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Asymptotic expansions for two-dimensional hypersingular integrals

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Summary. We define and examine two-dimensional hypersingular integrals on $[0, 1)^2$ and on $[0, \infty)^2$ and relate their Hadamard finite-part (HFP) values to Mellin transforms. These integrands have algebraic singularities of a possibly unintegrable nature on the axes and at the origin. Extending our work on one-dimensional integrals reported in 1998, we obtain variants of the classical Euler-Maclaurin expansion for various two-dimensional integrals. In many cases, the constant term in the expansion (which is not necessarily the leading term) provides the value of the HFP integral. These expansions may be used as the basis for the numerical evaluation of a class of HFP integrals by extrapolation.

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1 Introduction

In the last twenty years there have been an increasing use of (Hadamard) finite-part integrals (see [Ha52],[Mo94]) to formulate boundary value problems in terms of integral equations. These boundary integral equations are called hypersingular since their kernels have singularities of order higher than the dimension of the integrals. Several numerical approaches have been proposed to evaluate the finite-part integrals that are required for these applications. Most of these methods are based on either transforming the integral, or subtracting out the singulatities, in such a way that quadrature rules of Gaussian type may be applied. (See, for example, [ScWe92], [SaLa00], [KiScWe92], [PeSc97], [GuGi90], and [GuKRR92].) Extrapolation methods, based on asymptotic expansions of the Euler-Maclaurin type, have, until very recently, been confined to the numerical computation of integrals at most weakly singular. Only recently (see [Ly78],[MoLy98]) this approach has been extended to one-dimensional Cauchy and finite-part integrals.

In particular, in [MoLy98], using a technique based on Mellin transforms, we derived a generalization of the one-dimensional Euler-Maclaurin expansion for hypersingular integrals. Some of the results in that paper are listed in Section 2.

In this paper we show that this method can be extended to a family of two-dimensional finite-part integrals, whose integrands have singularities of a more general type than those that occur in present applications of hypersingular integral equations. We shall also establish that, in many cases, our finite part integral is simply the continuation in the complex plane of the Mellin Transform. This leads to a new definition of the finite-part integral, which includes also line (hyper)singularities. However, we are not presently aware of applications involving such singularities.

We treat the square $[0, 1]^2$ with a real integrand having a *full corner singularity* at the origin, that is, one that, in the unit square, takes the form $f(x_1, x_2)g(x_1, x_2)$ where g is smooth. For simplicity we treat $g \in C^\infty[0, 1]^2$; the "singular part" is

$$f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2} r_\rho(x_1, x_2), \tag{1.1}$$

Here, $r_\rho(x_1, x_2)$ is C^∞ in $[0, 1]^2$ except at $(0, 0)$. In addition r_ρ is homogeneous of degree ρ; that is for all $\lambda > 0$ and $(x_1, x_2) \neq (0, 0)$, $r_\rho(\lambda x_1, \lambda x_2) = \lambda^\rho r_\rho(x_1, x_2)$.

We note that in the hypersingular case we may have $\alpha_1 + \alpha_2 + \rho < -2$ and $\alpha_i < -1$. (These parameter values are conventionally excluded because they lead to divergent integrals.)

Several expansions for two-dimensional *regular* integrals on which extrapolation is based are mentioned briefly below. The purpose of the rest of this paper is to generalize some of these expansions to include two-dimensional *hypersingular* integrals.

One classical approach to cubature for *regular* integrals over a square is based on extrapolation. Let Q be any standard cubature rule over $[0, 1)^2$, and denote its m-copy version by $Q^{(m)}$. This is the rule obtained by partitioning the square into m^2 equal squares and applying a properly scaled version of Q in each. When $f(x, y)$ is Riemann integrable, $Q^{(m)} f$ is a discretization of If, the integral over $[0, 1)^2$, and If is the limit of $Q^{(m)} f$ as m becomes infinite.

In some cases one may write $Q^{(m)} f$ as an expansion in m. For example, when f is $C^{(p)}[0, \infty)^2$,we have

$$Q^{(m)} f = If + \sum_{s=1}^{p-1} B_s/m^s + R_p, \tag{1.2}$$

where B_s is *independent* of m and $R_p = O(m^{-p})$. This particular result is a two-dimensional version of a simple variant of the classical Euler-Maclaurin expansion, which is usually asymptotic.

When $f(x_1, x_2)$ is simply a homogeneous function $r_\rho(x_1, x_2)$ of degree ρ having no singularity in $[0, 1]^2$ other than at $(0, 0)$, a different expansion (see [Ly76]) is valid, namely,

$$Q^{(m)} f = If + (A_{\rho+2} + C_{\rho+2} \log m)/m^{\rho+2} + \sum_{s=1}^{p-1} B_s/m^s + R_p. \tag{1.3}$$

In this case, as in (1.2), simple integral representations for the coefficients B_s and C_s and for the remainder terms are known [Ly76]. ($C_{\rho+2} = 0$ unless $\rho + 2$ is a nonnegative integer.)

The extension of this result to the full corner singularity [LydD93] produced an expansion that included additional terms, some of the form $A^{[i]}_{n+\alpha_i}$ $m^{-n-\alpha_i}$ with positive integer n. In general, simple integral representations for these coefficients are not available.

The derivation of these expansions is not easy, and a separate long and detailed proof is required for each ([Ly76],[Si83],[LydD93].

It is well known that almost any rule may be used as a basis for extrapolation. Any such rule may be expressed as a linear combination of the offset trapezoidal rules

$$\overline{S}^m(\beta_1, \beta_2) f = \frac{1}{m^2} \sum_{j_1=0}^{m-1} \sum_{j_2=0}^{m-1} f\left(\frac{j_1+\beta_1}{m}, \frac{j_2+\beta_2}{m}\right), \tag{1.4}$$

with different parameters β_1, β_2. Thus, once an expansion for this offset trapezoidal rule (1.4) is available, the corresponding expansion for $Q^{(m)} f$ is readily obtained by linear superposition. In this paper we simply seek expansions for this offset trapezoidal rule.

We require little in the way of restrictions on the rule Q. For convenience only we insist that $Qf = If$ when f is constant and we require the abscissas satisfy $(\beta_1, \beta_2) \in [0, 1]^2$. The theory may be readily modified to remove these restrictions.

In 1993 Verlinden ([Ve93],[VeHa93]) introduced a new uniform approach, based on properties of the Mellin transform, for constructing these expansions. He treated the s-dimensional region $[0, \infty)^s$ and all s-dimensional monomial versions of our full corner singularity. He established that many expansions of this nature exist. While complicated in detail, his method deals with all these different cases in a uniform manner. The differences arise in a technical way and depend on the nature of the poles of an integrand function that depends, in turn, on the values of the parameters. In general, his treatment was limited to cases where the integral is regular. In this paper, we have followed his approach, but in the context of hypersingular integrals.

We approach the problem of divergent integrals from the following viewpoint. Suppose the parameters in (1.1) are such that $f(x, y)$ is a function for which the integral over $[0, 1)^2$ diverges. Then, while the quadrature rule sum $Q^{(m)} f$ exists for any finite m, it becomes unbounded with increasing m. In this paper, we show that, in many of these cases, there is still an expansion that (not unlike the Laurent expansion) starts with a few isolated terms involving positive powers of m and then continues with the familiar negative powers. In some cases (later defined as generic cases) the constant coefficient (which here is not the leading coefficient) coincides with the value of the Hadamard finite-part integral. But in other cases termed *non-generic* we have *not* been able to relate this constant term to the Hadamard finite part integral. Unfortunately, these cases, which we cannot handle, are of particular importance in boundary element methods. They include, for instance, the case $f(x_1, x_2) = r^{-2}$. These cases are currently under investigation.

We organize this paper as follows. In Section 2 we collect some of the results of our previous one-dimensional investigation. In Sections 3.1 and 3.2 we define two-dimensional Mellin transforms and Hadamard finite-part (HFP) integrals and note some of their elementary properties, including the connection between them.

In general, the approach based on the Mellin transform requires an integration region $[0, \infty)^2$. An integrand such as the basic full corner singularity specified in (1.1) does not usually converge over this region. We deal with this difficulty in Section 3, where we define *allowable* and *acceptable* functions and introduce neutralizer functions to mitigate the decay rate for large x_i. In Section 3.4, we present information about poles and residues of some Mellin transforms.

In Section 4, we substitute the expression for f given by the two-dimensional Mellin inversion formula (3.2) into the expression for the trapezoidal rule (4.1). This gives the basic relation on which the entire theory is based. This is a contour integral representation (4.3) of the trapezoidal rule, which can be developed into an expansion by moving contours to the left and including residues of those poles that are passed over. These residues depend on m and on the parameters α_1, α_2 and ρ. For many sets of parameters, all poles are simple; these are termed *generic* cases; see definition 4.1 and Theorem 5.1. In Section 5, we confine ourselves to these cases. Our principal result is Theorem 5.2, which gives the expansion for f, the pure full corner singularity over $[0, \infty)^2$. The extension to fg where g is a regular function is effected in Section 6, where the *form* of the expansions for $[0, \infty)^2$ and for $[0, 1)^2$ is given in Theorem 6.1 for generic integrands. The expansions for $[0, 1)^2$ are obtained by taking sums and differences of corresponding integrals over appropriate infinite regions (see (6.5); in these, neutralizer functions do not appear. In Section 6 we treat in more detail some special cases of the full corner singularity, such as those with $\alpha_i = 0$ or with $r_\rho = 1$. We are able to simplify the integral representations of the coefficients; in fact, some of these coefficients turn out to be HFP integrals, even when the original integral is regular. A straightforward way of obtaining the *form* of the expansion for nongeneric integrands is explained in Section 8. In Section 9 we remark that the asymptotic expansions for the unit square easily extend to smooth curved quadrangles. Section 10 is devoted to a few very simple numerical examples; these merely illustrate how the above expansions may be exploited numerically.

2 One-dimensional extrapolation for HFP integrals

Central to the one-dimensional theory treated in our earlier paper [MoLy98] is the Mellin transform of a function $f(x)$ together with the standard inversion formula. These are defined by

$$\begin{aligned} M_x(f(x); p) = M(f; p) &= \int_0^\infty f(x) x^{p-1} dx; \\ f(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M(f; p) x^{-p} dp. \end{aligned} \tag{2.1}$$

The path of integration in the second integral is $\mathrm{Re}(p) = c$, and c may take any real value p for which the first integral exists. In the many cases in which no confusion is likely to arise, we use the abbreviation $M(f; p)$. This is analytic in p and is generally defined by analytic continuation from regions where the integral representation is valid. The definition and properties of

Mellin Transforms with a discussion of their analytic continuation are nicely provided in [BlHa86].

In [MoLy98] we considered only functions $f(x) = x^\alpha g(x)$, where $g(x) \in C^{(m+1)}[0,\infty)$ has a decay rate at infinity exceeding that of any inverse power of x; that is,

$$\left|\int_0^\infty g^{(\kappa)}(x)x^j dx\right| < \infty, \quad \kappa = 0, 1, \dots, m+1,$$

for all $j \geq 0$. The functions $g(x)$ were termed "allowable" in $C^{(m+1)}[0,\infty)$. For these functions we established Theorem 2.1, which specifies the simple poles of $M(f; p)$.

Applying the conventional definition of the HFP integral (a one-dimensional version of definition 3.2 below), we established that

$$\int\!\!\!\!\!\!= _0^\infty f(x)dx = M(f; 1)$$

in all cases in which $f = gx^\alpha$ and g is allowable and $M(f, p)$ has no singularity at $p = 1$. This permitted us to derive several properties of the HFP integral by developing the (more robust and better established) Mellin transform.

When $M(f; p)$ has no poles in $\mathrm{Re}(p) > 0$, the analytic continuation of $M(f; p)$ into $\mathrm{Re}(p) > -m-1$, excluding the nonpositive integers, may be represented by

$$\begin{aligned} M(f; p) &= \int\!\!\!\!\!\!=_0^\infty f(x)x^{p-1}dx \\ &= \frac{(-1)^i}{p(p+1)\cdots(p+i)} \int_0^\infty f^{(i+1)}(x)x^{p+i}dx \end{aligned} \tag{2.2}$$

for all integers i for which the final integral exists.

Theorem 2.1 *When $f(x) = x^\alpha g(x)$ and $g(x)$ is an allowable function in $C^{(\infty)}[0,\infty)$, the analytic continuation $M(f; p)$ of the Mellin transform of $f(x)$ has simple poles at $p = -\alpha - n$, $n = 0, 1, 2, 3, \dots$, and*

$$\begin{aligned} M_t(t^\alpha g(t); -\alpha - n + \epsilon) = {} & \frac{g^{(n)}(0)/n!}{\epsilon} + \int\!\!\!\!\!\!=_0^\infty g(x)x^{-n-1}dx \\ & + \epsilon \int\!\!\!\!\!\!=_0^\infty g(x)(\log x)x^{-n-1}dx + O(\epsilon^2). \end{aligned} \tag{2.3}$$

Using the standard Riemann zeta function expansion

$$\zeta(p, x) = \sum_{k=0}^\infty (x+k)^{-p}, \qquad x \in (0,1], \qquad p > 1, \tag{2.4}$$

together with the Mellin inversion Theorem (second member of (2.1)), we derived a contour integral representation for the trapezoidal rule sum approximation

$$S^m(\beta)f = \frac{1}{m}\sum_{j=0}^{\infty} f\left(\frac{j+\beta}{m}\right). \tag{2.5}$$

This is the expression on the left-hand side of (2.6). Applying (2.3), we established the following (asymptotic) expansion.

Theorem 2.2 *Let $f(x) = x^\alpha g(x)$, let $g(x)$ be allowable in $C^{(N+1)}[0,\infty)$, and let α not be a negative integer. Then,*

$$\begin{aligned} &\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(f;p)\zeta(p,\beta)m^{p-1}dp \\ &= M(f;1) + \sum_{n=0}^{N} \frac{g^{(n)}(0)}{n!}\frac{\zeta(-n-\alpha,\beta)}{m^{n+\alpha+1}} \\ &\quad + \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} M(f;p)\zeta(p,\beta)\times m^{p-1}dp, \end{aligned} \tag{2.6}$$

where N is a nonnegative integer, $c > \alpha - 1$, $c' \in (-N-\alpha-2, -N-\alpha-1)$, and $M(f;p)$ is the (analytic continuation of the) Mellin transform of $f(x)$ (in the p-plane).

Since the first term on the right is an HFP integral, this expansion is a minor generalization of the classical Euler-Maclaurin asymptotic expansion. We note that this HFP integral is the constant term in an expansion that may contain terms of both higher and lower order.

Remark 2.1 When α is a negative integer, a variant of this Theorem pertains. The term in the summation having $n = -\alpha - 1$ is indeterminate as written. This term and with $M(f;1)$ must be replaced by a pair of terms of the form $C_0 \log m + D_0$. Details are given in [MoLy98]. We note that the same phenomenon, in a more complicated setting, occurs in the two-dimensional case. We refer to the cases covered by the Theorem as *generic* cases and the cases with negative integer α as *nongeneric* cases.

3 The two-dimensional mellin transform

This section is devoted to collecting together results about Mellin Transforms and Hypersingular integrals. We introduce neutraliser functions, which we use to partition the integrand function. This information is required in Sections 3 and 4, where we construct asymptotic expansions for cubature error functionals.

3.1 General definitions and properties

We define a double Mellin transform of $f(x_1, x_2)$ in a natural way.

For values of p_1 and p_2 for which the integral exists, we define

$$\begin{aligned} M_{x,y}(f(x,y); p_1, p_2) &= M(f; p_1, p_2) \\ &= \int_0^\infty \int_0^\infty f(x_1, x_2) x_1^{p_1-1} x_2^{p_2-1} dx_1 dx_2. \end{aligned} \tag{3.1}$$

For other values of p_1 and p_2, the transform may be defined by using analytic continuation. A double application of the one-dimensional inversion formula (2.1) gives the corresponding two-dimensional inversion formula, namely,

$$f(x_1, x_2) = \frac{1}{(2\pi i)^2} \int_{c_1-i\infty}^{c_1+i\infty} \int_{c_2-i\infty}^{c_2+i\infty} M(f; p_1, p_2) x_1^{-p_1} x_2^{-p_2} dp_1 dp_2. \tag{3.2}$$

Here c_1 and c_2 may take any real values for which the double integral $M(f; c_1, c_2)$ defined in (3.1) above exists in a regular sense.

We shall apply the Mellin transform only to functions that are *acceptable* according to the following straightforward generalization of the one-dimensional definition.

Definition 3.1 *A function* $g(x_1, x_2)$ *is an* allowable *function in* $C^{(n)}([0,\infty)^2)$, $n \geq 0$, *when it is a* $C^{(n)}$ *function in both variables and*

$$\left| \int_0^\infty \int_0^\infty g^{(i,j)}(x_1, x_2) x_1^k x_2^l dx_1 dx_2 \right| < \infty \tag{3.3}$$

for all integers $0 \leq i, j \leq n$*, all* $k, l > 0$.

An acceptable *function is one of the form* $g(x_1, x_2) x_1^{\alpha_1} x_2^{\alpha_2}$ *where* g *is allowable.*

When g is allowable, it is a simple matter to obtain a set of real integral representations for the Mellin transform, valid for all real noninteger p_1 and p_2. We start with the definition (3.1), assigning values of p_1 and p_2 sufficiently large that the integral exists. Then we carry out the process of integration by parts, i times in the x_1 variable and j times in the x_2 variable. The contributions from the lower limits contain factors of the form x_i^δ with $\delta > 0$ and so vanish; the decay rate of an allowable function ensures that the contributions from the upper limits vanish also. We are left with the following generalization of (2.2):

$$\begin{aligned} M(g; p_1, p_2) = & \frac{(-1)^{i+j}}{p_1(p_1+1)\cdots(p_1+i-1)p_2(p_2+1)\cdots(p_2+j-1)} \\ & \times \int_0^\infty \int_0^\infty g^{(i,j)}(x_1, x_2) x_1^{p_1+i-1} x_2^{p_2+j-1} dx_1 dx_2. \end{aligned} \tag{3.4}$$

The derivation of this relation is valid only for values of p_i for which integral representation (3.1) exists. But the right-hand side exists for a wider range of p_i and is analytic in p_i. An elementary application of the principle of analytic continuation produces the result

(3.5)
$$M(g; p_1, p_2) = \frac{(-1)^{i+j}(p_1-1)!(p_2-1)!}{(p_1+i-1)!(p_2+j-1)!} M(g^{(i,j)}; p_1+i, p_2+j),$$

valid for all allowable functions g, with all noninteger values of p_i.

We close this subsection with some standard rules for manipulating the two-dimensional Mellin transform.

Lemma 3.1 *Let f, ϕ, and h be functions of two variables, and let p_1 and p_2 be parameters such that the Mellin transform functions below exist. Then we have the following:*

(a) When $\phi(y_1, y_2) = f(y_1 y_2, y_2)$,

$$M_{x,y}(f(x, y); p_1, p_2) = M_{x,y}(\phi(x, y); p_1, p_1 + p_2).$$

(b) When $\phi(x_1, x_2) = x_1^{\gamma_1} x_2^{\gamma_2} h(x_1, x_2)$,

$$M_{x,y}(\phi(x, y); p_1, p_2) = M_{x,y}(h(x, y); p_1 + \gamma_1, p_2 + \gamma_2).$$

(c) When $f(x, y) = g(x)h(y)$,

$$M_{x,y}(f(x, y); p_1, p_2) = M_t(g(t); p_1) M_t(h(t); p_2)$$

(Bear in mind that x, y, t are dummy variables that may be renamed at will.) These textbook results are direct consequences of the definitions.

3.2 Definition of HFP integral and relation with the mellin transform

We now define a two-dimensional HFP integral and show that, in many circumstances, it coincides with $M(f; 1, 1)$.

Definition 3.2 *Let f be integrable over $(\epsilon, b)^2$, for all ϵ satisfying $0 < \epsilon < b \leq \infty$. Suppose there exists a strictly monotonic increasing sequence of nonpositive real numbers $\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_M \leq 0$ and a nonnegative integer J such that an expansion of the form*

(3.6)
$$\int_\epsilon^b \int_\epsilon^b f(x_1, x_2) dx_1 dx_2 = \sum_{k=0}^{K} \sum_{j=0}^{J} I_{k,j}(b) \epsilon^{\alpha_k} \log^j \epsilon + o(1)$$

exists. Then the corresponding finite-part integral may be defined as follows:

(3.7)

$$FP\int_0^b\int_0^b f(x_1,x_2)dx_1dx_2 = I_{i,0}(b),\ \ when\ \alpha_i = 0\ for some\ 0 \le i \le K$$
$$= 0\ otherwise.$$

(This is the unique term in the summation that is independent of ϵ.)

Remark 3.1 Other definitions are possible. One more general definition uses two independent parameters, say, ϵ_1 and ϵ_2, as lower limits in (3.6) together with a correspondingly more sophisticated expansion. This can lead to different results in some cases (see remark 3.2); however, the results in this paper would be unaffected by this change. The choice $\epsilon_1 = \epsilon_2 = \epsilon$ used here corresponds to a standard one adopted in hypersingular boundary integral equations where f is of form (1.1) with $\alpha_1 = \alpha_2 = 0$ and $\rho = -2$.

To our knowledge, until now, definitions of finite-part integrals have been given only with reference of integrand functions of type (1.1) with $\alpha_1 = \alpha_2 = 0$, that is, with a hypersingularity at the origin but otherwise regular (see [ScWe92]). In that case an expansion of form (3.6) may be obtained by taking out a circular or square neighborhood of the origin of "size" ϵ. In our definition, in order to allow line singularities along $x_1 = 0$ and $x_2 = 0$, we delete also a neighborhood of these lines. This strategy has the added advantage that we may readily exploit one-dimensional results about the Mellin transform. Nevertheless, it is not difficult to verify that when we have only a point singularity, that is, in (1.1) we have $\alpha_1 = \alpha_2 = 0$, our definition generally coincides with the one that takes out a square neighborhood of the origin. (We recall that the standard definition takes out a circular disc). The reason is that, unless $\rho = -2$, the two extra strips we delete do not contribute to the finite-part value. When $\rho = -2$, our transform may be obtained from the one with the square cut by subtracting the quantity

$$\int_1^\infty\left(\int_0^1 r_\rho(x,y)dy\right)dx + \int_0^1\left(\int_1^\infty r_\rho(x,y)dx\right)dy, \qquad \rho = -2. \tag{3.8}$$

In many applications, it is convenient to delete a circular disc, or to make some other shaped excision. We need hardly remind the reader that, in those applications, a *finite* contribution to the integral may well be generated by the difference between these different *infinitesimal* excisions and has to be taken into account. We now confine our attention to the finite-part integral

$$I[g;\alpha_1,\alpha_2] =: FP\int_0^\infty\int_0^\infty g(x_1,x_2)x_1^{\alpha_1}x_2^{\alpha_2}dx_1dx_2, \tag{3.9}$$

where $g(x_1, x_2)$ is integrable over $([0, \infty)^2)$. Clearly, when $\alpha_1 + 1$ and $\alpha_2 + 1$ are both positive, this is a regular integral and coincides with a Mellin transform (see definition (3.1)):

$$I[g; \alpha_1, \alpha_2] = M[g; \alpha_1 + 1, \alpha_2 + 1], \qquad \alpha_i > -1. \tag{3.10}$$

In fact, when $g(x_1, x_2)$ is an allowable function in $C^{(n)}([0, \infty)^2)$ with $n \geq 0$, this relation is valid for many other choices of α_1 and α_2, as specified in Theorem 3.3. The rest of this subsection is devoted to establishing this somewhat pedestrian Theorem in a straightforward manner; to this end we need the next two Theorems.

Theorem 3.1 *Let $g(x_1, x_2)$ be an allowable function in $C^{(n)}([0, \infty)^2)$, $n \geq 0$; let neither α_1 nor α_2 be a negative integer, and let i and j be nonnegative integers such that both $\alpha_1 + i$ and $\alpha_2 + j$ exceed -1. Then*

(3.11)
$$\int_\epsilon^\infty \int_\epsilon^\infty g(x_1, x_2) x_1^{\alpha_1} x_2^{\alpha_2} dx_1 dx_2 = T_{1,1} + T_{2,1}(\epsilon) + T_{1,2}(\epsilon) + T_{2,2}(\epsilon),$$

where

(3.12)
$$T_{1,1} = \frac{(-1)^{i+j} \alpha_1! \alpha_2!}{(\alpha_1 + i)! p(\alpha_2 + j)!} \int_0^\infty \int_0^\infty g^{(i,j)}(x_1, x_2) x_1^{\alpha_1 + i} x_2^{\alpha_2 + j} dx_1 dx_2$$

and

$$T_{2,1}(\epsilon) = \epsilon^{\alpha_1} U_{2,1}(\epsilon); \quad T_{1,2}(\epsilon) = \epsilon^{\alpha_2} U_{1,2}(\epsilon); \quad T_{2,2}(\epsilon) = \epsilon^{\alpha_1 + \alpha_2} U_{2,2}(\epsilon),$$

$U_{m,n}(\epsilon)$ being convergent power series in ϵ.

The finite part integral (3.9) is the constant coefficient of ϵ on the right of this equation. This is $T_{1,1}$ provided α_1 and α_2 are chosen so the other terms contain no constant terms. This leads to the following Theorem.

Theorem 3.2 *When none of $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$ are nonpositive integers and when $g(x_1, x_2)$ is an allowable function in $C^{(n)}([0, \infty)^2)$, $n \geq 0$,*

$$\begin{aligned} \text{(3.13)} \quad & FP \int_0^\infty \int_0^\infty g(x_1, x_2) x_1^{\alpha_1} x_2^{\alpha_2} dx_1 dx_2 \\ &= \frac{(-1)^{i+j} \alpha_1! \alpha_2!}{(\alpha_1 + i)! p(\alpha_2 + j)!} FP \int_0^\infty \int_0^\infty g^{(i,j)}(x_1, x_2) x_1^{\alpha_1 + i} x_2^{\alpha_2 + j} dx_1 dx_2 \end{aligned}$$

for all nonnegative integers i and j.

Naturally, the latter finite-part integral is regular when both $\alpha_1 + i$ and $\alpha_2 + j$ exceed -1. Theorem 3.3 is readily established from this equation and (3.5) by choosing i and j so that the finite-part integral is regular, setting $p_i = \alpha_i + 1$, and applying (3.10).

Theorem 3.3 *When none of* $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$ *are nonpositive integers, and when* $g(x_1, x_2)$ *is an allowable function in* $C^{(n)}([0, \infty)^2)$, $n \geq 0$

$$(3.14) \quad FP \int_0^\infty \int_0^\infty g(x_1, x_2) x_1^{\alpha_1} x_2^{\alpha_2} dx_1 dx_2 = I[g; \alpha_1, \alpha_2] = M[g; \alpha_1 + 1, \alpha_2 + 1]$$

Remark 3.2 If one were to adopt the more general definition of the finite part integral mentioned in Remark 2.1 above, one would recover Theorems 3.2 and 3.3 without the restriction on $\alpha_1 + \alpha_2$.

3.3 Full corner singularity with neutraliser function

The general theory above requires that the integrand function take the form $g(x_1, x_2) x_1^{\alpha_1} x_2^{\alpha_2}$, where $g(x_1, x_2)$ is an allowable function in $C^{(n)}([0, \infty)^2)$ for some finite $n \geq 1$. As written, our full corner singularity (1.1), namely $x_1^{\alpha_1} x_2^{\alpha_2} r_\rho(x_1, x_2)$, may fail on two counts. First, many choices of the parameters do not produce sufficient decay for large values of x_i. Second, the homogeneous function in general introduces a singularity at the origin that gives rise to a nonintegrable function in a subsequent integration. In this subsection, we address both counts by introducing a specially constructed two-dimensional neutralizer function $N(x_1, x_2)$, which we define in terms of one-dimensional neutralizer functions in such a way that fN coincides with the full corner singularity f in $[0, 1)^2$ and may be expressed as the sum of two independent acceptable functions.

The use of neutralizer functions for this purpose may seem to be artificial. Indeed many choices of prolungation functions are possible. However they all lead to the same result over the $[0, 1]^2$ region, since their contributions will later disappear. Our choice is one which simplifies the calculation

Definition 3.3 *A neutralizer function* $\nu(x, k_1, k_2)$ *is a* C^∞ *function of* x, *defined for all real arguments satisfying* $k_1 < k_2$, *that satisfies*

$$\begin{aligned} \nu(x, k_1, k_2) &= 1 \qquad \textit{for } x \leq k_1, \\ &= 0 \qquad \textit{for } x \geq k_2. \end{aligned}$$

Where no confusion is likely to arise, we abbreviate $\nu(x, k_1, k_2)$ as $\nu(x)$.

We now specify a neutralizer function

$$\overline{\nu}(x) = \overline{\nu}(x, k_1, k_2) \qquad \text{with } 1 < k_1 < k_2.$$

and construct a two dimensional neutraliser function

$$(3.15) \qquad \overline{N}(x_1, x_2) = \overline{\nu}(x_1, k_1, k_2) \overline{\nu}(x_2, k_1, k_2)$$

in terms of which we may define

$$\overline{f}(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2} r_\rho(x_1, x_2) \overline{N}(x_1, x_2). \tag{3.16}$$

This function is inconvenient to use when $\rho \neq 0$ because certain integrals that appear later do not converge. To continue, we need a second one-dimensional neutralizer function

$$\nu_0(x) = \nu_0(x, k_0^{-1}, k_0), \qquad \text{with } k_0 > 1.$$

It follows from the definition that the function

$$\tilde{\nu}_0(x) = \tilde{\nu}_0(x, k_0^{-1}, k_0) = 1 - \nu_0(x^{-1}, k_0^{-1}, k_0) \tag{3.17}$$

is also a neutralizer function. It is notationally convenient to choose ν_0 so that $\tilde{\nu}_0(x) = \nu_0(x)$.

For reasons we discuss later, we express $f(x_1, x_2)$ as the sum of two functions, one of which is zero in a sector including the x_1-axis and the other is zero in a sector including the x_2-axis. To this end, we define a two-dimensional neutralizer function

$$\begin{aligned} N(x_1, x_2) &= \nu_0\left(\frac{x_1}{x_2}, k_0^{-1}, k_0\right) \overline{\nu}(x_2, k_1, k_2) \\ &\quad + \left[1 - \nu_0\left(\frac{x_1}{x_2}, k_0^{-1}, k_0\right)\right] \overline{\nu}(x_1, k_1, k_2) \\ &=: N^{[1]}(x_1, x_2) + N^{[2]}(x_1, x_2). \end{aligned} \tag{3.18}$$

One may verify that $N(x_1, x_2) = 1$ for all $(x_1, x_2) \in [0, 1]^2$, that $N(x_1, x_2) = 0$ when either x_1 or x_2 exceeds $k_0 k_2$, and that $N \in C^{(\infty)}[0, \infty)^2$. This is a two-dimensional neutralizer function.

We now reintroduce our full corner singularity (1.1) as the function

$$f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2} r_\rho(x_1, x_2) N(x_1, x_2). \tag{3.19}$$

Here, $r_\rho(x_1, x_2)$ is homogeneous about the origin of degree ρ and has no singularity in the first quadrant other than possibly at the origin. Because of this singularity this function has a nonallowable component. We overcome this difficulty by expressing f as the sum of two parts, each of which is separately acceptable. These are

$$f^{[i]}(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2} r_\rho(x_1, x_2) N^{[i]}(x_1, x_2), \qquad i = 1, 2. \tag{3.20}$$

Since r_ρ is homogeneous, we have by definition

$$r_\rho(\lambda x_1, \lambda x_2) = \lambda^\rho r_\rho(x_1, x_2), \qquad \forall \lambda > 0, \tag{3.21}$$

and we may reexpress r_ρ in various ways including

$$r_\rho(x_1, x_2) = x_2^\rho r_\rho(x_1/x_2, 1) \tag{3.22}$$

$$r_\rho(x_1, x_2) = x_1^\rho r_\rho\,(1, x_2/x_1)\,. \tag{3.23}$$

Clearly, integral representation (3.1) may be synthesized; thus

$$M(f; p_1, p_2) = M(f^{[1]}; p_1, p_2) + M(f^{[2]}; p_1, p_2), \tag{3.24}$$

where

$$f^{[1]}(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2+\rho} r_\rho(x_1/x_2, 1) N^{[1]}(x_1, x_2) \tag{3.25}$$

and

$$f^{[2]}(x_1, x_2) = x_1^{\alpha_1+\rho} x_2^{\alpha_2} r_\rho(1, x_2/x_1) N^{[2]}(x_1, x_2). \tag{3.26}$$

We note that $f^{[1]}(x_1, x_2)$ and $f^{[2]}(x_1, x_2)$ become zero when $x_1 \geq k_0 x_2$ and when $x_1 \leq k_0^{-1} x_2$, respectively. Taking this into account, one can readily show that $r_\rho(x_1/x_2, 1)N^{[1]}(x_1, x_2)$ and $r_\rho(1, x_2/x_1)N^{[2]}(x_1, x_2)$ are allowable functions. Thus both $f^{[1]}$ and $f^{[2]}$ are acceptable functions.

Applying in turn several results in this section, we readily establish the following Theorem.

Theorem 3.4 *When* $f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2} r_\rho(x_1, x_2) N(x_1, x_2)$ *and none of* $\alpha_1, \alpha_2, \alpha_1 + \rho, \alpha_2 + \rho,$ *and* $\alpha_1 + \alpha_2 + \rho$ *are negative integers,*

$$FP \int_0^\infty \int_0^\infty f(x_1, x_2) dx_1 dx_2 = M(f; 1, 1). \tag{3.27}$$

Note: this extends the result in Theorem 3.3 so as to include the generally nonallowable function $g(x_1, x_2) = r_\rho(x_1, x_2)N(x_1, x_2)$.

Proof We treat the component $f^{[1]}$. In view of definition (3.25) we have

$$\begin{aligned} &FP \int_0^\infty \int_0^\infty f^{[1]}(x_1, x_2) dx_1 dx_2 \\ &\quad = FP \int_0^\infty \int_0^\infty x_1^{\alpha_1} x_2^{\alpha_2+\rho} g(x_1, x_2) dx_1 dx_2, \end{aligned} \tag{3.28}$$

where

$$g(x_1, x_2) = r_\rho(x_1/x_2, 1) N^{[1]}(x_1, x_2) \tag{3.29}$$

is an allowable function. In view of this, so long as none of $\alpha_1, \alpha_2 + \rho,$ and $\alpha_1 + \alpha_2 + \rho$ are negative integers, we may apply Theorem 3.3 to express this integral as a Mellin transform $M[g; \alpha_1 + 1, \alpha_2 + \rho + 1]$ which, in view of Lemma 3.1(b), coincides with $M[f^{[1]}; 1, 1]$.

Treating the component $f^{[2]}$ in an analogous way and combining the results for the two individual components, we obtain (3.28), establishing the Theorem. □

The idea of the partition of f, which is not acceptable, into the two functions $f^{[1]} + f^{[2]}$, each of which is acceptable, is of key importance. This was first used by Verlinden [Ve93] for regular integrals.

3.4 Development of $M[f^{[1]}; p_1, p_2]$

We next examine the behavior of these individual Mellin transforms in their domain of analyticity. We treat in detail only the first integrand; to reduce this, we change variables in the corresponding Mellin transform (3.1) using

$$y_1 = x_1/x_2; \qquad y_2 = x_2. \tag{3.30}$$

(this is a Duffy Transformation.) To effect this coordinate transformation (3.30), we apply part (a) of lemma 3.1 to the function $f^{[1]}$ in (3.25). This gives $M(f^{[1]}; p_1, p_2) = M(\phi; p_1, p_1 + p_2)$ with

$$\begin{aligned}\phi(y_1, y_2) = f^{[1]}(y_1 y_2, y_2) &= (y_1 y_2)^{\alpha_1} y_2^{\alpha_2+\rho} r_\rho(y_1, 1) N^{[1]}(y_1 y_2, y_2)\\ &= y_1^{\alpha_1} y_2^{\alpha_1+\alpha_2+\rho} r_\rho(y_1, 1)\nu_0(y_1)\overline{\nu}(y_2).\end{aligned}$$

Since ϕ turns out to be a product function, we may apply part (c) of the same lemma, giving the first part of the following Theorem.

Theorem 3.5 *For the functions $f^{[1]}$ and $f^{[2]}$ defined by (3.20), or by (3.25) and (3.26), we have*

$$\begin{aligned}M(f^{[1]}; p_1, p_2) &= M_t(t^{\alpha_1} r_\rho(t, 1)\nu_0(t), p_1) M_t(t^{\alpha_1+\alpha_2+\rho}\overline{\nu}(t), p_1 + p_2)\\ M(f^{[2]}; p_1, p_2) &= M_t(t^{\alpha_2} r_\rho(1, t)\tilde{\nu}_0(t), p_2) M_t(t^{\alpha_1+\alpha_2+\rho}\overline{\nu}(t), p_1 + p_2).\end{aligned}$$

The second part of this Theorem may be established in precisely the same way.

This factorization of the two-dimensional Mellin Transform into the product of two one-dimensional Mellin Transforms leads to major simplification in the subsequent development of the theory. It is a direct consequence of the particular form (3.18) of the two-dimensional neutralizer function $N(x_1, x_2)$. Other equally valid forms, some simpler, do not lead to this factorization.

To establish the asymptotic expansions in the next two sections, we require expressions for the poles and residues of these functions. To this end we apply Theorem 2.1 of the preceding section to the individual factors. Since for all our one-dimensional neutralizer functions we have $\nu(0) = 1$ and $\nu^{(n)}(0) = 0$ for all $n > 0$, this Theorem leads to the following two lemmas.

Lemma 3.2 $M_t(t^{\alpha_1} r_\rho(t, 1)\nu_0(t), p_1)$ *has a sequence of simple poles located at*

$$p_1 = -\alpha_1 - n_1, \quad n_1 = 0, 1, \dots, \tag{3.31}$$

with residues $r_\rho^{(n_1,0)}(0,1)/n_1!$ *respectively. The corresponding Laurent expansion is*

$$\begin{aligned} &M_t(t^\alpha r_\rho(t,1)\nu_0(t),\ -\alpha-n+\epsilon) \\ &\quad = \frac{r_\rho^{(n,0)}(0,1)}{\epsilon n!} + \sum_{j=0} \frac{\epsilon^j}{j!} \ ⨍_0^\infty r_\rho(t,1)t^{-n-1}\nu_0(t)\log^j t\ dt. \end{aligned} \tag{3.32}$$

Lemma 3.3 *$M_t(t^\gamma\overline{\nu}(t,k_1,k_2),p)$ has only one pole. This is a simple pole located at $p=-\gamma$ with residue 1.*

In fact there is a simple expression for this transform. When $p > -\gamma$, we may use the standard integral representation (2.1) . Remembering that $\overline{\nu}(t) = 1$ for $t < k_1$ and $\overline{\nu}(t) = 0$ for $t > k_2$, we find

$$M_t(t^\gamma\overline{\nu}(t,k_1,k_2),p) = \frac{k_1^{p+\gamma}}{p+\gamma} + \int_{k_1}^{k_2} t^{p+\gamma-1}\overline{\nu}(t)dt. \tag{3.33}$$

Analytic continuation extends this result to all $p \neq -\gamma$.

Since $k_1 > 1$, the Laurent expansion about this pole is

$$M_t(t^\nu\bar{\nu}(t,k_1,k_2),-\gamma+\epsilon) = \frac{1}{\epsilon} + \sum_{j=0} \frac{\epsilon^j}{j!}\int_1^{k_2} t^{-1}\bar{\nu}(t,k_1,k_2)\log^j t\ dt. \tag{3.34}$$

Note that the location of the poles and their residues do *not* depend on the details of the neutraliser functions.

4 Two-dimensional error expansion

After these preliminaries, we find an expansion for the double infinite sum

$$S^m(\beta_1,\beta_2)f^{[1]} = \frac{1}{m^2}\sum_{j_1=0}^\infty\sum_{j_2=0}^\infty f^{[1]}\left(\frac{j_1+\beta_1}{m}, \frac{j_2+\beta_2}{m}\right). \tag{4.1}$$

When this sum converges, it is clearly a discretization of the regular integral

$$M(f^{[1]};1,1) = \int_0^\infty\int_0^\infty f^{[1]}(x_1,x_2)dx_1dx_2. \tag{4.2}$$

Applying the two-dimensional Mellin inversion formula (3.2) to the function $f^{[1]}$ in (4.1) and simplifying by using the standard expansion (2.4) of the Riemann zeta function, we obtain a contour integral representation of the trapezoidal rule sum (4.1) of the form

$$\begin{aligned} S^m(\beta_1,\beta_2)f^{[1]} = \frac{1}{(2\pi i)^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty} M(f^{[1]};p_1,p_2)\zeta(p_1,\beta_1) \\ \times\zeta(p_2,\beta_2)m^{p_1+p_2-2}dp_1dp_2. \end{aligned} \tag{4.3}$$

We remark, at this stage, that this integrand has a pole (due to the zeta functions) at $p_1 = p_2 = 1$. This pole has residue $M(f^{[1]}; 1, 1)$ that is, in some cases, precisely the integral (4.2) to which the discretization (4.1) may converge. This circumstance motivates the rest of this paper. It suggests that, by moving the contours in (4.3), we may isolate the exact integral, leaving a remainder term. In this section, we put this suggestion on a proper mathematical footing. We find that, in many cases, other residues of the integrand function in (4.3) correspond to other terms in the Euler-Maclaurin expansion and, appropriately, in generalizations of this expansion.

In (4.3) the integration paths are along $\mathrm{Re}(p_1) = c_1$ and $\mathrm{Re}(p_2) = c_2$, respectively; and c_1 and c_2 are real numbers for which $M(f^{[1]}; c_1, c_2)$ is given by its standard integral representation of form (3.1). This implies that all poles of $M(f^{[1]}; p_1, p_2)$, as a function of p_1 with p_2 fixed and of p_2, with p_1 fixed, are on the left of the lines $\mathrm{Re}(p_1) = c_1$ and $\mathrm{Re}(p_2) = c_2$ respectively. The locations of these poles (for both $f^{[1]}$ and $f^{[2]}$) can be obtained from lemmas 3.2 and 3.3. We find these parameters need to satisfy

(4.4)
$$c_1, c_2 > 1; \qquad c_1 > -\alpha_1; \qquad c_2 > -\alpha_2; \qquad c_1 + c_2 > -(\alpha_1 + \alpha_2 + \rho).$$

To obtain an expansion, we employ precisely the technique used in [MoLy98], Section 4, in a one-dimensional context to establish Theorem 2.2. Here, we keep c_1 fixed and treat p_1 as an incidental parameter; we identify the poles of the integrand function in (4.3) in the p_2 plane. There are only two. The zeta function has a simple pole with residue 1 located at $p_2 = 1$. And, in Lemma 3.3, we noted that the second factor in the Mellin transform has a simple pole at $p_2 = -(p_1 + \alpha_1 + \alpha_2 + \rho)$, again with residue 1. We move the second contour to the left, passing over both these poles, including, in each case, a term that comprises the residue of the integrand function at that pole. Choosing $c_2' < min(1, -(p_1 + \alpha_1 + \alpha_2 + \rho))$, we find

(4.5)
$$\begin{aligned}
&\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} M(f^{[1]}; p_1, p_2)\zeta(p_2, \beta_2)m^{p_2-1}dp_2 \\
&\quad = M(f^{[1]}; p_1, 1) + \frac{\zeta(-(\alpha_1+\alpha_2+\rho+p_1), \beta_2)M_t(t^{\alpha_1}r_\rho(t,1)\nu_0(t), p_1)}{m^{(\alpha_1+\alpha_2+\rho+p_1)+1}} \\
&\qquad + \frac{1}{2\pi i}\int_{c_2'-i\infty}^{c_2'+i\infty} M(f^{[1]}; p_1, p_2)\zeta(p_2, \beta_2)m^{p_2-1}dp_2.
\end{aligned}$$

Naturally, the derivation above is invalid when p_1 is a pole of $M(f^{[1]}; p_1, p_2)$. Each term in (4.5) is an analytic function of p_1, however, and this fact is exploited below.

It may be helpful to remark that, since we may choose c_2' to be an arbitrarily large negative number, the final integral here vanishes (becomes part of the unstated but implied remainder term) in the asymptotic expansion (5.10) which is the result of these calculations. From this point on, the reader may mentally discard all terms involving c_2'.

Substituting (4.5) into (4.3) gives

(4.6)
$$\begin{aligned}
&S^m(\beta_1,\beta_2)f^{[1]}\\
&= \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} M(f^{[1]};p_1,1)\zeta(p_1,\beta_1)m^{p_1-1}dp_1 + \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\zeta(p_1,\beta_1)\\
&\quad\times\frac{\zeta(-(\alpha_1+\alpha_2+\rho+p_1),\beta_2)M_t(t^{\alpha_1}r_\rho(t,1)\nu_0(t),p_1)}{m^{\alpha_1+\alpha_2+\rho+2}}dp_1\\
&\quad+\frac{1}{(2\pi i)^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2'-i\infty}^{c_2'+i\infty} M(f^{[1]};p_1,p_2)\zeta(p_1,\beta_1)\\
&\quad\times\zeta(p_2,\beta_2)m^{p_1+p_2-2}dp_1dp_2.
\end{aligned}$$

We note that in the second integrand the part depending on m has turned out to be independent of p_1 and therefore may be taken outside the integral. This simplifying phenomenon allows us to set

$$(4.7)\quad \begin{aligned} A^{[1,0]}_{\alpha_1+\alpha_2+\rho+2} = \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} &\zeta(p_1;\beta_1)\zeta(-(\alpha_1+\alpha_2+\rho+p_1);\beta_2)\\ &\times M_t(t^{\alpha_1}r_\rho(t,1)\nu_0(t),p_1)dp_1,\end{aligned}$$

and (4.6) reduces to an expansion of the form

$$(4.8)\; \begin{aligned} S^m(\beta_1,\beta_2)f^{[1]} = &\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} M(f^{[1]};p_1,1)\zeta(p_1;\beta_1)m^{p_1-1}dp_1\\
&+\frac{A^{[1,0]}_{\alpha_1+\alpha_2+\rho+2}}{m^{\alpha_1+\alpha_2+\rho+2}} + \frac{1}{(2\pi i)^2}\\
&\times\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2'-i\infty}^{c_2'+i\infty} M(f^{[1]};p_1,p_2)\zeta(p_1;\beta_1)\\
&\times\zeta(p_2;\beta_2)m^{p_1+p_2-2}dp_1dp_2.\end{aligned}$$

We treat the first term on the right in (4.8). We move the integration contour $\operatorname{Re} p_1 = c_1$ to the left to a new location $\operatorname{Re} p_1 = c_1' < c_1$. In doing so, we have to addend the residue R_i of every pole P_i of the integrand function $\Phi^{[1]}(p_1) =: M(f^{[1]};p_1,1)\zeta(p_1;\beta_1)m^{p_1-1}$, which, as a result of the transfer, now appears to the right of the contour. Thus

$$(4.9)\quad \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+\infty} M(f^{[1]}; p_1, 1)\zeta(p_1, \beta_1)m^{p_1-1}dp_1$$
$$= \sum_{P_i>c_1'} R_i + \frac{1}{2\pi i}\int_{c_1'-i\infty}^{c_1'+i\infty} M(f^{[1]}; p_1, 1)\zeta(p_1, \beta_1)m^{p_1-1}dp_1.$$

To proceed, we need to locate these poles and find expressions for their residues. In view of Theorem 3.5, this integrand function may be written in the form

$$(4.10)\quad \Phi^{[1]}(p_1) = M_t(t^{\alpha_1}r_\rho(t, 1)\nu_0(t), p_1)$$
$$\times M_t(t^{\alpha_1+\alpha_2+\rho}\overline{\nu}(t), p_1 + 1)\zeta(p_1; \beta_1)m^{p_1-1}.$$

The zeta function has a simple pole with residue 1 at

$$(4.11)\qquad p_1 = p_1^{(0)} = 1.$$

the individual Mellin Transforms in this integrand have been treated in lemmas 3.3 and 3.2. The first Mellin Transform has a sequence of simple poles at

$$(4.12)\qquad p_1 = p_1^{(1)}(n_1) = -\alpha_1 - n_1, \quad n_1 = 0, 1, \dots ,$$

with residues $r_\rho^{(n_1,0)}(0, 1)/n_1!$ respectively, while the second has a simple pole with residue 1 at

$$(4.13)\qquad p_1 = p_1^{(2)} = -(\alpha_1 + \alpha_2 + \rho + 1).$$

The poles of each of the three factors of the integrand function (4.10) are simple. But, for some values of the parameters, a pole of one factor may coincide with a pole of another factor, giving rise to a multiple pole of $\Phi^{[1]}(p_1)$. The expression for the residue R_i depends on the multiplicity of the pole P_i.

Definition 4.1 *The set of parameters* α_1, α_2 *and* ρ *are termed* generic *when all the poles of* $\Phi^{[1]}$ *are simple and all the poles of* $\Phi^{[2]}$ *are simple.*

Here $\Phi^{[2]}$ is the integrand function in (4.10) when $f^{[2]}$ is treated in place of $f^{[1]}$. We note that a pole of $\Phi^{[1]}$ may coincide with a pole of $\Phi^{[2]}$. In several important cases, there are multiple poles. We discuss these *nongeneric* cases briefly in Section 7.

5 The two-dimensional error expansion in the generic case

So long as the locations of all the poles $p_1^{(j)}$ given in (4.11), (4.12), and (4.13) are distinct, the poles P_i of the integrand function $\Phi^{[1]}$ are simple. This fact, together with the corresponding remark concerning $\Phi^{[2]}$, allows us to state a sufficient condition for a generic case.

Theorem 5.1 *When none of the following five conditions are satisfied, a generic case occurs.*

(1) $\alpha_1 + \alpha_2 + \rho = -2$
(2) $\alpha_2 + \rho + 2 = \textit{positive integer} = m_1$.
(3) $\alpha_1 = \textit{negative integer} = -m_2$.
(4) $\alpha_1 + \rho + 2 = \textit{positive integer} = m_1'$.
(5) $\alpha_2 = \textit{negative integer} = -m_2'$.

Proof The reader may verify that the first of these five conditions reflects the coincidence of poles at $p_1^{(0)}$ and $p_1^{(2)}$ above. The second and third conditions reflect the coincidence of one of the poles $p_1^{(n)}$ with $p_1^{(0)}$ and with $p_1^{(2)}$, respectively. The fourth and fifth conditions arise from a corresponding inspection of $f^{[2]}$. □

We note that generic cases do occur, even when one or more of these conditions pertain. Such an occurrence arises when one of the simple poles listed above disappears as a result of a particular choice of parameters. The symptom is that the residue vanishes. (For example, when $\alpha_1 = \alpha_2 = \rho/2 = 0$, the poles $p_1^{(1)}(n_1) = -\alpha_1 - n_1$ disappear, except when $n_1 = 0$. None of the remaining poles coincide. This is a generic case, items (2) and (4) notwithstanding.)

We have given above the residues at $p_1^{(i)}$ of the individual components of the integrand function. We need the residues of the complete integrand function. In this generic case, the residue of the integrand function (4.10) at $p_1 = p_1^{(0)} = 1$ is simply $M_t(t^{\alpha_1} r_\rho(t, 1)\nu_0(t), 1)\times\ M_t(t^{\alpha_1+\alpha_2+\rho}\overline{\nu}(t), 2)$, which, in view of (3.31), reduces to $M(f^{[1]}; 1, 1)$.

The other residues are calculated in the standard way for simple poles. Collecting these terms, we find the specialization of (4.9) in the generic case to be

(5.1)
$$\begin{aligned}
&\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+\infty} M(f^{[1]}; p_1, 1)\zeta(p_1, \beta_1)m^{p_1-1}dp_1 \\
&\quad = M(f^{[1]}; 1, 1) + \sum_{n_1=0}^{N_1} A^{[1,1]}_{\alpha_1+n_1+1}/m^{\alpha_1+n_1+1} + A^{[1,2]}_{\alpha_1+\alpha_2+\rho+2}/m^{\alpha_1+\alpha_2+\rho+2} \\
&\qquad + \frac{1}{2\pi i}\int_{c_1'-i\infty}^{c_1'+i\infty} M(f^{[1]}; p_1, 1)\zeta(p_1, \beta_1)m^{p_1-1}dp_1,
\end{aligned}$$

where

$$\begin{aligned}
A^{[1,1]}_{\alpha_1+n_1+1} = &\frac{1}{n_1!} r_\rho^{(n_1,0)}(0, 1)M_t(t^{\alpha_1+\alpha_2+\rho}\overline{\nu}(t); -\alpha_1 - n_1 + 1) \\
&\times\zeta(-\alpha_1 - n_1, \beta_1),
\end{aligned} \tag{5.2}$$

$$
\begin{aligned}
A^{[1,2]}_{\alpha_1+\alpha_2+\rho+2} = {} & M_t(t^{\alpha_1} r_\rho(t,1)\nu_0(t); -\alpha_1-\alpha_2-\rho-1) \\
& \times\zeta(-\alpha_1-\alpha_2-\rho-1,\beta_1).
\end{aligned}
\tag{5.3}
$$

Here, N_1 is the number of poles to the right of the contour $\mathrm{Re}(p_1) = c_1$. Thus

$$
c_1 \in (\alpha_1 - N_1 - 2, \alpha_1 - N_1 - 1).
$$

Substituting the right-hand side of (5.1) into (4.8), we obtain

$$
\begin{aligned}
S^m(\beta_1,\beta_2) f^{[1]} = {} & M(f^{[1]};1,1) + \frac{\overline{A}^{[1]}_{\alpha_1+\alpha_2+\rho+2}}{m^{\alpha_1+\alpha_2+\rho+2}} + \sum_{n_1=0}^{N_1} \frac{A^{[1,1]}_{n_1+\alpha_1+1}}{m^{n_1+\alpha_1+1}} \\
& + \frac{1}{2\pi i}\int_{c_1'-i\infty}^{c_1'+i\infty} M(f^{[1]};p_1,1)\zeta(p_1,\beta_1)m^{p_1-1}dp_1 \\
& + \frac{1}{(2\pi i)^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2'-i\infty}^{c_2'+i\infty} M(f^{[1]};p_1,p_2)\zeta(p_1;\beta_1) \\
& \times\zeta(p_2;\beta_2)m^{p_1+p_2-2}dp_1dp_2,
\end{aligned}
\tag{5.4}
$$

with

$$
\overline{A}^{[1]}_{\gamma+2} = A^{[1,0]}_{\gamma+2} + A^{[1,2]}_{\gamma+2}.
\tag{5.5}
$$

Here, $M(f^{[1]};1,1)$ is the (double) analytic continuation of (4.2) with $f = f^{[1]}(x_1,x_2)$. If no continuation is necessary,

$$
M(f^{[1]};1,1) = \int_0^\infty\int_0^\infty f^{[1]}(x_1,x_2)dx_1dx_2.
\tag{5.6}
$$

Naturally, the sufficient conditions in Theorem 5.1 for this to be a generic case coincide with the condition that each term in the expansion (5.4) involves a distinct power of m. Further examination of the coefficients reveals that, when the conditions of that Theorem are violated, and two terms appear having the same power of m, the expressions as written for the coefficients may become indeterminate. For example, when condition (3) is violated, and α_1 is the negative integer $-m_2$, the zeta function in (5.2) is indeterminate when $n_1 = m_2 - 1$. However, if that particular term does not occur, for example when $r^{(n_1,0)}_\rho = 0$, the expansion is not affected. This fact reconfirms that condition (3) is only a *sufficient* condition for a nongeneric case and not a necessary one.

Condition (1) of the Theorem can be connected with the zeta function factor in (5.3) in the same way, while condition (2) is related to a pole of the Mellin transform factor in the same equation.

In the nongeneric cases one or more of these conditions are violated. The consequent modifications to the expansions required are treated briefly in Section 8. At this point, we rewrite (5.4) as an asymptotic expansion

$$S^m(\beta_1,\beta_2)f^{[1]} \sim M(f^{[1]};1,1) + \frac{\overline{A}^{[1]}_{\alpha_1+\alpha_2+\rho+2}}{m^{\alpha_1+\alpha_2+\rho+2}} + \sum_{n_1=0} \frac{A^{[1,1]}_{n_1+\alpha_1+1}}{m^{n_1+\alpha_1+1}} \tag{5.7}$$

for $f^{[1]}$, where

$$f^{[1]}(x_1,x_2) = x_1^{\alpha_1}x_2^{\alpha_2}r_\rho(x_1,x_2)N^{[1]}(x_1,x_2). \tag{5.8}$$

This function coincides with $f(x_1,x_2)$ in a region adjacent to the x_2-axis but tapers away and coincides with zero in a region adjacent to the x_1-axis. To obtain an expansion for $S^m(\beta_1,\beta_2)f$, we require the corresponding expansion for the function

$$f^{[2]}(x_1,x_2) = x_1^{\alpha_1}x_2^{\alpha_2}r_\rho(x_1,x_2)N^{[2]}(x_1,x_2).$$

This was introduced at the same time as $f^{[1]}$ in Section 3.3; we recall that $f = f^{[1]} + f^{[2]}$. The derivation of the expansion for $S^m(\beta_1,\beta_2)f^{[2]}$ corresponds in every respect to that for $S^m(\beta_1,\beta_2)f^{[1]}$ as described above. One obtains

$$S^m(\beta_1,\beta_2)f^{[2]} \sim M(f^{[2]};1,1) + \frac{\overline{A}^{[2]}_{\alpha_1+\alpha_2+\rho+2}}{m^{\alpha_1+\alpha_2+\rho+2}} + \sum_{n_2=0} \frac{A^{[2,1]}_{n_2+\alpha_2+1}}{m^{n_2+\alpha_2+1}}. \tag{5.9}$$

The major result of this paper is obtained by adding these expansions together, giving the following Theorem.

Theorem 5.2 *Let α_1,α_2 and ρ be a generic set of parameters, as specified in definition 4.1. Let $S^m(\beta_1,\beta_2)f$ be the offset trapeziodal rule approximation (4.1) to the full corner singularity function $f(x_1,x_2) = x_1^{\alpha_1}x_2^{\alpha_2}r_\rho(x_1,x_2)$ $\times N(x_1,x_2)$ as given in (3.19). Then there exists an asymptotic expansion of the form*

$$S^m(\beta_1,\beta_2)f \sim M(f;1,1) + \frac{A^{(0)}_{\alpha_1+\alpha_2+\rho+2}}{m^{\alpha_1+\alpha_2+\rho+2}} + \sum_{n_1=0} \frac{A^{[1,1]}_{n_1+\alpha_1+1}}{m^{n_1+\alpha_1+1}}$$
$$+ \sum_{n_2=0} \frac{A^{[2,1]}_{n_2+\alpha_2+1}}{m^{n_2+\alpha_2+1}}. \tag{5.10}$$

Here, we have set

$$M(f;1,1) = M(f^{[1]};1,1) + M(f^{[2]};1,1)$$

and

$$A^{(0)}_{\gamma+2} = \overline{A}^{[1]}_{\gamma+2} + \overline{A}^{[2]}_{\gamma+2} = A^{[1,0]}_{\gamma+2} + A^{[1,2]}_{\gamma+2} + A^{[2,0]}_{\gamma+2} + A^{[2,2]}_{\gamma+2}. \tag{5.11}$$

(Coefficients having superscript 1 are defined explicitly in (5.5), (4.7), (5.2), and (5.3).) As mentioned above, in the conventional case, the integral is regular and

$$M(f;1,1) = \int_0^\infty \int_0^\infty f(x_1,x_2)dx_1dx_2. \tag{5.12}$$

In general, when this does not exist, it is the analytic continuation of

$$M(f;p_1,p_2) = \int_0^\infty \int_0^\infty f(x_1,x_2)x_1^{p_1-1}x_2^{p_2-1}dx_1dx_2 \tag{5.13}$$

to $p_1 = p_2 = 1$. Since this is the generic case in which this meromorphic function has no poles at $p_1 = 1$ or $p_2 = 1$, the integral (5.13) does exist for some p_1 and p_2, and its continuation coincides with our definition of the Hadamard finite-part integral. Note that, in view of Theorem 3.3, there need be no relation of this kind if any one of α_1, α_2 and $\alpha_1 + \alpha_2$ is a nonpositive integer.

Remark 5.1 When α_i is a nonnegative integer, the term $x_i^{\alpha_i}$ plays no significant independent role in the theory, which can be rearranged to omit such factors with a consequent simplification. (In fact, with no loss in generality we can restrict the theory to the case $\alpha_i \neq$ nonnegative integer.) No such simplification occurs in general for special values of ρ. For example, a possible integrand is $\arctan(x_1/x_2)$, which is a function of form $r_\rho(x_1, x_2)$ with $\rho = 0$ and certainly gives rise to a corresponding term in the expansion.

Remark 5.2 Examination of the contour integral representation (4.7) of $A_{\gamma+2}^{[1,0]}$ shows that the integrand function has a pole at $p_1 = -1-\gamma$ whose residue turns out to be precisely $-A_{\gamma+2}^{[1,2]}$ as defined in (5.3). This implies that $\overline{A}_{\gamma+2}^{[1]}$ in (5.5) has a contour integral representation having this same integrand, but a different contour. Thus

$$\begin{aligned} \overline{A}^{[1]}_{\alpha_1+\alpha_2+\rho+2} &= \frac{1}{2\pi i}\int_{\overline{C}_1} \zeta(p_1;\beta_1)\zeta(-(\alpha_1+\alpha_2+\rho+p_1);\beta_2) \\ &\quad \times M_t(t^{\alpha_1}r_\rho(t,1)\nu_0(t), p_1)dp_1, \end{aligned} \tag{5.14}$$

where $\overline{C}_1$ is a modification of the contour $\mathrm{Re}(p) = c_1$; this modified contour passes to the left of the pole at $p = -1-\gamma$ but to the right of all the other poles. When $\alpha_1 > -1$ and $\gamma + 2 > 0$, the contour $\overline{C}_1$ may be taken to be the line $\mathrm{Re}(p_1) = 1$, indented to pass to the right of the pole at $p_1 = 1$.

6 The form of related expansions

Up to this point, the theory has been devoted to the asymptotic expansion of the trapezoidal rule sum approximation $S^m(\beta_1, \beta_2)f$ (introduced in (4.1)) of the basic integrand function

$$f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2} r_\rho(x_1, x_2) N(x_1, x_2) \tag{6.1}$$

over the first quadrant $[0, \infty)^2$. In this section we deal with the form of the corresponding expansions when the integration region is replaced by $[0, 1]^2$ and when the integrand function is generalized to fg with g regular in the integration region. We provide a simple framework for handling the somewhat tedious extensions to the theory required to obtain these variant expansions.

We denote various integration regions as follows:

$$\overline{H}_{0,0} = [0, 1)^2; \qquad H_{p,q} = [p, \infty) \times [q, \infty); \qquad p, q = 0, 1. \tag{6.2}$$

Specifically

$$H_{0,0} = [0, \infty)^2; \quad H_{0,1} = [0, \infty) \times [1, \infty); \quad H_{1,0} = [1, \infty) \times [0, \infty); \quad H_{1,1} = [1, \infty)^2.$$

We suppress dependence on β_1 and β_2 and denote by $S^m(H_{0,0})f$ the quantity $S^m(\beta_1, \beta_2)f$ defined in (4.1). The trapezoidal rule approximations corresponding to the regions specified above are denoted by

$$S^m(H_{p,q})f = \frac{1}{m^2} \sum_{j_1=mp}^{\infty} \sum_{j_2=mq}^{\infty} f\left(\frac{j_1+\beta_1}{m}, \frac{j_2+\beta_2}{m}\right), \qquad p, q = 0, 1, \tag{6.3}$$

and by

$$S^m(\overline{H}_{0,0})f = \frac{1}{m^2} \sum_{j_1=0}^{m-1} \sum_{j_2=0}^{m-1} f\left(\frac{j_1+\beta_1}{m}, \frac{j_2+\beta_2}{m}\right). \tag{6.4}$$

The sum over $\overline{H}_{0,0}$ can be expressed as

$$S^m(\overline{H}_{0,0})f = S^m(H_{0,0})f - S^m(H_{0,1})f - S^m(H_{1,0})f + S^m(H_{1,1})f, \tag{6.5}$$

and the expansion for $\overline{H}_{0,0}$ may be obtained as the sum of the four expansions, a different one for each region. The appropriate expansion in these different regions may differ from one another, depending on the extent (if any) to which the singularities of f penetrate that region.

In none of the theory or examples treated in this paper are there any integrand singularities in $H_{1,1}$, and the integral exists. Thus the standard Euler Maclaurin expansion may be applied.

The result of carrying out this process for the full singularity of Theorem 5.2, namely,

$$f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2} r_\rho(x_1, x_2) N(x_1, x_2), \tag{6.6}$$

is an asymptotic expansion including exclusively terms of the form A_γ/m^γ, where γ may take any value specified in items (1), (2), (3), and (4) of Theorem 6.1, together with $\gamma = \alpha_1 + \alpha_2 + \rho + 2$.

In the application to numerical quadrature, however, one needs an expansion for the more general function fg where $g(x_1, x_2)$ is a regular function. The standard approach is straightforward. One expands $g(x_1, x_2)$ as a Taylor expansion and applies corresponding results for each separate term and to the remainder term. The effect is to introduce for each γ already present a sequence $\gamma + j$ with $j = 1, 2, 3, \ldots$. If, as in the final summation on the right of (5.10), γ already belongs to such a sequence, the *form* of the expansion is not altered. On the other hand, the first term on the right of (5.10) is replaced by a sequence (5) below.

The second major result of this paper is an *umbrella* result for all expansions involving full singularities over these regions when the integrand function is *generic*.

Theorem 6.1 *Let α_1, α_2 and ρ be a generic set of parameters, as specified in definition 4.1. Let $S^m(\overline{H}_{0,0})fg$ and $S^m(H_{0,0})fg$ be the offset trapeziodal rule approximations (6.4) and (6.3) to the integral over $[0, 1)^2$ and over $[0, \infty)^2$, respectively, of fg, where $f(x_1, x_2)$ is the full corner singularity function as given in (6.6) and $g(x_1, x_2)$ is a regular function. Then there exists an asymptotic expansion for $S^m fg$ in powers of m containing exclusively terms of the form A_γ/m^γ for some or all of the following values of γ:*

(1) $\gamma = 0$;
(2) $\gamma = s$; $\quad s = 1, 2, 3, \ldots.$
(3) $\gamma = \alpha_2 + 1 + n_1$; $\quad n_1 = 0, 1, 2, 3, \ldots.$
(4) $\gamma = \alpha_1 + 1 + n_2$; $\quad n_2 = 0, 1, 2, 3, \ldots.$
(5) $\gamma = \alpha_1 + \alpha_2 + \rho + 2 + n$; $\quad n = 0, 1, 2, 3, \ldots.$

This large number of terms in the expansion is disappointing, if not unexpected. All expansions contain item (1), which is simply the (Hadamard finite-part) integral. The classical Euler-Maclaurin expansion includes additionally only sequence (2). The basic expansion (5.10) may include two sequences, however, and when g is included, this becomes three sequences.

7 Expressions for individual coefficients

While the form of the expansions derived in previous sections is relatively simple, some of the expressions for coefficients are forbiddingly complex. In many cases, these have a simple representation in special cases when one or two of the parameters do not appear. In this section we collect together several results of that nature.

7.1 The classical Euler-Maclaurin expansion

It is convenient to note the form of the classical Euler-Maclaurin expansion applied to $[0, \infty)^2$, of which (5.10) is a variant.

Theorem 7.1 *Let $f(x_1, x_2)$ be allowable and $C^p[0, \infty)^2$. Let $S^m(\beta_1, \beta_2)f$ be the off-set trapeziodal rule approximation (4.1) to this integral. Then there exists an asymptotic expansion of the form*

$$S^m(\beta_1, \beta_2)f = M(f; 1, 1) + \sum_{s=1}^{p-1} \frac{B_s}{m^s} + R_p, \tag{7.1}$$

where B_s is independent of m and $R_p = O(m^{-p})$.

Here, of course, $M(f; 1, 1)$ is a regular integral.

This form may be obtained from (1.2) by summing the corresponding result for the square $[K, K+1) \times [L, L+1)$ over all nonnegative integers K and L. We find that the coefficients B_s take the form

$$B_s = \sum_{k=0}^{s} c_k(\beta_1) c_{s-k}(\beta_2) \int_0^\infty \int_0^\infty f^{(k,s-k)}(x_1, x_2) dx_1 dx_2. \tag{7.2}$$

The coefficients here take the forms

$$c_k(\beta) = -\zeta(-k+1, \beta)/(k-1)! = B_k(\beta)/k! \tag{7.3}$$

where $B_k(\beta)$ is the Bernoulli polynomial. In view of the high-order continuity of f, expression (7.2) can be reduced to

$$\begin{aligned} B_s = &-c_s(\beta_1) \int_0^\infty f^{(s-1,0)}(0, x_2) dx_2 \\ &-c_s(\beta_2) \int_0^\infty f^{(0,s-1)}(x_1, 0) dx_1 \\ &+\sum_{k=1}^{s-1} c_k(\beta_1) c_{s-k}(\beta_k) f^{(k-1,s-k-1)}(0, 0). \end{aligned} \tag{7.4}$$

Thus, when $f(x_1, x_2)$ is allowable and $C^{(p)}[0, \infty)^2$, the coefficients B_s $(1 \leq s \leq p)$ depend only on the nature of $f(x_1, x_2)$ on the axes $x_1 = 0$ and $x_2 = 0$.

7.2 A simpler neutraliser function

This result can be used to simplify marginally some of the previous results by simplifying the dependence on neutralizer functions. The integrand functions of Sections 2 and 3 all involved a neutralizer function $N(x_1, x_2)$ given in (3.18). Examination of this function shows that it coincides with the simpler neutralizer function

$$\overline{N}(x_1, x_2) = \overline{\nu}(x_1, k_1, k_2)\overline{\nu}(x_2, k_1, k_2) \tag{7.5}$$

for all $0 \le x_1 \le k_1 k_0^{-1}$, $0 \le x_2 \le k_1 k_0^{-1}$. Consequently, the distinct full corner singularity functions

$$f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2} r_p(x_1, x_2) g(x_1, x_2) N(x_1, x_2) \tag{7.6}$$

and

$$\overline{f}(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2} r_p(x_1, x_2) g(x_1, x_2) \overline{N}(x_1, x_2) \tag{7.7}$$

coincide in a strip along the axes and are by definition $C^\infty[k_1 k_0^{-1}, \infty)^2$. The function $(f(x_1, x_2) - \overline{f}(x_1, x_2))$ is then $C^\infty[0, \infty)^2$ and so is one to which Theorem 7.1 applies, and examination of (7.4) shows that the coefficients B_s for this difference vanish. This gives immediately the following Theorem.

Theorem 7.2 *Theorem 4.2 is valid as it stands when $N(x_1, x_2)$ is replaced by $\overline{N}(x_1, x_2)$ in the definition of f.*

Naturally, $S^{(m)} f$ and $M(f; 1, 1)$ change when N is replaced by $\overline{N}$. The other coefficients, however, are identical. The expressions given for $A_\gamma^{[j,1]}$ are already independent of ν_0. However, the expression given for $A_\gamma^{(0)}$ includes several terms, some of which depend on ν_0. This dependence is in fact spurious.

7.3 The coefficient $A_\gamma^{[1,1]}$, general case

The coefficient $A^{[1,1]}_{\alpha_1+n_1+1}$ is given by (5.2) which involves the neutralizer function N. In view of Theorem 7.2, we may replace N by $\overline{N}$. This replacement allows a simple reexpression in terms of the cofactor function of $x_1^{\alpha_1}$ in f, defined by

$$h_1(x_1, x_2) = x_1^{-\alpha_1} f(x_1, x_2) = r_\rho(x_1, x_2) x_2^{\alpha_2} \overline{\nu}(x_1) \overline{\nu}(x_2). \tag{7.8}$$

Lemma 7.1 *The n_1th derivative of this cofactor function satisfies*

$$h_1^{(n_1,0)}(0, x_2) = r_\rho^{(n_1,0)}(0, x_2) x_2^{\alpha_2} \overline{\nu}(x_2) \tag{7.9}$$

$$= r_\rho^{(n_1,0)}(0, 1) x_2^{\alpha_2+\rho-n_1} \overline{\nu}(x_2). \tag{7.10}$$

Proof This is straightforward. Differentiating the right-hand side of (7.8) n_1 times with respect to x_1 using the Leibniz expansion leaves $n_1 + 1$ terms. Since $\overline{\nu}^{(s)}(0) = 0$ for all $s > 0$, when we set $x_1 = 0$, only one of these terms remains, this being the right-hand side of (7.9). The final equation is established by noting that the factor $r_\rho^{(n_1,0)}(0, x_2)$ is homogeneous in x_2 of degree $\rho - n_1$ and so can be reexpressed as required to establish the result. □

Minor rearrangement of (5.2) together with an application of this lemma give successively

$$A^{[1,1]}_{\alpha_1+n_1+1} = \frac{\zeta(-\alpha_1 - n_1, \beta_1)}{n_1!} M_t(r_\rho^{(n_1,0)}(0,1)t^{\alpha_2+\rho-n_1}\overline{\nu}(t); 1) \tag{7.11}$$

$$= \frac{\zeta(-\alpha_1 - n_1, \beta_1)}{n_1!} M_t(h_1^{(n_1,0)}(0,t); 1). \tag{7.12}$$

The final term here is a regular integral when $\rho+\alpha_2-n_1 > -1$ Otherwise it is a one-dimensional HFP integral, except when $\rho + \alpha_2 - n_1$ is a negative integer.

7.4 The coefficient $A^{[1,1]}_\gamma$, special case $r_\rho = 1$

We treat the special case

$$f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2} \overline{\nu}(x_1)\overline{\nu}(x_2). \tag{7.13}$$

Here, we may set $r_\rho(x_1, x_2) = 1$ and set $\rho = 0$ in any previously stated result. The set of poles in (4.12) may be replaced by a single pole $p_1^{(1)}(0)$. Consequently all the terms other than the initial term in the sums over n_1 and n_2 in (5.10) vanish, and only four individual terms remain on the right of (5.10). The set of five sufficient conditions for a generic case reduce to:

$$\alpha_1 \neq -1; \qquad \alpha_2 \neq -1; \qquad \alpha_1 + \alpha_2 \neq -2. \tag{7.14}$$

We then have the following Theorem.

Theorem 7.3 *Let $S^{(m)}(\beta_1, \beta_2)f$ be the offset trapezoidal rule approximation (4.1) to the corner singularity function (7.13) with parameters α_i satisfying (7.14). Then there exists an asymptotic expansion*

(7.15)

$$S^{(m)}(\beta_1, \beta_2)f = M(f; 1, 1) + \frac{A^{(0)}_{\alpha_1+\alpha_2+2}}{m^{\alpha_1+\alpha_2+2}} + \frac{A^{[1,1]}_{\alpha_1+1}}{m^{\alpha_1+1}} + \frac{A^{[2,1]}_{\alpha_2+1}}{m^{\alpha_2+1}} + O(m^{-p})$$

for all p.

Here

(7.16)

$$A^{[1,1]}_{\alpha_1+1} = M_t(t^{\alpha_2}\overline{\nu}(t); 1))\zeta(-\alpha_1, \beta_1); \quad A^{[1,2]}_{\alpha_2+1} = M_t(t^{\alpha_1}\overline{\nu}(t); 1))\zeta(-\alpha_2, \beta_2).$$

It is shown below that

$$A^{(0)}_{\alpha_1+\alpha_2+2} = \zeta(-\alpha_1, \beta_1)\zeta(-\alpha_2, \beta_2). \tag{7.17}$$

This special case may be treated without resort to the coordinate transformation of Section 3.3. Instead, we may exploit the circumstance that f in (7.13) is a product function, say $f = \phi_1\phi_2$. We apply Theorem 2.2 to the function $\phi_1(x_1)$, with $g(x_1) = \overline{\nu}(x_1)$, to obtain for the one-dimensional discretization (2.5)

$$S^{(m)}(\beta_1)\phi_1 = M(\phi_1, 1) + \zeta(-\alpha_1, \beta_1)/m^{\alpha_1+1} + O(m^{-p}), \tag{7.18}$$

valid so long as $\alpha_1 \neq 1$. Since $S^{(m)}(\beta_1, \beta_2)\phi_1\phi_2 = S^m(\beta_1)\phi_1 S^{(m)}(\beta_2)\phi_2$, we may take the product of two versions of asymptotic expansions (7.18), obtaining an independent proof of Theorem 7.3 that provides the expression for $A^{(0)}_{\alpha_1+\alpha_2+2}$ given above.

7.5 The coefficient $A^{[1,1]}_\gamma$, special case $\alpha_1 = \alpha_2 = 0$

We treat the special case

$$f(x_1, x_2) = r_\rho(x_1, x_2)\overline{v}(x_1)\overline{v}(x_2). \tag{7.19}$$

For this to be a generic case, we require that $\rho + 2$ not be a nonnegative integer, allowing us to apply (5.10) to obtain

$$S^m(\beta_1, \beta_2)f \sim M(f; 1, 1) + \frac{A^{(0)}_{\rho+2}}{m^{\rho+2}} + \sum_{s=1} \frac{B_s}{m^s}, \tag{7.20}$$

where

$$\begin{aligned} B_s &= A^{[1,1]}_s + A^{[2,1]}_s && (7.21)\\ &= \frac{\zeta(-s+1, \beta_1)}{(s-1)!} M_t(f^{(s-1,0)}(0, t); 1) \\ &\quad + \frac{\zeta(-s+1, \beta_2)}{(s-1)!} M_t(f^{0,(s-1)}(t, 0); 1) && (7.22)\\ &= -c_s(\beta_1) FP \int_0^\infty f^{(s-1,0)}(0, x_2)dx_2 \\ &\quad -c_s(\beta_2) FP \int_0^\infty f^{(0,s-1)}(x_1, 0)dx_1. && (7.23)\end{aligned}$$

These coefficients resemble closely the corresponding coefficients for the regular function given in (7.4). The only differences are that those integrals that, with the new integrand, do not converge, are replaced by HFP integrals and the final summation in (7.4) (the terms of which in this case are either zero or indeterminate) is omitted. The reduction of form (7.21) to one resembling (7.2) is not immediate. One requires the following lemma.

Lemma 7.2 *Let f be given by (7.19), and let t_1, t_2 and s be positive integers, and $\rho - s \neq -2$. Then*

$$FP \int_0^\infty\int_0^\infty f^{(t_1,t_2)}(x_1, x_2)dx_1dx_2 = 0 \tag{7.24}$$

$$FP \int_0^\infty\int_0^\infty f^{(0,s)}(x_1, x_2)dx_1dx_2 = FP \int_0^\infty f^{(0,s-1)}(x_1, 0)dx_1. \tag{7.25}$$

We omit our somewhat pedestrian proof of this elegant result.

In view of this, we may reexpress this coefficient as

$$B_s = \sum_{k=0}^{s} c_k(\beta_1)c_{s-k}(\beta_2) FP \int_0^{\infty}\int_0^{\infty} f^{(k,s-k)}(x_1, x_2)dx_1dx_2. \tag{7.26}$$

This is precisely the same form as the corresponding (7.2) except that regular integrals are consistently replaced by HFP integrals.

The expansion (7.20) bears a close resemblance to the corresponding expansion for a regular function described by Theorem 7.1. Apart from a single additional term $A^{(0)}_{\rho+2}/m^{\rho+2}$, the only differences are those required to modify integrals that would otherwise diverge to HFP integrals and to remove indeterminate quantities.

7.6 Special case $\alpha_1 = \alpha_2 = 0$*, region* $[0, 1)^2$

Here, the integrand function coincides with the integrand function treated in the previous subsection. We express the region $\overline{H}_{0,0} = [0, 1)^2$ as a linear combination of four regions $H_{p,q}$ as set out in Section 5 above. For $H_{0,0}$ we may employ (6.19) with B_s given by (6.25). Within the other three regions, $H_{0,1}$, $H_{1,0}$ and $H_{1,1}$ this integrand is regular and the standard Euler-Maclaurin expansion (7.1) may be used. The result is the following.

Theorem 7.4 *When $\rho + 2$ is not a nonnegative integer,*

$$S^m(\overline{H}_{0,0})f \sim M(f; 1, 1) + \frac{A^{(0)}_{\rho+2}}{m^{\rho+2}} + \sum_{s=1} \frac{B_s}{m^s}, \tag{7.27}$$

where

$$B_s = \sum_{k=0}^{s} c_k(\beta_1)c_{s-k}(\beta_2) FP \int_0^{1}\int_0^{1} f^{(k,s-k)}(x_1, x_2)dx_1dx_2. \tag{7.28}$$

Note that $A^{(0)}_{\rho+2}$ in this expansion is identical with the same coefficient in expansion (7.20).

8 The two-dimensional nongeneric expansions

In the preceding section, we developed (4.9) by finding expressions for the residues R_i in the case that all the poles P_i of $\Phi^{[1]}(p)$ are simple poles. In that case expressions for the residues are readily available, reducing (4.9) to (5.1) with accompanying expressions for the coefficients. These terms, together with terms arising from a corresponding development of $f^{[2]}$, appear in the final Theorem 5.2.

In a nongeneric case, some of the poles P_i are not simple; for these, a different residue calculation is required.

If we treat only $f^{[1]}$, to obtain a nongeneric case, two or more of the poles $p_1^{(j)}$ given in (4.11), (4.12), and (4.13) must coincide. This situation can happen in relatively few ways:

(1) $p^{(0)} = p^{(2)} = p^{(1)}(\overline{n}_1)$ for some nonnegative integer $\overline{n}_1$
(2) $p^{(0)} = p^{(2)} \neq p^{(1)}(n)$ for all nonnegative integers n
(3) $p^{(0)} = p^{(1)}(\overline{n}_3) \neq p^{(2)}$ for some nonnegative integer $\overline{n}_3$
(4) $p^{(2)} = p^{(1)}(\overline{n}_4) \neq p^{(0)}$ for some nonnegative integer $\overline{n}_4$

Case (1) is a triple pole. In this case all other poles of $\Phi^{[1]}$ are simple.

Case (2) is a double pole. In this case all other poles of $\Phi^{[1]}$ are simple.

Cases (3) and (4) are also double poles. They may both occur in the same expansion with $\overline{n}_3 \neq \overline{n}_4$, or possibly only one may occur. In either situation, all other poles of $\Phi^{[1]}$ are simple.

We note that $p^{(0)} = 1$. This pole gives rise to the term $M[f^{[1]}; 1, 1]$ in the expansion.

In all these cases, the *form* of the expansion can be readily obtained. The integrand function (4.10) is of the form $\Phi^{[1]}(p) = G(p)m^{p-1}$, where $G(p)$ contains the poles at $p = P_i$. Since there is no pole of $G(p)$ of order higher than 3, the Laurent expansion of $G(p)$ about any pole P can be written in the form

$$G(p) = c_{-3}(p-P)^{-3} + c_{-2}(p-P)^{-2} + c_{-1}(p-P)^{-1} + c_0 + \cdots \tag{8.1}$$

When P is a double pole, $c_{-3} = 0$. When P is a simple pole, $c_{-3} = c_{-2} = 0$. The factor of $\Phi^{[1]}(p)$ involving m may be expanded in the form

$$\begin{aligned} m^{p-1} &= m^{P-1}\exp((p-P)\log m) \\ &= m^{P-1}(1 + (p-P)\log m + ((p-P)\log m)^2/2 + \cdots). \end{aligned} \tag{8.2}$$

The residue of $\Phi^{[1]}(p) = G(p)m^{p-1}$ at the pole $p = P$ is simply the coefficient of $(p-P)^{-1}$ in the product of these two expansions. This is

$$R = (c_{-3}(\log m)^2/2 + c_{-2}(\log m) + c_{-1})/m^{1-P}. \tag{8.3}$$

Naturally, the principal Theorem of the preceding section 5, Theorem 5.2, requires modification before it may be applied to these nongeneric cases. This modification is minor, however. When P is a double pole of $\Phi^{[1]}(p)$, two of its factors have poles at $p = P$. If the residues are mistakenly calculated on the basis of this being a simple pole, the result contains an indeterminate factor. Thus, in the expansion (5.10) of that Theorem as written, when P is in fact a double pole, the two terms of the form A_{1-P}/m^{1-P} are both indeterminate. The proper residue to use in this case is of the form (8.3) with $c_{-3} = 0$; the two indeterminate terms should be replaced by a two-parameter term of the form $(C_{1-P}\log m + D_{1-P})/m^{1-P}$.

In the triple pole case, which occurs only when $P = 1$, one replaces three terms, each apparently constants in the expansion, by a three-parameter term of form

$$R = C_0'(\log m)^2 + C_0 \log m + D_0. \tag{8.4}$$

The derivation given above refers only to the terms in the final expansion arising from the component $f^{[1]}$. A similar treatment of $f^{[2]}$ is also required. This gives results of a precisely corresponding nature.

Thus it is straightforward to write down the *form* of the expansion in nongeneric cases. But formulas for the coefficients are cumbersome. The principal application is to numerical quadrature by extrapolation. There, expressions for the coefficients are needed only when the value of the integral is involved. In the generic case, this is the constant coefficient $M(f; 1, 1)$. In cases where there is a multiple pole at $p = 1$, we now have the terms in (8.4).

9 Application to curved quadrangles

The effects of a change of variables has been examined only in the special case where $\alpha_1 = \alpha_2 = 0$. This case occurs in many current boundary element method applications. The modification introduced by a smooth change of variable is known (see [Ki91]). However the sum over all boundary elements remains unchanged, as long as the hypersingularity is inside the global domain of integration. The reader should notice that in (1.1) we do not require $r_\rho(x_1, x_2)$ to have the form $(x_1^2+x_2^2)^{\rho/2}$. It merely need to be homogeneous of degree ρ. This assumption plays a key role when we have to extend the asymptotic expansions we have obtained, to include any smooth curved quadrangle. Indeed, if the integration region is a curved quadrangle, given by an analytic parametric representation defined on the unit square, the original kernel of the boundary integral equation, after having introduced this representation, is no longer homogeneous.

Nevertheless, the transformed kernel can be decomposed in the following form (see [ScWe92a])

$$\sum_{j=0}^{N} r^{\rho+j} f_j(\theta) + R_N(x_1, x_2), \;\; x_1, x_2 \in [0, 1]^2 \tag{9.1}$$

where $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \mathrm{atan}(x_2/x_1)$. The functions $f_j(\theta)$ are analytic in $[0, \frac{\pi}{2}]$, while the remainder term $R_N(x_1, x_2)$ can be made arbitrarily smooth by taking N sufficiently large. Since $r^{\rho+j} f_j(\mathrm{atan}(x_2/x_1))$ is homogeneous of degree $\rho + j$, our theory applies to each such term. All that is needed is the knowledge of the degree of homogeneity of r_ρ. We do not require, as in [KiScWe92] for example, the explicit determination of the functions $f_j(\theta)$ in (9.1). It follows that the expansions we have derived equally apply to finite-part integrals defined on curved quadrangles.

10 Numerical examples

Up to this point, this paper has been concerned exclusively with obtaining the proper error functional expansion for various cubature problems. In applications it is vital to use the appropriate expansion. But the choices of cubature rule and of mesh ratio sequence are important. Besides economy in cost, considerations concerning noise amplification and termination criteria may be critical. In the context of two-dimensional regular integrals, some of these problems are discussed in [Ly76a]. The examples in this section are not addressed to these issues. They should be taken only as illustrations of how extrapolation can be made to work in simple cases.

In these examples we integrate over $(0, 1)^2$ using the product mid point rule ($\beta_1 = \beta_2 = 1/2$) and a sequence of mesh ratios $m_1, m_2, m_3, ..., m_k$. The cost, in terms of function values, does not exceed the quantity $\sum \nu = m_1^2 + m_2^2 + \cdots + m_k^2$, where ν denotes m_i^2.

The numerical entries in the tables have been rounded. The noise level in function values used in these calculations is roughly 10^{-14}. We have noted in the tables the condition number associated with the final entry of each column.

Example 1 $f(x_1, x_2) = (x_1 x_2)^{-3/2}$

We used the mesh ratio sequence $m_j = 4, 7, 10, 13, ..., 28$ and did the problem twice, using two different expansions. The first is the one provided by direct application of the *umbrella* Theorem 6.1, namely

$$Q^{(m)} f \sim A_{-1} m + A_{-1/2} m^{1/2} + A_0 + \sum_{j=1} \frac{A_{j/2}}{m^{j/2}}. \tag{10.1}$$

This sequence would be required if treating a function $(x_1, x_2)^{-3/2} g(x_1, x_2)$. In our special case, $g(x_1, x_2) = 1$ so many coefficients vanish, leaving

$$Q^{(m)} f \sim A_{-1} m + A_{-1/2} m^{1/2} + A_0 + \sum_{k=1} \left(\frac{A_{2k-1/2}}{m^{2k-1/2}} + \frac{A_{2k}}{m^{2k}} \right). \tag{10.2}$$

The results, based on the same input, but using this expansion, are also shown. The fourth column of the Table 1 shows the data on which the results are based,

Table 1. Numerical results for Example 1

m	ν	$\sum \nu$	$Q^{(m)} f$	$A_0(9.1)$	$A_0(9.2)$
4	16	16	57.10		
7	49	65	113.18		
10	100	165	171.75	0.4279947E+01	0.4279947E+01
13	169	334	231.72	0.3643213E+01	0.4027182E+01
16	256	590	292.62	0.4048171E+01	0.3998226E+01
19	361	951	354.21	0.4062080E+01	0.3999871E+01
22	484	1435	416.32	0.3991173E+01	0.4000007E+01
25	629	2064	478.85	0.4000613E+01	0.4000001E+01
28	784	2848	541.73	0.4000711E+01	0.4000000E+01
Condition number of final entry				5.3E+05	3.6E+07

namely, $Q^{(m)} f$ for various values of m. The fifth gives the result of extrapolation using (10.1) and the final column the corresponding results using (10.2). In carrying out these calculations, the linear equation solver obtained approximations to

$$A_{-1} = (\zeta(3/2,\ 1/2))^2 \text{ and } A_{-1/2} = 4\zeta(3/2,\ 1/2). \tag{10.3}$$

Each of these appeared to be consistently more accurate than the corresponding approximation to $A_0 = If = 4$ by roughly four and two decimal places, respectively.

Example 2 $f(x_1, x_2) = r^{-2}$

This is not a generic case. Reference to Section 8 indicates that the proper expansion is

$$Q^{(m)} f \sim C_0 \log m + A_0 + \sum_{j=1} A_j/m^j.$$

Here $C_0 = \pi/2$ for all β_1, β_2, but A_0 depends on β_1 and β_2 and we know of no simple relation connecting A_0 with the Hadamard integral If. Nevertheless, as illustrated in Table 2, extrapolation provides an unambiguous result for A_0. There is no warning in the course of the calculation, that this is not the result being sought.

We remark that the same situation prevails in the nongeneric case $f(x_1, x_2) = r^{-3} x_1$.

Example 3 $f(x_1, x_2) = r^{-3}$

This happens to be a generic case. The expansion is

$$Q^{(m)} f \sim A_{-1} m + A_0 + \sum_{j=1} A_j/m^j.$$

Here,

$$A_0 = If = -\sqrt{2}.$$

Again, using extrapolation, one obtains excellent approximations to A_0. However, this result is of virtually no practical value. If the integrand is modified to $r^{-3}g(x_1, x_2)$, then it becomes a nongeneric case. We have remarked above, near the end of Example 2, that in the case $r^{-3}x_1$, extrapolation gives a result that does not coincide with If. It follows that extrapolation could be used generally for a function $r^{-3}g(x_1, x_2)$ only if $g^{(1,0)}(0, 0) = g^{(0,1)}(0, 0) = 0$.

Table 2. Numerical Results for Examples 2 and 3

			Example 2		Example 3	
m	ν	$\sum \nu$	$Q^{(m)}f$	A_0	$Q^{(m)}f$	A_0
1	1	1	2.00		2.83	
2	4	5	3.02	2.000000	6.88	−1.221442
3	9	14	3.65	1.916538	10.99	−1.397090
4	16	30	4.09	1.908571	15.11	−1.412839
5	25	55	4.44	1.907900	19.24	−1.414188
6	36	91	4.73	1.907868	23.37	−1.414208
7	49	140	4.97	1.907866	27.50	−1.414213
Condition Number of Final Entry				9.35×10^2		3.61×10^2

11 Concluding remarks

We are interested in integration over $[0, 1)^2$ and $[0, \infty)^2$. We have treated integrand functions having a full corner singularity. These are of the form

$$f(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2} r_\rho(x_1, x_2) g(x_1, x_2), \tag{11.1}$$

where r_ρ is homogeneous of degree ρ (see (3.21)) and has no singularity in $[0, 1)^2$ other than at $(0, 0)$, where $g(x_1, x_2)$ is $C^{(\infty)}[0, \infty)^2$ and where f is acceptable, that is, its decay rate for large x_1, x_2 is sufficient for the integral to converge there.

The overall result is this. For all values of the parameters α_1, α_2 and ρ there exists an asymptotic expansion of the offset trapezoidal rule of the form

$$S^{(m)}f \sim \sum_{i=0} (A_{\gamma_i} + C_{\gamma_i} \log m + D_{\gamma_i} (\log m)^2)/m^{\gamma_i}. \tag{11.2}$$

Here, the elements γ_i are distinct; only a finite number are nonpositive. For convenience we take γ_i in increasing order. In cases in which the integral converges (that is, $\alpha_1 + \alpha_2 + \rho > -2$), this result can be gleaned from several papers [Ly76], [LydD93], [VeHa93]. In this case $\gamma_i \geq 0$, $A_0 = If$, and $D_0 = C_0 = 0$.

The focus of our investigation has been on cases in which the integral does not converge. In the development of the theory, it became necessary to evaluate the residues at the poles of a function that depends on the parameters. In the cases in which all poles are simple, we have termed the set of parameters *generic*.

In generic cases, $C_{\gamma_i} = D_{\gamma_i} = 0$ for all γ_i, and $A_0 = If$, where If is the Hadamard finite part integral. The expansion reduces to

$$S^{(m)}f \sim \sum_{\gamma_i<0} A_{\gamma_i}/m^{\gamma_i} + If + \sum_{\gamma_i>0} A_{\gamma_i}/m^{\gamma_i}. \tag{11.3}$$

the required values of γ_i being given in Theorem 5.2. The first summation is finite, including only the negative values of γ (that is, positive powers of m). In the extrapolation context in the hypersingular cases, one seeks the constant term If, which is *not* the leading term. A list of conditions on the parameters that ensure a generic case is given in Theorem 5.1; however, these are only sufficient conditions. A generic case may occur, even if some of these conditions are violated. It is quite permissible to treat, in the first instance, a generic case as if it were nongeneric. One simply lengthens the calculation by introducing additional terms unnecessarily.

In the nongeneric cases, in which coefficients C_0 and D_0 exist, the integral is not given by A_0. In the corresponding one-dimensional case [Ly94], it is possible to extract If from the values of A_0 and C_0. In the two-dimensional case, expressions for A_0, C_0, and D_0 are much more complicated. At present, we have no evidence to the effect that one can extract If from these. This is under investigation.

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