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Quasi-Norm interpolation error estimates for the piecewise linear finite element approximation of p-Laplacian problems

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Summary. In this work, new interpolation error estimates have been derived for some well-known interpolators in the quasi-norms. The estimates are found to be essential to obtain the optimal a priori error bounds under the weakened regularity conditions for the piecewise linear finite element approximation of a class of degenerate equations. In particular, by using these estimates, we can close the existing gap between the regularity required for deriving the optimal error bounds and the regularity achievable for the smooth data for the 2-d and 3-d p-Laplacian.

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1 Introduction

In this work we study the continuous piecewise linear finite element approximation of the following quasi-linear equation with Dirichlet data:

$$\begin{aligned} &-\text{div}(k(|\nabla u|)\nabla u) = f \ \text{in } \Omega \\ &u = g \qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^d$ $(1 \leq d \leq 3)$ with a Lipschitz boundary $\partial\Omega$. Assumptions on the function k and the data f and g will be specified later. When k is smooth and satisfies the ellipticity and monotonicity conditions such as those given in [13] and [16], the problem (1) is

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known to be well-posed. Moreover, optimal error bounds for its finite element approximation have been established. For example, we have from [16] that if

(i) $(z, x) \to k(z, |x|)x \in C^2(\bar{\Omega} \times R^2)^2$, $f \in C^\alpha(\bar{\Omega})$ and $g \in C^{2,\alpha}(\bar{\Omega})$ for a $\alpha > 0$,
(ii) $\nabla_x(k(z, |x|)x)$ is a positive definite matrix for any $z \in \bar{\Omega}$ and $x \in R^2$,
(iii) $\int_\Omega (k(z, |\nabla w|)\nabla w - k(z, |\nabla v|)\nabla v, \nabla(w - v))dz \ \geq \ C_x \int_\Omega |\nabla(w - v)|^2 dz$,
for any $w, v \in W^{1,\infty}$ such that $(w - v) \in H_0^1(\Omega)$ and $\|w\|_{W^{1,\infty}(\Omega)} + \|v\|_{W^{1,\infty}(\Omega)} \leq X$, and furthermore if $u \in C^{2,\alpha}(\bar{\Omega})$ and u^t (the solution of the deformation equation in [16]) is uniformly bounded in $W^{1,\infty}(\Omega)$ then

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch, \ \ \|u - u_h\|_{L^2(\Omega)} \leq Ch^2$$

and

$$\|u - u_h\|_{L^\infty(\Omega)} \leq Ch^2 |\ln h|^{\frac{2}{3}},$$

where u_h is the piecewise linear finite element approximation of (1) based on a quasi-uniform triangulation.

For many physical models, k is degenerate, and therefore does not satisfy these conditions. A typical case is $k(t) = t^{p-2}$ with $p > 1$ and $p \neq 2$ (the p-Laplacian). In this case the above equation is not uniformly elliptic as it degenerates at unknown points $x \in \Omega$, where $|\nabla u(x)| = 0$. Such models arise in, for example, power-law materials, see [3], and non-Newtonian flows, see [4] and [27]. For such degenerate nonlinear systems many existing techniques (such as linearization or deformation procedure in [13] and [16]) in finite element method do not work well. For instance, a priori error estimates were suboptimal in all the earlier work, see [11] ,[12] and [17].

During the last decade, there has been significant progress in finite element approximation of this class of degenerate equations, see [1], [2], [10], [6]–[5], [9], [15], [20], and [21]–[31]. In particular, extensive research has been carried out for the finite element approximation of the p-Laplacian, since it is believed that this simpler equation contains most of the essential difficulties in the studies of finite element approximation of this class of degenerate systems. One of the significant developments is the introduction of the quasi-norm approach. One of the key ideas of this approach is to first estimate the approximation error in a quasi-norm, which naturally arises in degenerate problems of this type although it may not be equivalent to the norm of the underlaying Sobolev space. Then explicit sharp error bounds in the underlaying Sobolev space can be derived from the relationship between the quasi-norm and the Sobolev norm. The precise details will be seen later on.

With this approach, sharp a priori error estimates have been established for the conforming and non-conforming piecewise linear finite element approximation of the p-Laplacian and related equations, see [6], [24], and [26]. Recently, the quasi-norm techniques have been further developed, and improved a posteriori error estimates have also been derived, see [10], [25], and [26]. One can find some summaries of the quasi-norm techniques in [7], [25] and [26].

There are however many open problems in this area. Notably, there still exists a gap between the regularity required for deriving the optimal error bounds and the known regularity achievable for the data sufficiently smooth (to be specified later). For the p-Laplacian with $1 < p < 2$ for instance, it was proved in [6] that

$$\|u - u_h\|_{W^{1,p}} \leq ch, \tag{2}$$

where u_h is the continuous piecewise linear finite element approximation of the solution u, provided $u \in W^{3,1}(\Omega) \cap C^{2,1+2/p}(\bar{\Omega})$ and

$$\int_\Omega |\nabla u|^{p-2}|D^2u|^2 < \infty. \tag{3}$$

In [14] and [23], it was shown that **(3) is in general true for the p-Laplacian** with $1 < p < \infty$ and smooth data. However, the regularity $u \in W^{3,1}(\Omega) \cap C^{2,1+2/p}(\bar{\Omega})$ was only shown for the p-harmonic functions where $f = 0$, see [21]. Indeed it does not seem that such higher regularity is in general achievable for the p-Laplacian even with very smooth data. In [23], this regularity condition was weakened to: (3) and $u \in W^{1+2/p,p}(\Omega)$. Although it was shown in [22] that $u \in H^2(\Omega)$ on any 2-d smooth or convex domains for a class of degenerate equations including the p-Laplacian for $1 < p < 2$, it is still not clear whether or not this weakened regularity is in general achievable for smooth data, since $1 + 2/p > 2$ when $1 < p < 2$. Ideally one would expect the regularity required for deriving the optimal error bounds should be achievable for the data sufficiently smooth. Thus it is tempted to guess that **the condition (3) may be sufficient to ensure the optimal error bound**. However it has been difficult to prove such results due to the lack of a suitable interpolation error estimation theory in the quasi-norms. In all the existing work, the Taylor expansions are used in deriving interpolation error estimates in quasi-norms, and this seems to lead to unnecessarily higher requirements for the regularity of the solutions. It was unclear whether or not the elegant interpolation error theory in the Sobolev norms (see [12]) can be extended to the quasi-norm case, since the homogeneity of the norms plays a key role in the establishment of the theory.

It is the purpose of this work to extend some of the interpolation error estimation theory in the Sobolev norms to the quasi-norm case for some well-known interpolators like the Lagrange interpolator. Particularly for such interpolators P^1 we have:

Lemma *Let $v \in W^{2,p}(K)$. Then there is a constant c independent of v such that*

$$\left\| v - P^1 v \right\|^2_{(1,v,K)} \leq c |v|^2_{(2,v,K)}, \tag{4}$$

where K is a reference element and $\| \cdot \|_{(i,v,K)}$ are the quasi-norms to be defined later.

With the new framework established, essentially we can show, for example, that the **regularity (3) is indeed sufficient for the optimal error bounds for the p-Laplacian** with the data sufficiently smooth:

Theorem: *Let u and u_h be the solutions of the p-Laplacian and its piecewise linear finite element approximation respectively. Then*

$$|u - u_h|^2_{(1,u,\Omega)} \leq Ch^2 \int_\Omega |\nabla u|^{p-2} \left| D^2 u \right|^2 .$$

The precise details can be found in the Sections 3-5.

The plan of this work is as follows: In Section 2, we specify the conditions for the function k and the data, and give the weak formulation of the equation (1). Then we study its finite element approximation, and introduce some quasi-norms which are among the keys to our error estimation. In Sections 3-4 we establish a series of new interpolation error estimates in the quasi-norms for some well-known interpolators. In Section 5, we apply these new estimates to the finite element approximation of the p-Laplacian, and then obtain some optimal a priori error bounds with the regularity requirements achievable for sufficiently smooth data.

2 Finite Element Approximation of p-Laplacian and related equations

In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\| \cdot \|_{W^{m,q}(\Omega)}$ (or $\| \cdot \|_{m,q,\Omega}$ as a simplification) and semi-norm $| \cdot |_{W^{m,q}(\Omega)}$ (or $| \cdot |_{m,q,\Omega}$). We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with norm $\| \cdot \|_{m,\Omega}$ and semi-norm $| \cdot |_{m,\Omega}$. In addition c or C denotes a generic positive constant independent of h.

We first state the assumptions on k and the data for the weak formulation of the equation (1). As in [20], we make the following assumptions on the data:

Assumptions (A): Let Ω be a bounded domain in $\mathbb{R}^d$, $d \leq 3$, with a Lipschitz boundary $\partial\Omega$ in the case $d \geq 2$. We assume that $k \in C(\bar{\Omega} \times (0, \infty)$ and throughout the paper we will denote $k(x, s)$ by $k(s)$ for all $x \in \bar{\Omega}$.

Moreover, we assume that there exist constants $p \in (1, \infty)$, $\alpha \in [0, 1]$ and $C, M > 0$ such that for all $x \in \bar{\Omega}$

$$0 \le k(s)s \le C[s^{\alpha}(1+s)^{1-\alpha}]^{p-2}s \quad \forall s > 0,$$
$$|k(s_2)s_2 - k(s_1)s_1| \le C[(s_2+s_1)^{\alpha}(1+s_2+s_1)^{1-\alpha}]^{p-2}|s_2 - s_1|$$

$\forall s_2, s_1 \ge 0$, and

$$k(s_2)s_2 - k(s_1)s_1 \ge M[(s_2+s_1)^{\alpha}(1+s_2+s_1)^{1-\alpha}]^{p-2}(s_2 - s_1)$$

$\forall s_2 \ge s_1 \ge 0$. We note that many functions k met in practical problems satisfy (A) including the p-Laplacian. For example $k(s) \equiv [s^{\mu}(1+s)^{1-\mu}]^{p-2}$ with $p \in (1, \infty)$ and $\mu \in [0, 1]$ satisfies (A) with $\alpha = \mu$ the parameter α in (A) measures the degree of degeneracy in $k(\cdot)$ with $\alpha = 1$ implying full degeneracy (the p-Laplacian) and $\alpha = 0$ implying no degeneracy.

For ease of exposition we will only deal with homogeneous Dirichlet data. Given $f \in L^{p'}(\Omega)$, the weak formulation of the equation (1) reads: (WP) seek $u \in W_0^{1,p}(\Omega)$ with

$$a(u, v) = (f, v) \quad \text{for all } v \in W_0^{1,p}(\Omega). \tag{5}$$

where

$$a(u, v) = \int_{\Omega} k(|\nabla u|)\nabla u \cdot \nabla v, \quad (w, v) = \int_{\Omega} wv.$$

It is a simple matter to show that there exists a unique solution to (WP), since the functional: $J(v) = \int_{\Omega} k(|\nabla v|)\nabla v \cdot \nabla v - \int_{\Omega} fv$ is strictly convex, and $J(v) \to \infty$ as $\|v\|_{W^{1,p}} \to \infty$. There has been a great deal of work on the regularity of the solution u to (WP). For instance, for sufficiently regular data, global $C^{1,\alpha}$ regularity is established in [19] for the p-Laplacian. Higher order regularity (like $H^2(\Omega)$) is investigated in [21]–[23]. In the rest of the paper, we assume that the solution u is continuous.

Before studying the finite element approximation of (WP), we introduce the finite element spaces. Let T^h be a regular triangulation [12] of Ω^h into disjoint open regular triangles K, so that $\bar{\Omega} = \bigcup_{K \in T^h} \bar{K}$. Each element has at most one edge on $\partial\Omega$, and $\bar{K}$ and $\bar{K}'$ have either only one common vertex, or a whole edge if K and $K' \in T^h$. Let h_K denote the diameter of the element K in T^h and let ρ_K denote the diameter of the largest ball contained in K. We assume that there is a regularity constant R of T^h, independent of h, such that $1 \le \max_{K \in T^h}(h_K/\rho_K) \le R$. Let $h = \max_{K \in T^h} h_K$. Furthermore, we assume that there is a $C^1(\overline{\Omega})$–function $h(x)$ such that

$$c' h_K \le h(x) \le h_K \qquad \text{in } K \tag{6}$$

for all simplices $K \in T^h$ and some constant $c' > 0$ independent of K.

We shall only discuss the continuous piecewise linear element and a simple non-conforming element in this paper due to the limited higher order regularity for the solution of the p-Laplacian, see, for instance, [21] and [23], for the details. Basically even with very smooth data, any third order regularity is in general impossible for the solution of the p-Laplacian unless $f = 0$.

Conforming element: Associated with T^h is a finite dimensional subspace V^h of $C^0(\bar{\Omega}^h)$, such that $\chi|_K \in \mathcal{P}_1$ for all $\chi \in V^h$ and $K \in T^h$, where $\mathcal{P}_1$ is the linear functions space. Let

$$V_0^h = \{\chi \in V^h : \chi(x^k) = 0, \quad \text{for all vertices } x^k \in \partial\Omega^h\}.$$

Then the finite element approximation of (WP) is as follows $(WP)^h$: Find $u_h \in V_0^h$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_0^h,$$

where

$$a(u_h, v_h) = \int_{\Omega^h} k(|\nabla u_h|)\nabla u_h \cdot \nabla v_h,$$
$$(f, v_h) = \int_{\Omega^h} f v_h.$$

It is a simple matter to show that $(WP)^h$ has a unique solution u_h.

Non-conforming element: Associated with T^h is the Strang-Fix type finite dimensional subspace $\tilde{V}^h$ of $L^2(\Omega^h)$:

$$\tilde{V}^h = \{v \in L^2(\Omega^h) : v|_K \in \mathcal{P}_1, \forall K \in T^h, v \text{ is continuous on the midpoints of edges }\}.$$

Let

$$\tilde{V}_0^h = \{v \in \tilde{V}^h : v = 0, \text{ on the midpoints of edges on } \partial\Omega\}.$$

Note that $\tilde{V}_0^h \not\subset W_0^{1,p}(\Omega)$. The finite element approximation of (WP) is as follows $(WP)_n^h$: Find $u_h \in \tilde{V}_0^h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in \tilde{V}_0^h,$$

where

$$a_h(u_h, v_h) = \sum_K \int_K k(|\nabla u_h|)\nabla u_h \cdot \nabla v_h,$$
$$(f, v_h) = \int_{\Omega^h} f v_h.$$

It is also a simple matter to show that $(WP)_n^h$ has a unique solution u_h. Analysis of the finite element approximation of (WP) and a priori error bounds for this approximation are discussed in [26]. In particular, the optimal a priori error bounds have been established there.

One of the key ideas in our approach is to introduce some quasi-norms to handle the degeneracy of the p-Laplacian, in order to obtain sharp error bounds. We briefly introduce some quasi-norms and the relations between them and the standard Sobolev norms. Let us first examine the conforming approximation.

Let $x, y \in \mathbb{R}^d$ and $0 \le \alpha \le 1$. We define the function

$$\omega(x, y) = (|x| + |y|)^{\alpha}(1 + |x| + |y|)^{1-\alpha}. \tag{7}$$

For $k \in \{0, 1, \dots\}$ and $w \in W^{1,p}(\Omega)$ we set

$$|v|^2_{(k,w,\Omega)} = \int_\Omega \left[\omega(\nabla w, D^k v)\right]^{p-2} \left|D^k v\right|^2 ,$$

where ω is defined by (7) and $|D^k v| = (\sum_{|\gamma|=k} |\partial^\gamma v|^2)^{1/2}$. Further, let

$$\|v\|^2_{(k,w,\Omega)} = \sum_{j=0}^{k} |v|^2_{(j,w,\Omega)}.$$

Thus, if $\alpha = 1$ we have

$$|v|^2_{(k,w,\Omega)} = \int_\Omega (|\nabla w| + |D^k v|)^{p-2} \left|D^k v\right|^2 .$$

Given $v, w \in W^{1,p}(\Omega)$, set

$$|v|^2_{(w,p)} \equiv \int_\Omega (|\nabla w| + |\nabla v|)^{(p-2)} |\nabla v|^2. \tag{8}$$

Proposition 2.1 *(i) It holds $|v|_{(w,p)} \ge 0$, and, when $v \in W_0^{1,p}(\Omega)$, $|v|_{(w,p)} = 0$ if and only if $v = 0$.*
(ii) There holds $|v_1 + v_2|_{(w,p)} \le C(|v_1|_{(w,p)} + |v_2|_{(w,p)})$ for any $v_1, v_2 \in W^{1,p}(\Omega)$.
(iii) Furthermore, for $1 < p \le 2$, there holds

$$|v|_{W^{1,p}(\Omega)} \le C(|w|_{W^{1,p}(\Omega)}, |v|_{W^{1,p}(\Omega)})|v|_{(w,p)} \text{ and } |v|^2_{(w,p)} \le |v|^p_{W^{1,p}(\Omega)}. \tag{9}$$

(iv) For $2 \le p < \infty$, $s \in [2, p]$, $r = s(2 - p)/(2 - s)$, there holds

$$|v|^p_{W^{1,p}(\Omega)} \le |v|^2_{(w,p)} \le C(|w|_{W^{1,r}(\Omega)}, |v|_{W^{1,r}(\Omega)})|v|^2_{W^{1,s}(\Omega)}. \tag{10}$$

Proof The conclusion (ii) can be proved by the fact that for all $a \geq 0$ the function $t \to t^2(a+t)^{p-2}$ is monotonically increasing for $t \geq 0$, and the inequality (Lemma 5.4 in [25]):

$$(a+t+s)^{p-2}(t+s)^2 \leq C((a+t)^{p-2}t^2 + (a+s)^2 s^2).$$

The rest can be simply proved with Holder inequalities. The readers are referred to [8] and [25] for the details. □

For the non-conforming case, we modify the definitions of the above quasi-norms in the natural way. Let $W^{1,p}(\Omega, T^h) \equiv \{v \in L^p(\Omega) : v|_K \in W^{1,p}(K),\ \forall K \in T^h\}$. Let

$$\|v\|_{W^{1,p}(\Omega,T^h)} = (\sum_{K\in T^h} \|v\|^p_{W^{1,p}(K)})^{1/p}, \tag{11}$$

and

$$|v|_{W^{1,p}(\Omega,T^h)} = (\sum_{K\in T^h} |v|^p_{W^{1,p}(K)})^{1/p}. \tag{12}$$

Let $w \in W^{1,p}(\Omega, T^h)$. We define for any $v \in W^{1,p}(\Omega, T^h)$

$$|v|^2_{(k,w,\Omega,T^h)} = \sum_{K\in T^h} |v|^2_{(k,w,K)}, \tag{13}$$

and

$$\|v\|^2_{(k,w,\Omega,T^h)} = \sum_{j=0}^{k} |v|^2_{(j,w,\Omega,T^h)}. \tag{14}$$

It is easy to see that $\|v\|_{W^{1,p}(\Omega,T^h)} = \|v\|_{W^{1,p}(\Omega)}$, $|v|_{W^{1,p}(\Omega,T^h)} = |v|_{W^{1,p}(\Omega)}$, and

$$|v|^2_{(k,w,\Omega,T^h)} = |v|^2_{(k,w,\Omega)}, \quad \|v\|^2_{(k,w,\Omega,T^h)} = \|v\|^2_{(k,w,\Omega)}$$

if $w, v \in W^{1,p}(\Omega)$. Thus sometimes we do not distinguish them if no confusion is likely to cause.

The essential relationships between the quasi-norm and the equation are reflected in the following inequalities. If u solves (WP) and $v \in W^{1,p}(\Omega)$, then

$$C|u-v|^2_{(1,u,\Omega)} \leq a(u, u-v) - a(v, u-v). \tag{15}$$

For any $\theta > 0$ and $v, w \in W^{1,p}(\Omega)$, there exists a $\gamma > 0$, such that

$$|a(u,w) - a(v,w)| \leq C(\theta^\gamma |u-v|^2_{(1,u,\Omega)} + \theta |w|^2_{(1,u,\Omega)}). \tag{16}$$

The above two inequalities can be proved similarly as in [20] (Proposition 3.2) and [26]. The main tools for the proofs are some elementary inequalities, which were given in [20], [25], and [26].

Then it follows from (15)-(16) that, for any $u, v \in W^{1,p}(\Omega)$,

$$c(a(u, u-v) - a(v, u-v)) \le |u-v|^2_{(1,u,\Omega)} \le C(a(u, u-v) - a(v, u-v)).$$

Thus the quasi-norm is naturally related to the total energy difference. Let us note that the above equivalence does not hold in the $W^{1,p}$-norm for the degenerate case.

It then follows from the relations (15)-(16) that we have

Theorem 2.1 *Let u, u_h be the solutions of (WP) and $(WP)^h$ respectively. Then we have the following optimal a priori error estimate in the quasi-norm:*

$$|u-u_h|^2_{(1,u,\Omega)} \le C \min_{v_h \in V_0^h} |u-v_h|^2_{(1,u,\Omega)}. \tag{17}$$

Let u, u_h be the solution of (WP) and $(WP)^h_n$ respectively. Then

(18)
$$|u-u_h|^2_{(1,u,\Omega,T^h)} \le C \min_{v_h \in \tilde{V}_0^h} \Big(\sum_{K \in T^h} |u-v_h|^2_{(1,u,K)} + h_K |u-v_h|^2_{(1,u,\partial K)} \Big).$$

Proof For the conforming case, we have for any $v_h \in V_0^h$,

$$a(u, u-u_h) - a(u_h, u-u_h) = a(u, u-v_h) - a(u_h, u-v_h).$$

Then it follows from (15)-(16) that

$$\begin{aligned} c\|u-u_h\|^2_{(1,u,\Omega)} &\le a(u, u-u_h) - a(u_h, u-u_h) \\ &= a(u, u-v_h) - a(u_h, u-v_h) \\ &\le \theta C \|u-u^h\|^2_{(1,u,\Omega)} + \theta^{-\gamma} C \|u-v^h\|^2_{(1,u,\Omega)}. \end{aligned}$$

Let θ be small enough and we have (17). For the proof of (18), readers are referred to Theorem 4.1 in [26]. □

Explicit error bounds can then be obtained by estimating the subspace approximation error like $\min_{v_h \in V_0^h} |u-v_h|^2_{(u,p)}$ and exploiting Proposition 2.1. For example if u, u_h are the solutions of (WP) and $(WP)^h$ with $1 < p \le 2$, one has the optimal a priori error bound in the $W^{1,p}$ norm

$$\|u - u_h\|_{W^{1,p}(\Omega)} \le Ch,$$

provided u is smooth enough, see [20],[21] for the details. One can have similar sharp error bounds for the non-conforming approximation, see [26].

To bound up the subspace approximation error, one generally needs to estimate $|u - Pu|^2_{(1,u,\Omega)}$, where Pu is an interpolant of u. We wish to do so with the minimum regularity requirements of u, and this will be studied in the next section.

3 Interpolation Error Estimates for Piecewise Linear Elements in Sobolev Spaces

We first establish the estimates on a reference element. Let $\hat{K} \subset \mathbb{R}^d$ be the reference element. We assume that $\hat{K}$ is a d-simplex. For $p > 1$, let P^1: $W^{2,p}(\hat{K}) \to \mathcal{P}_1(\hat{K})$ be a linear continuous operator from $W^{2,p}(\hat{K})$ to $W^{1,p}(\hat{K})$. Furthermore assume that there is a constant c, independent of v (may depend on p), such that

$$\left\|v - P^1 v\right\|_{W^{1,p}(\hat{K})} \le c|v|_{W^{2,p}(\hat{K})} \qquad \text{for all } v \in W^{2,p}(\hat{K}). \tag{19}$$

This type of estimation is one of the key ingredients in establishing the finite element interpolation error estimation theory in the Sobolev norms. There have been some attempts to prove a quasi-norm analogous of (19). However it was not even clear what is the correct form of the possible extensions. Furthermore, the above estimate is often proved via the well-known contradiction arguments (see [12]), which seem to essentially utilize the homogeneity of the norms. Thus it has been found to be very difficult to prove such estimates in the quasi-norms using such arguments. It turns out that one has to assume the estimates like (19) in order to establish the finite element interpolation error estimation theory in the quasi-norms, as we will see below.

The following lemma states the basic estimate for the piecewise linear elements.

Lemma 3.1 *Let $p > \max(1, \frac{2d}{d+2})$ and $v \in W^{2,p}(\hat{K})$. Then there is a constant c independent of v such that*

$$\left\|v - P^1 v\right\|^2_{(1,v,\hat{K})} \le c|v|^2_{(2,v,\hat{K})}. \tag{20}$$

Proof Since the proof techniques are essentially different for the case $p < 2$ and $p > 2$, we divide our proofs into two parts.

The case p < 2. Let us prove there is a constant c independent of v such that

$$\int_{\hat{K}} \left[\omega(\nabla v, \widetilde{D}^1(v - P^1 v))\right]^{p-2} \left|\widetilde{D}^1(v - P^1 v)\right|^2 \leq c|v|^2_{(2,v,\hat{K})}, \tag{21}$$

where ω was defined in (7) and $|\widetilde{D}^1 v| = (\sum_{|\alpha|=0}^{1} |\partial^\alpha v|^2)^{1/2}$. Notice that the function

$$g(s) = [(c+s)^\alpha (1+c+s)^{1-\alpha}]^{p-2} s^2 \qquad (c \geq 0)$$

is monotone increasing for $s \geq 0$. Thus, we have for $k = 0, 1$

$$|v - P^1 v|^2_{(k,v,\hat{K})} \leq \int_{\hat{K}} \left[\omega(\nabla v, \widetilde{D}^1(v - P^1 v))\right]^{p-2} \left|\widetilde{D}^1(v - P^1 v)\right|^2.$$

Hence, from (21) the assertion follows. Let us note that $|\nabla P^1 v|$ is constant on $\hat{K}$. We set $\lambda = |\nabla P^1 v|$. We consider four cases.

Case 1: $|v|_{W^{2,p}(\hat{K})} > \lambda$ and $|v|_{W^{2,p}(\hat{K})} > 1$. First, we estimate $|v|^2_{(2,v,\hat{K})}$ from below. For $p < 2$ the Hölder inequality yields (with $q_1 = \frac{2}{2-p}$ and $q_2 = \frac{2}{p}$)

$$\begin{aligned} \int_{\hat{K}} |D^2 v|^p &= \int_{\hat{K}} [\omega(\nabla v, D^2 v)]^{\frac{p}{2}(2-p)} [\omega(\nabla v, D^2 v)]^{-\frac{p}{2}(2-p)} |D^2 v|^p \\ &\leq \left(\int_{\hat{K}} [\omega(\nabla v, D^2 v)]^p\right)^{\frac{2-p}{2}} \left(\int_{\hat{K}} [\omega(\nabla v, D^2 v)]^{p-2} |D^2 v|^2\right)^{\frac{p}{2}}. \end{aligned} \tag{22}$$

Notice that

$$\begin{aligned} \int_{\hat{K}} [\omega(\nabla v, D^2 v)]^p &\leq \int_{\hat{K}} \left(1 + |\nabla v| + |D^2 v|\right)^p \\ &\leq \int_{\hat{K}} \left(1 + |\nabla P^1 v| + |\nabla(v - P^1 v)| + |D^2 v|\right)^p \\ &\leq c\left(\left\|1 + |\nabla P^1 v|\right\|^p_{L^p(\hat{K})} + |v - P^1 v|^p_{W^{1,p}(\hat{K})} + |v|^p_{W^{2,p}(\hat{K})}\right). \end{aligned}$$

Due to (19) and the fact that $\left\|1 + |\nabla P^1 v|\right\|_{L^p(\hat{K})} \leq c|v|_{W^{2,p}(\hat{K})}$ we have

$$\int_{\hat{K}} [\omega(\nabla v, D^2 v)]^p \leq c\,|v|^p_{W^{2,p}(\hat{K})}.$$

Hence, from (22) we obtain

$$|v|^p_{W^{2,p}(\hat{K})} \leq c\,|v|^{\frac{p(2-p)}{2}}_{W^{2,p}(\hat{K})} |v|^p_{(2,v,\hat{K})};$$

that is,

$$|v|^p_{W^{2,p}(\hat{K})} \le c|v|^2_{(2,v,\hat{K})}. \tag{23}$$

Next, using (19) and the fact that

$$\omega(\nabla v, \widetilde{D}^1(v - P^1 v)) \ge \left|\widetilde{D}^1(v - P^1 v)\right|$$

we find

$$\begin{aligned}
&\int_{\hat{K}} \left[\omega(\nabla v, \widetilde{D}^1(v - P^1 v))\right]^{p-2} \left|\widetilde{D}^1(v - P^1 v)\right|^2 \\
&\quad \le \int_{\hat{K}} \left|\widetilde{D}^1(v - P^1 v)\right|^p \\
&\quad \le c \left\|v - P^1 v\right\|^p_{W^{1,p}(\hat{K})} \\
&\quad \le c|v|^p_{W^{2,p}(\hat{K})}.
\end{aligned} \tag{24}$$

Altogether, (23) and (24) imply (21). Thus, the assertion follows.

Case 2: $|v|_{W^{2,p}(\hat{K})} > \lambda$ and $|v|_{W^{2,p}(\hat{K})} \le 1$. The Hölder inequality entails (see (22))

$$|v|^p_{W^{2,p}(\hat{K})} \le \left(\int_{\hat{K}} [\omega(\nabla v, D^2 v)]^p\right)^{\frac{2-p}{2}} |v|^p_{(2,v,\hat{K})}. \tag{25}$$

Applying again the Hölder inequality (with $q_1 = \frac{1}{\alpha}$ and $q_2 = \frac{1}{1-\alpha}$) yields

$$\begin{aligned}
&\int_{\hat{K}} [\omega(\nabla v, D^2 v)]^p \\
&\quad = \int_{\hat{K}} \left(|\nabla v| + |D^2 v|\right)^{p\alpha} \left(1 + |\nabla v| + |D^2 v|\right)^{p(1-\alpha)} \\
&\quad \le \left\||\nabla v| + |D^2 v|\right\|^{p\alpha}_{L^p(\hat{K})} \left\|1 + |\nabla v| + |D^2 v|\right\|^{p(1-\alpha)}_{L^p(\hat{K})}.
\end{aligned} \tag{26}$$

Using (19) and the fact that $|P^1 v|_{W^{1,p}(\hat{K})} \le c|v|_{W^{2,p}(\hat{K})}$ we get

$$\begin{aligned}
&\left\||\nabla v| + |D^2 v|\right\|^{p\alpha}_{L^p(\hat{K})} \\
&\quad \le \left\||\nabla P^1 v| + |\nabla(v - P^1 v)| + |D^2 v|\right\|^{p\alpha}_{L^p(\hat{K})} \\
&\quad \le c\left(|P^1 v|^{p\alpha}_{W^{1,p}(\hat{K})} + |v - P^1 v|^{p\alpha}_{W^{1,p}(\hat{K})} + |v|^{p\alpha}_{W^{2,p}(\hat{K})}\right) \\
&\quad \le c\,|v|^{p\alpha}_{W^{2,p}(\hat{K})}.
\end{aligned} \tag{27}$$

Recalling that $|v|_{W^{2,p}(\hat{K})} \leq 1$ we obtain

$$\begin{aligned}
&\left\| 1 + |\nabla v| + |D^2 v| \right\|_{L^p(\hat{K})}^{p(1-\alpha)} \\
&\quad \leq c \left(\left\| 1 + |\nabla P^1 v| \right\|_{L^p(\hat{K})}^{p(1-\alpha)} + |v - P^1 v|_{W^{1,p}(\hat{K})}^{p(1-\alpha)} + |v|_{W^{2,p}(\hat{K})}^{p(1-\alpha)} \right) \\
&\quad \leq c \left(1 + |v|_{W^{2,p}(\hat{K})}^{p(1-\alpha)} \right) \\
&\quad \leq c. \qquad (28)
\end{aligned}$$

Altogether, it follows that

$$\int_{\hat{K}} [\omega(\nabla v, D^2 v)]^p \leq c\, |v|_{W^{2,p}(\hat{K})}^{p\alpha}.$$

In view of (25) we arrive at

$$|v|_{W^{2,p}(\hat{K})}^{p} \leq c\, |v|_{W^{2,p}(\hat{K})}^{p\alpha \frac{2-p}{2}} \, |v|_{(2,v,\hat{K})}^{p}.$$

This entails

$$|v|_{W^{2,p}(\hat{K})}^{2} \leq c\, |v|_{W^{2,p}(\hat{K})}^{\alpha(2-p)} \, |v|_{(2,v,\hat{K})}^{2};$$

thus,

$$|v|_{W^{2,p}(\hat{K})}^{2(1-\alpha)+p\alpha} \leq c\, |v|_{(2,v,\hat{K})}^{2}. \qquad (29)$$

Furthermore, due to

$$\begin{aligned}
&\omega(\nabla v, \widetilde{D}^1(v - P^1 v)) \\
&\quad = \left(|\nabla v| + |\widetilde{D}^1(v - P^1 v)| \right)^{\alpha} \left(1 + |\nabla v| + |\widetilde{D}^1(v - P^1 v)| \right)^{1-\alpha} \\
&\quad \geq \left| \widetilde{D}^1(v - P^1 v) \right|^{\alpha}
\end{aligned}$$

we conclude that

$$\begin{aligned}
J_0 &:= \int_{\hat{K}} \left[\omega(\nabla v, \widetilde{D}^1(v - P^1 v)) \right]^{p-2} \left| \widetilde{D}^1(v - P^1 v) \right|^2 \\
&\leq \int_{\hat{K}} \left| \widetilde{D}^1(v - P^1 v) \right|^{2-\alpha(2-p)} \\
&\leq c \left\| v - P^1 v \right\|_{W^{1,r}(\hat{K})}^{r},
\end{aligned}$$

where $r = 2(1-\alpha) + p\alpha$. Let us remark that $r \leq 2$ and $p \geq \max(1, \frac{2d}{2+d})$. Thus, the Sobolev embedding theorem implies that

$$\left\| v - P^1 v \right\|_{W^{1,r}(\hat{K})}^{r} \leq c \left\| v - P^1 v \right\|_{W^{2,p}(\hat{K})}^{r}. \qquad (30)$$

Due to (19) we have

$$\left\| v - P^1 v \right\|_{W^{2,p}(\hat{K})}^{r} \leq c\, |v|_{W^{2,p}(\hat{K})}^{r}. \qquad (31)$$

Collecting results (30)-(31) we have

$$J_0 \le c\,|v|^r_{W^{2,p}(\hat{K})}.$$

In view of (29) the assertion follows.

Case 3: $|v|_{W^{2,p}(\hat{K})} \le \lambda$ and $\lambda \le 1$. If $v \in \mathcal{P}_1$ the assertion is satisfied. Now let us assume that $v \notin \mathcal{P}_1$. Hence, $\lambda \ge |v|_{W^{2,p}(\hat{K})} > 0$. The triangle inequality yields

$$\lambda = |\nabla P^1 v| \le |\nabla v| + |\nabla(v - P^1 v)|.$$

We find

$$\begin{aligned}&\omega(\nabla v, \widetilde{D}^1(v - P^1 v))\\ &\quad= \left(|\nabla v| + |\widetilde{D}^1(v - P^1 v)|\right)^\alpha \left(1 + |\nabla v| + |\widetilde{D}^1(v - P^1 v)|\right)^{1-\alpha}\\ &\quad\ge \left(|\nabla v| + |\nabla(v - P^1 v)|\right)^\alpha\\ &\quad\ge \lambda^\alpha.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\int_{\hat{K}} \left[\omega(\nabla v, \widetilde{D}^1(v - P^1 v))\right]^{p-2} \left|\widetilde{D}^1(v - P^1 v)\right|^2\\ &\qquad \le \lambda^{\alpha(p-2)} \int_{\hat{K}} \left|\widetilde{D}^1(v - P^1 v)\right|^2. \end{aligned} \tag{32}$$

The Sobolev embedding theorem provides

$$\int_{\hat{K}} |\widetilde{D}^1(v - P^1 v)|^2 \le c\,\left\|v - P^1 v\right\|^2_{W^{1,2}(\hat{K})} \le c\,\left\|v - P^1 v\right\|^2_{W^{2,p}(\hat{K})}, \tag{33}$$

if $p \ge \max(1, \frac{2d}{2+d})$. Altogether, (19), (32), and (33) imply that

$$\int_{\hat{K}} \left[\omega(\nabla v, \widetilde{D}^1(v - P^1 v))\right]^{p-2} \left|\widetilde{D}^1(v - P^1 v)\right|^2 \le c\lambda^{\alpha(p-2)} |v|^2_{W^{2,p}(\hat{K})}. \tag{34}$$

Next, estimate (22) implies that

$$|v|^2_{W^{2,p}(\hat{K})} \le \left(\int_{\hat{K}} [\omega(\nabla v, D^2 v)]^p\right)^{\frac{2-p}{p}} |v|^2_{(2,v,\hat{K})}. \tag{35}$$

From (26), (27), and (28) we get

$$\begin{aligned}&\int_{\hat{K}} [\omega(\nabla v, D^2 v)]^p\\ &\quad\le c\left(|P^1 v|^{p\alpha}_{W^{1,p}(\hat{K})} + |v - P^1 v|^{p\alpha}_{W^{1,p}(\hat{K})} + |v|^{p\alpha}_{W^{2,p}(\hat{K})}\right) \times\\ &\qquad \times \left(\left\|1 + |\nabla P^1 v|\right\|^{p(1-\alpha)}_{L^p(\hat{K})} + |v - P^1 v|^{p(1-\alpha)}_{W^{1,p}(\hat{K})} + |v|^{p(1-\alpha)}_{W^{2,p}(\hat{K})}\right).\end{aligned}$$

Using $|v|_{W^{2,p}(\hat{K})} \le \lambda$ we have

$$\left(|P^1 v|^{p\alpha}_{W^{1,p}(\hat{K})} + |v - P^1 v|^{p\alpha}_{W^{1,p}(\hat{K})} + |v|^{p\alpha}_{W^{2,p}(\hat{K})}\right) \le c\lambda^{p\alpha}$$

and

$$\left(\left\|1 + |\nabla P^1 v|\right\|^{p(1-\alpha)}_{L^p(\hat{K})} + |v - P^1 v|^{p(1-\alpha)}_{W^{1,p}(\hat{K})} + |v|^{p(1-\alpha)}_{W^{2,p}(\hat{K})}\right) \le c(1+\lambda)^{p(1-\alpha)}.$$

Altogether, we obtain

$$\int_{\hat{K}} [\omega(\nabla v, D^2 v)]^p \le c\lambda^{p\alpha}(1+\lambda)^{p(1-\alpha)}. \tag{36}$$

In view of $\lambda \le 1$ it follows that $(1+\lambda)^{p(1-\alpha)} \le c$. We conclude that

$$\int_{\hat{K}} [\omega(\nabla v, D^2 v)]^p \le c\lambda^{p\alpha}.$$

Using (35) we arrive at

$$|v|^2_{W^{2,p}(\hat{K})} \le c\lambda^{\alpha(2-p)}|v|^2_{(2,v,\hat{K})}. \tag{37}$$

From (34) and (37) we now have

$$\int_{\hat{K}} \left[\omega(\nabla v, \widetilde{D}^1(v - P^1 v))\right]^{p-2} \left|\widetilde{D}^1(v - P^1 v)\right|^2 \le c|v|^2_{(2,v,\hat{K})}.$$

That is, we have proven (21). Hence, the assertion follows.

Case 4: $|v|_{W^{2,p}(\hat{K})} \le \lambda$ and $\lambda > 1$. We have

$$\begin{aligned}
&\omega(\nabla v, \widetilde{D}^1(v - P^1 v)) \\
&\quad \ge \left(|\nabla v| + |\widetilde{D}^1(v - P^1 v)|\right)^{\alpha} \left(|\nabla v| + |\widetilde{D}^1(v - P^1 v)|\right)^{1-\alpha} \\
&\quad \ge \left(|\nabla v| + |\nabla(v - P^1 v)|\right) \\
&\quad \ge \lambda.
\end{aligned}$$

Thus, proceeding as above (see (32)-(34)) we may conclude that

$$\int_{\hat{K}} \left[\omega(\nabla v, \widetilde{D}^1(v - P^1 v))\right]^{p-2} \left|\widetilde{D}^1(v - P^1 v)\right|^2 \le c\lambda^{p-2}|v|^2_{W^{2,p}(\hat{K})}. \tag{38}$$

In view of $\lambda > 1$ and estimate (36) we have

$$\int_{\hat{K}} [\omega(\nabla v, D^2 v)]^p \le c\lambda^{p\alpha}(1+\lambda)^{p(1-\alpha)} \le c\lambda^p.$$

Thus, using (35) we obtain

$$
\begin{aligned}
|v|^2_{W^{2,p}(\hat{K})} &\le \left(\int_{\hat{K}} [\omega(\nabla v, D^2 v)]^p\right)^{\frac{2-p}{p}} |v|^2_{(2,v,\hat{K})} \\
&\le c\lambda^{2-p} |v|^2_{(2,v,\hat{K})}. \qquad (39)
\end{aligned}
$$

Now (38) and (39) entail

$$
\int_{\hat{K}} \left[\omega(\nabla v, \widetilde{D}^1(v - P^1 v))\right]^{p-2} \left|\widetilde{D}^1(v - P^1 v)\right|^2 \le c|v|^2_{(2,v,\hat{K})}.
$$

Hence, the assertion follows.

The case p > 2. First, we estimate $\left\|v - P^1 v\right\|^2_{(1,v,\hat{K})}$. Let $k \in \{0, 1\}$. Using the triangle inequality $|\nabla v| \le |\nabla P^1 v| + |\nabla(v - P^1 v)|$, we have

$$
\begin{aligned}
&\left[\omega(\nabla v, D^k(v - P^1 v))\right]^{p-2} \\
&\quad = \left(|\nabla v| + |D^k(v - P^1 v)|\right)^{\alpha(p-2)} \times \\
&\quad\quad \times \left(1 + |\nabla v| + |D^k(v - P^1 v)|\right)^{(1-\alpha)(p-2)} \\
&\quad \le c\left(|\nabla P^1 v|^{\alpha(p-2)} + |\nabla(v - P^1 v)|^{\alpha(p-2)} + |D^k(v - P^1 v)|^{\alpha(p-2)}\right) \times \\
&\quad\quad \times \left(1 + |\nabla P^1 v|^{(1-\alpha)(p-2)} + |\nabla(v - P^1 v)|^{(1-\alpha)(p-2)} \right. \\
&\quad\quad \left. + |D^k(v - P^1 v)|^{(1-\alpha)(p-2)}\right).
\end{aligned}
$$

Applying the Young inequality we find

$$
\begin{aligned}
&|v - P^1 v|^2_{(k,v,\hat{K})} \\
&\quad = \int_{\hat{K}} \left[\omega(\nabla v, D^k(v - P^1 v))\right]^{p-2} \left|D^k(v - P^1 v)\right|^2 \\
&\quad \le c\left(\int_{\hat{K}} |\nabla P^1 v|^{\alpha(p-2)} |D^k(v - P^1 v)|^2 \right. \\
&\quad\quad + \int_{\hat{K}} |\nabla(v - P^1 v)|^{\alpha(p-2)} |D^k(v - P^1 v)|^2 \\
&\quad\quad + \int_{\hat{K}} |D^k(v - P^1 v)|^{\alpha(p-2)+2} + \int_{\hat{K}} |\nabla P^1 v|^{(p-2)} |D^k(v - P^1 v)|^2 \\
&\quad\quad \left. + \int_{\hat{K}} |\nabla(v - P^1 v)|^{(p-2)} |D^k(v - P^1 v)|^2 + \int_{\hat{K}} |D^k(v - P^1 v)|^p\right).
\end{aligned}
$$

Due to (19) and the Young inequality we obtain

$$
\begin{aligned}
&\int_{\hat{K}} |\nabla(v - P^1 v)|^{\alpha(p-2)} |D^k(v - P^1 v)|^2 + \int_{\hat{K}} |D^k(v - P^1 v)|^{\alpha(p-2)+2} \\
&\le c\int_{\hat{K}} |D^2 v|^{\alpha(p-2)+2} \qquad (40)
\end{aligned}
$$

and

$$\int_{\hat K} |\nabla(v - P^1 v)|^{(p-2)} |D^k(v - P^1 v)|^2 + \int_{\hat K} |D^k(v - P^1 v)|^p \le c \int_{\hat K} |D^2 v|^p. \tag{41}$$

Further, noting that $|\nabla P^1 v|$ is constant in $\hat K$ we obtain

$$\int_{\hat K} |\nabla P^1 v|^{\alpha(p-2)} |D^k(v - P^1 v)|^2 \le c \int_{\hat K} |\nabla P^1 v|^{\alpha(p-2)} |D^2 v|^2. \tag{42}$$

Using the fact that $|\nabla P^1 v| \le |\nabla v| + |\nabla(v - P^1 v)|$, (19), and the Young inequality it follows that

$$\begin{aligned} &\int_{\hat K} |\nabla P^1 v|^{\alpha(p-2)} |D^2 v|^2 \\ &\quad \le c \left(\int_{\hat K} |\nabla v|^{\alpha(p-2)} |D^2 v|^2 + \int_{\hat K} |D^2 v|^{\alpha(p-2)+2} \right). \end{aligned} \tag{43}$$

Similarly, we estimate

$$\begin{aligned} &\int_{\hat K} |\nabla P^1 v|^{p-2} |D^k(v - P^1 v)|^2 \\ &\quad \le c \int_{\hat K} |\nabla P^1 v|^{p-2} |D^2 v|^2 \\ &\quad \le c \left(\int_{\hat K} |\nabla v|^{(p-2)} |D^2 v|^2 + \int_{\hat K} |D^2 v|^p \right). \end{aligned} \tag{44}$$

Collecting results (40)-(44) we arrive at

$$\begin{aligned} &\|v - P^1 v\|^2_{(1,v,\hat K)} \\ &\quad \le c \left(\int_{\hat K} |D^2 v|^{\alpha(p-2)+2} + \int_{\hat K} |D^2 v|^p \right. \\ &\qquad \left. + \int_{\hat K} |\nabla v|^{\alpha(p-2)} |D^2 v|^2 + \int_{\hat K} |\nabla v|^{p-2} |D^2 v|^2 \right). \end{aligned} \tag{45}$$

Next, we estimate $|v|^2_{(2,v,\hat K)}$ from below. We find

$$\begin{aligned} \omega(\nabla v, D^2 v) &= \left(|\nabla v| + |D^2 v|\right)^\alpha \left(1 + |\nabla v| + |D^2 v|\right)^{1-\alpha} \\ &\ge \frac{1}{4} \left(|D^2 v|^\alpha + |D^2 v| + |\nabla v|^\alpha + |\nabla v|\right). \end{aligned} \tag{46}$$

We now conclude from (46)

$$
\begin{aligned}
|v|^2_{(2,v,\hat{K})} &= \int_{\hat{K}} \left[\omega(\nabla v, D^2 v)\right]^{p-2} |D^2 v|^2 \\
&\geq c \left(\int_{\hat{K}} |D^2 v|^{\alpha(p-2)+2} + \int_{\hat{K}} |D^2 v|^p \right. \\
&\quad \left. + \int_{\hat{K}} |\nabla v|^{\alpha(p-2)} |D^2 v|^2 + \int_{\hat{K}} |\nabla v|^{p-2} |D^2 v|^2 \right).
\end{aligned} \tag{47}
$$

From (47) and (45) the estimate follows. □

We now consider the conforming piecewise linear finite elements. By $\Pi^1 v$ we denote the pointwise Lagrange interpolant of a function v. That is, $\Pi^1 v$ is the piecewise linear function defined by

$$\Pi^1 v(x^k) = v(x^k) \qquad \text{for all vertices } x^k \in T^h.$$

This interpolator is well-defined if $p > d/2$ and furthermore (19) holds for any $p > d/2$ [12]. Then we can prove the following interpolation error estimates.

Theorem 3.1 *Suppose $p > \frac{d}{2}$. Then there is a constant $c > 0$, independent of v, such that*

$$\left|h^{-1}(v - \Pi^1 v)\right|^2_{(0,v,\Omega)} + \left|D^1(v - \Pi^1 v)\right|^2_{(0,v,\Omega)} \leq c \left|h\, D^2 v\right|^2_{(0,v,\Omega)} \tag{48}$$

for all $v \in W^{2,p}(\Omega)$.

Proof (of Theorem 3.1) Let $K \in T^h$ be a d–simplex, and let $\hat{K}$ be the reference element. There is an affine equivalent mapping, which maps K onto $\hat{K}$, and v onto $\tilde{v}$. It follows that

$$
\begin{aligned}
&|v - P^1 v|^2_{(1,v,K)} \\
&\quad = \int_K \left[\omega(\nabla v, D^1(v - P^1 v))\right]^{p-2} \left|D^1(v - P^1 v)\right|^2 \\
&\quad \leq c h_K^d \int_{\hat{K}} \left[\omega(h_K^{-1} \nabla \tilde{v}, h_K^{-1} D^1(\tilde{v} - \Pi^1 \tilde{v}))\right]^{p-2} \left|h_K^{-1} D^1(\tilde{v} - \Pi^1 \tilde{v})\right|^2 \\
&\quad = c h_K^{d-p} |\tilde{v} - \Pi^1 \tilde{v}|^2_{(1,\tilde{v},\hat{K})}
\end{aligned} \tag{49}
$$

and

$$
\begin{aligned}
&|h_K^{-1}(v - P^1 v)|^2_{(0,v,K)} \\
&\quad = \int_K \left[\omega(\nabla v, h_K^{-1}(v - P^1 v))\right]^{p-2} \left|h_K^{-1}(v - P^1 v)\right|^2 \\
&\quad \leq c h_K^d \int_{\hat{K}} \left[\omega(h_K^{-1} \nabla \tilde{v}, h_K^{-1}(\tilde{v} - \Pi^1 \tilde{v}))\right]^{p-2} \left|h_K^{-1}(\tilde{v} - \Pi^1 \tilde{v})\right|^2 \\
&\quad = c h_K^{d-p} |\tilde{v} - \Pi^1 \tilde{v}|^2_{(0,\tilde{v},\hat{K})}.
\end{aligned} \tag{50}
$$

Due to Lemma 3.1, (49), and (50) we now obtain

$$(51)\qquad |h_K^{-1}(v - P^1 v)|^2_{(0,v,K)} + |v - P^1 v|^2_{(1,v,K)} \le c h_K^{d-p} |\tilde{v}|^2_{(2,\tilde{v},\hat{K})}.$$

Moreover, we have

$$\begin{aligned} |\tilde{v}|^2_{(2,\tilde{v},\hat{K})} &= \int_{\hat{K}} \left[\omega(\nabla\tilde{v}, D^2\tilde{v})\right]^{p-2} \left|D^2\tilde{v}\right|^2 \\ &\le c h_K^{-d} \int_K \left[\omega(h_K \nabla v, h_K^2 D^2 v)\right]^{p-2} \left|h_K^2 D^2 v\right|^2 \\ (52)\qquad &\le c h_K^{p-d} |h_K D^2 v|^2_{(0,v,K)}. \end{aligned}$$

Notice that, due to (6),

$$(53)\qquad |h_K D^2 v|^2_{(0,v,K)} \le c |h D^2 v|^2_{(0,v,K)}$$

and

$$(54)\qquad |h_K^{-1}(v - P^1 v)|^2_{(0,v,K)} \ge c |h^{-1}(v - P^1 v)|^2_{(0,v,K)}.$$

Thus, in view of (51)-(52) and (53)-(54) we find

$$(55)\ |h^{-1}(v - P^1 v)|^2_{(0,v,K)} + |D^1(v - P^1 v)|^2_{(0,v,K)} \le c |h\, D^2 v|^2_{(0,v,K)}.$$

Let us note that $\bar{\Omega} = \bigcup_{K \in T^h} \bar{K}$. Hence, summing (55) over all triangles K we obtain the assertion. □

Likewise we may consider other kinds of interpolation operators which do not require the continuity of v. Then we obtain analogous of (48) under the weaker assumption $p > \frac{2d}{2+d}$. As an example, we consider the well-known interpolation operator π_h for the Strang-Fix type element space V^h for $d = 2$ (see [12] for the definition): For any $w \in W^{1,1}(\Omega)$ and any $K \in T^h$, let $\pi_h w|_K \in \mathcal{P}_1$, and

$$\int_{l_i} \pi_h w = \int_{l_i} w, \quad i = 1, 2, 3,$$

where l_i (i=1,2,3) are the three edges of the element K. For the above interpolator, (19) holds, see [12]. Then similarly we can prove the following interpolation error estimate for the Strang-Fix type element space.

Theorem 3.2 *Let $d \le 2$ and $1 < p < \infty$. Then there is a constant $c > 0$ such that*

(56)

$$\left|h^{-1}(v - \pi_h v)\right|^2_{(0,v,\Omega,T^h)} + \left|D^1(v - \pi_h v)\right|^2_{(0,v,\Omega,T^h)} \le c \left|h\, D^2 v\right|^2_{(0,v,\Omega,T^h)}$$

for all $v \in W^{2,p}(\Omega)$.

Remark 3.1 Similar results hold for the case of piecewise constant elements. For $p > 1$, assume that P^0: $W^{1,p}(\hat{K}) \to \mathcal{P}_0(\hat{K})$ is a linear continuous operator from $W^{1,p}(\hat{K})$ to $W^{0,p}(\hat{K})$. Furthermore assume that there is a constant c, independent of v, such that

$$\|v - P^0 v\|_{W^{0,p}(\hat{K})} \le c|v|_{W^{1,p}(\hat{K})} \qquad \text{for all } v \in W^{1,p}(\hat{K}). \tag{57}$$

For instance, we have:

Let $1 < p < \infty$. Then there is a constant c such that

$$\left|h^{-1}(v - P^0 v)\right|^2_{(0,v,\Omega)} \le c\left|D^1 v\right|^2_{(0,v,\Omega)} \tag{58}$$

for all $v \in W^{1,p}(\Omega)$.

The estimates can be similarly established. In fact, the proofs are trivial. For instance, if $1 \le p \le 2$ and $\alpha = 1$,

$$\begin{aligned}\|v - P^0 v\|^2_{(0,v,\hat{K})} &\le c|v - P^0 v|^p_{W^{0,p}(\hat{K})}\\ &\le c|v|^p_{W^{1,p}(\hat{K})} \qquad \forall v \in W^{1,p}(\hat{K})\end{aligned}$$

Thus

$$\|v - P^0 v\|^2_{(0,v,\hat{K})} \le c|v|^2_{(1,v,\hat{K})}.$$

For the case $p > 2$, the above estimate can also be easily proved. Thus (58) can be easily proved. □

4 Interpolation Error Estimates for Piecewise Linear Elements in Nikolskii Spaces

If the solution $u \in W^{2,p}(\Omega)$ and (3) holds, then one can apply Lemma 3.1 to estimate the interpolation error as in the above section. This is indeed true for the p-Laplacian with $1 < p < 2$. For the case $p > 2$, in general one cannot expect that $u \in W^{2,p}(\Omega)$, though (3) holds, see [14]. In this case even $W^{2,2}(\Omega)$–regularity is only known for the p-harmonic functions (where $f = 0$), see [22]. Sharp regularity results of the degenerate equations, in particular of the p-Laplacian, are likely held in some fractional order Sobolev spaces, or even some Nikolskii spaces where the derivatives are replaced by the difference quotients. For example for the solution of the p-Laplacian with smooth data, it was shown in [14] for $p > 2$ that the regularity (3) holds, and

$$u \in \mathcal{N}^{1+\frac{2}{p},p}(\Omega), \tag{59}$$

Here, $\mathcal{N}^{1+\frac{2}{p},p}(\Omega)$ denotes the Nikolskii space consisting of all functions $\phi \in L^p(\Omega)$ for which the norm

$$\|\phi\|_{\mathcal{N}^{s,p}(\Omega)} = \left(\|\phi\|^p_{L^p(\Omega)} + \sup_{\substack{\delta>0 \\ 0<|z|<\delta}} \int_{\Omega_\delta} \frac{|\nabla\phi(x+z) - \nabla\phi(x)|^p}{|z|^2}\, dx \right)^{\frac{1}{p}}$$

is finite, where $z \in \mathbb{R}^d$ and $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq \delta\}$, see [18].

Let us note that (59) and the embedding theorem of the Nikolskii spaces into the Sobolev spaces imply that $u \in W^{1+\frac{2}{p}-\varepsilon,p}(\Omega)$ for all $\varepsilon > 0$, see [18].

Thus instead of assuming that $v \in W^{2,p}(\Omega)$ we shall only assume that v satisfies (3) and (59) for $p > 2$. We then prove an interpolation error estimate only requiring (3) and (59), by utilize some techniques in Nikolskii spaces. Let us emphasize that although we used Nikolskii sapces, the final estimate does not refer to the spaces. Since the regularity theory of the p-Laplacian is far clearer than that of the other degenerate cases, we shall only study the interpolation estimates in the quasi-norms with $\alpha = 1$, which corresponds to the degeneracy type of the p-Laplacian. We shall only consider Π^1 as it is always well-defined when $u \in \mathcal{N}^{1+2/p,p}(\Omega)$ for $p > 2$. Let

$$|v|^2_{(k,w,\Omega)} = \int_\Omega (|\nabla w| + |D^k v|)^{p-2} \left|D^k v\right|^2$$

for $k = 0, 1, 2$. Furthermore, let $z \in \mathbb{R}^d$ and $0 < |z| < 1$. We define

$$\Delta_z \nabla v(x) = \frac{|\nabla v(x+z) - \nabla v(x)|}{|z|},$$

$$\Delta_z^{2/p} \nabla v(x) = \frac{|\nabla v(x+z) - \nabla v(x)|}{|z|^{2/p}},$$

$$\mu_z(\nabla v(x)) = \int_0^1 |t\nabla v(x+z) + (1-t)\nabla v(x)|^{p-2} dt.$$

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, be a convex polyhedron. Further, let $2 < p < \infty$. Then there is a constant c such that*

$$\left|h^{-1}(v - \Pi^1 v)\right|^2_{(0,v,\Omega)} + \left|D^1(v - \Pi^1 v)\right|^2_{(0,v,\Omega)} \leq Ch^2 \int_\Omega |\nabla v|^{p-2} \left|D^2 v\right|^2 \tag{60}$$

for all v such that the right-hand side of (60) is bounded.

Let us note that the integral on the right-hand side of (60) is bounded, if v is the solution of (WP); see [14]. In order to prove Theorem 4.1 we need the following lemma.

Lemma 4.1 *Let* $2 < p < \infty$ *and* $v \in \mathcal{N}^{1+\frac{2}{p},p}(\Omega)$. *There is a constant* c *independent of* v *such that*

$$\|v - \Pi^1 v\|^2_{(1,v,\hat{K})} \le c \sup_{\substack{\delta>0 \\ 0<|z|<\delta}} \int_{\hat{K}_\delta} \left(|\nabla v| + \left|\Delta_z^{2/p}\nabla v\right|\right)^{p-2} \left|\Delta_z^{2/p}\nabla v\right|^2.$$

Proof It can be seen that there is an analogous of (19) in the Nikolskii spaces, which may be proved in the same way as in [12]. Thus there is a constant c independent of v such that

$$\|v - \Pi^1 v\|^p_{W^{1,p}(\hat{K})} \le c|v|^p_{\mathcal{N}^{1+\frac{2}{p},p}(\hat{K})} \equiv c \sup_{\substack{\delta>0 \\ 0<|z|<\delta}} \int_{\hat{K}_\delta} \left|\Delta_z^{2/p}\nabla v\right|^p. \tag{61}$$

Proceeding as in the proof of Lemma 3.1 and using (61) instead of (19) the assertion follows. □

Proof (of Theorem 4.1) Let $0 < |z| < \delta \le 1$ and $\bar{v}(\cdot) = v(\cdot + z)$. Using (15) we estimate

$$\begin{aligned}
&\int_{\hat{K}_\delta} \left(|\nabla v| + \frac{|\nabla\bar{v} - \nabla v|}{|z|^{2/p}}\right)^{p-2} \left|\frac{\nabla\bar{v} - \nabla v}{|z|^{2/p}}\right|^2 \\
&\le \int_{\hat{K}_\delta} \frac{1}{|z|^2}(|\nabla v| + |\nabla\bar{v} - \nabla v|)^{p-2}|\nabla\bar{v} - \nabla v|^2 \\
&\le c\int_{\hat{K}_\delta} \frac{1}{|z|^2}(|\nabla\bar{v}|^{p-2}\nabla\bar{v} - |\nabla v|^{p-2}\nabla v)(\nabla\bar{v} - \nabla v) \\
&=: J_0.
\end{aligned}$$

Notice that for any $p > 2$ there is a constant c such that

$$\left(|s|^{p-2}s - |\bar{s}|^{p-2}\bar{s}\right)(s - \bar{s}) \le c\left(|s|^{\frac{p}{2}-1}s - |\bar{s}|^{\frac{p}{2}-1}\bar{s}\right)^2 \tag{62}$$

for all $s, \bar{s} \in \mathbb{R}^d$. Let $0 < |z| < \delta$, $\bar{z} = \frac{z}{|z|}$, and $\tau = |z|$. Putting $s = \nabla v(x + \tau\bar{z})$ and $\bar{s} = \nabla v(x)$ we find by applying (62)

$$\begin{aligned}
J_0 &\le c\int_{\hat{K}_\delta} \left|\frac{1}{\tau}\left(|\nabla v(x+\tau\bar{z})|^{\frac{p}{2}-1}\nabla v(x+\tau\bar{z}) - |\nabla v(x)|^{\frac{p}{2}-1}\nabla v(x)\right)\right|^2 dx \\
&= c\int_{\hat{K}_\delta} \left|\frac{1}{\tau}\int_0^\tau \frac{\partial}{\partial t}\left[|\nabla v(x+t\bar{z})|^{\frac{p}{2}-1}\nabla v(x+t\bar{z})\right] dt\right|^2 dx \\
&\le c\int_{\hat{K}_\delta} \frac{1}{\tau}\int_0^\tau \left|\frac{\partial}{\partial t}\left[|\nabla v(x+t\bar{z})|^{\frac{p}{2}-1}\nabla v(x+t\bar{z})\right]\right|^2 dt\, dx \\
&\le c\int_{\hat{K}} \left|\nabla[|\nabla v|^{\frac{p}{2}-1}\nabla v]\right|^2 \\
&\le c\int_{\hat{K}} |\nabla v|^{p-2}\left|D^2 v\right|^2.
\end{aligned}$$

Now let us consider a d-simplex $K \in T^h$, and an affine equivalent mapping which maps K onto the reference element $\hat{K}$. Arguing as in the proof of Theorem 3.1 the assertion follows. □

5 Applications to the p-Laplacian

In this section we use the new interpolation error estimates obtained in Sections 3-4 to derive some explicit a priori error bounds for the finite element approximation of (WP). We shall only discuss the p-Laplace equation since it is one of the most interesting examples of (WP). Let $u : \Omega \to \mathbb{R}$ be a weak solution of

$$\begin{aligned} -\mathrm{div}\left(|\nabla u(x)|^{p-2}\nabla u(x)\right) &= f(x) \qquad \text{in } \Omega \\ u(x) &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{63}$$

where $1 < p < \infty$ and $\Omega \subset \mathbb{R}^d$ $(1 \le d \le 3)$ is a convex polyhedron. We now assume that f is sufficiently smooth. Then there is a unique weak solution u of (63) fulfilling

$$\int_\Omega |\nabla u|^{p-2} \left|D^2 u\right|^2 < \infty \tag{64}$$

see [14]. Let us note that (64) was shown in [23] for $1 < p < 2$ on 2-d smooth domains, and in [14] for $p > 2$ on convex domains. In view of estimate (64) we have the following result.

Lemma 5.1 *Let $1 < p < 2$. Then the weak solution u of (63) satisfies*

$$u \in W^{2,p}(\Omega).$$

Proof Let $\Omega_0 = \{x \in \Omega : |\nabla u(x)| \le |D^2 u(x)|\}$. We find

$$\int_{\Omega_0} |D^2 u|^p = \int_{\Omega_0} |D^2 u|^{p-2} |D^2 u|^2 \le \int_{\Omega_0} |\nabla u|^{p-2} |D^2 u|^2 < \infty$$

and

$$\int_{\Omega \setminus \Omega_0} |D^2 u|^p \le \int_{\Omega \setminus \Omega_0} |\nabla u|^p < \infty.$$

This yields $D^2 u \in L^p(\Omega)$. □

Thus for the case $1 < p < 2$, we have

Theorem 5.1 *Let u and u_h be the solutions of (WP) and $(WP)^h$ respectively with $d/2 < p < 2$. Then*

$$c|u-u_h|^2_{W^{1,p}(\Omega)} \le |u-u_h|^2_{(1,u,\Omega)} \le Ch^2 \int_\Omega |\nabla u|^{p-2} \left|D^2 u\right|^2 .$$

Let u and u_h be the solutions of (WP) and $(WP)^h_n$ respectively with $1 < p < 2$ and $d \le 2$. Then

$$c|u-u_h|^2_{W^{1,p}(\Omega,T^h)} \le |u-u_h|^2_{(1,u,\Omega,T^h)} \le Ch^2 \int_\Omega |\nabla u|^{p-2} \left|D^2 u\right|^2 .$$

Proof It follows from Proposition 2.1 and Theorem 2.1 that

$$c|u-u_h|^2_{W^{1,p}(\Omega)} \le |u-u_h|^2_{(1,u,\Omega)} \le C|u-\Pi^1 u|^2_{(1,u,\Omega)}. \tag{65}$$

Then the first estimate follows from Theorem 3.1 and (64). Similarly we have

$$\begin{aligned} &c|u-u_h|^2_{W^{1,p}(\Omega,T^h)} \\ &\le |u-u_h|^2_{(1,u,\Omega,T^h)} \\ &\le \min_{v_h \in \tilde{V}_0^h} C\Big(\sum_{K\in T^h} |u-\pi_h u_h|^2_{(1,u,K)} + h_K|u-\pi_h u_h|^2_{(1,u,\partial K)}\Big). \end{aligned} \tag{66}$$

By Theorem 4.1 in [26],

$$\sum_{K\in T^h} h_K|u-\pi_h u_h|^2_{(1,u,\partial K)} \le Ch^2 \int_\Omega |\nabla u|^{p-2} \left|D^2 u\right|^2 . \tag{67}$$

Thus it follows from Theorem 3.2, (64) and (66)-(67) that

$$c|u-u_h|^2_{W^{1,p}(\Omega,T^h)} \le |u-u_h|^2_{(1,u,\Omega,T^h)} \le Ch^2 \int_\Omega |\nabla u|^{p-2} \left|D^2 u\right|^2 . \tag{68}$$

The theorem has been proved completely. □

For the case $p > 2$, we have.

Theorem 5.2 *Let u and u_h be the solutions of (WP) and $(WP)^h$ respectively with $2 < p < \infty$. Then*

$$|u-u_h|^2_{(1,u,\Omega)} \le Ch^2 \int_\Omega |\nabla u|^{p-2} \left|D^2 u\right|^2 .$$

Proof The estimate follows from the similar techniques used above, Theorem 2.1, and Theorem 4.1. □

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