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A posteriori error estimators for obstacle problems – another look

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Abstract We show that a posteriori estimators for the obstacle problem are easily obtained from the theory for linear equations. The theory would be even simpler if the Lagrange multiplier does not have a nonconforming contribution as it has in actual finite element computations.

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1 Introduction

Elliptic obstacle problems often lead to the minimization of a quadratic functional on a subspace $V \subset H^1(\Omega)$,

$$\Pi(v) := \frac{1}{2}\, a(v, v) - (f, v) \tag{1.1}$$

subject to the constraint

$$v(x) \geq \chi(x) \quad \text{for almost all } x \in \Omega. \tag{1.2}$$

The solution u can be characterized by the linear equations

$$a(u, v) = (f, v) + \langle \sigma, v \rangle \quad \text{for all } v \in V, \tag{1.3}$$

where $\sigma \in V^*$ is the Lagrange multiplier associated to the constraint (1.2). Similarly, when the discrete solution u_h in a finite element space V_h has been computed, one has

$$a(u_h, v) = (f + \sigma_h, v) \quad \text{for all } v \in V_h. \tag{1.4}$$

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Obviously, u_h is also the finite element approximation of the solution of the linear elliptic problem

$$a(z, v) = (f + \sigma_h, v) \quad \text{for all } v \in V. \tag{1.5}$$

There is a well-established theory and a large number of a posteriori error estimators for linear equations, and $z - u_h$ can be controlled e. g. by residual based estimators, by local problems or by hierarchical estimators. More estimators for linear problems are found in the monographs [1, 10].

We will show that all the estimators for $\|z - u_h\|_1$ are also suited for the obstacle problem and the control of $\|u - u_h\|_1$. The a posteriori error analysis in [2, 5–9] is well understood in this framework. The crucial inequality is

$$\|u - u_h\|_1 \leq \|z - u_h\|_1 + \text{ higher order terms.} \tag{1.6}$$

It will not be necessary to repeat here the manipulations known from the theory of linear equations. Of course, one must be aware of traps when looking for a general theory. This will be indicated for the estimators in the literature cited above. We only have to add terms of higher order known from [2, 9]. The extra terms are not intrinsic for obstacle problems; they are caused by the fact that the constraint (1.2) is often modeled by conditions with point functionals in finite element computations and that $L_\infty(\Omega) \not\subset H^1(\Omega)$. – Remark 1(1) below justifies that the extra terms in (1.6) are labelled as *higher order terms*.

The efficiency of the estimators is also true if the Lagrange multiplier is incorporated into the estimation process; cf. [9]. This will be elucidated for residual based estimators by a general argument that is also satisfied by estimators based on local Dirichlet problems, Neumann problems or averaging techniques.

2 The Lagrange multiplier in the estimator

To be specific, let $a(\cdot, \cdot)$ be the quadratic form (associated to the Poisson equation)

$$a(u, v) := \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx$$

that is coercive on $V := H_0^1(\Omega)$, and let $(\cdot, \cdot)$ refer to the inner product in $L_2(\Omega)$. Moreover, let $V_+ := \{v \in V;\ v(x) \geq 0 \text{ a.e.}\}$. The obstacle is assumed to be given by a function $\chi \in V$.

The existence of the solution $u \in \mathcal{K} := \{v \in V;\ v(x) \geq \chi(x) \text{ for } x \in \Omega \text{ a.e.}\}$ is clear for the obstacle problem, and there is the well-known characterization

$$a(u, v - u) \geq (f, v - u) \quad \text{for all } v \in \mathcal{K}. \tag{2.1}$$

The associated Lagrange multiplier σ is defined by (1.3), i.e. $\langle \sigma, v \rangle := a(u, v) - (f, v)$. Here $\langle \cdot, \cdot \rangle$ refers to the pairing of H^{-1} and H_0^1, but we will also write (σ, v) whenever σ can be understood as an L_2 function. We rewrite (2.1) as

$$\langle \sigma, v - u \rangle \geq 0 \quad \text{for all } v \in \mathcal{K}. \tag{2.2}$$

By applying this inequality to $v := \chi$, $v := 2u - \chi$ and any $v \geq u$ we obtain

$$\langle \sigma, u - \chi \rangle = 0 \quad \text{and} \quad \langle \sigma, w \rangle \geq 0 \quad \text{for all } w \in V_+ \,, \tag{2.3}$$

which together with (2.2) provides the complemementarity conditions.

The discretization of the obstacle problem means that the linear space is replaced by the finite element space V_h and, in the case of interest, the constraint (1.2) is replaced by

$$v(x_i) \geq \chi(x_i), \quad i = 1, 2, \dots, \dim V_h,$$

where the x_i's are the nodal points of a basis of V_h. The Lagrange multiplier σ_h is given by the non-negative residues of the finite element equations in the contact zone, and

$$\langle \sigma_h, v_h \rangle \geq 0 \quad \text{for all } v_h \in V_h,\ v_h \geq 0. \tag{2.4}$$

Specifically, σ_h provides a unique functional on V_h, but the extension of σ_h from V_h^* to $H^{-1}(\Omega)$ is not unique. Following the literature, we may represent σ_h by an L_2 function, and we will understand σ_h in this way. Since σ_h arises from point functionals on V_h, the inequality $(\sigma_h, v) \geq 0$ cannot be guaranteed for all $v \in V_+$. For this reason, an approximation of σ by a nonnegative L_2 function is constructed in [2,9] by considering $\langle \sigma_h, v_h \rangle$ as a lumped integral. In this way, one gets a splitting,

$$\sigma_h = \sigma_h^+ + \sigma^{ce} \quad \text{with } (\sigma_h^+, v) \geq 0 \quad \text{for all } v \in V_+.$$

The complement σ^{ce} carries the *consistency error* of the extension.

We note that the sign of the Lagrange multipliers has been chosen here as usual in optimization theory, and σ_h^+ coincides with $-\sigma_h(u_h)$ in [9] and with $-\rho_h$ in [2].

For convenience, we refer to the energy norm and the dual norm that are equivalent to $\|\cdot\|_1$ and $\|\cdot\|_{-1}$, resp., and extend the symbols also to subsets $\omega \subset \Omega$,

$$|||v|||_\omega := \left(\int_\omega \nabla v(x) \cdot \nabla v(x)\, dx \right)^{1/2},$$
$$|||\lambda|||_{*,\omega} := \sup\{\langle \lambda, v \rangle;\ v \in H_0^1(\omega),\ |||v|||_\omega = 1\}.$$

The subscript ω will be omitted if $\omega = \Omega$. By definition, $|||w||| = |||\lambda|||_*$ if w is the solution of the auxiliary equation

$$a(w, v) = \langle \lambda, v \rangle \quad \text{for all } v \in V. \tag{2.5}$$

3 Reliable estimators

First we compare the finite element solution of the obstacle problem with the error of an associated linear elliptic equation. In this way it will be possible to apply all the well-known a posteriori error estimators for linear problems, and Lemma 1 will be crucial for our investigation.

For convenience, we assume that the obstacle is defined by a function $\chi = \chi_h \in V_h$. (Otherwise there are additional terms as given in [9, (4.1)].)

Lemma 1 *Let u_h be the finite element solution of the obstacle problem and z be the solution of the linear problem (1.5) for which u_h is also the finite element solution. Then*

$$|||u - u_h||| \le |||z - u_h||| + |||\sigma_h - \sigma_h^+|||_* + (\sigma_h^+, u_h - \chi)^{1/2}. \tag{3.1}$$

Proof We start with

$$\begin{aligned} a(u_h - u, v) - a(u_h - z, v) &= a(z - u, v) \\ &= (f, v) + \langle \sigma_h, v \rangle - [(f, v) + \langle \sigma, v \rangle] \\ &= \langle \sigma_h^+ - \sigma, v \rangle + \langle \sigma^{ce}, v \rangle. \end{aligned} \tag{3.2}$$

We recall that σ and σ_h^+ are nonnegative functionals and that also $\langle \sigma, u - \chi \rangle = 0$ holds according to (2.3). The first term in (3.2) is evaluated for $v := u_h - u$,

$$\begin{aligned} \langle \sigma_h^+ - \sigma, u_h - u \rangle &= \langle \sigma_h^+, u_h - \chi \rangle - \underbrace{\langle \sigma_h^+, u - \chi \rangle}_{\ge 0} - \underbrace{\langle \sigma, u_h - \chi \rangle}_{\ge 0} + \underbrace{\langle \sigma, u - \chi \rangle}_{=0} \\ &\le (\sigma_h^+, u_h - \chi). \end{aligned}$$

Combining the relations above we obtain

$$\begin{aligned} |||u_h - u|||^2 &= a(u_h - u, u_h - u) \\ &\le a(u_h - z, u_h - u) + \langle \sigma^{ce}, u_h - u \rangle + (\sigma_h^+, u_h - \chi) \\ &\le (|||u_h - z||| + |||\sigma^{ce}|||_*)\, |||u_h - u||| + (\sigma_h^+, u_h - \chi). \end{aligned}$$

Note that an inequality for positive numbers of the form $x^2 \le ax + b$ implies that $x \le a + b^{1/2}$, and the proof of (3.1) is complete. □

Remark 1 (1) While $(\sigma_h, u_h - \chi) = 0$, the term $(\sigma_h^+, u_h - \chi)$ does not vanish in general. It can be evaluated after u_h has been computed. If σ_h^+ is fixed as described in [5,9], only those elements of the triangulation contribute which have one node but not all its nodes in the contact area. We note that $(\sigma_h^+, u_h - \chi)$ was expressed in [9, Proposition 3.7] by quantities that occur in the residual error estimator anyway. In this framework the term $|||\sigma - \sigma^+|||_*$ is also expressed in terms of computable quantities. From the Bramble–Hilbert lemma it follows that it can be estimated by $ch\|\sigma^+\|_0$ or $ch^2\|\nabla\sigma^+\|_0$.

(2) The term $(\sigma_h^+, u_h - \chi)$ is treated in [2] by a different technique.

(3) The relation $(\sigma_h^+, u_h - \chi) = -(\sigma^{ce}, u_h - \chi)$ shows that we have an effect of the point functionals. The representation on the left hand side of the formula, however, is better suited for numerical computations of an estimator.

(4) An alternative to (3.1) is an estimator that refers to z^+, i.e. the solution of

$$a(z^+, v) = (f + \sigma_h^+, v) \quad \text{for all } v \in V. \tag{3.3}$$

We obtain by the same arguments as above

$$|||u - u_h||| \le |||z^+ - u_h||| + (\sigma_h^+, u_h - \chi)^{1/2}. \tag{3.4}$$

At first glance, this estimate seems to be preferable since there is one term less. This conceals the fact that u_h is the finite element solution of the linear problem (1.5), but not of (3.3). The standard a posteriori error estimators can be directly applied to estimate $|||u_h - z|||$ while modifications are required for bounds of $|||u_h - z^+|||$ since u_h is not the finite element approximation of z^+.

(5) The extra terms do not occur if the constraint (1.2) is modeled by inequalities in $L_2(\Omega)$ as e. g. in [8].

Following [9] we also want an estimate that includes the discretization error of the Lagrange multiplier.

Corollary 1 *Let u_h be the finite element solution of the obstacle problem and z be the solution of the auxiliary problem (1.5). Then*

$$|||u - u_h||| + |||\sigma - \sigma_h|||_* \le 3|||z - u_h||| + 2|||\sigma_h - \sigma_h^+|||_* + 2(\sigma_h^+, u_h - \chi)^{1/2}. \tag{3.5}$$

Proof Recalling (2.5) we have by the definition of the dual norm and the triangle inequality

$$|||\sigma - \sigma_h|||_* = |||u - z||| \le |||u - u_h||| + |||u_h - z|||.$$

Combining this inequality with (3.1) we obtain (3.5). □

For completeness and in order to be specific in the next section, we provide the application to residual based estimators; cf. [1,10]. Let Γ_h be the set of all inter-element boundaries of the triangulation $\mathcal{T}_h$ that is assumed to be shape-regular. When we look at u_h as the finite element solution of (1.5), there are the area-based residuals and the edge-based jumps,

$$\begin{aligned} R_T &:= R_T(u_h) := \Delta u_h + f + \sigma_h && \text{for } T \in \mathcal{T}_h\,, \\ R_e &:= R_e(u_h) := [\![\tfrac{\partial u_h}{\partial n}]\!] && \text{for } e \in \Gamma_h\,. \end{aligned}$$

They build the local estimator

$$\eta_T := \left\{ h_T^2 \|R_T\|_{0,T}^2 + \frac{1}{2} \sum_{e \in \partial T} h_e \|R_e\|_{0,e}^2 \right\}^{1/2} \quad \text{for } T \in \mathcal{T}_h. \tag{3.6}$$

Theorem 1 *Let u_h be the finite element solution of the obstacle problem. Then there exists a constant c such that*

$$|||u - u_h||| + |||\sigma - \sigma_h|||_* \le c \left\{ \sum_{T \in \mathcal{T}_h} \eta_{T,R}^2 \right\}^{1/2} + 2|||\sigma_h - \sigma_h^+|||_* \\ + 2(\sigma_h^+, u_h - \chi)^{1/2}.$$

Theorem 1 contains the error estimates in [5,8,9] if σ_h^+ is fixed as in the cited papers. Similarly, Corollary 1 can be used to derive the a posteriori estimates in [2,6,7]. We also drop the application of Corollary 1 to further estimators and note that the direct application is not always visible at first glance. So the estimates in [2] follow directly from [4], but the different treatments of the extra terms conceal this fact.

4 Efficiency of the estimators

When the efficiency of the estimator (3.6) is investigated, it turns out that the origin of the Lagrange multipliers (i.e. point functionals) formally does not imply complications as in the upper estimates. As a point of departure we recall that we know from the linear theory [1, 10] that

$$\eta_T \le c \left\{ |||z - u_h|||^2_{\omega_T} + \sum_{T' \subset \omega_T} h^2_{T'} \| f - f_h \|^2_{0,T'} \right\}^{1/2} . \tag{4.1}$$

Here, as usual, $\omega_T := \cup\{T' \in \mathcal{T}_h;\ T \text{ and } T' \text{ have a common edge or } T' = T\}$. Moreover, f_h is an approximation of f by piecewise linear (or quadratic) polynomials, and σ_h is assumed to be chosen as a function in the same polynomial space.

Although we have $|||u_h - z||| \le |||u_h - u||| + |||u - z||| = |||u_h - u||| + |||\sigma_h - \sigma|||_*$, we cannot use the triangle inequality for replacing $|||u_h - z|||_\omega$ by $|||u_h - u|||_\omega + |||\sigma_h - \sigma|||_{*,\omega}$. Fortunately, (4.1) is obtained in the literature by estimates with test functions in $H^1_0(\omega_T)$. Specifically, lower bounds with residual error estimators are usually established by a result that can be formulated as follows; see e.g. [1, pp. 28–31], [3, pp. 174–175] or [10, pp. 16–17]. (Here $\mathcal{P}_2$, the space of polynomials of degree lower or equal 2, may be replaced by another space of polynomials.)

Lemma 2 *Let w be the solution of*

$$a(w, v) = (p, v)_{0,\Omega} + \langle q, v \rangle + \sum_e (r_h, v)_{0,e} \quad \textit{for all } v \in H^1_0(\Omega),$$

where $p \in L_2(\Omega)$, $q \in H^{-1}(\Omega)$ and $r_h|_e \in \mathcal{P}_2$ for all edges e. Moreover, let p_h be an approximation of p such that $p_h|_T \in \mathcal{P}_2$ for all T. Then we have for each T,

$$h_T \|p\|_{0,T} + \sum_{e \in \partial T} h_e^{1/2} \|r_h\|_{0,e} \le c \left(\|w\|_{\omega_T} + h_T \|p - p_h\|_{0,\omega_T} + |||q|||_{*,\omega_T} \right) . \tag{4.2}$$

Now we recall that

$$a(u_h - z, v) = \sum_T (R_T, v)_{0,T} + \sum_e (R_e, v)_{0,e} , \tag{4.3}$$

$$a(u_h - u, v) = \sum_T (R_T, v)_{0,T} + \sum_e (R_e, v)_{0,e} + \langle \sigma - \sigma_h, v \rangle . \tag{4.4}$$

When comparing (4.3) and (4.4), we see that the residues can be estimated by $|||u_h - z|||$ or by $|||u_h - u|||$ as main terms. The first case is well known, and the difference arises from the last term in (4.4). Thus we obtain

Theorem 2 *Let u_h be the finite element solution of the obstacle problem and η_T be the estimator (3.6). Then*

$$\eta_T \le c \left(|||u - u_h|||_{\omega_T} + |||\sigma - \sigma_h|||_{*,\omega_T} + \sum_{T' \subset \omega_T} h_{T'} \| f - f_h \|_{0,T'} \right) . \tag{4.5}$$

The efficiency of estimators based on local problems is also clear; cf. [6,7]. Since test functions with local support are used, we have an analogue of Lemma 2. Moreover, the equivalence of various estimators can be found in [10].

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