\ell^{-1} \text{ for all } T \in \mathcal{T}_\ell \text{ and}$$
$$\|\nabla\phi_{mz}\|_{L_1(T)} \leq \mathcal{L}^2(T)\,\|\nabla\phi_{mz}\|_{L_\infty(T)} \leq Ch_m \text{ for all } T \in \mathcal{T}_m.$$

Moreover, setting $\mathcal{N}_{m,\ell z} := \tilde{\mathcal{N}}_m \cap \mathcal{E}_{\ell z}$ and using Cauchy-Schwarz, we get

$$\begin{aligned}\sum_{m\geq\ell}\sum_{y\in\mathcal{N}_{m,\ell z}} \gamma_{my}q^{m-\ell} &= \sum_{m\geq\ell}\sum_{y\in\mathcal{N}_{m,\ell z}} q^{3(m-\ell)/4}\big[\gamma_{my}q^{(m-\ell)/4}\big] \\ &\leq \Big[\sum_{m\geq\ell}\sum_{y\in\mathcal{N}_{m,\ell z}} q^{3(m-\ell)/2}\Big]^{1/2}\Big[\sum_{m\geq\ell}\sum_{y\in\mathcal{N}_{m,\ell z}} q^{(m-\ell)/2}\gamma_{my}^2\Big]^{1/2}.\end{aligned}$$

Since each $\mathcal{N}_{m,\ell z}$ with $m \geq \ell$ has at most $Ch_\ell/h_m \leq Cq^{\ell-m}$ elements and there holds $\sum_{m\geq\ell} q^{(m-\ell)/2} = 1/(1-\sqrt{q})$, the first factor is bounded in terms of $\alpha_{\min}$ and q whence

$$\sum_{m\geq\ell}\sum_{y\in\mathcal{N}_{m,\ell z}} \gamma_{my}q^{m-\ell} \leq C\Big[\sum_{m\geq\ell}\sum_{y\in\mathcal{N}_{m,\ell z}} q^{(m-\ell)/2}\gamma_{my}^2\Big]^{1/2}.$$

Inserting (4.27) into (4.26) and the last inequality after another Cauchy-Schwarz, we derive

$$\begin{aligned}\|\nabla\chi\|^2 &\leq C\sum_{\ell\geq 0}\sum_{z\in\tilde{\mathcal{N}}_\ell}\gamma_{\ell z}\sum_{m\geq\ell}\sum_{y\in\mathcal{N}_{m,\ell z}}\gamma_{my}q^{(m-\ell)} \\ &\leq C\gamma\Big[\sum_{\ell\geq 0}\sum_{z\in\tilde{\mathcal{N}}_\ell}\sum_{m\geq\ell}\sum_{y\in\mathcal{N}_{m,\ell z}} q^{(m-\ell)/2}\gamma_{my}^2\Big]^{1/2}.\end{aligned}$$

It therefore remains to estimate the term in the square root by γ^2. Thanks to Fubini, this term equals

$$\sum_{m\geq 0}\sum_{y\in\tilde{\mathcal{N}}_m}\gamma_{my}^2\sum_{\ell\leq m}\sum_{z:y\in\mathcal{N}_{m,\ell z}} q^{(m-\ell)/2},$$

where '$z : y \in \mathcal{N}_{m,\ell z}$' stands for those $z \in \tilde{\mathcal{N}}_\ell$ with $y \in \mathcal{N}_{m,\ell z}$. Since the number of edges in $\mathcal{T}_m$ containing y is bounded by C, the same holds for the number of these z. Consequently, we have $\sum_{\ell\leq m}\sum_{z:y\in\mathcal{N}_{m,\ell z}} q^{(m-\ell)/2} \leq C/(1-\sqrt{q})$, which finishes the proof. □

Finally, we prove the global counterpart of Theorem 4.2.

Theorem 4.3 (Global lower bound) *Denoting the constant in the upper bound of Theorem 4.1 by C_1, we have the complementing global lower bound*

$$\zeta + C_1(\overline{\rho} + \overline{\gamma}) \leq C_2\kappa_\Omega^2 \Big[\|\nabla(u_S - u)\|_{A;\Omega} + \operatorname{osc}_{\mathcal{T}}\big(f;\Omega\big) + \operatorname{osc}_{\mathcal{T}}(g;\Omega) \Big],$$

where C_2 depends only on α_{min} and μ in (2.6) *and κ_Ω satifies* (4.21)*, where T is replaced by Ω.*

Proof In view of Proposition 4.3, we first show that the smoothing procedure σ is (globally) nondeteriorating. To this end, we square the inequality in Proposition 4.2 and sum over all $T \in \mathcal{T}$. Since during the summation the number of hits for a triangle is bounded uniformly in terms of the minimum angle of $\mathcal{T}_0$, we obtain

$$\|Gu_S - \nabla u\|_A \leq C\kappa_\Omega \Big[\|\nabla(u_S - u)\|_{A;\Omega} + \operatorname{osc}_{\mathcal{T}}\big(f;\Omega\big) \Big].$$

We conclude by inserting this estimate in the ones of Proposition 4.3. □

5 Adaptive algorithm

This section proposes an adaptive algorithm based upon §4 and briefly describes the implementation used for the numerical results in §6. Main feature of the algorithm is that it benefits from superconvergence of the smoothed gradient in stopping and adaptivity.

5.1 Adaptivity and superconvergence

Let us first formulate the main steps of our algorithm and then discuss the role of the a posteriori bounds in §4.

Algorithm 5.1 *Suppose that a macro triangulation $\mathcal{T}_0$ satisfying* (2.5)*, an approximation $\tilde{C}$ of the constant in Theorem 4.1, a tolerance* tol > 0*, and a parameter $\theta \in (0, 1]$ are given. Set $k := 0$ and iterate*

(1) Obtain u_k by solving (2.7) *on $\mathcal{T}_k$ approximately and compute the corresponding error indicators.*
(2) If $\zeta_k + \tilde{C}(\overline{\rho}_k + \overline{\gamma}_k + \overline{\delta}_k) \leq$ tol*, then* STOP.
(3) With the marking indicators $\mu_{kz} := \zeta_{kz} + \underline{\rho}_{kz} + \underline{\gamma}_{kz} + \overline{\delta}_{kz}$, $z \in \mathcal{N}_k$ and $\mu_{k,\max} := \max_{z\in\mathcal{N}_k} \mu_{kz}$, define

$$\mathcal{M}_k := \{z \in \mathcal{N}_k \mid \mu_{kz} \geq \theta\mu_{k,\max}\}.$$

(4) Obtain a new (conforming) triangulation $\mathcal{T}_{k+1}$ from $\mathcal{T}_k$ by bisecting at least all the triangles of the marked stars ω_z, $z \in \mathcal{M}_k$, and increment k.

In order to analyze the stopping test, let us observe that the global bounds in §4.1 and §4.3 imply

$$\begin{aligned}\|\nabla(u_k - u)\|_A &\le \zeta_k + C(\overline{\rho}_k + \overline{\gamma}_k + \overline{\delta}_k)\\ &\le \|\nabla(u_k - u)\|_A + C\kappa_\Omega\big\{\, \|Gu_k - \nabla u\|_A + \epsilon_k\big\}\\ &\le C\kappa_\Omega^2\big\{\, \|\nabla(u_k - u)\|_A + \epsilon_k\big\}\end{aligned}$$

with $\epsilon_k := \operatorname{osc}_{\mathcal{T}_k}(f;\Omega) + \operatorname{osc}_{\mathcal{T}_k}(g;\Omega) + \overline{\delta}_k$, which is expected to converge faster than the error. This inequality chain immediately yields the following statements.

Corollary 5.1 (Stopping test) *The stopping test in step (2) is reliable whenever $\tilde{C}$ overestimates the constant in Theorem 4.1 and it is efficient with constant $\tilde{C}C^{-1}\kappa_\Omega^2$ up to higher order terms.*

In addition, regardless of the gap κ_Ω^2 and the choice of $\tilde{C}$, the stopping test is asymptotically exact if $\|Gu_k - \nabla u\|_A + \epsilon_k = o\big(\, \|\nabla(u_k - u)\|_A\big)$ for $k \to \infty$.

Section 4.2 leads to the following local inequality chain: for any $z \in \mathcal{N}_k$,

$$\begin{aligned}\mu_{kz} &\le \|\nabla(u_k - u)\|_{A\phi_z} + C\big\{\, \|Gu_k - \nabla u\|_{A\phi_z} + \epsilon_{kz}\big\}\\ &\le C\kappa_{\mathcal{B}(\omega_z)}\big\{\, \|\nabla(u_k - u)\|_{A;\mathcal{B}(\omega_z)} + \epsilon_{kz}\big\},\end{aligned}$$

where $\epsilon_{kz} := \operatorname{osc}_{\mathcal{T}_k}\big(f;\mathcal{B}(\omega_z)\big) + \operatorname{osc}_{\mathcal{T}_k}\big(g;\mathcal{B}(\omega_z)\big) + \overline{\delta}_{kz}$, which is expected to be asymptotically smaller than the local error. We thus have the following corollary.

Corollary 5.2 (Marking indicators) *In any event, each marking indicator μ_{kz}, $z \in \mathcal{N}_k$, is efficient with constant $C\kappa_{\mathcal{B}(\omega_z)}$ up to higher order terms.*

Moreover, if $\|Gu_k - \nabla u\|_{A\phi_z} + \epsilon_{kz}$ is negligible with respect to $\|\nabla(u_k - u)\|_{A\phi_z}$, it is efficient with constant close to 1.

The indicator $\tilde{\mu}_{kz} := \zeta_{kz} + \overline{\rho}_{kz} + \overline{\gamma}_{kz} + \overline{\delta}_{kz}$, which appears in the stopping test, has similar properties. However, its 'security part' $\overline{\rho}_{kz} + \overline{\gamma}_{kz}$ offers less protection from overestimation than $\underline{\rho}_{kz} + \underline{\gamma}_{kz}$ of μ_{kz}.

5.2 Implementation in ALBERTA and parameters

We have implemented Algorithm 5.1 within the framework of the finite element toolbox ALBERTA [22] of Schmidt/Siebert. The theory in §4 neglects numerical integration. For nonpolynomial integrands, we therefore use the quadrature formula with the highest available order, except for §6.1 where we integrate exactly.

The computation of the error indicators ζ_k, $\overline{\rho}_k$, $\overline{\delta}_k$, and their variants is explicit and quite simple: only standard operations on the current triangulation are needed. For the computation of $\overline{\gamma}_k$ and variants, one can exploit the hierarchical structure of the ALBERTA mesh in order to simplify its computation; the resulting cost is thus below one multigrid iteration.

In view of (4.18), we actually implemented the following form of the estimator:

$$\mathcal{E}_k := \Big[\big(\zeta_k + C_1\overline{\rho}_k + C_2\overline{\gamma}_k\big)^2 + C_3\overline{\delta}_k^2\Big]^{1/2}.$$

For the simulations in §6, we chose $C_1 = 1/5$, $C_2 = 1/3$, and $C_3 = 1$; the first two values are numerically equilibrated for the used macro triangulations, while the third one is an ad hoc choice. We used $\theta = 0.5$ for the parameter of the maximum strategy.

6 Numerical results

This last section is devoted to numerical tests that confirm, complement, and apply the theory in §4. In particular, the numerical examples illustrate

- that the principal part ζ_k alone is in general not reliable and thus should not be used in a 'black box algorithm',
- that the superconvergence takes place also for unstructured inital meshes and singular problems,
- the relevance of asymptotic exactness for stopping.

6.1 Oscillatory load

We consider (2.4) with the following data: $\Omega = (0, 1)^2$, $\Gamma_D = \partial\Omega$, $A = \mathrm{Id}$, $v = 0$, and the load term is 0 in $(0, 0.5) \times (0, 1)$ and has a checkerboard structure in $(0.5, 1) \times (0, 1)$:

$$f(x_1, x_2) = 10^4 \cdot \begin{cases} 0 & \text{if } x_1 \in (0, 0.5) \\ \chi(x_1)\chi(x_2) & \text{if } x_1 \in (0.5, 1) \end{cases} \tag{6.1}$$

where $\chi(s) = \operatorname{sgn}\sin(32\pi s)$, $s \in \mathbb{R}$ is a 'characteristic' function. Clearly, $u \neq 0$.

Figure 3 depicts approximate solutions and meshes of selected iterations, where the initial mesh consists of the four congruent triangles induced by the diagonals of the unit square.

Note that the load function (6.1) is $L_2(\Omega)$-orthogonal to the finite element space over the initial and (at least) 4 next triangulations. Consequently, $u_k = 0$ and thus

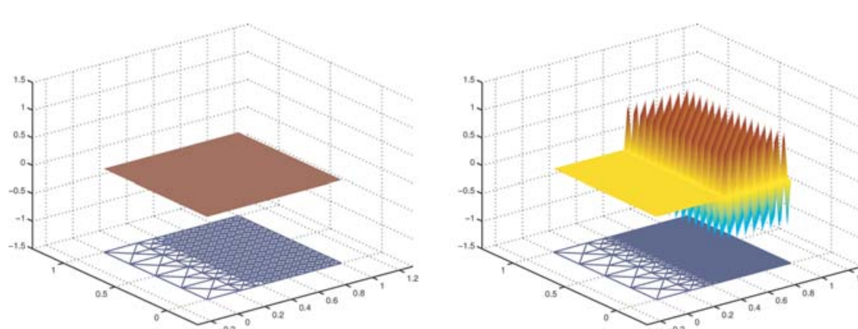

Fig. 3 Example (6.1): approximate solutions and meshes for iterations $k = 4$ (left) and $k = 5$ (right).

$\zeta_k = 0$ for $k = 0, \ldots, 4$. This shows that the principal part ξ_k alone is not reliable and may not provide any information for marking. Nevertheless, here thanks to the security parts $\overline{\rho}_k$ and $\underline{\rho}_{kz}$, Algorithm 5.1 does not stop and introduces a reasonable mesh grading during these first iterations.

A similar failure of the principal part ζ_k, which is then cured by $\overline{\delta}_k$ and $\overline{\delta}_{kz}$, takes place for oscillatory Dirichlet boundary values; cf. also the next example §6.2.

6.2 Interior and 'boundary' layer

The following example, which is taken from [19], illustrates that the extreme and somehow artificial situation of §6.1 may be 'partially' encountered in practice, entailing underestimation of the error. Let Ω be as in Figure 4, $\Gamma_D = \partial\Omega$, $A = \mathrm{Id}$, as well as v and f such that

$$u(x) = \arctan\left(60(|x|^2 - 1)\right) \tag{6.2}$$

is the exact solution. Notice that the initial mesh is unstructured, that the layer $\{x \in \Omega \mid |x| = 1\}$ crosses the boundary $\partial\Omega$, and that the load term oscillates across it. Correspondingly, relevant parts of it are $L_2(\Omega)$-orthogonal to the finite element space as long as the triangles in that region are relatively coarse.

As a consequence and as can be seen from Table 2, the principal part ζ_k underestimates the error in the beginning but its effectivity index improves with refinement and even reaches values quite close to 1. The latter is due to the superconvergence of the smoothed gradient and thus, in agreement with Corollary 5.1, the effectivity index of the total estimator $\mathcal{E}_k$ approaches 1 from above.

Notice that the effectivity index of principal term ζ_k gets closer to 1 than that of the total estimator $\mathcal{E}_k$. It may therefore be possible to improve the latter one by deriving a sharper upper bound that, in contrast to Theorem 4.1, takes the angle between $\nabla u_k - Gu_k$ and $Gu_k - \nabla u$ into account.

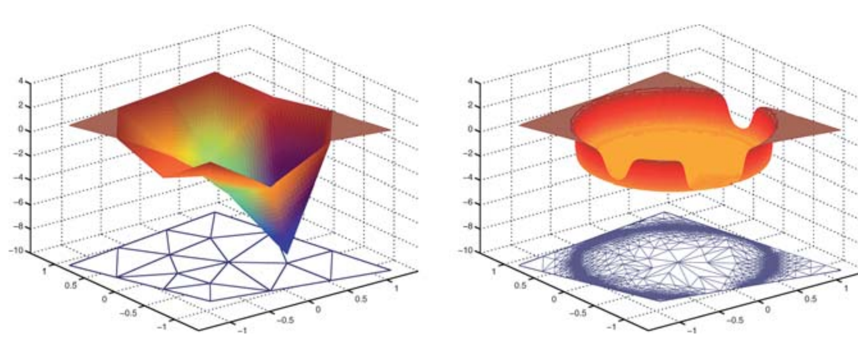

Fig. 4 Example (6.2): solution and mesh of iteration $k = 0$ (left) and $k = 9$ (right).

Table 2 Example (6.2): number of DOFs, error of untreated and smoothed gradient, and effectivity indices for selected iterations.

k	#DOFs	$\|\nabla(u_k - u)\|_A$	$\|Gu_k - \nabla u\|_A$	$\frac{\zeta_k}{\|\nabla(u_k - u)\|_A}$	$\frac{\mathcal{E}_k}{\|\nabla(u_k - u)\|_A}$
0	18	3.090e+01	2.754e+01	0.478	10.174
3	276	2.694e+01	2.705e+01	0.357	2.931
6	2227	1.362e+01	1.430e+01	0.737	2.174
9	12253	4.636e+00	3.438e+00	0.994	2.034
12	54226	1.971e+00	7.797e−01	0.996	1.560
15	391889	7.060e−01	1.860e−01	0.992	1.338
17	1350407	3.791e−01	8.787e−02	0.993	1.290

6.3 Crack problem

The solution (6.2) is in $H^2(\Omega)$, while the solution of the next example has a singularity. Let $\Omega = \{|x_1|+|x_2| < 1\}\setminus\{0 \le x_1 < 1, x_2 = 0\}$, $\Gamma_D = \partial\Omega$, $A = \mathrm{Id}$, $f = 1$ on Ω, and v such that the exact solution is in polar coordinates (r, θ) given by

$$u(r,\theta) = r^{1/2}\sin(\theta/2) - \tfrac{1}{2}r^2\sin^2\theta. \tag{6.3}$$

Due to the $r^{1/2}$-singularity, $u \notin H^2(\Omega)$. The second derivatives of u however exist in $L_1(\Omega)$ and so the error of nonlinear approximation decays with #DOFs$^{-1/2}$.

Figure 5 and Table 3 show that, in spite of the presence of the singularity, the smoothed gradient is superconvergent, whence the effectivity index of $\mathcal{E}_k$ approaches again 1 from above. Notice also that the decay of $\|\nabla(u_k - u)\|_A$ is optimal in that it coincides with the one of nonlinear approximation. This confirms the efficiency of the marking indicators stated in Corollary 5.2.

6.4 Anisotropic ellipticity

We conclude with an example where the condition number of the coefficient matrix A is large. Let $\Omega \in (0, 1)^2$ and

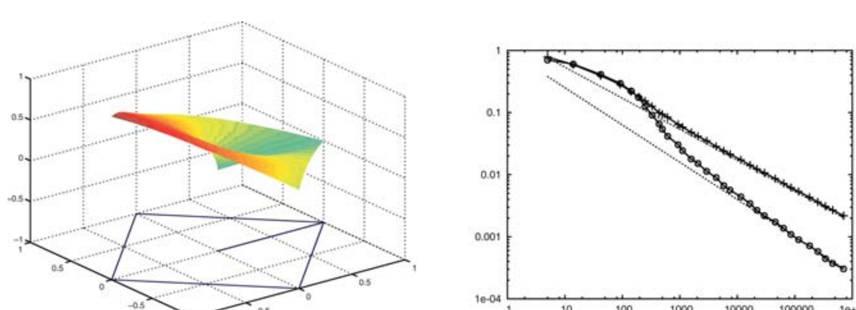

Fig. 5 Example (6.3): domain and solution of iteration $k = 17$ (left). Log-log plot of error of the untreated ('+') and smoothed ('∘') gradient versus number of DOFs; the decay rates −0.5 and −0.6 are indicated by dashed lines (right).

Table 3 Example (6.3): number of DOFs, error of untreated and smoothed gradient, and effectivity indices for selected iterations.

k	#DOFs	$\|\nabla(u_k - u)\|_A$	$\|Gu_k - \nabla u\|_A$	$\frac{\zeta_k}{\|\nabla(u_k - u)\|_A}$	$\frac{\mathcal{E}_k}{\|\nabla(u_k - u)\|_A}$
0	5	7.839e−01	6.996e−01	0.690	1.347
5	192	1.791e−01	1.746e−01	1.043	1.662
10	614	8.397e−02	4.149e−02	1.008	1.370
15	2 930	3.508e−02	1.131e−02	0.997	1.278
20	16 643	1.425e−02	3.423e−03	0.997	1.230
25	86 203	6.175e−03	1.090e−03	0.998	1.178
30	456 913	2.693e−03	3.738e−04	0.999	1.143

$$A = \begin{pmatrix} 0.1 & 0 \\ 0 & 10 \end{pmatrix} \tag{6.4a}$$

and consider two exact solutions

$$u^i(x_1, x_2) = 10^{-(i-1)} \sin(\pi x_i), \qquad i = 1, 2, \tag{6.4b}$$

combined with appropriate homogeneous Dirichlet and Neumann conditions so that the discretization is conforming. The solutions u^1 and u^2 have the same energy norm and profile, but depend, respectively, only on the direction associated to the eigenvalue 0.1 or 10. One thus expects that the difference $u^i_S - u^i$ between approximate and exact solution depends 'mainly' on the direction x_i, in particular when the triangulation has a suitable structure.

To approximate both exact solutions u^1 and u^2, we employed Algorithm 5.1 and the standard adaptive algorithm of ALBERTA using the explicit residual estimator (1.2) multiplied by $C_1 = 1/5$ and the maximum strategy. The results of

Table 4 Example 6.4: number of DOFs, error and effectivity indices of $\mathcal{E}_k$ (left subcolumns) and η_k (right subcolumns) related to minimum ($i = 1$) and maximum ($i = 2$) eigenvalue of A.

$i = 1$

k		#DOFs		$\|\nabla(u_S - u)\|_A$		$\frac{\mathcal{E}_k \text{ or } \sqrt{10}\,\eta_k}{\|\nabla(u_S - u)\|_A}$	
0	0	5	5	7.002e−01	7.002e−01	0.582	0.671
3	3	145	121	7.917e−02	7.917e−02	1.260	1.063
6	9	6 849	6 393	1.025e−02	1.021e−02	1.217	1.088
9	34	98 683	106 757	2.678e−03	2.540e−03	1.175	1.176
12	305	456 245	446 585	1.255e−03	1.251e−03	1.111	1.097

$i = 2$

k		#DOFs		$\|\nabla(u_S - u)\|_A$		$\frac{\mathcal{E}_k \text{ or } \sqrt{10}\,\eta_k}{\|\nabla(u_S - u)\|_A}$	
0	0	5	5	4.153e−01	4.153e−01	9.381	10.643
3	3	145	121	4.643e−02	4.689e−02	3.920	12.213
6	6	6 805	6 521	5.968e−03	6.011e−03	2.195	12.178
9	9	97 267	101 675	1.575e−03	1.516e−03	2.497	12.186
12	11	456 055	413 225	7.388e−04	7.528e−04	1.942	12.175

these four simulations, which all started from the initial mesh in §6.1, are given in Table 4.

We observe that, on comparable meshes, the energy norm error for $i = 1$ is a bit larger than the one for $i = 2$. This seems to be plausibile, because the case $i = 1$ of the minimum eigenvalue is less well-conditioned.

For the latter case $i = 1$, both estimators have effectivity indices that are bigger but close to 1, while the standard residual estimator needs more iterations to reach a given tolerance. The higher number of iterations seems to be partially due to the maximum strategy: for example, with the fixed fraction strategy of Dörfler [13], the errors of $1.624e{-}03$ and $8.053e{-}04$ are reached with 267 905 and 1 070 657 DOFs in 46 and 47 iterations, respectively. In any case, the lower iteration numbers with the marking indicators μ_{kz}, $z \in \mathcal{N}_k$, underlines the significance of Corollary 5.2.

For the case $i = 2$ of the maximum eigenvalue, both estimators start with an effectivity index of about 10. However, the effectivity index of $\mathcal{E}_k$ improves with refinement, while the one of the residual estimator $\sqrt{10}\,\eta_k$ remains about the same. In view of the other case $i = 1$, the large effectivity indices of $\sqrt{10}\,\eta_k$ are not due to a bad equilibration of the constant C_1 in (1.3). The improving effectivity indices of $\mathcal{E}_k$ are again a consequence of the superconvergence of the smoothed gradient and Corollary 5.1. For given tolerances, the consequences in terms of CPU time and #DOFs of these different effectivity indices are illustrated in Table 1 of the introduction §1; the CPU times correspond to a gcc compilation on an IBM Think Pad R40 with a 2 GHz Pentium 4M processor and 512 MB.

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